Ruelle-Taylor spectrum for Anosov actions

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Introduction

Let $A : D(A) \subset L^2(M) \to L^2(M)$ be an elliptic self-adjoint differential operator on M compact manifold (ex: Dirac op, Laplacian)

The spectrum is discrete (compact Sobolev embedding):

$$\mathrm{Sp}(A) = \{\cdots \leq \lambda_{-n} \cdots \leq \lambda_1 \leq \lambda_2 \leq \ldots \lambda_n \leq \ldots\} \subset \mathbf{R}$$

with a basis of eigenvectors ψ_j.

Eigenvectors/eigenvalues of Laplacian are related to:

- Functional calculus: for $F \in L^\infty(\mathbf{R})$, $u \in L^2$

$$F(A)u = \sum_j F(\lambda_j)\langle u, \psi_j\rangle \psi_j$$

  with ψ_j eigenvectors. Ex: $F(x) = e^{itx}, \cos(tx) \longrightarrow$ long time dynamics.
- Geometric invariants: heat trace invariants (curvature), Weyl asymptotics (volume)
- Topological invariants: Hodge theory, analytic torsion
- Trace formula and dynamical data (length of closed orbits): Selberg trace formula, Colin de Verdière/Duistermaat-Guillemin
- Quantum ergodicity: equidistributions of ψ_j.

Commuting family

If $A = (A_1, \dots, A_\kappa)$ is a family of commuting differential operators on M compact, with one of them being elliptic, one can define a discrete joint spectrum:

$$\lambda = (\lambda_1, \dots, \lambda_\kappa) \in \mathrm{Spec}(A) \subset \mathbf{R}^\kappa \iff \exists \psi_\lambda \in L^2, \ \forall i = 1, \dots, \kappa, \quad A_i \psi_\lambda = \lambda_i \psi_\lambda$$

Functional calculus: for $F \in L^\infty(\mathbf{R}^\kappa), u \in L^2$,

$$F(A_1, \dots, A_\kappa)u = \sum_{\lambda \in \mathrm{Spec}(A)} F(\lambda_1, \dots, \lambda_\kappa) \langle u, \psi_\lambda \rangle \psi_\lambda$$

Anosov Flows

X a smooth vector field with flow φ_t, and assume that there is $d\varphi_t$ invariant splitting

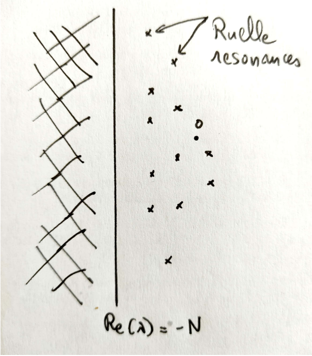
$$TM = \mathbf{R}X \oplus E_u \oplus E_s$$

where, $\exists \nu > 0$, $\|d\varphi_t|_{E_s}\| \leq Ce^{-\nu t}$ for $t > 0$ and $\|d\varphi_t|_{E_u}\| \leq Ce^{-\nu|t|}$ for $t < 0$.

X is **NOT** elliptic and has non-discrete spectrum on L^2 (in general).

Ruelle resonances

There is a way to define natural discrete spectrum for X:
Ruelle, Pollicott, Butterley, Liverani, Baladi, Gouezel, Faure, Sjöstrand, Dyatlov, Zworski...:
There are family of Hilbert spaces $\mathcal{H}_N$ (for $N > 0$) such that $-X$ has only discrete spectrum with finite multiplicity in $\{\lambda \in \mathbf{C}; \mathrm{Re}(\lambda) > -N\}$. Note: iX **NOT** self-adjoint on $\mathcal{H}_N$



Wavefront

To get discrete spectrum in $\mathbf{C}$, it is convenient to define

$$C^{-\infty}_{E_u^*}(M) := \{f \in C^{-\infty}(M); \mathrm{WF}(f) \subset E_u^*\}$$

where $E_u^* \subset T^*M$ is the subbundle defined by

$$E_u^*(E_u \oplus \mathbf{R}X) = 0.$$

The wavefront set $\mathrm{WF}(f)$ describes singularities of the distribution f (decay of Fourier transform):

Ex: let V smooth vector field non-vanishing at $x \in M$, and assume $V^n f \in L^2$ near x for all $n \geq 0$, then $(x, \xi) \notin \mathrm{WF}(f)$ for all $\xi(V) \neq 0$

Roughly speaking: $f \in C^{-\infty}_{E_u^*}(M)$ means that f is smooth along weak-unstable derivatives.

