The Smooth Realization Problem: Area Preserving Diffeomorphisms with Polynomial Decay of Correlations

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The Smooth Realization Problem

The smooth realization problem in dynamics asks whether there is a diffeomorphism of a compact smooth manifold, which has a prescribed collection of ergodic properties with respect to a natural invariant measure such as the Riemannian volume (or a smooth measure). Another interesting measure to consider is the MME.

A yet more interesting but substantially more difficult version of the smooth realization problem is to construct a volume preserving diffeomorphism with prescribed ergodic properties on any given smooth compact manifold. In other words, does the topology of the manifold can provide an obstruction for realizing a given ergodic property.

Some History

Starting with the basic ergodic property – ergodicity – Anosov and Katok (1970) constructed an example of a volume preserving ergodic C^∞ diffeomorphism.

Katok (1979) gave an example of an area preserving C^∞ map with non-zero Lyapunov exponents on any surface which is Bernoulli.

Brin, Feldman, and Katok (1981) extended this result by constructing a volume preserving C^∞ diffeomorphism, which is Bernoulli, on any Riemannian manifold of dimension ≥ 5.

In their example the map has all but one non-zero Lyapunov exponents. Dolgopyat and Pesin (2002) constructed a volume preserving C^∞ Bernoulli diffeomorphism with all non-zero Lyapunov exponents on any Riemannian manifold of dimension ≥ 2.

It is natural to ask if a compact smooth manifold admits a volume preserving Bernoulli diffeomorphism with non-zero Lyapunov exponents that enjoys other important statistical properties such as exponential or polynomial decay of correlations (that is rate of mixing), the Central Limit Theorem, and the Large Deviations property; all three with respect to a natural class of observables, e.g., functions which are Hölder continuous.

The Correlations Function

Let X be a measurable space and $T : X \to X$ a measurable invertible transformation preserving a measure μ.
Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two classes of real-valued functions on X called *observables*. For $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$ define the correlation function

$$\mathrm{Cor}_n(h_1, h_2) := \int h_1(T^n(x))h_2(x)\,d\mu - \int h_1(x)\,d\mu \int h_2(x)\,d\mu.$$

Decay of Correlations

1. T has polynomial decay of correlations (more precisely, admits polynomial upper bound on correlations) w.r.t classes $\mathcal{H}_1$ and $\mathcal{H}_2$ if there exists $\gamma_1 > 0$ s.t. for any $h_1 \in \mathcal{H}_1$, $h_2 \in \mathcal{H}_2$, and any $n > 0$,

$$|\mathrm{Cor}_n(h_1, h_2)| \leq C n^{-\gamma_1},$$

where $C = C(h_1, h_2) > 0$ is a constant.
2. T admits a polynomial lower bound on correlations w.r.t classes $\mathcal{H}_1$ and $\mathcal{H}_2$ of observables if there exists $\gamma_2 > 0$ s.t. for any $h_1 \in \mathcal{H}_1$, $h_2 \in \mathcal{H}_2$, and any $n > 0$,

$$|\mathrm{Cor}_n(h_1, h_2)| \geq C' n^{-\gamma_2},$$

where $C' = C'(h_1, h_2) > 0$ is a constant.
3. T has exponential decay of correlations w.r.t classes $\mathcal{H}_1$ and $\mathcal{H}_2$ if there exists $\gamma_3 > 0$ s.t. for any $h_1 \in \mathcal{H}_1$, $h_2 \in \mathcal{H}_2$, and any $n > 0$,

$$|\mathrm{Cor}_n(h_1, h_2)| \leq C'' e^{-\gamma_3 n},$$

where $C'' = C''(h_1, h_2) > 0$ is a constant.

Main Theorem

There are numbers $\gamma_2 > \gamma_1 > 0$ and $\beta > 0$ such that any compact smooth connected and oriented surface M admits an area preserving $C^{1+\beta}$ diffeomorphism $f = f_M$ satisfying:

1. f has the Bernoulli property.
2. f has non-zero Lyapunov exponents almost everywhere with respect to area m.
3. f admits a polynomial upper bound on correlations with respect to the exponent γ_1 and the class C^ρ of all Hölder continuous functions on M.
4. There is a nested sequence of subsets $\{M_j\}$ that exhaust M such that f admits a polynomial lower bound on correlations with respect to the exponent γ_2 and the class $\tilde{C}^\rho$ of all Hölder continuous functions h for which there is $k = k(h)$ such that $\text{supp}(h) \subset M_k$.

