Exponential mixing of geodesic flow for geometrically finite manifolds with cusps

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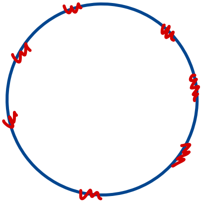
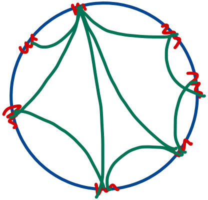
Setting and some basics

- $\mathbb{H}^n$ hyperbolic n-space
- $\Gamma < \mathrm{Isom}_+(\mathbb{H}^n)$ torsion-free discrete subgroup
- $\Lambda(\Gamma)$ limit set of Γ
  - the set of accumulation points of $\Gamma \cdot o$ ($o \in \mathbb{H}^n$)
  - $\subset \partial\mathbb{H}^n$
- $\mathrm{Hull}(\Gamma)$ smallest convex subset in $\mathbb{H}^n$ which contains all the geodesics connecting any two points in $\Lambda(\Gamma)$
- convex core C_Γ of $\Gamma = \Gamma\backslash\,\mathrm{Hull}(\Gamma) \subset \Gamma\backslash\mathbb{H}^n$

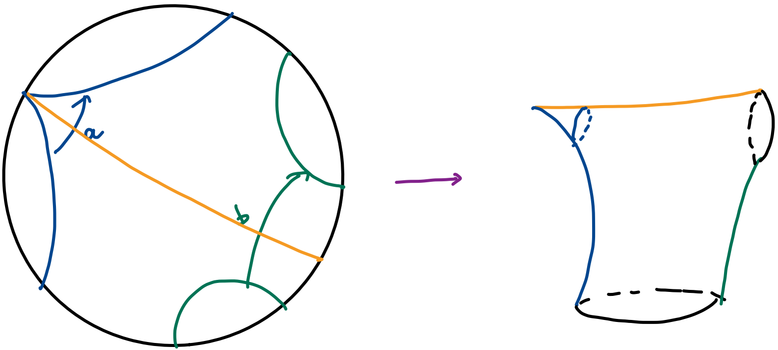




Γ is called geometrically finite if $\mathrm{Vol}(1\text{-nbhd of } C_\Gamma) < \infty$

Example

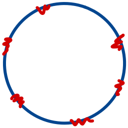
$\mathbb{H}^2$, $\mathrm{Isom}_+(\mathbb{H}^2) = PSL_2(\mathbb{R})$



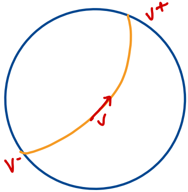
$\Gamma = \langle a, b \rangle$

$\Gamma \backslash \mathbb{H}^2$

Assume Γ geometrically finite



- Patterson-Sullivan measures (PS measures) $\{\mu_x\}_{x\in\mathbb{H}^n}$ a family of finite measures on $\Lambda(\Gamma)$
- Bowen-Margulis-Sullivan measure on $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$ (Hopf parametrization)



$v = (v^+, v^-, t)$

$$(\partial\mathbb{H}^n \times \partial\mathbb{H}^n\backslash\Delta)\times\mathbb{R} \to \mathrm{T}^1(\mathbb{H}^n)$$

$$T^1(\mathbb{H}^n) : d\tilde{m}^{BMS} = \frac{d\mu_x(v^+)\, d\mu_x(v^-)\, dt}{D(v^+, v^-)^{\delta_\Gamma}}$$

μ_x is Γ-quasi-inv

$$T^1(\Gamma\backslash\mathbb{H}^n) : dm^{BMS}$$

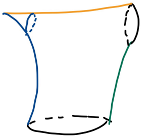
δ_Γ: critical exponent of Γ

$D(v^+, v^-)$: visual distance

Thm(Sullivan, Otal-Peigné) m^{BMS} is the unique measure supported on the nonwandering set for the geodesic flow which has the maximal entropy.

