A HYPERBOLIC DAY ONLINE

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SMOOTH CONJUGACY CLASSES OF 3D AXIOM A CONTACT FLOWS

RIGIDITY QUESTIONS

given 2 smooth dynamical systems which are topologically conjugate, when are they actually smoothly conjugate?

- CIRCLE DIFFEOMORPHISMS: Arnold, Herman, Yoccoz, ...
  (RENORMALIZATION TECHNIQUES)

- HYPERBOLIC SYSTEMS: each periodic orbit carries a possible obstruction to smooth conjugation

Anosov systems (de la Llave - Marco - Moriyon)

Hyperbolic sets for surface diffeomorphisms (Pinto - Rand, Bedford - Fisher, ...)

CONTACT 3D AXIOM A FLOWS. what are they?

Contact structure.

M oriented 3-manifold, α contact form on M

$\xi = \operatorname{Ker}(\alpha)$ contact structure.

Reeb vector field X associated to α s.t. $\begin{cases} i_X d\alpha \equiv 0 \\ i_X \alpha \equiv 1 \end{cases}$

Reeb flow ϕ^t is flow generated by X

EXAMPLE

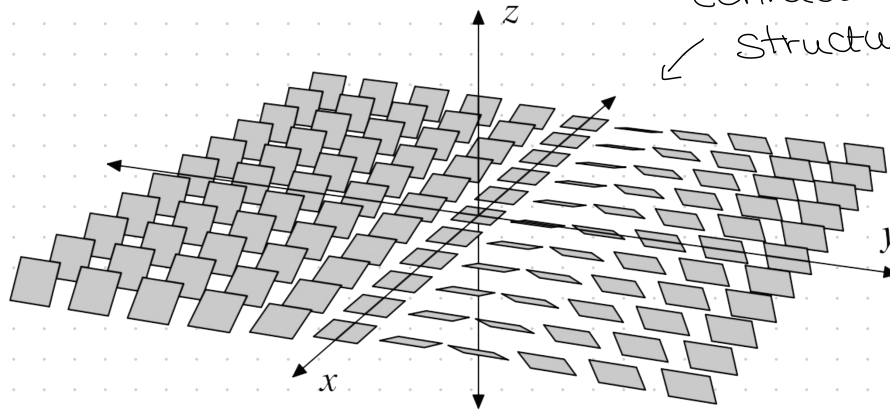
$M = \mathbb{R}^3$

$\alpha = -y\,dx + dz$

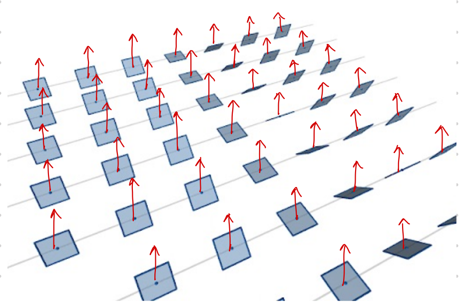
$\xi = \langle \partial_y, \partial_x + y\partial_z \rangle$

$X = \partial_z$

Contact structure



Reeb vector field



Axiom A flows.

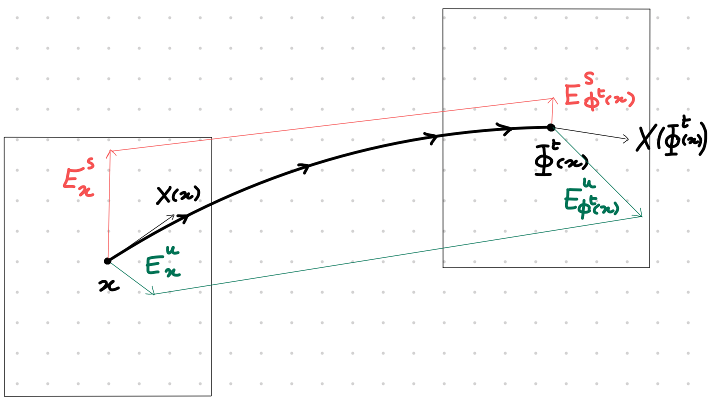
Let $\phi : M \times \mathbb{R} \to M$ be a $\mathcal{C}^1$ flow on a 3D mnfd M.

