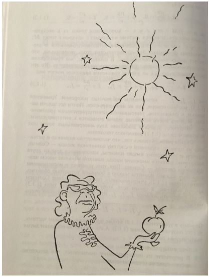
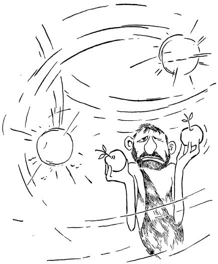
Holomorphic dynamics of low complexity and polynomial entropy

Olga Paris-Romaskevich

based on the joint work with Serge Cantat





What dynamical systems are simple ?...

Let $f : X \to X$ be a homeomorphism of a compact metric space X.

One may say that its dynamics is *simple* or *of low complexity* if

1. f has zero topological entropy : $h_{\mathrm{top}}(f) = 0$
2. f is integrable (Arnold-Liouville integrability)
3. the growth rate of the derivative $||Df^n||$ is slow enough
4. f^n is equicontinuous on large open subsets
5. f is minimal (all of its orbits are dense)
6. f has no periodic orbit
7. your variant

To give some of these definitions, one needs higher regularity.

Quantifying complexity : topological entropy

Let (X, d) be a compact metric space, $f : X \to X$ be a continuous map.

- the Bowen metric : $d_n^f(x,y) := \max_{0 \leq j \leq n-1} d\left(f^j(x), f^j(y)\right)$;
- the (n,ε)-covering number $\mathrm{Cov}_\varepsilon(n)$ is the minimal number of balls of radius ε in the metric d_n^f that cover X.

The topological entropy $h_{\mathrm{top}}(f) \in \mathbf{R} \cup \{+\infty\}$ is the double limit

$$h_{\mathrm{top}}(f) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log\left(\mathrm{Cov}_\varepsilon(n)\right).$$

Polynomial entropy : definition and examples

Many classes of (not so simple...) dynamical systems belong to the set $\{f \mid h_{\rm top}(f) = 0\}$.

- elliptic and polygonal billiards (Katok)
- billiard maps in convex polyhedrae in $\mathbf{R}^3$ (Bédaride)
- harmonic oscillator and simple pendulum
- 2-body keplerian problem
- $\mathrm{Homeo}^+(\mathbb{S}^1)$
- Brouwer homeomorphisms = plane homeomorphisms without fixed points
- many (but not all !) integrable geodesic flows : ellipsoid

Their complexity can be measured via *polynomial entropy*

$$\mathrm{h_{pol}}(f) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{\log n} \log\left(\mathrm{Cov}_\varepsilon(n)\right).$$

Polynomial entropy : some remarks

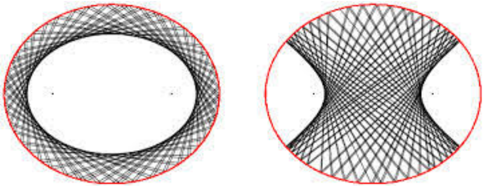
Polynomial entropy has been introduced, in various contexts, by Marco and Katok-Thouvenot. In general, it is very difficult to calculate both topological and polynomial entropy of continuous or smooth maps of manifolds.

- $\mathrm{h_{pol}}(f)$ is a conjugacy invariant, doesn't depend on the choice of the (topologically equivalent to d) metric on X
- If f is an isometry, $h_{\mathrm{top}}(f) = \mathrm{h_{pol}}(f) = 0$
- (!) $h_{\mathrm{top}}(f^n) = n h_{\mathrm{top}}(f)$ but $\mathrm{h_{pol}}(f^n) = \mathrm{h_{pol}}(f)$

Polynomial entropy is not very well understood (calculate polynomial entropy for more examples ? analogue of variational principle ? Pesin entropy formula ?...)