Comments

1. In the two dimensional case Liverani and Martens (2005) constructed an example of an area preserving C^∞ diffeomorphism of the 2-torus with non-zero Lyapunov exponents which has polynomial decay of correlations with respect to the class of smooth observables.

2. The proof of the Main theorem is based on the work of Katok (1979) but requires some new ideas. I believe that modifying our argument and using the approach of Brin, Feldman, and Katok (1981), and of Dolgopyat and P. one can extend the Main theorem to manifolds of dimension ≥ 2.

3. One can show that the map f in the Main theorem satisfies the Central Limit Theorem and has the Large Deviation property. Moreover, f has a unique MME which is Bernoulli, has non-zero Lyapunov exponents almost everywhere, exponential decay of correlations, satisfies the Central Limit Theorem and has the Large Deviation property.

4. In view of our Main theorem it is interesting to know whether any smooth compact Riemannian manifold admits a volume preserving diffeomorphism with exponential decay of correlations and whether it admits a diffeomorphism with a unique MME with respect to which it has polynomial decay of correlations. Both questions seem to be quite difficult.

The Katok Map

Consider the automorphism of the two-dimensional torus T^2 given by the matrix $A := \begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix}$. It has four fixed points $x_1 = (0,0)$, $x_2 = (\frac{1}{2}, 0)$, $x_3 = (0, \frac{1}{2})$, and $x_4 = (\frac{1}{2}, \frac{1}{2})$. For $i = 1,2,3,4$ let $D_r^i = \{(s_1, s_2) : {s_1}^2 + {s_2}^2 \leq r^2\}$ be the disk of radius r centered at x_i and $D_r = \bigcup_{i=1}^4 D_r^i$. Here (s_1, s_2) is the coordinate system obtained from the eigendirections of A and originated at x_i. Let $\lambda > 1$ be the largest eigenvalue of A.

Choose $0 < \alpha < 1$ and a slow-down function $\psi : [0,1] \mapsto [0,1]$ satisfying:

1. $\psi(u) = 1$ for $u \geq r_0$ and some $0 < r_0 < 1$;
2. $\psi'(u) > 0$ for every $0 < u < r_0$;
3. $\psi(u) = (u/r_0)^\alpha$ for $0 \leq u \leq \frac{r_0}{2}$.

There exists an area preserving $C^{2+\kappa}$, $\kappa = \frac{2\alpha}{1-\alpha}$, diffeomorphism f_{T^2} of T^2 such that

1. It is topologically conjugate to A.
2. It has two continuous, uniformly transverse, invariant one-dimensional stable $E^s(x)$ and unstable $E^u(x)$ distributions; for a.e. $x \in T^2$ the Lyapunov exponent along these distributions are negative and respectively, positive; the Lyapunov exponents at the fixed points x_i are zero;
3. It has two continuous, uniformly transverse, invariant one-dimensional foliations with smooth leaves called stable W^s and unstable W^u foliations.
4. It is isomorphic to a Bernoulli diffeomorphism.
5. In D_r^i, $i = 1, 2, 3, 4$, the map f_{T^2} is – up to a smooth coordinate change – the time-1 map of the flow given by

$$\dot{s}_1 = s_1 \psi({s_1}^2 + {s_2}^2) \log \lambda, \quad \dot{s}_2 = -s_2 \psi({s_1}^2 + {s_2}^2) \log \lambda.$$

The map f_{T^2} is called the Katok map.

