A COMPLETE CLASSIFICATION OF TRANSITIVE, TOTALLY CARTAN ACTIONS OF $\mathbb{R}^k \times \mathbb{Z}^\ell$

Joint w/ Ralf Spatzier

Setting

$\mathbb{R}^k \times \mathbb{Z}^\ell \curvearrowright X$ locally free

$X = C^\infty$-manifold

$a \in \mathbb{R}^k \times \mathbb{Z}^\ell$ is Anosov if $\exists$ continuous a-inv. splitting $TX = E^s_a \oplus T\mathcal{O} \oplus E^u_a$ s.t

$$\|a_* v\| < \lambda \|v\| \quad \forall v \in E^s_a$$

$$\|a_*^{-1} v\| < \lambda \|v\| \quad \forall v \in E^u_a$$

For some fixed $\lambda = \lambda(a) < 1$.

A rank one factor of an action $\mathbb{R}^k \times \mathbb{Z}^\ell \curvearrowright X$ consists of the following data:

- A group action $A \curvearrowright Y$, where $A \cong$ a compact extension of $\mathbb{R}$ or $\mathbb{Z}$
- A homomorphism $\sigma : \mathbb{R}^k \times \mathbb{Z}^\ell \to A$
- A projection $\pi : X \to Y$

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ Y \arrow[r, "\sigma(a)"] & Y \end{tikzcd}$$

Conjecture

Every $\mathbb{R}^k \times \mathbb{Z}^\ell \curvearrowright X$ with an Anosov element without C^∞-rank one factors has a finite cover smoothly conjugate to an affine action.

Affine Actions

Ex 1 (Toral Automorphisms)

$$A_1, \ldots, A_\ell \in SL(d, \mathbb{Z})$$

$$\rightsquigarrow$$

$$\mathbb{Z}^\ell \curvearrowright \mathbb{T}^d$$

Ex 2 (Weyl chamber flows)

$$\mathbb{R}^{d-1} \cong \left\{ \begin{pmatrix} e^{t_1} & & \\ & \ddots & \\ & & e^{t_d} \end{pmatrix} : \sum t_i = 0 \right\} \subset SL(d, \mathbb{R})$$

$$\mathbb{R}^{d-1} \curvearrowright SL(d, \mathbb{R})/\Gamma$$

$$t : g\Gamma \mapsto \begin{pmatrix} e^{t_1} & & \\ & \ddots & \\ & & e^{t_d} \end{pmatrix} g\Gamma$$

Lyapunov "Exponents" without measures

An action $\mathbb{R}^k \times \mathbb{Z}^\ell \curvearrowright X$ is **totally Anosov** if the set of Anosov elements of $\mathbb{R}^k \times \mathbb{Z}^\ell$ are dense in [redacted] $\mathbb{R}P^{k+\ell-1}$

Lemma Assume $\mathbb{R}^k \times \mathbb{Z}^\ell \curvearrowright X$ is totally Anosov.

$\exists !$ splitting $T_x X = T\mathcal{O} \oplus \bigoplus_{\chi \in \Delta} E^\chi$

