Measure rigidity for Anosov flows via the factorization method

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Introduction

Let M be a closed manifold. Let $g_t : M \to M$ be a smooth flow.

Definition
g_t is *Anosov* if for all $x \in M$, we have a continuous splitting

$$T_x M = E^s \oplus E^0 \oplus E^u$$

such that there exist $C > 0, \lambda < 1$ with

$$\|g_t|_{E^s}\| \leq C \cdot \lambda^t \text{ for all } t \geq 0$$
$$\|g_t|_{E^u}\| \leq C \cdot \lambda^{|t|} \text{ for all } t \leq 0.$$
$$\dim E^0 = 1.$$

Basic Examples

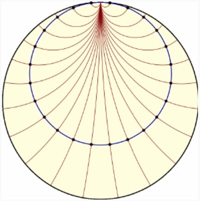
We define the unstable leaf W^u by $T(W^u) = E^u$. Similarly, W^s - the stable leaf is defined by $T(W^s) = E^s$.

Main Example

$M = T^1(\text{Riemann surface}) = SL_2(\mathbb{R})/\Gamma$ for $\Gamma \leq SL_2(R)$ is a *uniform* lattice.

Then the flow defined by the action of $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ is an Anosov flow.

Here W^u, W^s amount to horospheres.



Oseledets theorem

Assume now that M is endowed with a g_t-invariant and ergodic probability measure μ.

Theorem (Oseledets)
Assume V is a vector bundle over M, then for almost every $x \in M$, there exists a g_t-equivarient splitting of $V(x)$ and numbers $\lambda_i \in \mathbb{R}$ such that

$$V = \oplus_i V^{\lambda_i}$$

and for all $0 \neq v \in V^{\lambda_i}$ we have

$$\lim_{t\to\infty} \frac{1}{t} \log\|g_t.v\| = \lambda_i.$$

Oseledets splitting

Applying the theorem to the tangent bundle TM, we may refine the Anosov splitting to the Oseledets splitting:

$$E^s \oplus E^0 \oplus E^u = \oplus_i E^{\lambda_i}$$

with $E^s = \oplus_{\lambda_i<0} E^{\lambda_i}$, $E^u = \oplus_{\lambda_i>0} E^{\lambda_i}$ and E^0.

We order the exponents as $\ldots < \lambda_{-1} < \lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \ldots$.

Cont.

In general we have the backwards flag

$$0 \leq E^{\lambda_k} \leq E^{\lambda_k} \oplus E^{\lambda_{k-1}} \leq \ldots \leq \oplus_{i>\ell} E^{\lambda_i} \leq \ldots \leq TM,$$

where E^{λ_k} - most expanding subspace.

We may define distributions according to the splitting, which are leading to the fast-unstable leaf $T^1(W^{uu}_{loc}) = E^{\lambda_k}$ and the generalized fast-unstable $T^1(W^{>1}_{loc}) = \oplus_{i\geq 2} E^{\lambda_i}$ where $0 = \lambda_0 < \lambda_1$.

Basic examples - Cont.

Consider the group $\mathbf{ASL}_2 = \mathbf{SL}_2 \ltimes \mathbf{G}_a^2$. This group can be embedded into $\mathbf{SL}_3$ as $\begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix}$.
One may form the quotient space $Y = ASL_2(\mathbb{R})/ASL_2(\mathbb{Z})$.
This is a toral bundle over $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$.
Define $g_t = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{pmatrix} \leq ASL_2(\mathbb{R})$.
Then the flow induced over Y by g_t is Anosov as

$$g_t \cdot \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix} \cdot g_{-t} = \begin{pmatrix} a & e^{2t}b & e^t x \\ e^{-2t}c & d & e^{-t}y \\ 0 & 0 & 1 \end{pmatrix}$$

with $E^u = E^{\lambda_2} \oplus E^{\lambda_1}, \ E^s = E^{\lambda_{-1}} \oplus E^{\lambda_{-2}}$

u-Gibbs states and SRB measures

From now on, let μ be a g_t-invariant and ergodic probability measure, defined over M.

Definition

- The measure μ is called an *SRB measure* if its conditionals $\mu|_{W^u}$ are a.c.
- The measure μ is called a (generalized) *u-Gibbs state* if its conditionals $\mu|_{W^{uu}_{loc}}$ ($\mu|_{W^{>1}_{loc}}$) are a.c.

Sinai and Pesin ($\sim$ 82) gave a construction of u-Gibbs states by averaging densities over W^u. One may show that u-Gibbs states are weak-$\star$ closed and convex.

Measure classification

It is clear that every SRB measure is a u-Gibbs state.

Question - Is the other direction true?

