Continuity of Lyapunov exponents for non-uniformly fiber-bunched linear cocycles

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Linear Cocycle

Let

M compact metric space

$f\colon M \to M$ continuous,

$F\colon \mathcal{E} \to \mathcal{E}$ a vector bundle automorphism

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{F} & \mathcal{E} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{f} & M \end{array}$$

$F_x\colon \mathcal{E}_x \to \mathcal{E}_{f(x)}$ is a linear isomorphism.

Examples

1) M be parallelizable manifold and f a local diffeomorphism.

$$F : TM \to TM$$

$$F(x, v) = (f(x), Df(x)v).$$

Examples

2)Let

$$F\colon \{0,1\}^{\mathbb{Z}} \times \mathbb{R}^2 \mapsto \{0,1\}^{\mathbb{Z}} \times \mathbb{R}^2 \quad F(x,v) = (f(x), A(x)v),$$

$f\colon \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ be the full shift.

$$A(x) = \begin{cases} \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 3 \end{pmatrix} \text{ if } x_0 = 0 \\ \\ \begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \text{ if } x_0 = 1 \end{cases}$$

Lyapunov exponents

$$F\colon M\times\mathbb{R}^2\to M\times\mathbb{R}^2 \quad F^n(x,v)=(f^n(x),A^n(x)v)$$

$$A^n(x)=A(f^{n-1}(x))\dots A(x).$$

By Furstenberg-Kesten, for μ ergodic measure $f-$invariant

$$\lambda_+(A)=\lim_{n\to\infty}\frac{1}{n}\log\|A^n(x)\|$$ is upper semi-continuous

$$\lambda_-(A)=\lim_{n\to\infty}\frac{1}{n}\log\|(A^n(x))^{-1}\|^{-1}$$ is lower semi-continuous

for μ-a.e are called Lyapunov exponents.

Examples

2)Let

$$f\colon \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}} \text{ be the full shift.}$$

$$A(x) = \begin{cases} \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 3 \end{pmatrix} \text{ if } x_0 = 0 \\ \\ \begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \text{ if } x_0 = 1 \end{cases}$$

Take the measure

$$\mu = \left(p_0 \delta_{\{x_0=0\}} + p_1 \delta_{\{x_0=1\}}\right)^{\mathbb{Z}} \text{ with } p_1 \neq p_0 \text{ and } p_1 + p_0 = 1.$$

$$\lambda_+(A) = |p_0 - p_1| \log(3)$$

We use cocycles in $SL(2)$ because is satisfied $\lambda_+(A) + \lambda_-(A) = 0$. Denote

$$\mathcal{S}_0(M, 2) = \{A\colon M \mapsto SL(2) \text{ continuous map}\}.$$

We are interested in the continuity points for the map

$$\lambda_+ \colon \mathcal{S}_0(M, 2) \mapsto \mathbb{R}$$

$$A \mapsto \lambda_+(A)$$

and which is going to depend on the topology on $\mathcal{S}_0(M, 2)$.

History of the problem

Theorem - Mañé-Bochi [2002]

If f is an homeomorphism and μ is aperiodic. Then A is a continuity point in the C^0 topology if and only if

a) A is uniformly hyperbolic,

b) $\lambda_+(A) = 0 = \lambda_-(A)$.

History of the problem

Recently, Viana and Yang (2016) prove that when the base is non-invertible Mañé-Bochi is not true: there exists a C^0 continuity point with $\lambda_+(A) > 0$ and non-uniformly hyperbolic.

Example

3)Let

$$f\colon S^1 \to S^1 \text{ as } f(x) = 3x$$

and

$$A\colon S^1 \to SL(2,\mathbb{R}) \text{ as } A(x) = \begin{pmatrix} \frac{1}{\gamma} & 0 \\ 0 & \gamma \end{pmatrix} R_x$$

-Not uniformly hyperbolic.
-Positive Lyapunov exponents.
-C^0 continuity point.

Example

3) Let $f\colon S^1 \to S^1$ as $f(x) = 3x$ and

$$A\colon S^1 \to SL(2,\mathbb{R}) \text{ as } A(x) = \begin{pmatrix} \frac{1}{\gamma} & 0 \\ 0 & \gamma \end{pmatrix} R_x$$

Define the natural extension

$$\hat{M} = \{\hat{x} = (\dots, f^{-2}(x), f^{-1}(x), x)\},$$

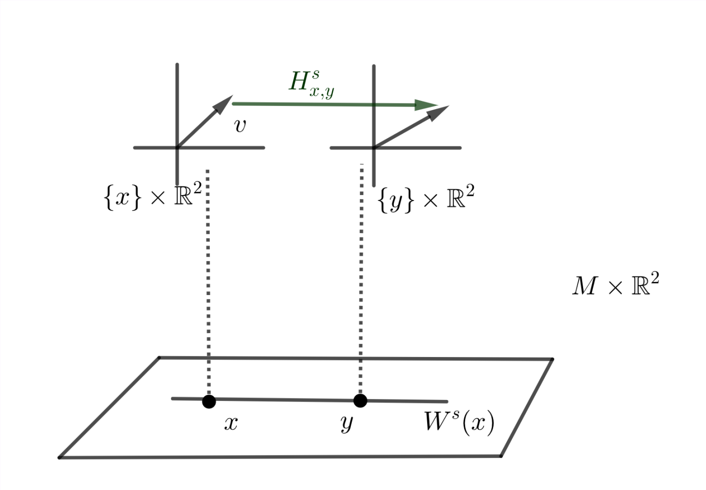
$$\hat{f}\colon \hat{M} \to \hat{M}, \quad \hat{f}(\hat{x}) = (\dots, f^{-1}(x), x, f(x)),$$