Young tower for the Katok map

Since the map f_{T^2} is topologically conjugate to the hyperbolic automorphism A of the torus, it possesses a Markov partition of arbitrary small diameter. Let $\tilde{P}$ be an element of the partition which is far away from the neighborhood D_r of the four fixed points. Let also $\tau = \tau(x)$ be the first return time of x to $\tilde{P}$. Consider the set $P \subset \tilde{P}$ consisting of points x for which $\tau(x) < \infty$. Since f_{T^2} preserves area, P has full measure in $\tilde{P}$. Let $F(x) = f^{\tau(x)}(x)$ be the first return time map. Thus we obtain a tower with base P, hight τ, and the base map F.

In fact, this tower is a Young tower (P., Senti, Zhang, 2019):

1. for each $n > 0$ the level set $\{x \in P : \tau(x) = n\}$ is a disjoin union of s-sets $P_{n,i}$, $i = 1, \dots, k(n)$ called partition elements; for $x \in P_{n,i}$, the full length stable curve through x lies in $P_{n,i}$; $f^n_{T^2}(P_{n,i})$ is a u-set;
2. $k(n) \le C \exp(hn)$ where $C > 0$ and $h > 0$ are constants and $h < h_{\text{top}}(f_{T^2}) = \log \lambda$;
3. the induced map F has the bounded distortion property with respect to the partition elements;
4. the inducing time is integrable:

$$S = \sum_{n=1}^{\infty} m(\{x \in P : \tau(x) = n\}) = \sum_{n=1}^{\infty} \sum_{i=1}^{k(n)} m(P_{n,i}) < \infty.$$

Our goal now is to obtain polynomial lower and upper bounds on the rate of decay of the tale in the above sum, i.e.,

$$S_n = m(\{x \in P : \tau(x) > n\}).$$

Lemma

There is $C_1 > 0$ *such that for any* $0 < \alpha < \frac{1}{4}$

$$m(\{x \in P : \tau(x) > n\}) > C_1 n^{-(\gamma-1)},$$

where $\gamma = \frac{1}{2\alpha} + 5 \cdot 2^{\alpha-3} + \frac{1}{24}$ *(note that* $\gamma > 2$*).*

To see this write

$$m(\{x \in P : \tau(x) > n\}) = \sum_{k=n+1}^{\infty} m(\{x \in P : \tau(x) = k\})$$

$$= \sum_{k=n+1}^{\infty} \sum_{P_{n,i}:\tau(P_{n,i})=k} m(P_{n,k}) > \sum_{k=n+1}^{\infty} m(P_{n,\ell}),$$

where for large n, $P_{n,\ell}$ is chosen such that it travels some time outside D_r, then enters D_r only once and travels there for a long time and finally travels some time before entering P.

Claim 1. *There is* $Q > 0$ *such that for any* $N > 0$ *one can find an s-set* $P_{n,\ell}$ *with* $n > N$ *and numbers* $0 < m_1 < m_2$ *satisfying* $m_1 < Q$, $n - m_2 < Q$, $f^k(P_{n,\ell}) \cap D_r = \emptyset$ *for* $0 \le k < m_1$ *or* $m_2 < k \le n$ *and* $f^k(P_{n,\ell}) \cap D_r \ne \emptyset$ *for* $m_1 \le k \le m_2$*.*

To show this it suffices to find $Q > 0$ such that for sufficiently large n there is an admissible word of length n of the form $P\bar{W}_1\bar{P}_i\bar{W}_2P$ where the words $\bar{W}_j$ are of length $l(\bar{W}_j) < Q$, $j = 1, 2$, and do not contain any of the symbols P or P_k (the element of the Markov partition containing the fixed point x_k for $k = 1, 2, 3, 4$), and the word $\bar{P}_i$ consists of the symbol P_i which is repeated $n - 2 - l(\bar{W}_1) - l(\bar{W}_2)$ times.

Claim 2. *There is* $C_2 > 0$ *such that* $m(P_{n,\ell}) > C_2 n^{-\gamma}$*.*

The proof requires some technically involved estimates on the solutions of differential equations

$$\dot{s}_1 = s_1\psi({s_1}^2 + {s_2}^2)\log\lambda, \quad \dot{s}_2 = -s_2\psi({s_1}^2 + {s_2}^2)\log\lambda,$$

which describe the dynamics in the slow-down domains.