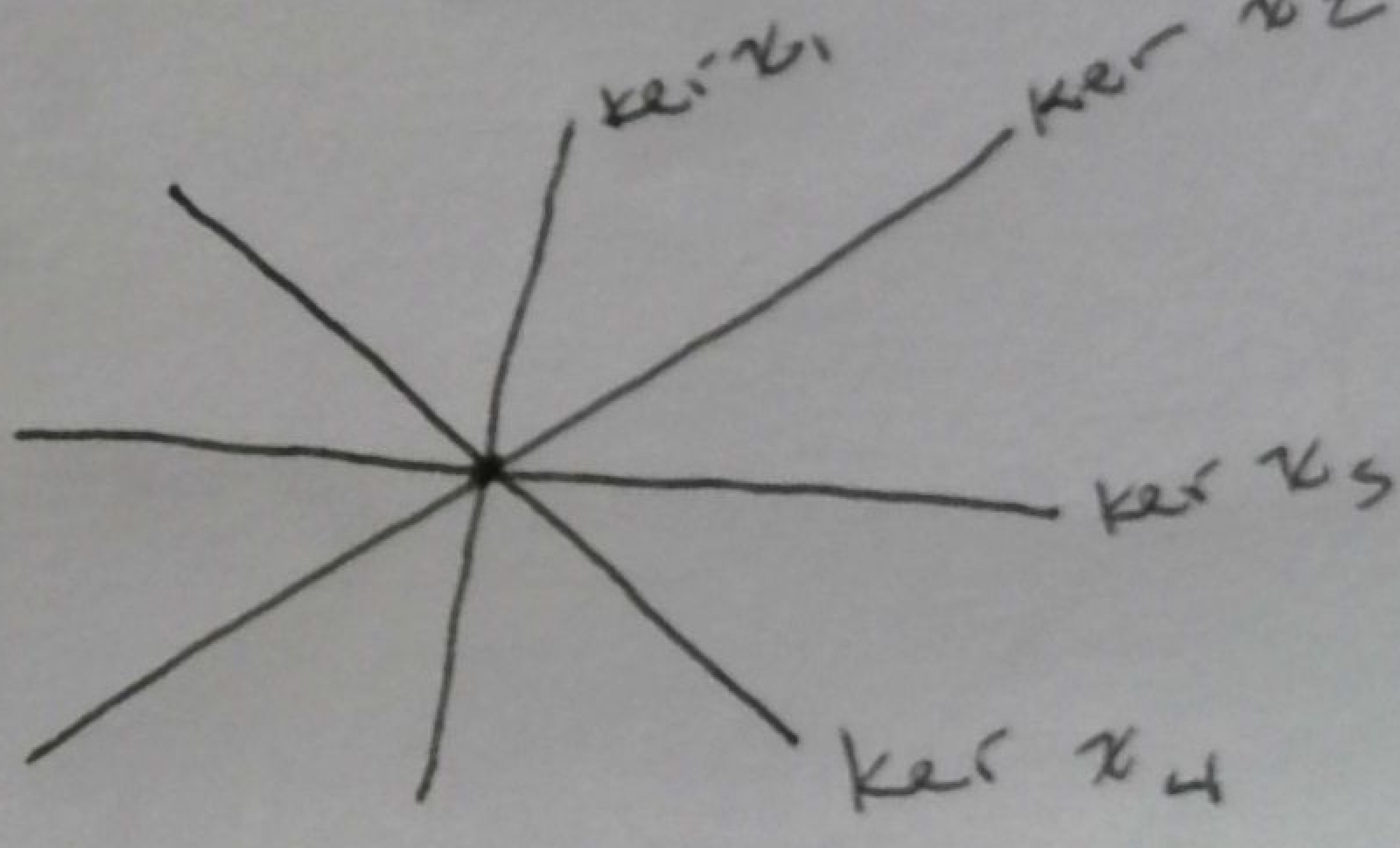
satisfying the following:

- Each $\chi \in \Delta$ is a functional $\chi : \mathbb{R}^k \times \mathbb{Z}^\ell \to \mathbb{R}$
- The set of Anosov elements is [redacted] $\left(\bigcup \ker \chi\right)^c$
- If a is Anosov, then

$$E_a^s = \bigoplus_{\chi(a)<0} E^\chi$$

$$E_a^u = \bigoplus_{\chi(a)>0} E^\chi$$

- Each E^χ is integrable to a foliation W^χ



$\mathbb{R}^k \times \mathbb{Z}^\ell \curvearrowright X$ is totally Cartan if it is totally Anosov and E^χ is 1-dimensional $\forall \chi \in \Delta$

THEOREM Let $\mathbb{Z}^\ell \curvearrowright X$ be a transitive, totally Cartan action without a C^∞ rank one factor. Then the action is C^∞ conjugate to an automorphism action of a nilmanifold.