A compact set $\Lambda \subset M$ is a HYPERBOLIC SET for ϕ if

(i) Λ is ϕ-invariant,

(ii) $\forall x \in \Lambda$

$$T_x M = \underbrace{E^s_x}_{\text{unif. contraction under } D\phi^1} \oplus \underbrace{E^u_x}_{\text{unif contraction under } (D\phi^1)^{-1}} \oplus \underbrace{\mathbb{R}X(x)}_{\text{flow vector field}}$$



A point $x \in M$ is NON-WANDERING if

$\forall U$ nghbd of x and $\forall T > 0$, $\exists t \geq T$ s.t $U \cap \phi^t(U) \neq \emptyset$

$\Omega(\phi) = \{$ non-wandering pts $\}$

A flow $\phi : M \times \mathbb{R} \to M$ is **AXIOM A** if

(i) $\Omega(\phi)$ is hyperbolic;

(ii) periodic points are dense in $\Omega(\phi)$.

SPECTRAL DECOMPOSITION THM (Smale)

If ϕ is Axiom A, then $\Omega(\phi) = \bigcup_i \Lambda_i$

where

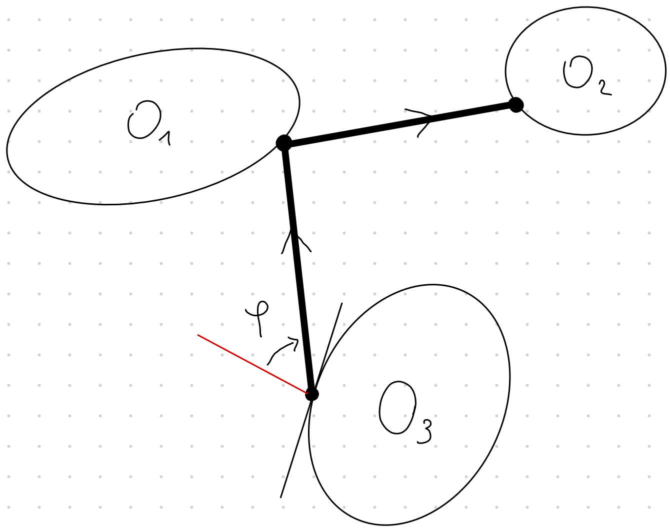
$\{\Lambda_i\}_i$ is a finite collection of BASIC SETS

- transitive
- locally maximal hyperbolic set
- periodic pts dense in Λ_i

EXAMPLE

- Transitive Anosov flow
- Suspension of Axiom A diffeomorphisms

EXAMPLE OF CONTACT 3D AXIOM A FLOW $\longrightarrow$ open dispersing billiards



$O_1 \cup O_2 \cup O_3 \subseteq \mathbb{R}^2$

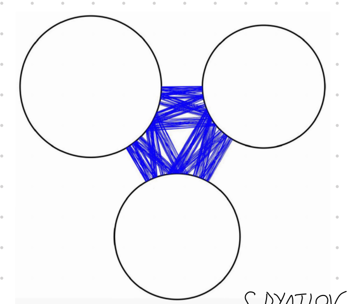
O_i convex domain

Coordinates (r, s, t)

$r = \sin \varphi$

$s =$ arc length parameter

$t =$ distance to next collision



TRAPPED SET

Contact form: $-r\,ds + dt$

Axiom A: hyperbolicity on $\Omega(\phi)$, i.e orbits not escaping to ∞

THM (F-LEGUIL)

Let ϕ_1, ϕ_2 be AXIOM A flows of class $\mathcal{C}^K$, $K \geq 2$, on 3D manifolds M_1, M_2

Assume that ϕ_i preserves a contact form α_i $i = 1, 2$.