Then, one can show for an Anosov flow:

$$\boxed{\lambda \in \mathbf{C} \text{ Ruelle resonance } \iff \exists f \in C^{-\infty}_{E_u^*}(M), \quad Xf = -\lambda f}$$

ie. Ruelle resonances= spectrum of $-X$ in $C^{-\infty}_{E_u^*}(M)$, $\mathcal{H}_N$ approximates $C^{-\infty}_{E_u^*}(M)$ as $N \to \infty$.

Same applies to the Lie derivative $\mathcal{L}_X = \iota_X d + d\iota_X$ acting on differential $k-$forms, we call Ruelle resonances on k-forms the spectrum of $\mathcal{L}_X$ on $C^{-\infty}_{E_u^*}(M; \Lambda^k T^*M)$

Applications of the theory

- Butterley-Liverani: SRB measures are the eigen(co)vectors at $\lambda = 0$ and there is no Ruelle resonance for X in $\mathrm{Re}(\lambda) > 0$.
- Dolgopyat, Liverani, Tsujii, Faure-Tsujii, Nonnenmacher-Zworski: contact Anosov flows have a spectral gap, resonances $\mathrm{Res}(X) \subset \{0\} \cup \{\mathrm{Re}(\lambda) > -\varepsilon\}$. Exponential mixing.
- Giulietti-Liverani-Pollicott, Dyatlov-Zworski, Dyatlov-G: (twisted) Ruelle zeta function $\zeta_\rho(\lambda) = \prod_\gamma \det(1 - \rho(\gamma)e^{-\lambda\ell(\gamma)})$ admits meromorphic extension to $\mathbb{C}$, zeros/poles are the resonances on (twisted) forms
- Dyatlov-Zworski, Dang-G-Riviere-Shen: in dim 3, multiplicities of Ruelle resonances for $\mathcal{L}_X$ on forms are Betti numbers, and $\zeta_\rho(0)$ is Reidemeister torsion if $\rho : \pi_1(M) \to U(m)$ unitary representation – Fried conjecture.

Anosov $\mathbf{R}^\kappa$ actions

$\mathbf{A} \simeq \mathbf{R}^\kappa$ Abelian group, $\tau : \mathbf{A} \to \mathrm{Diffeo}(M)$ locally free action. Let

$$X : \mathbf{a} \to C^\infty(M; TM), \quad X_A := \partial_t|_{t=0}(\tau(e^{tA})).$$

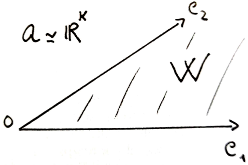
If $A_1, \dots, A_\kappa$ basis of $\mathbf{a} = \mathrm{lie}(\mathbf{A})$, one has $[X_{A_j}, X_{A_k}] = 0$.
Anosov: there exists a transversely hyperbolic element $A_1 \in \mathbf{a}$: ie. there is $d\varphi_t^{X_{A_1}}$ invariant splitting

$$TM = E_0 \oplus E_s \oplus E_u$$

with $E_0 = \mathrm{span}(X_{A_1}, \dots, X_{A_\kappa})$, and $\|d\varphi_t^{X_{A_1}}|_{E_s}\| \leq Ce^{-\nu t}$ for $t > 0$ and $\|d\varphi_t^{X_{A_1}}|_{E_u}\| \leq Ce^{-\nu |t|}$ for $t < 0$.
Define the positive Weyl chamber (is convex open cone)

$$\mathcal{W} := \{A \in \mathbf{a}; X_A \text{ transversely hyp with same splitting as } X_{A_1}\}$$

Examples, rigidity



Main examples: Weyl chamber flows on $\Gamma\backslash G/M$, $G = KAN$ with A of rank $\kappa > 1$, $\Gamma \subset G$ co-compact, $M = \mathrm{stab}_K(A)$

Conjecture (Katok-Spatzier): such actions (without rank-1 factor) are smoothly conjugate to Weyl chamber flows and variations of those. Proved locally near algebraic cases. See Vinhage's talk.

