Stable ergodicity beyond partial hyperbolicity

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Dynamics on the screen
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1 introduction

1 setting

setting

- M compact Riemannian manifold without boundary
- m volume probability measure
- $\mathrm{Diff}^r_m(M)$: C^r-diffeomorphisms preserving m

1 stable ergodicity

stable ergodicity

- $f \in \mathrm{Diff}^1_m(M)$ is stably ergodic
- if there exists $f \in \mathcal{U} \subset \mathrm{Diff}^1_m(M)$ open
- such that

$$g \in \mathcal{U} \cap \mathrm{Diff}^2_m(M) \quad \Rightarrow \quad g \quad \text{ergodic}$$

1 question

question

- does there exist $\mathcal{U} \subset \mathrm{Diff}^1_m$ open
- such that

$$g \in \mathcal{U} \quad \Rightarrow \quad g \quad \text{ergodic?}$$

1 mechanisms that activate stable ergodicity

hyperbolicity

$f \in \mathrm{Diff}^1_m(M)$ is Anosov or hyperbolic if

- there exists a Df-invariant splitting $TM = E^s \oplus E^u$
- such that

$$\begin{array}{ccccc} TM & = & E^s & \oplus & E^u \\ & & \downarrow & & \downarrow \\ & & Df - \text{contracting} & & Df - \text{expanding} \end{array}$$

1 mechanisms that activate stable ergodicity

Anosov-Sinai (1967)

- $f \in \mathrm{Diff}^{1+\alpha}_m(M)$ hyperbolic
- $\Rightarrow$ ergodic
- $\Rightarrow$ stably ergodic

① mechanisms that activate stable ergodicity

Grayson - Pugh - Shub (1995)

- $\exists$ a non-hyperbolic stably ergodic diffeomorphism

1 mechanisms that activate stable ergodicity

partial hyperbolicity

$f \in \mathrm{Diff}^1_m(M)$ is partially hyperbolic if

- there exists a Df-invariant splitting $TM = E^s \oplus E^c \oplus E^u$
- such that

$$
\begin{array}{ccccccc}
TM & = & E^s & \oplus & E^c & \oplus & E^u \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \text{contracting} & & \text{intermediate} & & \text{expanding}
\end{array}
$$

1 mechanisms that activate stable ergodicity

conjecture: pugh - shub (1995)

stable ergodicity is C^r-dense among partially hyperbolic diffeomorphisms

1 mechanisms that activate stable ergodicity

Avila - Crovisier - Wilkinson (2016)

stable ergodicity is C^1-dense among partially hyperbolic diffeomorphisms

Hertz - H. - Ures (2008)

stable ergodicity is C^∞-dense among partially hyperbolic diffeomorphisms ($c = 1$)

1 mechanisms that activate stable ergodicity

Tahzibi (2004)

- $\exists$ a non-partially hyperbolic stably ergodic diffeomorphism

1 mechanisms that activate stable ergodicity

dominated splitting

$f \in \mathrm{Diff}^1_m(M)$ admits a dominated splitting

- there exists a Df-invariant splitting $TM = E \oplus F$
- such that

$$TM = \underset{\text{more contracting}}{\underset{\downarrow}{E}} \oplus \underset{\text{more expanding}}{\underset{\downarrow}{F}}$$

① mechanisms that activate stable ergodicity

it is a necessary condition:

stable ergodicity

$$\Downarrow$$

dominated splitting

1 mechanisms that activate stable ergodicity

pugh - shub (1995)

a little hyperbolicity goes a long way toward guaranteeing stable ergodicity

1 conjecture

conjecture: JRH. (2012)

generically in $\mathrm{Diff}^1_m(M)$

dominated splitting

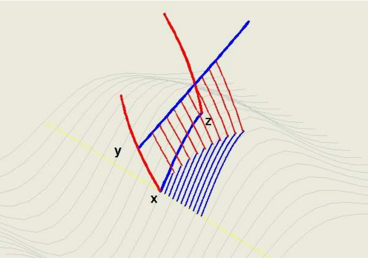
$\Downarrow$

stable ergodicity

2 exploring new mechanisms for SE

2 known mechanism for PHD

accessibility

- for partially hyperbolic diffeomorphisms
- Pugh - Shub proposed accessibility
- 

2 known mechanism for PHD

pugh - shub program

- for partially hyperbolic diffeomorphisms in $\mathrm{Diff}^2_m(M)$
- stable accessibility is C^r-dense
- accessibility $\Rightarrow$ ergodicity

2 other mechanisms activating SE

other mechanisms

- in dimension 3
- dominated splitting $\Rightarrow$ one hyperbolic bundle
- $\Rightarrow$ either $E = E^s$ or $F = E^u$
- suppose $TM = E \oplus E^u$
- there is an invariant foliation $\mathcal{F}^u$ tangent to E^u

2 other mechanisms activating SE

minimality

- a foliation is minimal
- if every leaf is dense

② other mechanisms activating SE

program in dimension 3

- for $f \in \mathrm{Diff}^1_m(M)$ with a dominated splitting
- (stable) minimality of the $\mathcal{F}^u$ is C^1-dense
- C^1-generically, minimality of $\mathcal{F}^u \Rightarrow$ stable ergodicity

② other mechanisms activating SE

theorem (G. Nuñez, JRH 2019)

generically in $\mathrm{Diff}^1_m(M^3)$,

- $\mathcal{F}^u$ minimal

$\Rightarrow$ stable ergodicity

2 question

question (2014)

generically in $\mathrm{Diff}^1_m(M^3)$,

- dominated splitting
- $\Rightarrow \mathcal{F}^u$ or $\mathcal{F}^s$ minimal?