Some calculations of polynomial entropy

- (Marco) integrable Hamiltonian systems and billiards (strictly convex with analytic boundary) : $\mathrm{h_{pol}}(f_\Omega) = 1 \iff \Omega$ is a disk. Ω is an ellipse $\Rightarrow \mathrm{h_{pol}} = 2$. *Birkhoff-Marco conjecture :* the only billiard table with strictly convex and analytic boundary such that $\mathrm{h_{pol}}(f_\Omega) = 2$, is that in the ellipse.



- (Bernard-Labrousse) if g is a C^2-metric on $\mathbb{T}^2$ and $\mathrm{h_{pol}}(g) < 2$ then g is flat and $\mathrm{h_{pol}}(g) = 1$. If g is a metric of revolution then $\mathrm{h_{pol}}(g) = 2$.

Some calculations of polynomial entropy

- (Labrousse) $\mathrm{Homeo}^+(\mathbb{S}^1)$: $\mathrm{h_{pol}}(f) \in \{0,1\}$ and is equal to 0 if and only if f is conjugate to a rotation
- (Hauseux-Le Roux) Brouwer homeomorphisms : $\mathrm{h_{pol}}(f) = 1$ if and only if f is conjugated to a translation, $\mathrm{h_{pol}}(f) \in [2, +\infty]$ otherwise and any value is attained
- (Kanigowski) Kochergin flows on $\mathbb{T}^2$ (irrational vector fields with one singularity of type $(x^{-\gamma}, \gamma \in (0,1))$: $\mathrm{h_{pol}}(f) = 1 + \gamma$

Holomorphic setting

The following is the joint work with Serge Cantat.

Setting : Let X be a compact Kähler manifold of dimension k and $f \in \mathrm{Aut}(X)$. What can one say about f with slow dynamics ?

Theorem (Gromov 03', Yomdin 87')

Let $f : X \to X$ be a holomorphic endomorphism of a compact Kähler manifold X. Then $h_{\mathrm{top}}(f) = \log \lambda(f)$, where $\lambda(f)$ is the spectral radius of the action of $f^* : H^*(X; \mathbf{C}) \to H^*(X; \mathbf{C})$.

Yomdin : lower bound for any $f \in C^\infty$
Gromov : upper bound for holomorphic maps of Kähler manifolds)

Gromov's upper bound on entropy

Gromov's beautiful upper bound is easily adapted for the polynomial entropy (Yomdin's lower bound doesn't survive !).

The map f^* preserves the Dolbeaut partition $H^*(X;\mathbf{C}) = \sqcup H^{p,q}(X;\mathbf{C})$.
Denote by f_j^* the action of f on $H^{j,j}(X;\mathbf{C})$ or $H^{j,j}(X;\mathbf{R})$. The **polynomial growth rate** $s_j(f) \in \mathbf{R}_+ \cup \{+\infty\}$ is the number

$$s_j(f) := \lim_{n\to+\infty} \frac{\log \| (f^n)_j^* \|}{\log(n)}.$$

Let $s(f) := \sum_{j=0}^{k} s_j(f)$.

Theorem (Cantat, P.-R. 19')

X- compact Kähler manifold of dimension k, $f \in \mathrm{Aut}(X)$, $h_{\mathrm{top}}(f) = 0$. Then $\mathrm{h}_{\mathrm{pol}}(f)$ is finite and bounded from above by the following integers $k + s(f)$, $k \times (s_1(f) + 1)$, $k \times b_2(X)$.

Why $s_j(f)$ is an integer when $h_{\mathrm{top}}(f) = 0$?

Proposition

Let M be a compact manifold. There is an integer $m = m(\dim(H^*(M;\mathbf{R}))$ such that for any $\mathcal{C}^\infty$-diffeomorphism g of M with $h_{\mathrm{top}}(g) = 0$, all eigenvalues of $(g^m)^* \colon H^*(X;\mathbf{R}) \to H^*(X;\mathbf{R})$ are equal to 1 (i.e. g^* is virtually unipotent).