Joint spectrum

Question: $\exists$ discrete natural joint spectrum for the family of vector fields $X_{A_1}, \dots, X_{A_\kappa}$?

Definition: say that $\lambda \in \mathbf{a}_{\mathbf{C}}^* \simeq \mathbf{C}^\kappa$ is a joint Ruelle resonance if there is $f \in C^{-\infty}_{E_u^*}(M)$ non-zero such that for all $A \in \mathbf{a}$

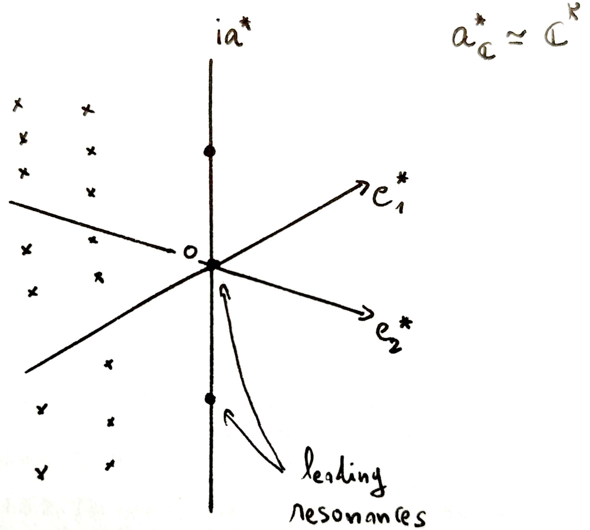
$$X_A f = -\lambda(A) f.$$

Theorem (Guedes Bonthonneau-G-Hilgert-Weich 2020)

If τ is an Abelian Anosov action, the set of joint Ruelle resonances is discrete, with finite multiplicity and contained in

$$\bigcap_{A \in \mathcal{W}} \{\lambda \in \mathbf{a}_{\mathbf{C}}^* ; \mathrm{Re}(\lambda(A)) \leq 0\}.$$

Rem: the proof strongly uses the notion of Taylor joint spectrum and Koszul complexes.


ia^*
$a^*_{\mathbb{C}} \simeq \mathbb{C}^\kappa$
e_1^*
e_2^*
0
leading
resonances

Taylor spectrum of X

Define an exterior derivative in the direction E_0 as follows:

$$d_X : C^\infty(M)\otimes \Lambda^k \mathbf{a}^* \to C^\infty(M)\otimes \Lambda^{k+1}\mathbf{a}^*$$

$$(d_X u)(A) := X_A u, \quad , d_X(u\otimes \omega) := (d_X u)\wedge \omega$$

$$\Big(\text{ or equivalently} \quad d_X(u\, e^*_{i_1}\wedge \dots e^*_{i_k}) = \sum_{j=1}^k (X_{A_j}u)e^*_j \wedge e^*_{i_1}\wedge \dots e^*_{i_k}\Big).$$

if $u\in C^\infty(M)$, $\omega\in \Lambda^k\mathbf{a}^*$ and $(e^*_j)_j$ basis of $\mathbf{a}^*_{\mathbf{C}}$ dual to $(A_j)_j$. It satisfies

$$\boxed{d_X\circ d_X = 0}$$

Same, for $\lambda\in \mathbf{a}^*_{\mathbf{C}}$, we define $d_{(X-\lambda)}$, satisfies $d_{(X-\lambda)}\circ d_{(X-\lambda)} = 0$

