Open Problem Session

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Problem

Does the entropy functional of an Anosov map on a torus with respect to its SRB measure have any non-trivial local extrema?

More details: Let $\mathcal{A}(\mathbb{T}^n)$ denote the family of all $C^r, r \geq 3$ Anosov maps on the tours $\mathbb{T}^n$ topologically conjugating to a linear transitive Anosov map L.

$\rho = \rho_f$: the SRB measure of $f \in \mathcal{A}(\mathbb{T}^n)$

$$h(f) = h_\rho(f) = \int \log J^u f \; d\rho.$$

($=$ sum of positive Lyapunov exponents of f), $J^u f > 1$, the Jacobian of f along the (unstable) expanding direction.

$h(f) : \mathcal{A}(\mathbb{T}^n) \to \mathbb{R}$ is a differentiable. Assume that $h(f) < h(L)$. Is it true that $\frac{d}{d\epsilon}\big|_{\epsilon=0} h(f + \epsilon g) \neq 0$ in some direction g?

Properties of the entropy functional

(Ruelle 1997) The entropy $h_\rho(f)$ is differentiable.

(Hu-J-Jiang 2008) $\inf_{\mathcal{A}(\mathbb{T}^n)} h_\rho(f) = 0$ (2015: for volume preserving Anosov systems)

Conjecture

$h_\rho(f)$ has no non-trivial critical values: the only critical value is a linear Anosov map on T^n where the global maximum is attained.

Connection to Statistical Mechanics

Gallavotti-Cohen's Chaotic Hypothesis (1995)

The motions of a chaotic system develop asymptotically on *an attracting set* on which dynamics can be regarded as a transitive hyperbolic ("Anosov") evolution.

Original statement:

A reversible many-particle system in a stationary state can be regarded as a transitive Anosov system for the purpose of computing the macroscopic properties of the system.

Gallavotti: Uniformly hyperbolic (Anosov) systems model thermodynamic systems both at equilibrium and not at equilibrium.

Second Law of Thermodynamics

Entropy Maximization Hypothesis:

A thermodynamic system not at equilibrium will evolve in the direction in which the entropy increases the fastest,i.e., the evolution of a system not at equilibrium will flow the gradient flow of the entropy.

In an isolated environment, as a thermodynamic system not at equilibrium evolves to its equilibrium, its entropy strictly increases.

Thus, under the entropy maximization hypothesis, the no-nontrivial critical value conjecture of the KS-entropy is equivalent to the second law of thermodynamics.

Known Cases

(1) (J-2019) Markov transformation on the unit interval; Expanding maps on the unit circle.

(2) 2d Volume preservation Anosov: (2019 Bochi-Katok-R. Hertz, 2020 Saghin-Valenzuela-Henrsìquez-Vasquez)