2 other mechanisms activating SE

question - in any dimension

$\mathcal{F}^u$ minimal

$\Downarrow$

stable ergodicity

2 conjecture

conjecture JRH (2019)

generically in $\mathrm{Diff}^1_m(M)$
$\mathcal{F}^u$ minimal

$$\Downarrow$$

stable ergodicity

② theorem

G. Núñez - JRH (2020)

for a generic map $f \in \mathrm{Diff}^1_m(M)$, if

- $\mathcal{F}^u$ minimal

$\Rightarrow \exists\, \mathcal{U}(f) \subset \mathrm{Diff}^1_m(M)$: $\forall g \in \mathrm{Diff}^2_m(M) \cap \mathcal{U}$

- $\exists$ ergodic component $\mathrm{Phc}(q)$
- $\overline{\mathrm{Phc}(q)}^{ess} = M$

2 another mechanism for SE

G. Núñez - D. Obata - JRH (2020)

generically in $\mathrm{Diff}^1_m(M^3)$

- dominated splitting
- $+$ a little partial hyperbolicity $(*)$
- $\Rightarrow$ stable ergodicity

2 another mechanism for SE

G. Núñez - D. Obata - JRH (2020)

- in the isotopy class of every partially hyperbolic $f \in \mathrm{Diff}^1_m(M^3)$
- there is a stably ergodic diffeomorphism
- which is not (strictly) partially hyperbolic

3 ergodic homoclinic classes

3 Pesin invariant manifolds

Pesin stable/unstable manifolds

$$W^-(x) = \left\{ y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f^n(x), f^n(y)) < 0 \right\}$$

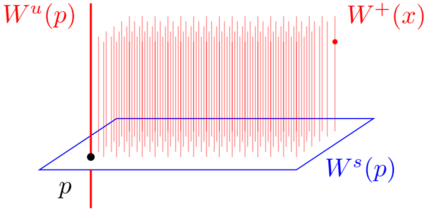
$$W^+(x) = \left\{ y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f^{-n}(x), f^{-n}(y)) < 0 \right\}$$

3 ergodic homoclinic class

ergodic homoclinic class (HHTU11)

$p \in \mathrm{Per}_H(f)$

$$\mathrm{Phc}^+(p) = \{x : W^+(x) \pitchfork W^s(o(p)) \neq \emptyset\}$$

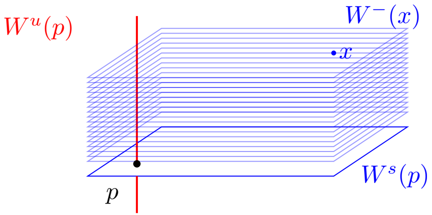


3 ergodic homoclinic class

ergodic homoclinic class (HHTU11)

$p \in \mathrm{Per}_H(f)$

$$\mathrm{Phc}^-(p) = \{x : W^-(x) \pitchfork W^u(o(p)) \neq \emptyset\}$$



3 criterion of ergodicity

criterion of ergodicity (HHTU11)

- $p \in \mathrm{Per}_H(f)$, $f \in \mathrm{Diff}^2_m(M)$
- $m(\mathrm{Phc}^-(p)) > 0$ and $m(\mathrm{Phc}^+(p)) > 0$

then

1. $\mathrm{Phc}^-(p) \overset{\circ}{=} \mathrm{Phc}^+(p) :\overset{\circ}{=} \mathrm{Phc}(p)$
2. $f|\,\mathrm{Phc}(p)$ ergodic

4 strategy

4 strategy

strategy

prove that C^1-densely, for $f \in \mathrm{Diff}^1_m(M)$

- $\exists\, p \in \mathrm{Per}_H(f)$
- $\exists\, \mathcal{U} \subset \mathrm{Diff}^1_m(M)$
- such that $g \in \mathcal{U} \cap \mathrm{Diff}^2_m(M) \Rightarrow$

$$m(\mathrm{Phc}(p)) = 1$$

4 strategy

setting

- $f \in \mathrm{Diff}^1_m(M^3)$
- with $TM = E \oplus E^u$ dominated splitting
- $\exists\ p \in \mathrm{Per}_H(f)$ with $u(p) = \dim(E^u) = 1$