Answer - In general, NO! (i.e. for ASL_2, consider the measure coming from the Haar measure on $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$[1]).

[1] or any orbit supported on the set torsion points of some CM-curve, c.f. Elkies-McMullen.

Measure classification

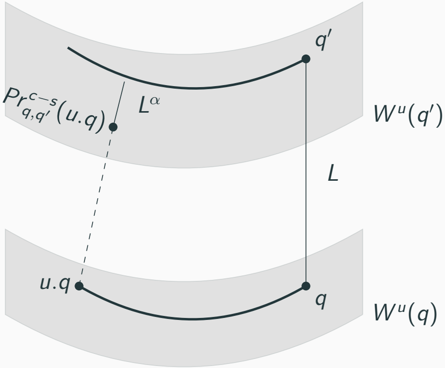
Theorem (K.)
Let (M, g_t, μ) *be an Anosov system. Assume that* μ *is a (generalized) u-Gibbs state, with simple least positive Lyapunov exponent, satisfying QNI, then* μ *is SRB.*

QNI - Quantitative Non-Integrability condition.

Morally - the measure is *not supported* over g_t-invariant embedded submanifolds. Such submanifolds form obvious obstructions to measure classification, due to Pesin-Sinai.

QNI

QNI - There exists a positive measure set of generic points q such that for many points $q' \in W^s(q)$ with $d(q, q') \approx L$, there are generic points $u.q \in W^{uu}_{loc}(q)$ with $d(q, u.q) \approx L$ such that the deviation of the central-stable projection from the fast unstable is polynomial in L.



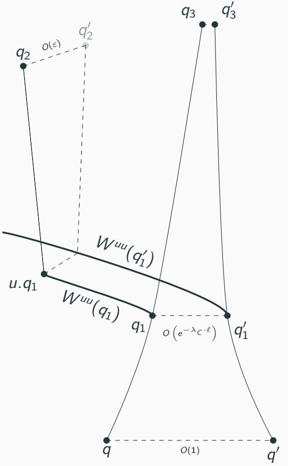
Standing assumptions

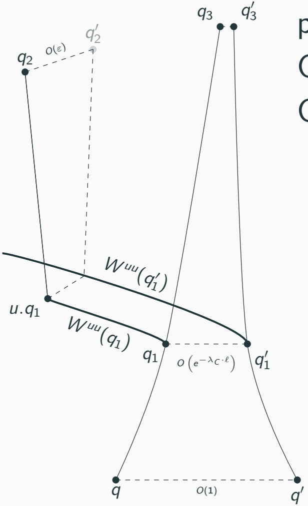
Assume from now (for simplicity) M is 4-dimensional, with the following Lyapunov spectrum

$$\lambda_{-} < 0 < \lambda_1 < \lambda_2.$$

We denote $W^{uu}_{loc} = W^{>1}_{loc}$ by $T(W^{uu}_{loc}) = E^{\lambda_2}$.

The Eskin-Mirzakhani Scheme





Fix some auxiliary $\epsilon > 0$. Assume that all points in this diagram are generic.
Consider the conditional measure $\mu\mid_{W^u}$.
Gives a function f_1 by $x \mapsto \mu_x^u$. We get:

$$f_1(q_2) = f_1(g_\ell.u.g_{-t}.q_3) = e^{\lambda_\star(u.q_1,\ell)} \cdot e^{-\lambda_\star(q_1,t)}.f_1(q_3).$$

Hence for suitable choice of ℓ, t, we get $f_1(q_3) = f_1(q_2)$ and similarly $f_1(q_3') = f_1(q_2')$.
But we have that $q_2 \approx u_1.q_2'$ with $dist(q_2, q_2') \approx \epsilon$ in the direction of E^{λ_1}!

Problems

1. ϵ is an external parameter - so it is not dynamically defined. How to ensure all points are generic?

   Answer - FACTORIZATION.

2. How to choose suitable q'?

   Answer - ENTROPY.

3. W^u does not carry an homogeneous structure???

   Answer - Normal forms coordinates over W^u (Kalinin-Sadovaskaya).

Idea of Eskin-Mirzakhani

Assume that there exists a **magical linear operator**

$$A(q, u, \ell, t) : E^s(q) \to \mathbb{R}$$

which measures the distance between $g_t.u.q_1$ and $g_t.u.q_1'$ in the Eskin-Mirzakhani scheme. This leads to a factorization of the E-M scheme by considering $A(q, u, \ell, t).v_{q'}$.

