Hyperbolic models of transitive topological Anosov flows.

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Topological Anosov flows

Topological Anosov flows.

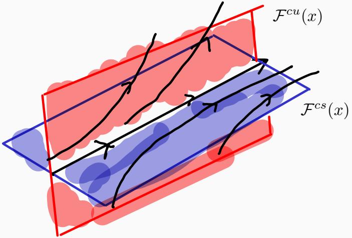
- M a closed, smooth, Riemannian, 3-mfld,
- $\phi_t : M \to M$ a non-singular C^r flow, $r \geq 0$.

Definition (topological Anosov)

The partitions of M into stable/unstable sets of the orbits induce a pair of transverse foliations $\mathcal{F}^{cs}$, $\mathcal{F}^{cu}$ that intersect along the ϕ_t-orbits.

Main Difference with Anosov $\to$ Lack of Hyperbolic Splitting

top. Anosov $\Longleftrightarrow$ $\begin{cases} \text{① Orbitally Expansive} \\ \text{② Global shadowing} \end{cases}$



Hyperbolic models

Question

Is every topological Anosov flow in a 3-manifold orbitally equivalent to some smooth Anosov flow?

HYPERBOLIC MODELS

CONSTRUCTION OF ANOSOV FLOWS ON 3-MFLDS.

Hyperbolic models

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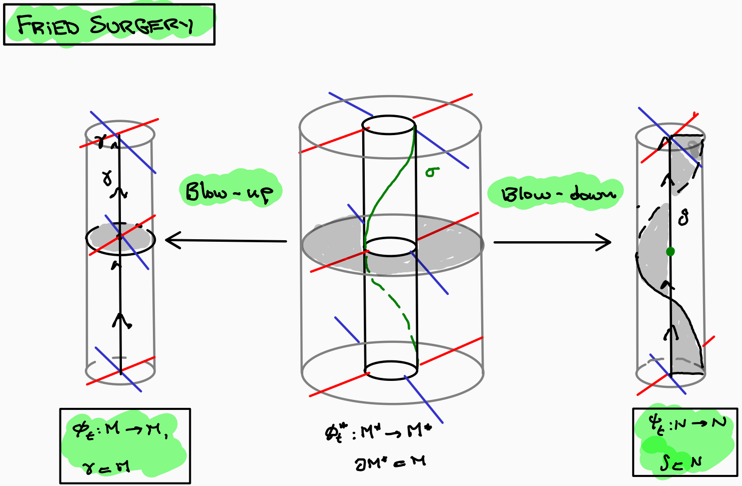
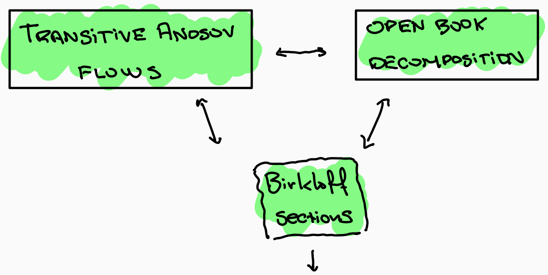


Theorem A (S., PhD thesis)

If (ϕ_t, M) *is a transitive topological Anosov flow on a* 3*-manifold, then it is orbitally equivalent to an Anosov flow* (ψ_t, N).



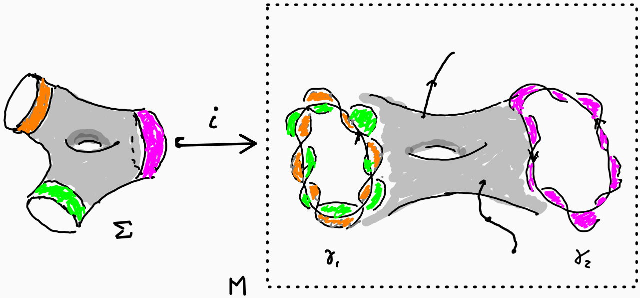
- Closed surface with $\partial\Sigma \neq \emptyset$.
- pseudo-Anosov homeo,
- combinatorial Data.