4 strategy

strategy

on the one hand, use geometric approach to find

- $\exists\, \mathcal{U} \subset \mathrm{Diff}^1_m(M)$ such that
- $g \in \mathcal{U} \cap \mathrm{Diff}^2_m(M) \Rightarrow$

$$m(\mathrm{Phc}^u(p)) = 1$$

- $\Rightarrow m(\mathrm{Phc}^+(p)) = 1$

4 strategy

strategy

on the other hand, use generic approach to guarantee

- $\exists\, \mathcal{U} \subset \mathrm{Diff}^1_m(M)$ such that
- $g \in \mathcal{U} \cap \mathrm{Diff}^2_m(M) \Rightarrow$

$$m(\mathrm{Phc}^-(p)) > 0$$

- then

$$\mathrm{Phc}^+(p) \overset{\circ}{=} \mathrm{Phc}^-(p) \overset{\circ}{=} \mathrm{Phc}(p) \overset{\circ}{=} M$$

4 strategy

generic mechanism - M - B - H - ACW - AB

for a generic $f \in \mathrm{Diff}^1_m(M)$ with dominated splitting

- f ergodic
- $\exists q \in \mathrm{Per}_H(f)$ such that $\mathrm{Phc}(q) \overset{\circ}{=} M$
- Oseledets splitting is dominated $u(q) = \#\{LE > 0\}$
- $\exists \mathcal{U}(f)$ such that $m(\mathrm{Phc}(q)) > 0$ for all $g \in \mathcal{U}$

4 strategy

geometric mechanism - minimality

assume $f \in \mathrm{Diff}^1_m(M^3)$ with dominated splitting $TM = E \oplus E^u$ generic

- $\Rightarrow \exists p \in \mathrm{Per}_H(f)$ with $u(p) = \dim E^u = 1$
- if $\mathcal{F}^u$ is minimal
- $\exists\, \mathcal{U}(f)$ such that for all $g \in \mathcal{U}$

$$\mathrm{Phc}^u(p) = M$$

4 strategy

geometric mechanism - minimality

assume $f \in \mathrm{Diff}^1_m(M^3)$ with dominated splitting $TM = E \oplus E^u$ generic

- if $u(p) = u(q)$
- generically $\mathrm{Phc}(p) = \mathrm{Phc}(q)$ (HHTU - AC)
- on one hand $\mathrm{Phc}^+(p) = M$ for all $g \in \mathcal{U}$
- on the other hand $\mathrm{Phc}^-(p) > 0$ for all $g \in \mathcal{U}$
- $\Rightarrow f$ stably ergodic

4 strategy

geometric mechanism - minimality

assume $f \in \mathrm{Diff}^1_m(M^3)$ with dominated splitting $TM = E \oplus E^u$ generic

- if $u(q) > u(p)$
- $TM = E^s \oplus E^c \oplus E^u$
- $\Rightarrow f$ partially hyperbolic
- $\Rightarrow f$ stably ergodic

4 strategy

geometric mechanism - a little partial hyperbolicity

assume $f \in \mathrm{Diff}^1_m(M^3)$ with dominated splitting $TM = E \oplus E^u$ generic

- $\exists\, p \in \mathrm{Per}_H(f)$ with $u(p) = \dim(E^u) = 1$
- $\Rightarrow \mathrm{Phc}^u(p)$ is open
- assume
$$\Lambda(f) = M \setminus \mathrm{Phc}^u(p)$$
is partially hyperbolic $(*)$
- (NOH hypothesis)

4 strategy

geometric mechanism - a little partial hyperbolicity

- $g \mapsto \Lambda(g)$ continuous in f
- $\Lambda(g)$ PH
- with a generic argument,
- we see $m(\Lambda(g)) = 0$ for all C^2 $g \in \mathcal{U}$
- $\Rightarrow \mathrm{Phc}^u(p) \overset{\circ}{=} M$ for all $g \in \mathcal{U}$

4 strategy

geometric mechanism - a little partial hyperbolicity

- $u(p) = u(q)$
- generically $\mathrm{Phc}(p) = \mathrm{Phc}(q)$ (HHTU - AC)
- on one hand $\mathrm{Phc}^+(p) \overset{\circ}{=} M$ for all $g \in \mathcal{U}$
- on the other hand $\mathrm{Phc}^-(p) > 0$ for all $g \in \mathcal{U}$
- $\Rightarrow$ f stably ergodic $\square$

4 strategy

generic argument (if there is time)

- $m(\Lambda(f_n)) > 0$ with $f_n \to f$ $f_n \in C^2$
- $\Rightarrow \Lambda(f_n)$ *su*-saturated (P-JRH)
- $\Rightarrow \Lambda(f)$ *su*-saturated
- $\Rightarrow \exists\, q_1, q_2 \in \mathrm{Per}_H(f)$ such that

$$W^{ss}(q_1) \pitchfork_q W^{uu}(q_2) \neq \emptyset \qquad (H)$$

- $\Rightarrow$ non-generic situation (Kupka-Smale argument)

4 question

question

stable ergodicity

$\Downarrow$?

mixing

thank you!

thank you!

谢谢！