Proof.

g^* preserves $H^*(M;\mathbf{Z}) \Rightarrow \chi_{g^*}(t) \in \mathbf{Z}[t]$ with leading coefficient 1. Yomdin's lower bound $\log \lambda(g) \leq h_{\mathrm{top}}(g) = 0$ shows that all roots of $\chi_{g^*}(t)$ have modulus ≤ 1, and hence (Kronecker lemma) they are roots of 1, moreover, $\deg(\chi_{g^*}(t)) = \dim H^*(X;\mathbf{R})$. □

This implies that f^* is (virtually) unipotent and then f_j^* is a diagonal matrix of Jordan blocks. Hence $s_j(f)$ is the size (minus 1) of the largest Jordan block of f_j^* !

Reminder of the proof of Gromov's upper bound

Goal : bound topological (and polynomial) entropy by linear algebraic invariants related to the action of f^* on cohomology.

Step 1 : relate the entropies to volumes of iterated graphs $\Gamma(n)$
Step 2 : calculate the volumes via cohomological computation

- The **iterated graph** $\Gamma(n) := \{\boldsymbol{x} = (x_0, x_1, \ldots, x_n) \in X^{n+1} \mid \quad x_j = f(x_{j-1})\}$;
- the **distance** d_n^X on X^{n+1} is defined via $d_n^X(x, x') := \max_{j=0,\ldots,n} d(x_j, x_j') = d_n^f(x_0, x_0')$;
- $\pi_j : X^{n+1} \to X$ the projection on the j-th factor, let $\varkappa_n := \sum_j \pi_j^* \varkappa$ for $\varkappa$ - some fixed Kähler metric on X. Define $\mathrm{Vol}\Gamma(n)$ to be the k-dimensional volume with respect to $\varkappa_n$.

Step 1. Bounding entropy by the volumes $\mathrm{Vol}\Gamma(n)$

- Let W be a submanifold of X^n, $\dim_{\mathbf{C}}(W) = d$.

$$\mathrm{Dens}_\varepsilon(W, z) := \mathrm{Vol}_{2d}\left(W \cap B_z(\varepsilon)\right)$$
$$\mathrm{Dens}_\varepsilon(W) := \inf_{z \in W} \mathrm{Dens}_\varepsilon(W, z).$$

- Set $\mathrm{Dens}_\varepsilon(n) := \mathrm{Dens}_\varepsilon(\Gamma(n))$. Then, $\mathrm{Cov}_\varepsilon(n)\mathrm{Dens}_{\frac{\varepsilon}{2}}(n) \leq \mathrm{Vol}(\Gamma(n))$ for any $\varepsilon > 0$. By taking a logarithm : $\log \mathrm{Cov}_{2\varepsilon}(n) \leq \log \mathrm{Vol}(\Gamma(n)) - \log \mathrm{Dens}_\varepsilon(n)$.
- (!)

Theorem (Federer)

For a fixed Kähler metric $\varkappa$ on X, and fixed $\varepsilon > 0$. There exists $C = C(\varepsilon, \varkappa)$ such that $\mathrm{Dens}_\varepsilon(n) \geq C > 0$.

Step 2. Calculation of the volumes $\mathrm{Vol}\Gamma(n)$

The bound from Step 1. gives
$h_{\mathrm{top}}(f) \leq \limsup_n n^{-1} \log \mathrm{Vol}(\Gamma(n))$. For polynomial entropy, the division by $\log(n)$ gives

$$\mathrm{h_{pol}}(f) \leq \limsup_{n\to\infty} \frac{\log \mathrm{Vol}(\Gamma(n))}{\log n}.$$

Now we calculate (!) the volumes following Wirtinger's theorem :

$$\mathrm{Vol}(\Gamma(n)) = \int_{\Gamma(n)} \kappa_n^k = \int_X (\sum_j \pi_j^* \kappa)^k = \int_X (\sum_{j=0}^{n} (f^j)^* \kappa)^k.$$