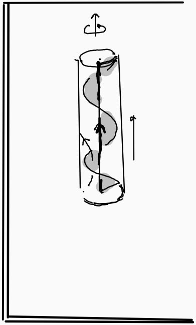
Birkhoff sections

Definition (Birkhoff section)

Is an immersion $\iota : (\Sigma, \partial\Sigma) \to (M, \Gamma)$ *of a compact surface* Σ *such that:*

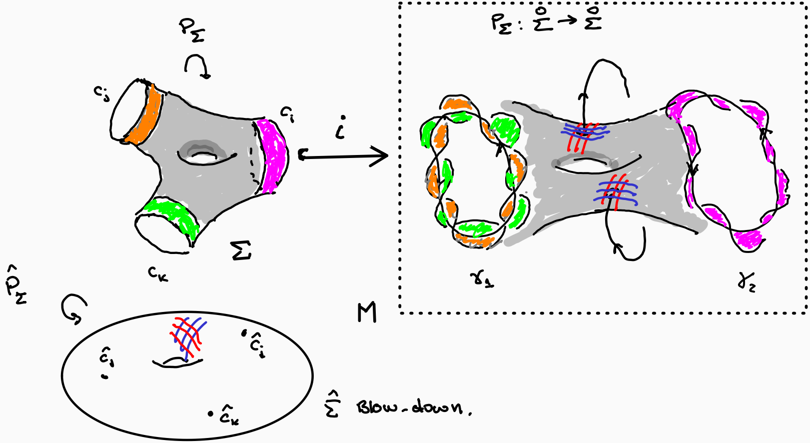
1. *On* $\mathring{\Sigma} = \Sigma \backslash \partial\Sigma$ *the map* ι *is an embedding, transverse to the flow lines;*
2. $\Gamma = \iota(\partial\Sigma)$ *is a finite set of periodic orbits of* ϕ_t*;*
3. $\exists\, T > 0$ *s.t.* $\phi_{[0,T]}(x) \cap \Sigma \neq \emptyset$, $\forall\, x \in M$.





Birkhoff sections: Properties

1. $\exists$ a well-defined first return map $P_\Sigma : \mathring{\Sigma} \to \mathring{\Sigma}$ (homeomorphism).
2. **Blow-down**: Let $\hat{\Sigma}$ be the closed surface obtained by collapsing each component of $\partial\Sigma$. $\Rightarrow \exists$ an induced homeomorphism $\hat{P}_\Sigma : \hat{\Sigma} \to \hat{\Sigma}$ that is *pseudo-Anosov*.



Birkhoff sections and transitivity

Theorem (Fried-Brunella)

Every topological Anosov flow on a closed 3-manifold admits a Birkhoff section.

TRANSITIVE

Consequence (almost-Anosov)

In $M_\Gamma = M \backslash \Gamma$ there exists a smooth atlas such that:

- *The foliation by ϕ_t-orbits is tangent to a smooth vector field $X \in \chi(M_\Gamma)$,*
- *$DX_t : TM_\Gamma \to TM_\Gamma$ preserves a uniformly hyperbolic splitting (in an open manifold).*

Existence of hyperbolic models

Idea for constructing hyperbolic models

- $\phi_t : M \to M$ a transitive topological Anosov flow;

Two steps:

1. To construct a smooth Anosov flow $\psi_t : N \to N$, generated by a smooth vector field Y in a Riemannian manifold N, expected to be equivalent with (ϕ_t, M).

2. To show that $(\phi_t, M) \simeq (\psi_t, N)$.

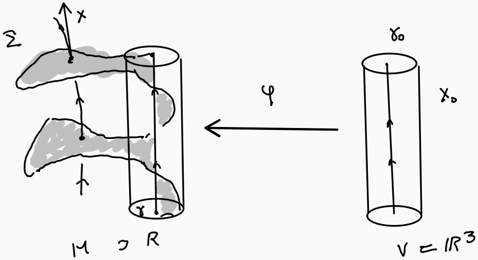
1. Construction of the model

- $\iota : (\Sigma, \partial\Sigma) \to (M, \Gamma)$ a Birkhoff section;
- In M_Γ: The orbits of ϕ_t are tg to a smooth v. f. X, preserving a unif. hyp. splitting.

Idea: to make a *Goodman-like* surgery around the singularity locus Γ, in order to produce an Anosov flow.

