Locally moving groups acting on the real line

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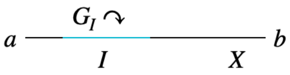
August 4, 2020

Locally moving groups

Main setting. Let $X = (a, b) \subseteq \mathbb{R}$ be an open interval.

Let $G \leq \mathrm{Homeo}_+(X)$. For $I \subset X$ let $G_I \leq G$ be the subgroup of elements supported on I.



Definition

The group $G \leq \mathrm{Homeo}_+(X)$ is **locally moving** if for every open interval $I \subset X$ the group G_I acts minimally on I.

Main theme. Locally moving groups detect the topology.

- Special case of a thm of Rubin: any two locally moving faithful actions of G on intervals are conjugate.
- Special case of a thm of Mann–Wolff: locally moving groups recognize maps.

What about the other actions† (if any) of G on $\mathbb{R}$?

Is it possible to describe the space of actions†?

† without global fixed points

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Is it possible to describe the space of actions?

Sample results.

(Assumptions are quite general, but slightly different for every result.)

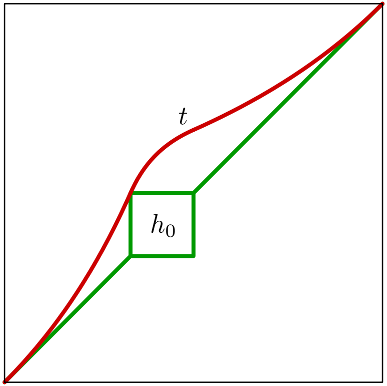
- C^1 *actions:* nothing interesting (standard or from quotients).
- *Local rigidity* of standard action.
- *Non-standard actions exist* (from germs at endpoints).

Toy example (not locally moving)

Lamplighter group (or wreath product$^{(\star)}$):
$\bigoplus_{\mathbb{Z}} \mathbb{Z} \rtimes \mathbb{Z} = \langle h_0, t \mid [t^n h_0 t^{-n}, h_0] \quad (n \in \mathbb{Z}) \rangle$

- *Abelian actions.* Blow-up actions of the abelianization $\mathbb{Z}^2$:
- *Affine actions.* h_0 unit translation, h scalar multiplication.
- *Plante action.* Action with no quasi-invariant Radon measure.

($\star$) in Spanish: *producto en corona*.



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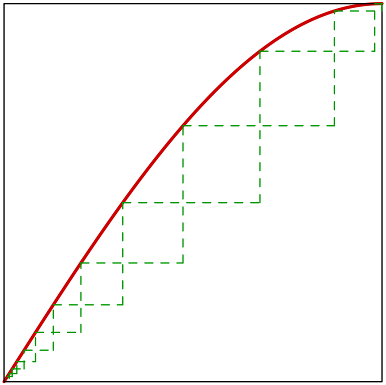

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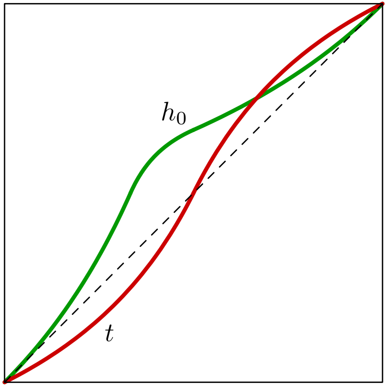

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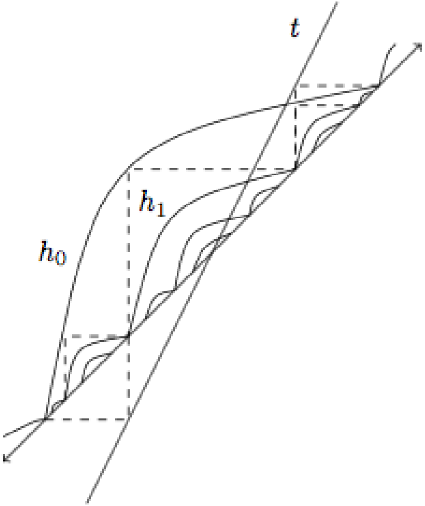