$\mathbb{R}^k$-actions: More Malleable.

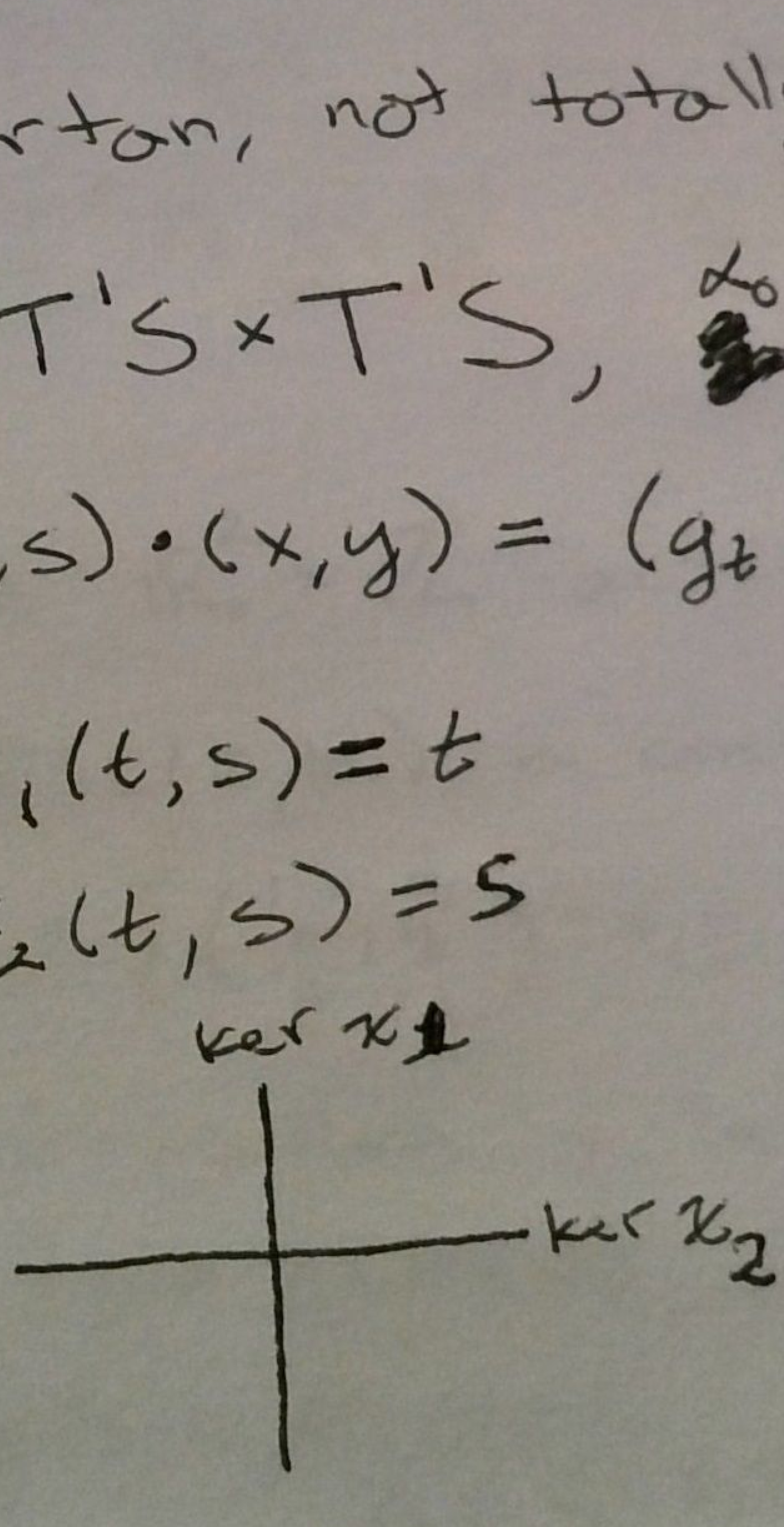
Ex 3 (Cartan, not totally Cartan)