Main result

- $\mathbb{H}^n$
- $\Gamma < \mathrm{Isom}_+(\mathbb{H}^n)$ geometrically finite with parabolic elements
- $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n) \circlearrowleft$ geodesic flow $\mathcal{G}_t$, m^{BMS}



Thm (Jialun Li-P.) There exists $\eta > 0$ such that for any $u, v \in C^1(\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n))$, we have

$$\int_{\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)} u(\mathcal{G}_t x)v(x)dm^{\mathrm{BMS}}(x)$$
$$= \int_{\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)} udm^{\mathrm{BMS}}(x) \int_{\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)} vdm^{\mathrm{BMS}} + O(\|u\|_{C^1}\|v\|_{C^1}e^{-\eta t}).$$

Some history

- Babillot proved the geodesic flow is mixing
- Γ convex cocompact: Stoyanov, built on Dolgopyat's framework ($n = 2$, Naud)
  (Γ convex cocompact, Sarkar-Winter: frame flow)
- Γ geometrically finite and $\delta_\Gamma > \frac{n-1}{2}$: Mohammadi-Oh (frame flow), Edwards-Oh

Lax-Phillips: Δ negative of the Laplace operator on $\Gamma\backslash\mathbb{H}^n$

- $\delta_\Gamma > \frac{n-1}{2}$: there are finitely many eigenvalues of Δ on $L^2(\Gamma\backslash\mathbb{H}^n)$ in the interval $[\delta_\Gamma(n-1-\delta_\Gamma), (n-1)^2/4) \to$ representation theory

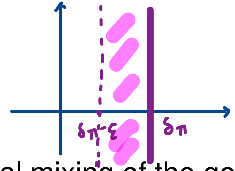
- $\delta_\Gamma \leq \frac{n-1}{2}$: L^2-spectrum of Δ is purely continuous

$\delta_\Gamma \leq \frac{n-1}{2}$: Resolvent $\mathcal{R}_s$ of $\Delta := (\Delta - s(n-1-s))^{-1}$, $s \in \mathbb{C}$

$\Re s \gg 1.$

- $\mathcal{R}_s$ has a meromorphic continuation to $\mathbb{C}$: convex-compact (Mazzeo-Melrose), geometrically finite (Guillarmou-Mazzeo)

- (Patterson) $\Gamma(s - \frac{n-1}{2} + 1)\mathcal{R}_s$ has a simple pole at δ_Γ and no further poles on $\operatorname{Re} s = \delta_\Gamma$



- Using exponential mixing of the geodesic flow, $\mathcal{R}_s$ has no poles in the strip $\delta_\Gamma - \epsilon < \operatorname{Re} s < \delta_\Gamma$
  - Effective orbit counting $\#\{\gamma \in \Gamma : d(x, \gamma y) < T\}$ (mixing $\rightarrow$ orbit counting: Margulis; Roblin (geo. fin.))
  - Meromorphic extension of the Poincaré series $P(s, x, y) = \sum_\gamma e^{-sd(x,\gamma y)}$
  - (Guillarmou-Mazzeo) relate $P(s, x, y)$ with $\mathcal{R}_s$

Ideas of the proof

- Code the geodesic flow
- Prove a Dolgopyat-like spectral estimate for the corresponding transfer operator: Dolgopyat, Avila-Gouëzel-Yoccoz, Araújo-Melbourne, Naud, Stoyanov (non-wandering set of the geodesic flow is a fractal set: non-integrability condition; how to get the contraction of transfer operator)

Coding

Γ geometrically finite

- $\Lambda(\Gamma) = \Lambda_r \sqcup \Lambda_{bp}$
- A parabolic fixed point $\xi \in \Lambda(\Gamma)$ is said to be bounded if

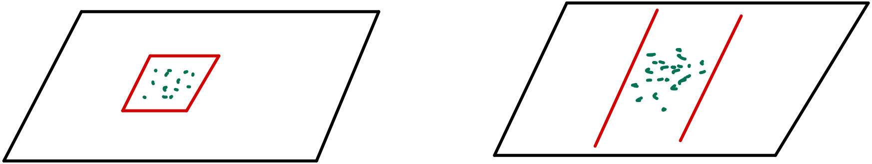
$$\mathrm{Stab}_\Gamma(\xi)\backslash\Lambda(\Gamma) - \{\xi\}$$

is compact.

$\mathbb{H}^3$, ∞ bounded parabolic fixed pt, $\Gamma_\infty = \mathrm{Stab}_\Gamma(\infty)$

(i) rank 2, $\Gamma_\infty \overset{\sim}{\cong} \mathbb{Z}^2$ up to a finite index subgp