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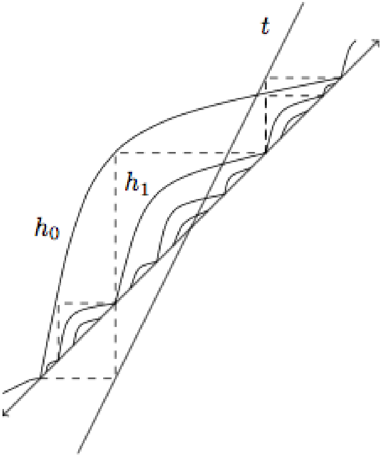
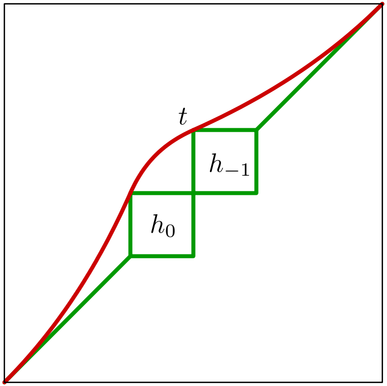
Plante action.

Write $h_k = t^{-k} h_0 t^k$, $L_n = \langle h_k \rangle_{k=n}^{\infty}$, then $t L_n t^{-1} = L_{n-1} \supset L_n$, $\bigcup_{n \in \mathbb{Z}} L_n = \bigoplus_{\mathbb{Z}} \mathbb{Z}$.





Sample results.

- C^1 *actions:* nothing interesting (standard or from quotients).
- *Local rigidity* of standard action.[A+B]
- *Non-standard actions exist* (from germs at endpoints).[A+C]

A: There exists $f \in G$ with *no fixed point*.
B: There exist *finitely generated* subgroups $H_a, H_b \subset G$ and $x < y$ in X such that

$$G_{(a,x)} \subset H_a \subset G_{(a,y)} \text{ and } G_{(x,b)} \supset H_b \supset G_{(y,b)}.$$

C: "Nice" group of germs at one endpoint of X

(nice: abelian + isolated fixed points)

For very large groups: Uniqueness of action up to conjugacy for $G = \mathrm{Homeo}_c(\mathbb{R})$ (Militon) and $G = \mathrm{Diff}^r_c(\mathbb{R})$ (Chen–Mann).

Actions from quotients: Let $G_c \leq G$ be the subgroup of **compactly supported elements**.

Proposition (this goes back to Higman)

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For countable groups: ?

Actions from orders: when G is countable, an action $G \hookrightarrow \mathsf{Aut}(S, \leq)$ on a totally ordered space determines a left-invariant order $\preceq$ on G, and thus an action on $\mathbb{R}$. This is called *dynamical realization*.

Dynamical realization

(you can do this more directly...)

Enumerate $G = \{g_0, g_1, \ldots\}$. Given $G \hookrightarrow \mathrm{Aut}(S, \leq)$, take $\{s_n\} \subset S$ such that $g(s_n) \neq s_n$.

Declare $g \succ \mathrm{id}$ if $g(s_k) > s_k$, where $k = \min\{n \mid g(s_n) \neq s_n\}$. This defines a *left-invariant order* $\preceq$ on G.

Define a *monotone embedding* $\iota : (G, \preceq) \hookrightarrow ((0,1), \leq)$, using the enumeration: at each step take the good middle point.

Fact: the left-action of G on itself induces an *action by homeomorphisms* on $\overline{\iota(G)} \subset [0,1]$.

Problem: is this giving *new* actions?

A significant example

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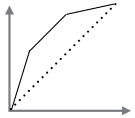
Thompson's group F.

This is the group of all piecewise linear homeomorphisms g of $X = (0,1)$ such that

1. g is piecewise $x \mapsto 2^k x + \frac{p}{2^q}$
2. the (finitely many) discontinuity points of derivatives are dyadic rationals.



