Expansive flow-models for geodesic flows

Katrin Gelfert

Federal University of Rio de Janeiro (UFRJ), Brazil

Anosov flows

X compact Riemannian manifold, $f^t\colon X \to X$ C^2 **Anosov flow**

- invariant splitting $TX = E^{\mathrm{u}} \oplus E^0 \oplus E^{\mathrm{s}}$ and $C, \lambda > 0$:

$$\frac{d}{dt} f^t(x) \in E^0_x, \quad \|df^t|_{E^{\mathrm{s}}_x}\| \le Ce^{-\lambda t}, \quad \|df^{-t}|_{E^{\mathrm{u}}_x}\| \le Ce^{-\lambda t}$$

- $x \mapsto E^\star_x$, $\star \in \mathrm{s}/\mathbf{0}/\mathrm{u}$, Hölder continuous,
  bundle integrates to foliation $\mathscr{W}^\star$, $\star \in \mathrm{s}/\mathbf{0}/\mathrm{u}$
  foliations have local product structure

[Bowen, Ruelle, Sinai 1970s]

ergodic properties for mixing Anosov systems
thermodynamic properties (measures of maximal entropy (mme), equilibrium states)
statistical properties

Geodesic flow for curvature $K < 0$ *surface*

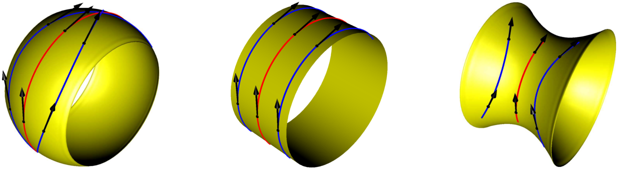
... is Anosov flow

M compact connected C^∞ Riemannian surface, negative Euler characteristic

- $\gamma_v(\cdot)$ geodesic curve with $\dot{\gamma}_v(0) = v$
- $G = \{g^t\}$ geodesic flow $g^t \colon T^1M \to T^1M$ acts by $g^t(v) = \dot{\gamma}_v(t)$

for $v \approx w$, $d(t)$ "distance" between $\gamma_v(t), \gamma_w(t)$ and with $\kappa(t) = K(\gamma_v(t))$

$$\ddot{d} \approx -\kappa d \quad \text{(Jacobi equation)}$$



Positive curvature **concave** | Zero curvature **linear** | Negative curvature **convex**

[Figure borrowed from V. Climenhaga's talk]

Geodesic flow for curvature $K \leq 0$ *surface*

... very-close-to-hyperbolic structure

$t \mapsto J(t) \in T_{\gamma(t)}TM$ **Jacobi field** if

$$J''(t) = -K(\gamma_v(t))\, J(t)$$

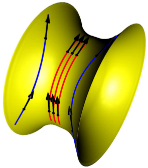
- $\{$**orthogonal** $_{J(0),\, J'(0) \in v^\perp}$ Jacobi fields$\}$ spans 2-dim. vector space
- $\|dg^t(\xi)\|^2 = \|J_\xi(t)\|^2 + \|J'_\xi(t)\|^2$.
- naturally defined *continuous* sub-bundles
  - $F^0_v \stackrel{\text{def}}{=} \langle \frac{d}{dt} g^t(v)|_{t=0} \rangle$ flow direction
  - $F^{\mathrm{s}}_v \stackrel{\text{def}}{=} \{\xi \in T_v T^1 M \colon \|J_\xi\|$ non-decreasing$\}$,
    $F^{\mathrm{u}}_v \stackrel{\text{def}}{=} \{\xi \in T_v T^1 M \colon \|J_\xi\|$ non-increasing$\}$, $F^{\mathrm{c}\star} \stackrel{\text{def}}{=} F^0 \oplus F^\star$, $_{\star = \mathrm{s}/\mathrm{u}}$
- integrate to invariant foliations $g^t \mathscr{W}^\star_v = \mathscr{W}^\star_{g^t(v)}$, $_{\star = \mathrm{s}/\mathrm{cs}/\mathbf{0}/\mathrm{u}/\mathrm{cu}}$ **[Eberlein 1970s]**

Geodesic flow for curvature $K \leq 0$ *surface*

... very-close-to-hyperbolic structure

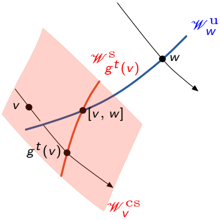
$\|J(t)\|$ constant $\quad\Leftrightarrow\quad F^{\mathrm{s}}_v = F^{\mathrm{u}}_v \quad\Leftrightarrow\quad K(\gamma_v(\cdot)) \equiv 0$

$F^{\mathrm{s}}_v \neq F^{\mathrm{u}}_v \quad\Leftrightarrow\quad v$ **rank-1 vector**



Flat Strip Theorem ([Eberlein-O'Neil 1973])

Geodesics ***in universal cover*** $\tilde{M}$ *within bounded Hausdorff distance* ***(bi-asymptotic)*** *are edges of* ***flat strip*** ***(isometrically totally embedded*** $[0, \varepsilon] \times \mathbb{R}$***).***



Lemma ([Coudene-Schapira 2014])

v not bounding flat strip has local product structure.

Geodesic flow for curvature $K \leq 0$ *surface*

... only "obstruction" to expansivity: **flat strips**.

IF **expansive**

$$\text{if for } v, w \text{ exists } \rho\colon \mathbb{R} \to \mathbb{R} \text{ surjective continuous, } \rho(0) = 0, d(g^t(v), g^{\rho(t)}(w)) \leq \varepsilon$$
$$\text{then } g^t(v) = w \text{ for some } |t| \leq \varepsilon,$$

THEN no **flat strips**.

Geodesic flow for curvature $K \leq 0$ *surface*

... only "obstruction" to expansivity: **flat strips**.

$\Rightarrow$ Get rid of obstructions [G-Ruggiero '19]:

$v, w \in T^1 M$ **related** $\sim$ if γ_v, γ_w are bi-asymptotic (in universal cover $\tilde{M}$).

$X \stackrel{\text{def}}{=} T^1 M/_\sim$ **quotient space**

$\psi^t \colon X \to X, \quad \psi^t([v]) \stackrel{\text{def}}{=} [g^t(v)]$ **quotient flow**.

$$X \overset{\text{def}}{=} T^1 M/\sim, \quad \psi^t\colon X \to X, \quad \psi^t([v]) \overset{\text{def}}{=} [g^t(v)].$$

$K \leq 0 \Rightarrow$ **no focal points** $\Rightarrow$ **no conjugate points**

Theorem ([G-Ruggiero 2019])

M ***be compact surface without focal points and genus greater than one.***

Quotient space X is compact topological 3-manifold with smooth structure where quotient flow ψ^t is continuous.

g^t is time-preserving semi-conjugate to ψ^t,

$$\chi \circ g^t = \psi^t \circ \chi, \quad \chi\colon T^1 M \to X \text{ quotient}$$

ψ^t is expansive, topologically mixing, has local product structure.

$\Rightarrow$ (1) g^t unique measure of maximal entropy

[G-Ruggiero 2019, G-Ruggiero]

$\Rightarrow$ (2) g^t unique equilibrium state for $t\varphi^{\mathrm{u}}(v)$, $t < 1$ $\quad$ $(\varphi^{\mathrm{u}}(v) = -\frac{d}{dt} \log \| dg^t|_{F_v^{\mathrm{u}}} \| |_{t=0})$

[G-Kwietniak]