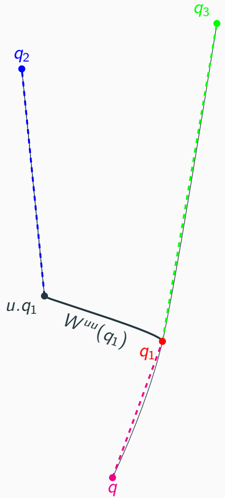
One can define the *stopping time* of the scheme as

$$\tau(q, u, \ell) = \sup \left\{ t \geq 0 \mid \|A(q, u, \ell, t)\|_{op} \leq \epsilon \right\}.$$

Magic(Eskin-Mirzakhani): τ is bilipschitz in ℓ.

Hint - $A(q, u, \ell + r, t + s) = g_s.A(q, u, \ell, t).g_r$

Choice of points



By the ergodic theorem, if q_1 is generic, for many times t (=positive density) the points $q_2 = g_t.u.q_1$ are in a good set.

Bilipschitz estimate gives that many points $q = g_{-\ell}.q_1$ are in a good set.

Stopping time for $q_3 = g_t.q_1$ is defined by cocycle with q_2, hence again many points are in a good set, by bilipschitz estimate.

Choice of q'

Second idea of Eskin-Mirzakhani

Choosing q' amounts to choosing a "good vector" $v \in E^s(q)$ such that

$$A(q,u,\ell,t).v \approx \|A(q,u,\ell,t)\|_{op}.$$

Done by entropy considerations similar to the low entropy technique (à la Einsiedler-Lindenstrauss).

a.k.a Case 1.

a.k.a not Case 2.

Major Issue - Construction of $A(q, u, \ell, t)$

While all the involved manifolds are smooth, the W^u foliation is in general *only Hölder-continuous* (c.f. Hasselblatt-Wilkinson).

Hence the map $Proj_{q_1'}^{c-s}(u.q_1)$ is not smooth but only Hölder.

Factorization overcomes this issue, by approximating it (polynomially), as-long as we have an a-priori upper-bound over the stopping time (due to QNI). This makes the construction of the operator $A(q, u, \ell, t)$ pretty complicated in the non-homogeneous case.

Examples

Example I - $ASL_2(\mathbb{R})/ASL_2(\mathbb{Z})$ - the space of affine lattices

Consider $Y = ASL_2(\mathbb{R})/ASL_2(\mathbb{Z})$. Define $g_t = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Then g_t is Anosov, $\lambda_1 = e^t, \lambda_2 = e^{2t}$ with $W^{\lambda_2} = \begin{pmatrix} 1 & \star & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $W^{\lambda_1} = \begin{pmatrix} 1 & 0 & \star \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

So u-Gibbs state means W^{λ_2}-invariant. One may show that if $h_\mu(g_1) > 3$, then μ is QNI , as the leafwise measure through $W^{\lambda_{-1}}$ is non-trivial.

Example II - Borel-Smale constructions

Let $\{N_i\}_{i=1,2}$ be two copies of the Heisenberg group, $Lie(N_i) = span\{x_i, y_i, z_i\}$ with $[x_i, y_i] = z_i$.

[Borel-Smale] There exists a rationally-defined map $A : Lie(N_1 \times N_2) \to Lie(N_1 \times N_2)$ such that

$$A \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{pmatrix} = \begin{pmatrix} \lambda^a.x_1 & \lambda^{-a}.x_2 \\ \lambda^b.y_1 & \lambda^{-b}.y_2 \\ \lambda^{a+b}.z_1 & \lambda^{-(a+b)}.z_2 \end{pmatrix}.$$

A is clearly Anosov diffeo if $a, b, a+b \neq 0$. Consider M - the suspension of $N_1 \times N_2/\Gamma$.

Take $a = 3, b = -2$. Notice that $h_\mu(A) \geq 5\ln(\lambda)$ for every generalized u-Gibbs state μ. One may show (by entropy considerations) that *every u-Gibbs state* is QNI.

Application to equidistribution

A system (admitting a dominant splitting) is *HQNI* if every generalized u-Gibbs state is *QNI*.

Theorem (K.)
Assume (M, μ, g_t) is HQNI, where $W^{>1} \subset W^u$ is isomorphic to $\mathbb{R}^n$ (by means of its subresonant group, i.e. narrow-band spectrum). Then for every $x \in M$, for almost every $u.x \in W^{>1}(x)$ we have that $\frac{1}{T}\int_{t=0}^{T} \delta_{g_t.u.x} dt \to \mu$ in the weak-$\star$ topology, where μ is some SRB measure on M.

Analogous to results of Eskin-Chaika (Translation surfaces) and Kleinbock-Shi-Weiss (Homogeneous dynamics).

Related to a conjecture of Gogolev about convergence of push-forwards of u-Gibbs states.