Upper Bound of the Tale

Lemma

There is $C_3 > 0$ *such that for any* $\frac{1}{9} < \alpha < \frac{1}{4}$

$$m(\{x \in P : \tau(x) > n\}) < C_3 n^{-(\gamma'-1)},$$

where $\gamma' = \frac{1}{2\alpha} + \frac{1}{2^{\alpha+4}}$ *(note that* $\gamma > \gamma' > 2$*).*

To prove this given an s-set $P_{n,i}$ with $\tau(P_{n,i}) = n$, choose any numbers $k = k(P_{n,i})$, $p = p(P_{n,i})$, and two finite collections of numbers $\{k_m \geq 0\}_{m=1,\dots,p}$ and $\{l_m \geq 0\}_{l=0,\dots,p}$ such that

1. $k_1 + k_2 + \cdots + k_p = k$ and $l_1 + l_2 + \cdots + l_{p+1} = n - k$;
2. the trajectory of the set $P_{n,i}$ under $f^j_{T^2}$, $0 \leq j \leq n$, consecutively spends l_m-times outside D_r and k_m-times inside D_r.

Given $0 < p < k < n$, consider the collections

$$\mathcal{S}_{k,n,p} = \{P_{n,i} : \tau(P_{n,i}) = n,\ k = k(P_{n,i}),\ p = p(P_{n,i})\}.$$

Note that

$$m(\{x \in P : \tau(x) = n\}) \le \sum_{k=1}^{n} \sum_{p=1}^{k} \max_{P_{n,i} \in \mathcal{S}_{k,n,p}} \{m(P_{n,i})\} \text{Card } \mathcal{S}_{k,n,p}.$$

Claim 1. *There are $0 < h < h_{top}(f)$, $\varepsilon_0 > 0$, and $C_4 > 0$ such that $\varepsilon_0 < h_{top}(f) - h$ and*

$$\text{Card } \mathcal{S}_{k,n,p} \le C_4 p^{-2} e^{(h+\varepsilon_0)(n-k)}.$$

Claim 2. *There exists $\varepsilon_0 > 0$ such that for any $P_{n,i} \in \mathcal{S}_{k,n,p}$,*

$$m(P_{n,i}) \le C_5 k^{-\gamma'} e^{(-\log \lambda + \varepsilon_0)(n-k)},$$

where $C_5 > 0$ is a constant.

Carrying the Katok Map to a Surface

Theorem

There are maps

$$\varphi_1 : T^2 \to S^2,\ \varphi_2 : S^2 \to D^2,\ \varphi_3 : D^2 \to M$$

such that the maps

$$f_{S^2} = \varphi_1 \circ f_{T^2} \circ \varphi_1^{-1} : S^2 \to S^2,$$
$$f_{D^2} = \varphi_2 \circ f_{S^2} \circ \varphi_2^{-1} : D^2 \to D^2$$

are area preserving $C^{2+\kappa}$ diffeomorphisms, f_{D^2} is identity on the boundary of the disk and the map

$$f_M = \varphi_3 \circ f_{D^2} \circ \varphi_1^{-1} : M \to M$$

is area preserving and of class $C^{1+\beta}$ for some $\beta > 0$.

1. φ_1 is a double branched covering, is one-to-one on each branch, and C^∞ everywhere except at the points x_i, $i = 1, 2, 3, 4$ where it branches;
2. φ_2 unfolds S^2 onto the unit disk D^2 and is C^∞;
3. φ_1 and φ_2 preserve area, i.e., $(\varphi_1)_* m_{T^2} = m_{S^2}$ and $(\varphi_2)_* m_{S^2} = m_{D^2}$.

The maps φ_1 and φ_2 were constructed by Katok. However, constructing the map φ_3 in our case is quite difficult, since we have to deal with finite regularity of the map f_{D^2}.