We have $\| (f^n)_1^*[\kappa] \| \leq C \| [\kappa] \| n^{s_1(f)}$. Hence the norm of the class $[\sum_{j=0}^n (f^j)^* \kappa]$ is no more than $C' \| \kappa \| n^{s_1(f)+1}$, for some $C' > 0$, and we get $\mathrm{Vol}(\Gamma(n)) \leq C'' n^{k(s_1(f)+1)}$ for some $C'' > 0$. This proves the second and third upper bounds. $\square$

Theorem (Cantat, P.-R. 19')

X- compact Kähler manifold of dimension k,
$f \in \mathrm{Aut}(X), h_{\mathrm{top}}(f) = 0$. Then $\mathrm{h}_{\mathrm{pol}}(f)$ is finite and bounded from above by the following integers $k + s(f)$, $k \times (s_1(f) + 1)$, $k \times b_2(X)$.

- birational case : $g \in \mathrm{Bir}(X)$ then $\mathrm{h}_{\mathrm{pol}}(g) \leq k + s(g)$
- in dimensions $k = 2, 3$, one proves that $\mathrm{h}_{\mathrm{pol}}(f) \leq k^2$
- for k-dimensional tori one has $\mathrm{h}_{\mathrm{pol}}(f) \leq k(k-1)$ for all k
- surfaces : $\mathrm{h}_{\mathrm{pol}}(f) \in \{0, 1, 2, \infty\} \cup$ ...?...
- is $\mathrm{h}_{\mathrm{pol}}(f) \in \mathbf{Z}$ for any $f \in \mathrm{Aut}(X)$ with $h_{\mathrm{top}}(f) = 0$?...
  (discrete spectrum of polynomial entropy ?)

Growth of derivatives and polynomial entropy

$\| Df^n \| := \max_{x \in M} \| D(f^n)_x \|$. If $\| Df^n \| \geq \exp(\eta n), n \to \infty$ then $\mathrm{h}_{\mathrm{pol}}(f) \geq 1/2$ (maybe 1?...). If $\max \mathrm{Lip}(f, \ldots, f^n) = o(n^\alpha)$ for all α then $\mathrm{h}_{\mathrm{pol}}(f) = 0$.

Theorem (Cantat, P.-R, 19')

There exists a real analytic and area preserving diffeomorphism f of the torus $\mathbb{T}^2$ satisfying the following four properties

1. *f is minimal;*
2. *its iterates f^n do not form an equicontinuous family;*
3. *$\| Df^n \| = o(n^\varepsilon)$ for every $\varepsilon > 0$;*
4. *the polynomial entropy of f vanishes.*

This answers the question by Artigue, Carrasco-Oliveira, Monteverde. (!) For $f \in Aut(X)$: sublinear growth implies equicontinuity.

Minimality : naive approach to simplicity

"All orbits look the same" – all orbits are dense.
Does minimality imply zero topological enropy ?

- Herman: NO ! $f \in C^{\infty}$, dim $= 4$, no
- Katok: YES ! $f \in C^{1+\varepsilon}$, dim $= 2$
- Rees : NO ! $f \in C^{0}$, dim $= 2$
- Cantat, P.-R., Xie : NO ! $f \in \mathrm{Aut}(X)$, $\dim_{\mathbf{C}} \leq 3$
- **Question.** Does there exist $f \in \mathrm{Aut}(X)$, $f-$minimal with $h_{\mathrm{top}}(f) > 0$?

Question. Let $f \in \mathrm{Aut}(X)$-minimal. What can one say about the geometry of X?
Dimension 2 : we obtain the classification of all $f \in \mathrm{Aut}(X)$ of surfaces, with no periodic orbits, all orbits Zariski dense, all orbits euclidian dense. Moreover, minimality implies that X is a complex torus. Is it true in any dimension ?...
Dimension 3 and higher : seems to be a hard algebraic geometry question...

Thank you for your attention !