Definition: λ is **not** in Taylor joint spectrum of X on functional space $\mathcal{H}$ iff for each $k=0,\dots,\kappa$, the cohomology of Taylor complex is trivial

$$\ker d_{(X-\lambda)}|_{\mathcal{H}\otimes\Lambda^k\mathbf{a}^*_{\mathbf{C}}} = \mathrm{Im} d_{(X-\lambda)}|_{\mathcal{H}\otimes\Lambda^{k-1}\mathbf{a}^*_{\mathbf{C}}}$$

Taylor spectrum

Remark: the cohomology of degree 0 is non-trivial for λ iff $\ker d_{X-\lambda} \neq 0$, ie. there is $u \in \mathcal{H}$ so that

$$(X_{A_1} - \lambda(A_1))u = 0, \quad \dots, \quad (X_{A_\kappa} - \lambda(A_\kappa))u = 0$$

ie. usual notion of joint spectrum.

The cohomology of degree κ is trivial if for each $f \in \mathcal{H}$, there is $u_1, \dots, u_\kappa \in \mathcal{H}$ st

$$(X_{A_1} - \lambda(A_1))u_1 + \cdots + (X_{A_\kappa} - \lambda(A_\kappa))u_\kappa = f.$$

Remark: In finite dimension, this notion of joint spectrum is equivalent to the usual one.

But it has deeper content in infinite dimension (functional calculus).

We say that λ is **not** in essential Taylor spectrum if the cohomologies

$$\ker d_{(X-\lambda)}|_{\mathcal{H}\otimes\Lambda^k \mathbf{a}^*_{\mathbf{C}}} / \mathrm{Im} d_{(X-\lambda)}|_{\mathcal{H}\otimes\Lambda^{k-1}\mathbf{a}^*_{\mathbf{C}}}$$

are finite dimensional.

V. Muller: The Taylor spectrum, in region where it is not essential, is an analytic submanifold of $\mathbf{C}^k$.
J. Taylor: there is a functional calculus for commuting families $A_1, \ldots A_k$ of bounded operators on Hilbert space $\mathcal{H}$.

Definition: $\lambda \in \mathbf{a}^*_{\mathbf{C}}$ is a Ruelle-Taylor resonance if there is k

$$\ker d_{(X+\lambda)}|_{C^{-\infty}_{E^*_u}(M)\otimes\Lambda^k \mathbf{a}^*_{\mathbf{C}}} \neq \mathrm{Im} d_{(X+\lambda)}|_{C^{-\infty}_{E^*_u}(M)\otimes\Lambda^{k-1}\mathbf{a}^*_{\mathbf{C}}}$$

Note: Ruelle joint resonances $\subset$ Ruelle-Taylor resonances (a priori)

Theorem (Bonthonneau-G-Hilgert-Weich 2020)

The set of joint Ruelle-Taylor resonances for an Anosov abelian action is discrete and equal to the set of Ruelle joint resonances. More precisely, for each $N > 0$ there are Hilbert spaces $\mathcal{H}_N$ so that the Taylor spectrum of X on $\mathcal{H}_N$ is discrete outside

$$\bigcap_{A \in \mathcal{W}} \{\lambda \in \mathfrak{a}^*_{\mathbb{C}}; \mathrm{Re}(\lambda(A)) \leq -N\}$$

and these discrete joint eigenvalues are the Ruelle-Taylor resonances.

The resonance $\lambda = 0$ - Physical measure

Definition: μ is called physical measure if there is a smooth Lebesgue type measure ω s.t. for any proper open a cone $\mathcal{C} \subset \mathcal{W}$

$$\mu(f) = \lim_{T\to\infty} \frac{1}{\mathrm{Vol}(\mathcal{C}_T)} \int_{\mathcal{C}_T} \int_M f(\varphi_1^{-X_A}(x)) d\omega(x) dA$$

for all $f \in C^0(M)$, where

$$\mathcal{C}_T := \{A \in \mathcal{C}; |A| \leq T\}.$$

ie. μ is the weak Cesaro limit (Birkhoff average) of a Lebesgue measure.