If $\exists \Lambda_i \subset M_i$ basic set for ϕ_i, $i = 1, 2$, and

$\exists \Psi_0 : \Lambda_1 \to \Lambda_2$ **ORBIT EQUIVALENCE** such that

$$\forall x \in \Lambda_1 \cap \mathrm{Per}(\phi_1) \qquad L_{\phi_1}(x) = L_{\phi_2}(\Psi_0(x))$$

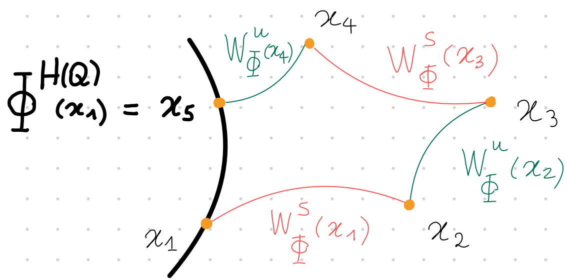
then

- $\phi_1|_{\Lambda_1}$ and $\phi_2|_{\Lambda_2}$ are conjugated by Ψ of class $\mathcal{C}^K$ (in whitney sense)
- $\Psi^* \alpha_2|_{\Lambda_1} = \alpha_1|_{\Lambda_1}$

Sketch of the proof

1) LIVSIC THM : we obtain a flow conjugacy

2) QUADRILATERALS :
(OTAL)



Lemma. temporal displacement $H(Q)$ determined by periodic data $\Rightarrow$ same for Φ_1, Φ_2

3) $H(Q)$ = Area of surface bounded by the quadrilateral Q

4) $\exists$ $\partial_s \Psi, \partial_u \Psi$ + Ψ is C^k along stable/unstable direction
(thermodynamical formalism)

5) Journé Lemma (C^k in whitney sense)

just in case

More ideas of the proof

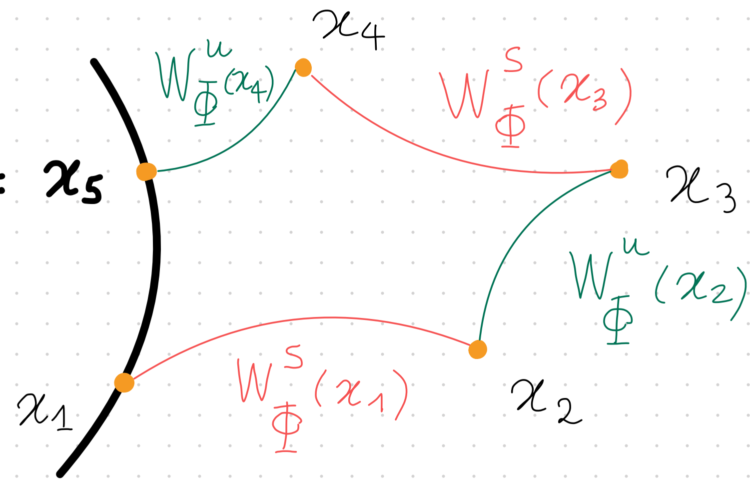
(1) LIVSIC THM : we obtain a flow conjugacy Ψ

(2) BUILDING QUADRILATERALS (OTAL)