$$\hat{A}(\hat{x}) = A(x).$$

In $W^s(\hat{f}, \hat{x}) = \{\hat{y} \in \hat{M}, x = y\}$, $\hat{A}(\hat{x}) = A(x) = A(y) = \hat{A}(\hat{y})$.

History of the problem

$f : M \to M$ sub-shift



History of the problem

$f\colon M \to M$ a sub-shift

Definition

An *uniform stable holonomy* for A over f are isomorphisms $H^s_{x,y}\colon \mathbb{R}^2 \to \mathbb{R}^2$

- $H^s_{y,z} \circ H^s_{x,y} = H^s_{x,z}$ e $H^s_{x,x} = Id$;
- $H^s_{x,y} = (A(y))^{-1} \circ H^s_{f(x),f(y)} \circ A(x)$;
- $(x, y, \xi) \mapsto H^s_{x,y}(\xi)$ is continuous.

Example

3) Let $f\colon S^1 \to S^1$ as $f(x) = 3x$ and

$$A\colon S^1 \to SL(2,\mathbb{R}) \text{ as } A(x) = \begin{pmatrix} \frac{1}{\gamma} & 0 \\ 0 & \gamma \end{pmatrix} R_x$$

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$$H^s = Id$$

History of the problem

We fix

$$f : \Sigma \to \Sigma \text{ sub-shift of finite type,}$$

$$A \colon \Sigma \to SL(2),$$

μ an ergodic f-invariant measure fully supported with local product structure.

History of the problem

Theorem (Backes, Brown e Butler, 2018)

If $(A_k, H^{s,k}, H^{u,k}) \xrightarrow{C^0} (A, H^s, H^u)$, then $\lambda_+(A_k) \to \lambda_+(A)$.

Conjecture (M.Viana)

If $(A_k, H^{s,k}) \xrightarrow{C^0} (A, H^s)$, then $\lambda_+(A_k) \to \lambda_+(A)$.

Results

For σ expansion rate of f, $\lambda_+(A) < \frac{\log \sigma}{2}$ is called non-uniformly fiber-bunched.

Theorem A (.-Marin)

If

$$A_k \xrightarrow{Lip} A \text{ and } H^{s,k} \xrightarrow{C^0} H^s$$

then

$$\lambda_+(A_k) \to \lambda_+(A)$$

2) Let

$$f:\{0,1\}^{\mathbb{Z}}\to\{0,1\}^{\mathbb{Z}} \text{ be the full shift.}$$

$$A(x)=\begin{cases}\begin{pmatrix}\frac{1}{3} & 0\\ 0 & 3\end{pmatrix} & \text{if } x_0=0\\ \\ \begin{pmatrix}3 & 0\\ 0 & \frac{1}{3}\end{pmatrix} & \text{if } x_0=1\end{cases}$$

$$\lambda_+(A)=|p_0-p_1|\log(3)<\frac{\log\sigma}{2}$$

Bocker- Viana proved that it can be Lipschitz approximated by a cocycle of Lyapunov exponent zero, thus, is a discontinuity point for λ_+.

Results

A locally constant cocycle A is *irreducible* if there is no proper subspace of $\mathbb{R}^2$ invariant by $A(x)$ for μ-a.e.p.

Theorem B (.-Marin)

Let A be irreducible locally constant and non-uniformly fiber-bunched. If $A_k \xrightarrow{Lip} A$, then $\lambda_+(A_k) \to \lambda_+(A)$

Example

4) Let $f\colon \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ the shift, $\mu = \left(\frac{1}{2}\delta_{\{0\}} + \frac{1}{2}\delta_{\{1\}}\right)^{\mathbb{Z}}$.

$$A(x) = \begin{cases} \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 3 \end{pmatrix} \text{ if } x_0 = 0 \\ \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ if } x_0 = 1 \end{cases}$$

Thus $\lambda_+(A) = 0$ is a continuity point. Take $B_n \to A$ as

$$B_n(x) = \begin{cases} \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 3 \end{pmatrix} \text{ if } x_0 = 0 \\ \\ \begin{pmatrix} \cos\theta_n & \sin\theta_n \\ -\sin\theta_n & \cos\theta_n \end{pmatrix} \text{ if } x_0 = 1 \end{cases}$$

with $\theta_n \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $\theta_n \to \frac{\pi}{2}$.

By Thm. B, B_n is a continuity point for the Lyapunov exponents.

¡Muchas gracias!
Muito obrigada!
Thanks you very much!!