$$X = T^1S \times T^1S, \quad \alpha_0 : \mathbb{R}^2 \curvearrowright X$$

$$\alpha_0 : (t,s)\cdot(x,y) = (g_t x, g_s y)$$

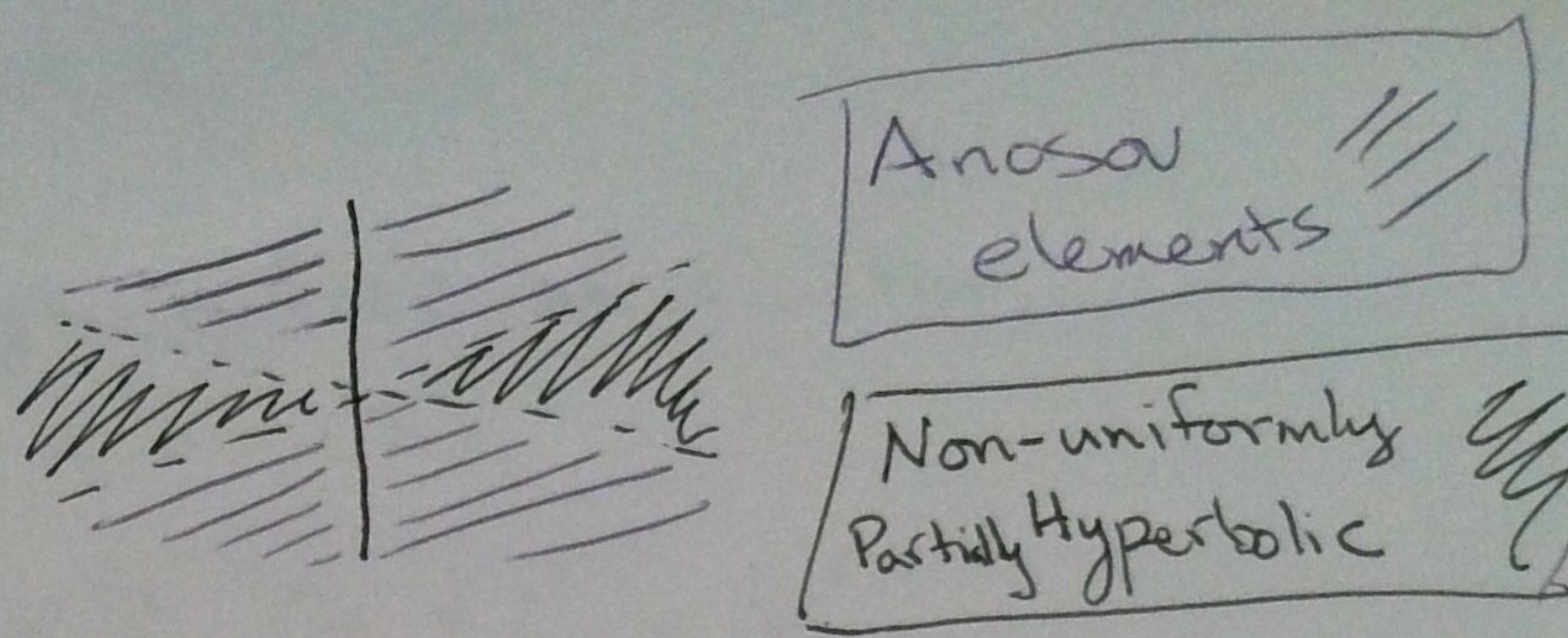
$$\chi_1(t,s) = t$$

$$\chi_2(t,s) = s$$



$\varphi : T^1S \to \mathbb{R}$ nontrivial cocycle

$$\alpha : (t,s)\cdot(x,y) \longmapsto \left(g_t(x),\ g_{s+\varphi(t,x)}(y)\right)$$