(ii) rank 1, $\Gamma_\infty \overset{\sim}{\cong} \mathbb{Z}$

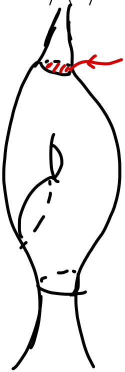


Coding

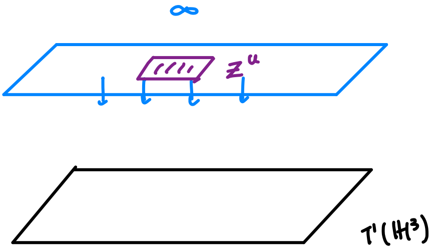
- $\mathbb{H}^3$
- $\Gamma\backslash\mathbb{H}^3$ has one full rank cusp
- ∞: a representative
- Starting idea: Poincaré section Λ for the geodesic flow

suspension flow

suspension space $\quad \Lambda \times \mathbb{R}/\langle r\rangle \to \mathrm{T}^1(\Gamma\backslash\mathbb{H}^3)$ geodesic flow



Poincaré section



- unstable horosphere based at ∞
- $\Gamma_\infty = \mathrm{Stab}_\Gamma(\infty)$

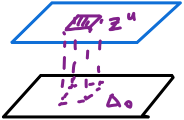
- Poincaré section Λ: thickening of Z^u in the stable direction
- Reduction: get rid of the stable direction

$$\begin{tikzcd}
\Lambda \times \mathbb{R}/\langle R \rangle \circlearrowleft \text{ suspension flow} \arrow[d, "\text{Avila-Gouëzel-Yoccoz, Araújo-Melbourne}"] \\
Z^u \times \mathbb{R}/\langle R \rangle \circlearrowleft \text{ suspension flow}
\end{tikzcd}$$

(in this argument, require return time const. on each stable manifold)

Proposition There exist a countable collection of disjoint, open subsets $\Delta_j \subset \Delta_0$ and an expanding map T defined on the union $\sqcup_j \Delta_j$ such that:

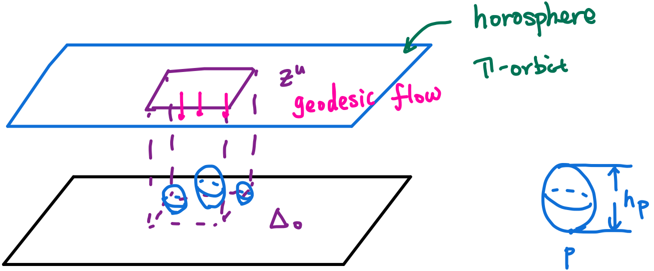
1. $\sum_j \mu(\Delta_j) = \mu(\Delta_0)$.
2. For each j, $T : \Delta_j \to \Delta_0$ is a diffeomorphism and there exists $\gamma_j \in \Gamma$ such that $\Delta_j = \gamma_j \Delta_0$ and $T = \gamma_j^{-1}$ on Δ_j. Denote by $\mathcal{H} = \{\gamma_j\}_j$ the set of inverse branches of T.
3. There exists $\lambda \in (0,1)$ such that for every $\gamma \in \mathcal{H}$, $|\gamma'(x)| \leq \lambda$ for all $x \in \Delta_0$.
4. There exists $C > 0$ such that for every $\gamma \in \mathcal{H}$, $|(\log|\gamma'(x)|)'|_\infty < C$.
5. (Exponential tail) Let R be the roof function given by $R(x) = \log|T'(x)|$ for $x \in \Delta_0$. There exists $\epsilon_0 > 0$ such that



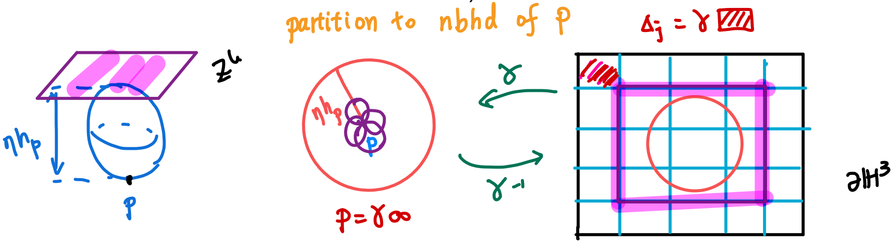
$$\int e^{\epsilon_0 R} d\mu < \infty.$$

(compare with Bowen-Series coding)

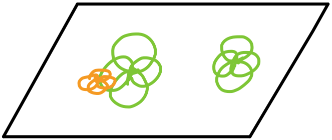
Elementary Version



"nice" partition to Z^u (Lai-Sang Young, Burns-Masur-Matheus-Wilkinson)