Infinite vs. Finite Differentiability and Flatness

If the slow-down function ψ is C^∞, then so is the map f_{D^2}. Moreover, if the function ψ^{-1} is infinitely flat at zero, then f_{D^2} is infinitely flat at the boundary of the disk. In this case a construction of the map φ_3 was given by Katok.
In our case the slow-down function ψ is polynomial and as a result, the map f_{D^2} is only of class $C^{2+\kappa}$ and is finitely flat at the boundary: there is a sequence of open domains $V_n \subset D^2$ s.t.

$$V_n \subset \overline{V}_n \subset V_{n+1}, \ \cup_{n\geq 1} V_n = D^2, \ V_{n-1} \subset f_{D^2}(V_n) \subset V_{n+1}$$

and for every $0 < \beta < 2+\kappa$,

$$\|f_{D^2} - \mathrm{Id}\|_{C^{1+\beta}(V_{n+1}\setminus V_{n-1})} \leq (r_{n-1})^{2+\kappa-\beta},$$

where $r_n = \mathrm{dist}(V_n, \partial D^2)$. This requires us to develop a specific construction of the map φ_3 which guarantees that the map f_M is an area preserving diffeomorphism of class $C^{1+\beta}$ for some $\beta > 0$.

The Embedding Theorem

Given a smooth compact connected oriented surface M, for any $\frac{1}{9} < \alpha < \frac{1}{4}$ there exist $\beta = \beta(\alpha)$, $0 < \beta < 2 + \kappa$, and a continuous map $\varphi_3 \colon \overline{D^2} \to M$ such that

1. the restriction $\varphi_3|\text{int } D^2$ is a diffeomorphic embedding;
2. $\varphi_3(\overline{D^2}) = M$;
3. φ_3 preserves area; more precisely, $(\varphi_3)_* m_{D^2} = m_M$ where m_M is the area in M; moreover, $m_M(M \setminus \varphi_3(\text{int } D^2)) = 0$;
4. the map $f_M := \varphi_3 \circ f_{D^2} \circ \varphi_3^{-1}$ is a $C^{1+\beta}$ area preserving diffeomorphism of the surface.

Completing the Proof of the Main theorem

The map f_{T^2} is a Young diffeomorphism and has a collection of s-subsets $P_{n,i}$ as well as the return time τ. Define $P_{n,i,M} := \varphi_3(\varphi_2(\varphi_1(P_{n,i})))$ and the return time $\tau_M(x) = \tau(y)$ where $x = \varphi_3(\varphi_2(\varphi_1(y))),\ y \in T^2$. This represents f_M as a Young diffeomorphism.

By the above two lemmas

$$\frac{C_1}{n^{\gamma-1}} < m(\{x\colon \tau(x) > n\}) < \frac{C_2}{n^{\gamma'-1}}.$$

To establish lower and upper bounds on the decay of correlations we need the following result by Gouëzel, Sarig, Shahidi and Zelerowicz.

Theorem. Assume that (M, m, f) is a Young diffeomorphism for which the greatest common denominator of numbers $\{\tau_i\}$, $\gcd\{\tau_i\} = 1$ and for which $m(\tau > n) = \mathcal{O}(\frac{1}{n^\nu})$ for some $\nu > 1$. Assume also that for some $C > 0$ and all $x, y \in P_{n,i}$,

$$d(f^j(x), f^j(y)) \leq C \max\{d(x, y), d(f^{\tau_i}(x), f^{\tau_i}(y))\}.$$

Then for any $\sigma > 0$ and $h_1, h_2 \in C^\rho(M)$:

1. $\mathrm{Cor}_n(h_1, h_2) = O(\frac{1}{n^{\nu-1}})$.
2. There exists a nested sequence of sets $M_1 \subset M_2 \cdots \subset M$ such that if h_1, h_2 are supported in M_k for some $k > 0$ then

$$\mathrm{Cor}_n(h_1, h_2) = \sum_{n>N}^{\infty} m(\{x \colon \tau(x) > N\}) \int_M h_1 \, dm \int_M h_2 \, dm + r_\nu(n),$$

where $r_\nu(n) = \mathcal{O}(R_\nu(n))$ and

$$R_\nu(n) = \begin{cases} \frac{1}{n^\nu} & \text{if } \nu > 2, \\ \frac{\log n}{n^2} & \text{if } \nu = 2, \\ \frac{1}{n^{2\nu-2}} & \text{if } 1 < \nu < 2. \end{cases}$$