Ex 4 (Starkov component)

$\mathbb{R}^k \times \mathbb{Z}^\ell \curvearrowright X$ any totally Cartan action

$\mathbb{R} \times \mathbb{R}^k \times \mathbb{Z}^\ell \curvearrowright X \times S^1$ is also totally Cartan

A special $\mathbb{R}^2 \times \mathbb{Z}$ action (Starkov)

$\Gamma \subset PSL(2,\mathbb{R})$ a cocompact surface group

$\text{Heis} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$

$\text{Heis}(\mathbb{Z}) \subset \text{Heis}$ a lattice, $A : \text{Heis} \circlearrowleft$ Auto st

$\langle A, Z(\text{Heis}) \rangle \cong \mathbb{R} \times \mathbb{Z}$

$\psi : \Gamma \to Z(\text{Heis})$

$\Lambda \subset PSL(2,\mathbb{R}) \times \text{Heis}$

$\Lambda = \langle (\gamma, \psi(\gamma)), \text{Heis}(\mathbb{Z}) \rangle$

THEOREM

Let $\mathbb{R}^k \curvearrowright X$ be a totally Cartan, transitive action. Then

$$S = \bigcap_{\chi \in \Delta} \ker \chi$$

factors through a free torus action, and if $k' = k - \dim(S)$,

$$\mathbb{R}^{k'} = \mathbb{R}^k/S \curvearrowright X/S = X'$$

is a totally Cartan action.

$\exists$ a homogeneous action

$$\mathbb{R}^{k_1} \curvearrowright G/\Gamma$$

and finitely many Anosov flows on 3-manifolds

$$\varphi_t^{(1)} \curvearrowright Y_1$$

$$\varphi_t^{(2)} \curvearrowright Y_2$$

$$\vdots$$

$$\varphi_t^{(k_2)} \curvearrowright Y_{k_2}$$

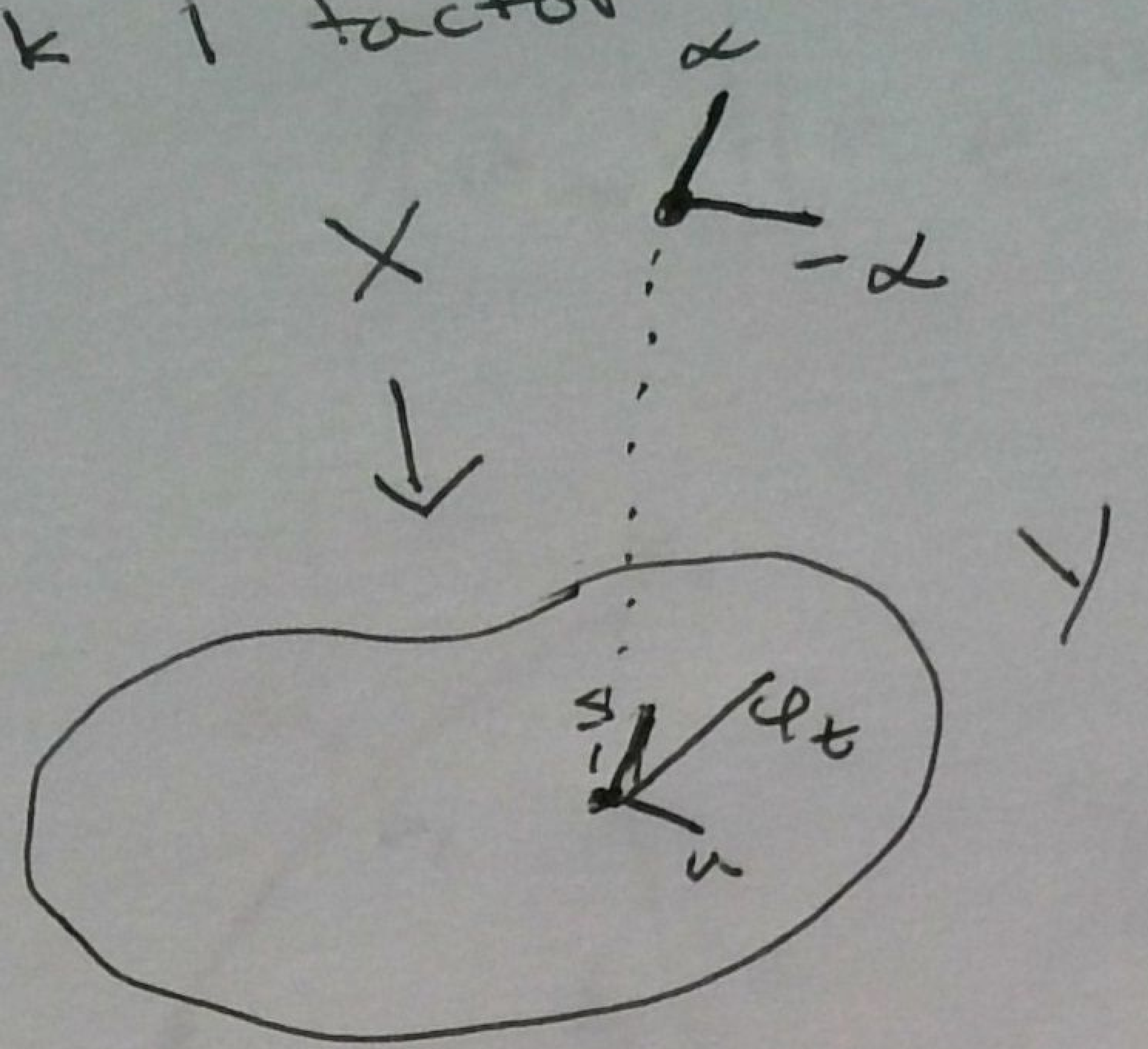
$$\sigma : \mathbb{R}^{k'} \hookrightarrow \mathbb{R}^{k_1+k_2}$$