Remark: For Anosov flows, physical measures are SRB measures.

Theorem (Bonthonneau-G-Hilgert-Weich 2020)

- *If μ is a smooth invariant measure with full support, then*

$$\dim\ker d_X|_{C^{-\infty}_{E_u^*}(M)\otimes\Lambda^j\mathbf{a}^*_{\mathbb{C}}}/\mathrm{Im} d_X|_{C^{-\infty}_{E_u^*}(M)\otimes\Lambda^{j-1}\mathbf{a}^*_{\mathbb{C}}} = \binom{\kappa}{j}$$

- *The space of physical measures has dimension* $\dim\ker d_X|_{C^{-\infty}_{E_u^*}(M)} \geq 1$, *the dimension of joint Ruelle resonant states at* $\lambda = 0$.
- *The physical measures are the invariant measures* μ *with* $\mathrm{WF}(\mu) \subset E_s^*$.
- *Assume there is a unique physical measure* μ *then the following are equivalent:*
  1) the only joint Ruelle resonance on $i\mathbf{a}^*$ *is* 0
  2) there is $A \in \mathbf{a}$, *s.t.* $\varphi_t^{X_A}$ *is weak-mixing*
  3) for all $A \in \mathcal{W}$, $\varphi_t^{X_A}$ *is strong-mixing*

SRB measures

The physical measures are SRB measures:

there is $B \subset \mathcal{M}$ of positive Lebesgue measure such that for all $f \in C^0(M)$, all proper open subcones $\mathcal{C} \subset \mathcal{W}$ and all $x \in B$,

$$\mu(f) = \lim_{T \to \infty} \frac{1}{\mathrm{Vol}(\mathcal{C}_T)} \int_{A \in \mathcal{C}_T,} f(e^{-X_A}(x)) dA.$$

and their Rohlin diseintegrations along stable manifolds are absolutely continuous wrt Lebesgue.

In progress: μ should be obtained in terms of measures on periodic tori (Bowen type formula).

Applications in mind

- Classical/Quantum correspondance for locally symmetric spaces
- Exponential mixing under certain assumptions via spectral gap. Apply to local rigidity to more general classes (such as contact case for rank 1).
- Use SRB measures for rigidity problem.
- Counting periodic orbits/tori
- Trace formula, zeta functions ?

Ideas of proof

First step: For $A_0 \in \mathcal{W}$, construct parametrix $Q(\lambda) : C^\infty(M) \otimes \Lambda \mathbf{a}^* \to C^\infty(M) \otimes \Lambda \mathbf{a}^*$ so that

$$d_{(X+\lambda)} Q(\lambda) + Q(\lambda) d_{(X+\lambda)} = \mathrm{Id} + K(\lambda)$$

s.t. there is a Hilbert space $C^\infty(M) \subset \mathcal{H}_N \subset C^{-\infty}(M)$ with

$$Q(\lambda) : \mathcal{H}_N \otimes \Lambda \mathbf{a}^* \to \mathcal{H}_N \otimes \Lambda \mathbf{a}^* \text{ bounded}$$

$$K(\lambda) : \mathcal{H}_N \otimes \Lambda \mathbf{a}^* \to \mathcal{H}_N \otimes \Lambda \mathbf{a}^* \text{ compact}^1$$

when $\mathrm{Re}(\lambda(A_0)) > -N$.
Let Π_0 be the (finite rank) spectral projector at 0 of $\mathrm{Id} + K(\lambda)$. Then

$$u \mapsto \Pi_0 u$$

factors to an isomorphism

$$\ker d_{(X+\lambda)}|_{\mathcal{H}_N \otimes \Lambda \mathbf{a}^*} / \mathrm{Im}\, d_{(X+\lambda)}|_{\mathcal{H}_N \otimes \Lambda \mathbf{a}^*} \to \ker d_{(X+\lambda)}|_{\mathrm{Im} \Pi_0} / \mathrm{Im}\, d_{(X+\lambda)}|_{\mathrm{Im} \Pi_0}$$

[1] or more generally if $r_{\mathrm{ess}}(K(\lambda)) < 1$

Conclusion: all λ with $\mathrm{Re}(\lambda(A_0)) > -N\}$ is **not** in the essential Taylor spectrum.