Properties:

$[F,F] = F_c$ is simple, $F/F_c \cong \mathbb{Z}^2$

$$F = \langle f, g \mid [f, (gf)g(gf)^{-1}] = [f, (gf)^2 g(gf)^{-2}] = 1 \rangle$$

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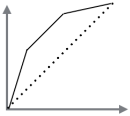

Proposition (this goes back to Brin)

Every locally moving group G of homeomorphisms of an interval contains a copy of F (with its standard action).

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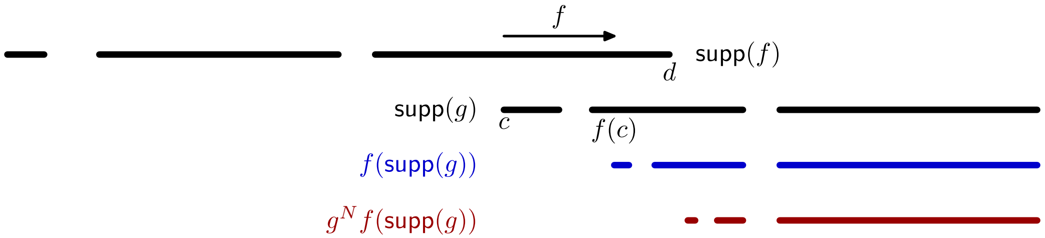
Lemma (2-chain lemma)

Take $f, g \in \mathrm{Homeo}_+(X)$, *write* $d = \sup \mathrm{Supp}(f)$, $c = \inf \mathrm{Supp}(g)$, *and assume the following:*

1. $c < d$,
2. $c \notin \mathrm{Fix}(f)$ *and* $d \notin \mathrm{Fix}(g)$,
3. d *and* $f(c)$ *are in the same connected component of* $\mathrm{Supp}(g)$.

Then $\langle f, g \rangle$ *contains a copy of* F.



$(g^N f) g^N (g^N f)^{-1}$ commutes with f (and so does $(g^N f)^2 g^N (g^N f)^{-2}$).

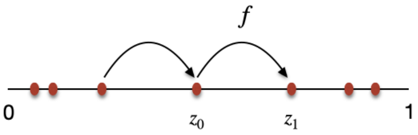
A non-standard action of F

Here $X = (0,1)$, $F \subset \mathrm{Homeo}_+(X)$.
Let $\tau_+ \colon F \to \mathrm{Germ}(F,1) \simeq \mathbb{Z}$ be the germ morphism ($\sim$ derivative).

Convention: $\tau_+(g) > 0$ if 1 is an attractive fixed point.

Choose:

- $f \in F$ with $\tau_+(f) = 1$ and $f(x) > x$ for $x \in (0,1)$.
- An orbit $(z_n) \subset X$ of f ($z_n = f^n(z_0)$).



We let F act on the set of sequences $X^{\mathbb{Z}} = \{(x_n)_{n\in\mathbb{Z}} \subset X\}$ by

$$g \cdot (x_n) = (g x_{n-\tau_+(g)}).$$

(Note that $f \cdot (z_n) = (z_n)$.)

Let $\Omega \subset X^{\mathbb{Z}}$ be the F-orbit of (z_n) w.r.t. this action.

Key point: Every $(x_n) \in \Omega$ coincides with (z_n) for $n >> 0$.

Define an order $\prec$ on Ω:

$(x_n) \prec (y_n)$ if $x_m < y_m$ for $m = \max\{n \colon x_n \neq y_n\}$.

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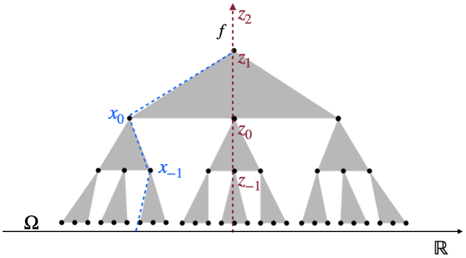
Then $F \hookrightarrow \mathrm{Aut}(\Omega, \prec)$ and by taking the dynamical realization we obtain $\varphi \colon F \to \mathrm{Homeo}_+(\mathbb{R})$.

Fact: φ is minimal and faithful, with $\varphi(f)$ conjugate to the homothety $x \mapsto 2x$.