$$\tilde{X}' \hookrightarrow G/\Gamma \times Y_1 \times \ldots \times Y_2$$

The toolbox

- Consider the way the coarse Lyapunov foliations W^x interact with one another (Holonomies)

- Rank 1 factor



The fiber should have tangent space $\underbrace{\ker\alpha \oplus \bigoplus_{\beta \neq \pm t} E^\beta}$

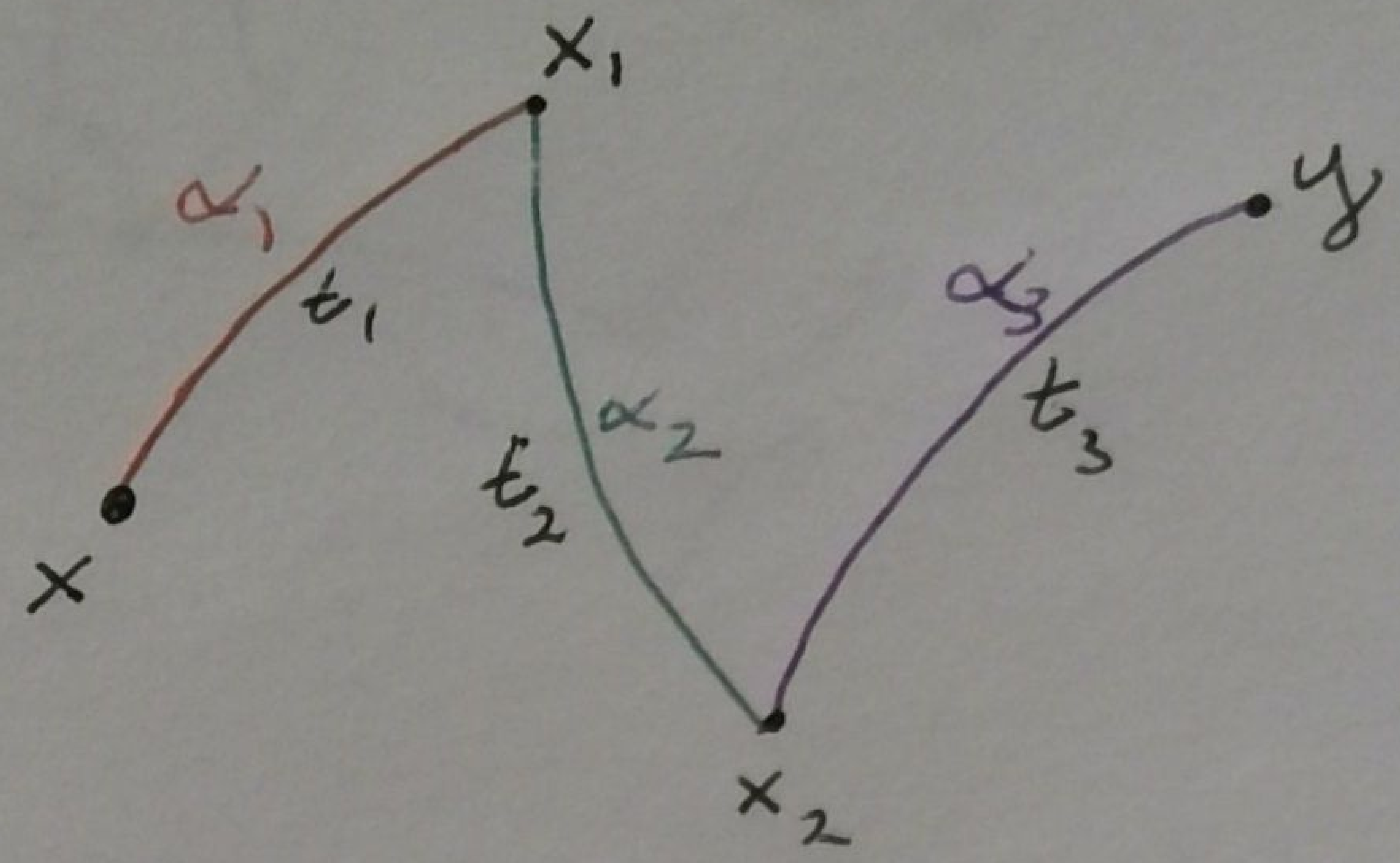
PROPOSITION Either $\ker\alpha$ has a dense orbit, or this distribution integrates to a foliation, and induces a factor.

Very difficult

The Hammer

If $\ker \alpha$ has a dense orbit $\forall \alpha \in \Delta$, $\exists$ metric on E^α st

$$\|a_* v\| = e^{\alpha(a)} \|v\| \quad \forall v \in E^\alpha$$



$$y = t_3^{(\alpha_3)} * t_2^{(\alpha_2)} * t_1^{(\alpha_1)} \cdot x$$

$$\hat{P} = \mathbb{R}^k \ltimes \underbrace{(\mathbb{R} * \mathbb{R} * \mathbb{R} \dots * \mathbb{R})}_{\text{One copy for each } \alpha \in \Delta}$$