Second step: Assume $Q_1, \dots Q_\kappa$ are continuous operators on $C^\infty(M)$, commuting one to each other and commuting with X_{A_j}. Define the divergence

$$\delta_Q(u\, e^*_{i_1} \wedge \dots \wedge e^*_{i_k}) = \sum_{j=1}^{k} (-1)^j (Q_{i_j} u) e^*_{i_1} \wedge \dots \wedge \hat{e^*_{i_j}} \wedge \dots e^*_{i_k}$$

for $u \in C^\infty(M)$, e^*_i fixed basis of $\mathbf{a}^*$. Then

$$d_{(X+\lambda)}\delta_Q + \delta_Q d_{(X+\lambda)} = -\sum_{j=1}^{\kappa} ((X_{A_j} + \lambda_j)Q_j) \otimes \mathrm{Id}.$$

Proposition

If the remainder $-\sum_{j=1}^{\kappa}((X_{A_j} + \lambda_j)Q_j) = 1 + K(\lambda)$ for some $K(\lambda)$ compact on $\mathcal{H}_N$ near $\lambda = \lambda_0$, then the Taylor spectrum is discrete near λ_0.

Let $A_j \subset \mathcal{W}$ basis of **a** and $\chi \in C_c^\infty(\mathbf{R})$ equal to 1 near 0. Set

$$Q_j'(\lambda) := \int_0^\infty \chi(t) e^{-t(X_{A_j}+\lambda_j)} dt, \quad R_j(\lambda) := \int_0^\infty \chi'(t) e^{-t(X_{A_j}+\lambda_j)} dt,$$

cummuting with each other and with X_{A_k}. Then set

$$Q_j(\lambda) := (-1)^j Q_j'(\lambda) \prod_{k=1}^{j-1} R_j(\lambda).$$

We get, with $Q(\lambda) = (Q_1(\lambda), \dots, Q_\kappa(\lambda))$

$$\boxed{d_{(X+\lambda)}\delta_{Q(\lambda)} + \delta_{Q(\lambda)} d_{(X+\lambda)} = (\mathrm{Id} - K(\lambda)) \otimes \mathrm{Id}}$$

where the remainder $K(\lambda)$ is quasi-compact on $\mathcal{H}_N$ in $\mathrm{Re}\lambda(A_0) > -N$ and given by

$$\boxed{K(\lambda)u = (-1)^\kappa \int_0^\infty \cdots \int_0^\infty (e^{\sum_{j=1}^\kappa t_j(X_{A_j}+\lambda_j)} u) \chi'(t_1) \dots \chi'(t_\kappa) dt_1 \dots dt_\kappa}$$

Quasi-compact means: the essential spectrum is contained in a disc of radius $r < 1$.

Where does compactness come from ?

1) smoothing in the directions X_{A_j} due to integration in t

2) smoothing in E_u^* and E_s^* directions due to the action of $e^{-tX_{A_j}}$ on the anisotropic Sobolev space $\mathcal{H}_N$. This space allows for: H^N regularity far from E_u^* and H^{-N} regularity in an open cone near E_u^*.

Physical measures: The spectral projector of $K(0)$ at eigenvalue 1 (which is the leading one) can be obtained as limit $K(0)^n$ as $n \to \infty$. This amounts to study the operator

$$K(0)^n = \prod_{j=1}^{\kappa} \int_{\mathbf{R}} e^{-t_j X_{A_j}} \psi^{*^n}(t_j) dt_j$$

as $n \to \infty$, where $\psi(x) = -\chi'(x)$ is a non-negative bump function supported near $x = 1$.

Merci!