$Q = (x_1, x_2, x_3, x_4)$

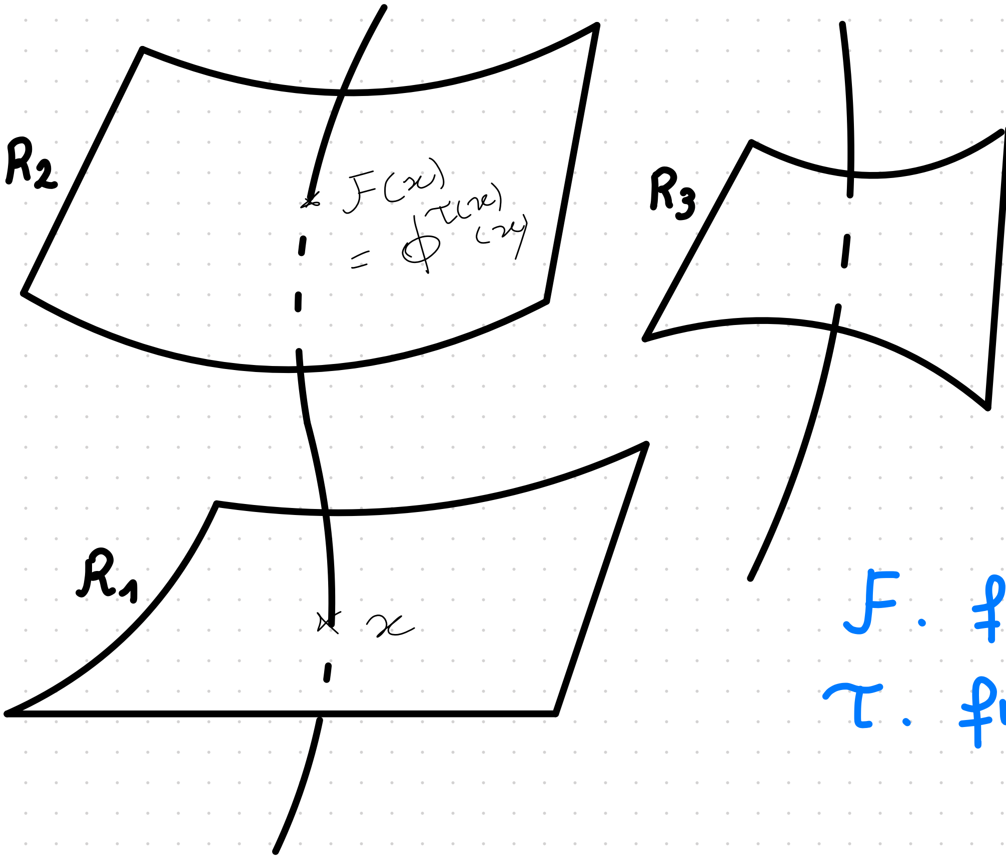
$x_i \in \Lambda$

$$\Phi^{H(Q)}(x_1) = x_5$$



$H(Q)$ temporal displacement

(3) Stable/unstable holonomies (Markov families):



$S = \{ R_j \}_j$

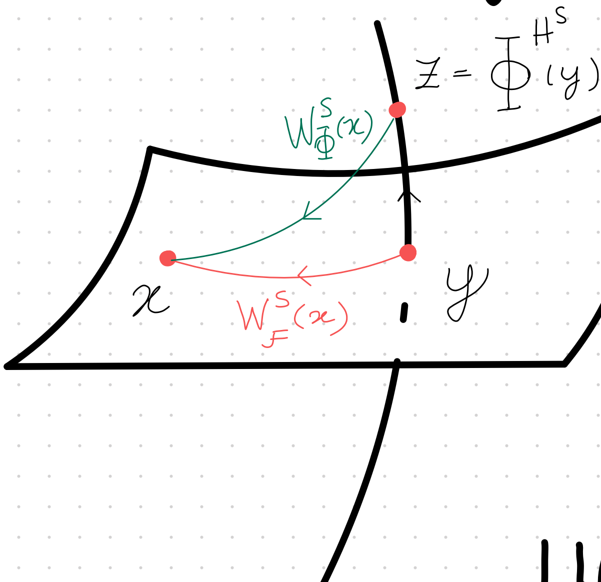
R_j rectangles transverse to Φ

(Markovian)

F. first return map on S

τ. first return time

Stable holonomy.