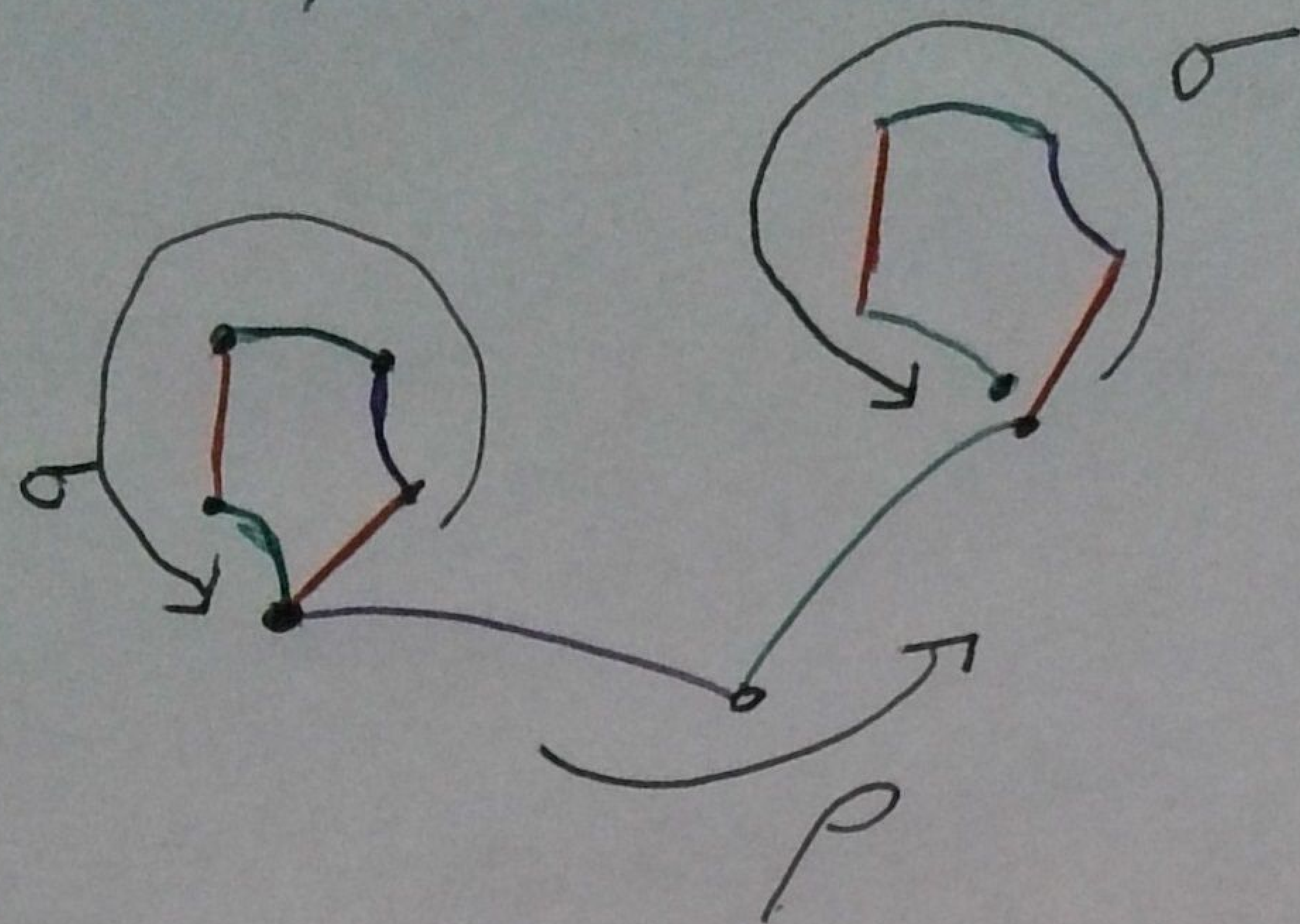
$$\hat{P} \curvearrowright X$$

Our method: Find a co-finite dimensional normal subgroup of $\mathrm{stab}(x)$

Normality of subgroups

$\sigma \in \mathrm{Stab}(x)$, $\rho \in \hat{\mathcal{P}}$

$\rho \sigma \rho^{-1} \in \mathrm{Stab}(x)$?



Normality $\longleftrightarrow$ $\sigma \in \mathrm{Stab}(y)\ \forall y$

THREE CRUCIAL STEPS

I. "Commutator" relations are independent of x (Difficult dynamical argument)

II. "Symplectic" relations are independent of x (Standard dynamical argument)

III. Such relations generate a co-finite dimensional subgroup of $\hat{\mathcal{P}}$ (Difficult algebraic argument)