Theorem (Dynamical trichotomy for actions of F)

Every action $\varphi\colon F \to \mathrm{Homeo}_+(\mathbb{R})$, *without global fixed points, satisfies one of the following.*

1. *(Abelian) It is semi-conjugate to an action by* translations *of* $F^{ab} \simeq \mathbb{Z}^2$.
2. *(Standard) It is semi-conjugate to the natural action of* F *on* $(0,1)$.
3. *It is of* focal type *with domination of the right germ morphism* $\tau_+\colon F \to \mathrm{Germ}(F,1) \simeq \mathbb{Z}$:
   - *For every* g *such that* $\tau_+(g) \neq 0$ *the element* $\varphi(g)$ *has a compact set of fixed points which is repelling if* $\tau_+(g) > 0$ *and attractive if* $\tau_+(g) < 0$.
   - *(*Dynamics of $F^+ = \ker \tau_+$*). Write* $F^+ = \bigcup_{x\in(0,1)} F_{(0,x)}$. *For every* x *the image of* $F_{(0,x)}$ *has global fixed points which accumulate on both* $\pm\infty$. *In contrast the image of* $F^+_{(x,1)}$ *has no global fixed point.*
4. *it is of focal type with domination of the left germ morphism* $\tau_-\colon F \to \mathrm{Germ}(F,0) \simeq \mathbb{Z}$.

Harmonic actions and local rigidity

Let $G = \langle S \rangle$ with finite symmetric $S = S^{-1}$.
Fix a symmetric prob measure μ on S.

Definition

An action $\varphi\colon G \to \mathrm{Homeo}_+(\mathbb{R})$ is μ-**harmonic** if for every $x \in \mathbb{R}$

$$\sum_{s \in S} \mu(s)\varphi(s)(x) = x.$$

Theorem (Deroin–Kleptsyn–Navas–Parwani)

Fix (G, μ) as before. Then every action $\varphi\colon G \to \mathrm{Homeo}_+(\mathbb{R})$ w/o global fixed points, is semi-conjugate to a μ-harmonic action $\bar{\varphi}$, which is unique up to conjugation by an affine map $x \mapsto ax + b$.

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In the space of actions $\mathcal{R}_G$, one can consider the subspace $\mathcal{H}^1_{(G,\mu)}$ of **normalized** harmonic actions, which

- is **compact**,
- contains a representative of every semi-conjugacy class, unique up to conj. by translations $T_t\colon x \mapsto x + t$ and by the reflection $x \mapsto -x$,
- is endowed with the **translation flow** $\Psi\colon \mathbb{R} \times \mathcal{H}^1_{(G,\mu)} \to \mathcal{H}^1_{(G,\mu)}$,

$$\Psi^t(\varphi)(g) = T_t \varphi(g) T_{-t}.$$

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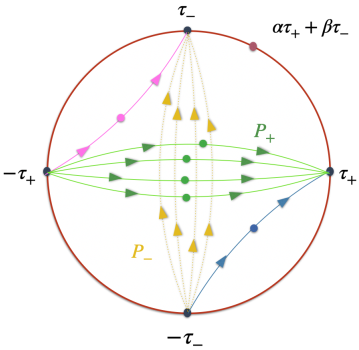
Theorem

A normalized harmonic action $\varphi \in \mathcal{H}^1_{(G,\mu)}$ is locally rigid iff transversely isolated in $\mathcal{H}^1_{(G,\mu)}$.

The space of harmonic actions of F

The space $\mathcal{H}^1_{F,\mu}$ of normalized harmonic actions of F consists of:

1. actions by translations $\varphi\colon F \to (\mathbb{R}, +)$, $\varphi = \alpha\tau_+ + \beta\tau_-$, $\alpha^2 + \beta^2 = 1$,
2. the standard action, and its reflection,
3. the subsets P_+ and P_- of focal actions dominated by τ_+ and τ_- (resp.).



Theorem

The subsets P_+ and P_- admit compact transversals *intersecting every Ψ-orbit in exactly one point, which converge* uniformly *to the limits shown in the picture as $t \to \pm\infty$.*

Corollary

The standard action of F is locally rigid.