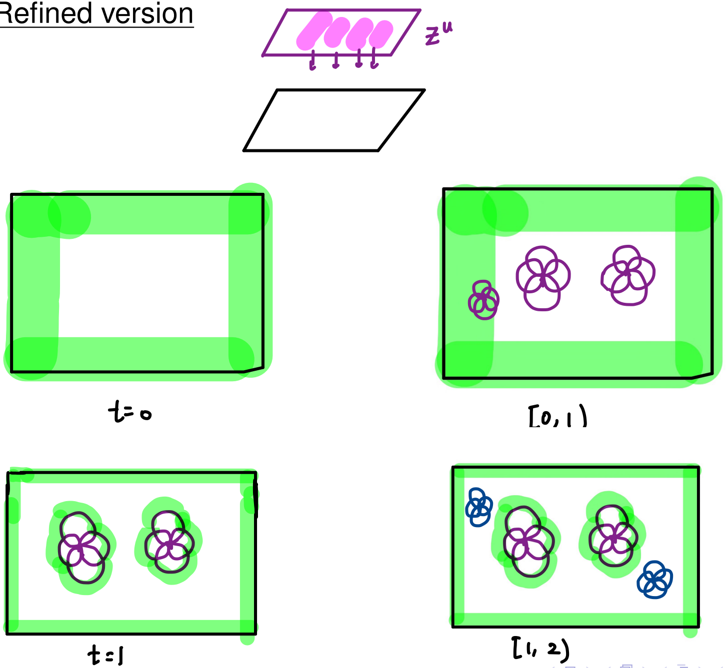


Issue about the boundary



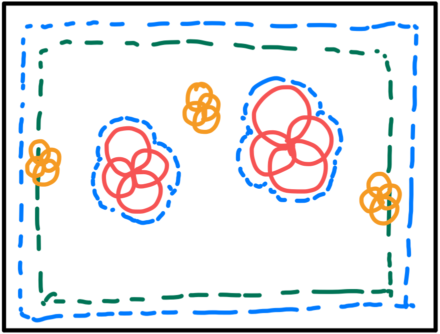
Form a forbidden region

Refined version



Refined Version

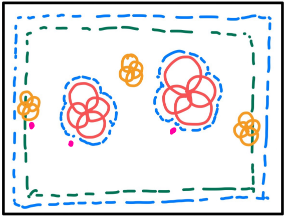
- $\Omega_0 = Z^u$
- $\Omega_n = \Omega_{n-1} - \bigcup_{p \in P_n}$
- $P_{n+1} = \{p$ parabolic fixed pts in Λ_0 : $\mathcal{G}^{[n,n+1)}\Omega_n \cap H_p(\eta h_p) \neq \emptyset, \;\; d(p, \partial\Omega_n) > l_n\}$
  ($l_n \to 0$ as $n \to \infty$)



Thm $\exists \epsilon_0 > 0$ and $N > 0$ such that for all $n > N$, we have

$$\mu(\Omega_n) \leq (1 - \epsilon_0)^n.$$

Ω_n : remaining part at $t = n$



Introduce a finer structure on Ω_n: inspired by Lai-Sang Young

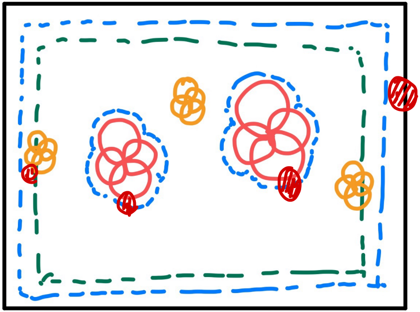
- $Q_n = \{p : \mathcal{G}^{[n-1,n)}\Omega_{n-1} \cap H_p(\eta h_p) \neq \emptyset,\ d(p, \partial\Omega_{n-1}) < l_{n-1}\}$
- $B_n = \Omega_n \cap \left(\cup_{p \in \cup_{1 \leq k \leq n} Q_k} B(p, r_{p,n})\right)$, $r_{p,n} \to 0$ as $n \to \infty$
- $A_n = \Omega_n - B_n$

(a key ingredient: prove two versions of doubling property of PS measure: use the doubling property proved by Stratmann-Velani, Das-Fishman-Simmons-Urbánski)

Energy exchange argument

- $\mu(B_n \cap A_{n+1})$
- $\mu(A_n \cap (B_{n+1} \cup \bigcup_{n+1} \quad))$
- $A'_n := \{\xi \in A_n : \xi$ away from $\partial\Omega_n\}$

$$\mu\Big(\bigcup_{n+1 \le k \le n+L} \quad \Big) \ge c\mu(A'_n)$$



Thank you!