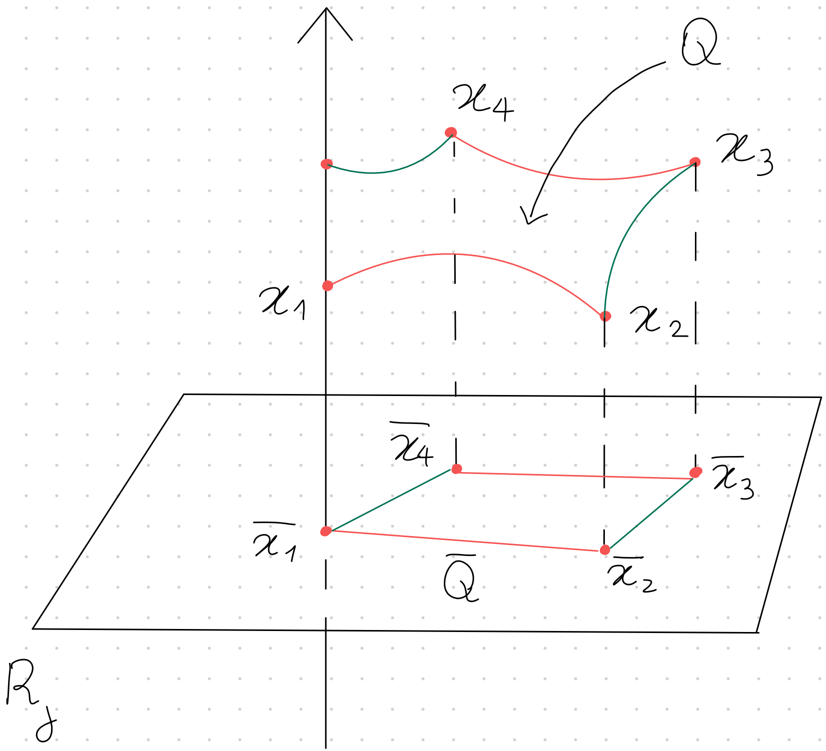
$$H^s = H^s(x,y)$$
$$= \sum_{j=0}^{+\infty} \left[\tau(F^j(y)) - \tau(F^j(x)) \right]$$

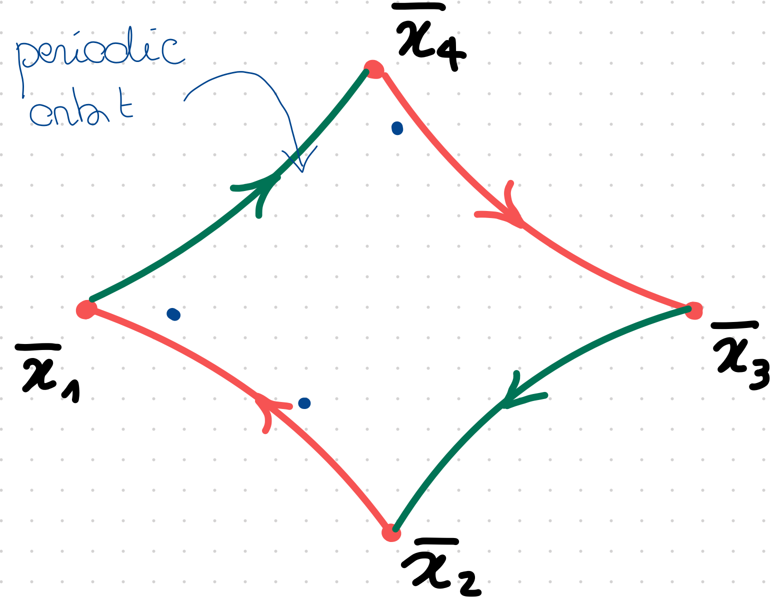
temporal displacement

$$H(Q) =$$

$$= \lim_{m \to +\infty} \sum_{j=-m}^{m} \left[-\tau(F^j(x_1)) + \tau(F^j(x_2)) - \tau(F^j(x_3)) + \tau(F^j(x_4)) \right]$$

(4) $H(Q)$ in terms of lengths of periodic orbits:

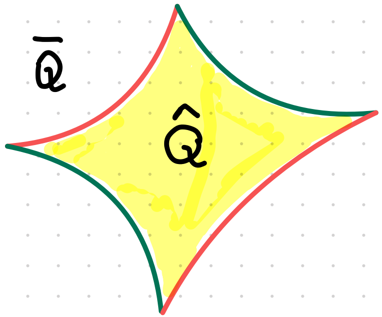




$\bar{x}_1, \bar{x}_3$ fixed points

$\Rightarrow$ We can describe $H(Q)$ in terms of periodic data

(5) Stokes : $H(Q) = \mathrm{Area}(\hat{Q})$



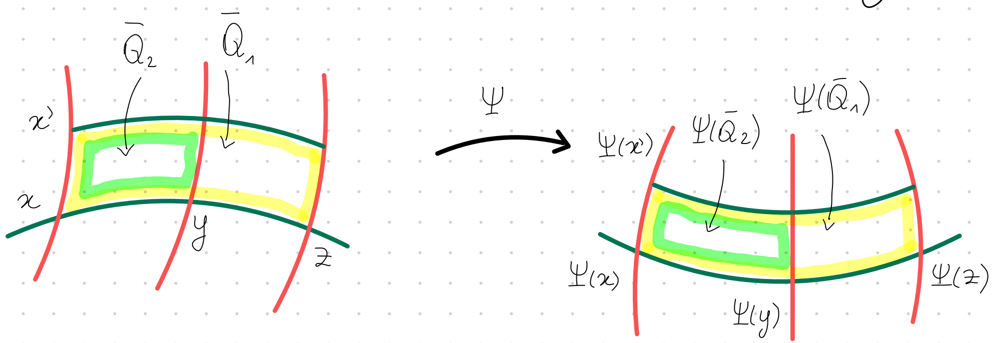
Φ_1 and Φ_2 have same length data

$\Downarrow$

$$H(Q) = H(\Psi(Q))$$

(6) Existence of $\partial_s \Psi$, $\partial_u \Psi$

$x, y, z, x' \in \bar{\Lambda}_1$



$$\frac{d(x,z)}{d(x,y)} \approx \frac{H(Q_1)}{H(Q_2)} = \frac{H(\Psi(Q_1))}{H(\Psi(Q_2))} \approx \frac{d(\Psi(x),\Psi(z))}{d(\Psi(x),\Psi(y))}$$

$$\Rightarrow \quad \frac{d(\Psi(x),\Psi(y))}{d(x,y)} \approx \frac{d(\Psi(x),\Psi(z))}{d(x,z)}$$

if $x_n \in \Lambda_1 \to x$

$\left(\dfrac{d(\Psi(x),\Psi(x_n))}{d(x,x_n)}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence $\Rightarrow$ $\partial_s \Psi$ is $\mathcal{C}^{\beta}$ $\quad \beta\in(0,1)$

(7) Journé lemma (Nicol-Török)

Ψ is $\mathcal{C}^{1,\beta}$ in Whitney sense

(8) Preservation of contact structures

$$X_1 \longrightarrow X_2$$

$$\langle E^u_{1}, E^s_1 \rangle \to \langle E^u_{2}, E^s_2 \rangle \quad \text{contact plane}$$

(9) Ψ is $\mathcal{C}^k$ $\to$ thermodynamical formalism

Potential $\phi^u_i := -\delta^u \log \| DF_i|_{E^u_{F_i}} \|$ with $\delta^u := \dim_H \left(W^u_{F_i, loc}(x) \cap \Lambda_i \right)$

for $x \in \Lambda_i$

$\to$ equilibrium state μ^u_i

ϕ^u_1 and $\Psi^* \phi^u_2$ are cohomologous $\Rightarrow$ $\Psi^* \mu^u_2 = \mu^u_1$

It holds:

(de la Llave)

$$\partial_u \Psi(x) = \left(\frac{\rho^u_{1,x}(x)}{\rho^u_{2,\Psi(x)} \circ \Psi(x)} \right)^{1/\delta^u}$$

with $\mathcal{C}^{k-1}$ densities ρ^u_i

(Bootstrap) $\blacksquare$