Hyperbolic Outside R

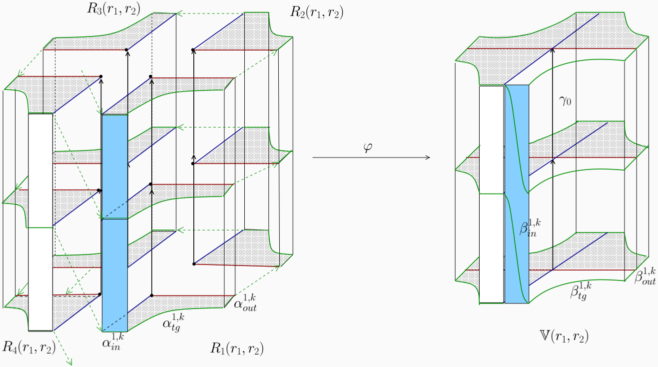
Hyperboli saddle



- $N = M \setminus \mathring{R} \sqcup_\varphi V$
- SMOOTH V.f. $Y = \begin{cases} X & \text{on } M \setminus \mathring{R} \\ X_0 & \text{on } V \end{cases}$
- $\Psi_t : N \to N$ flow

- To check that (Ψ_t, N) is Hyperbolic $\longrightarrow$ **CONE FIELD CRITERION**

1. Construction of the model



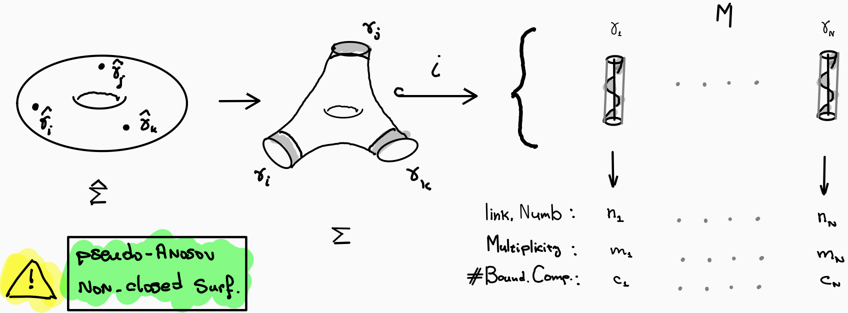
$$\boxed{\text{NORMAL FORM ON } R\setminus\gamma} \xrightarrow{\text{Glueing = twist Map}} \boxed{\text{Hyp. SADDLE TYPE PER. ORBIT}}$$

2. A criterion for orbital equivalence

Theorem (S., PhD thesis)

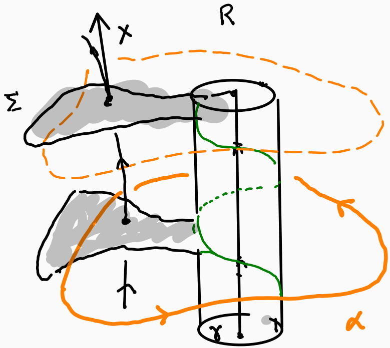
Let (ϕ_t, M) be a topological Anosov flow and let $\iota : (\Sigma, \partial\Sigma) \to (M, \Gamma)$ be a Birkhoff section. Then, the orbital equivalence class of (ϕ_t, M) is uniquely determined by:

- *The action $(P_\Sigma)_* : \pi_1(\widehat{\Sigma}; \hat{\gamma}_1, \ldots, \hat{\gamma}_N) \to \pi_1(\widehat{\Sigma}; \hat{\gamma}_1, \ldots, \hat{\gamma}_N)$ (marked surface);*
- *The combinatorial parameters of the embedding ι around each boundary component:* **linking-number** *and* **multiplicity***.*

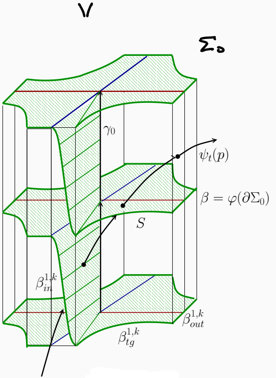


To check that φ_t satisfies the Criterion

$$(\Sigma \setminus \mathring{R} \sqcup \Sigma_0) \longrightarrow \begin{cases} \text{BIRKHOFF SECTION } T \simeq \Sigma \\ P_T : \mathring{T} \to \mathring{T} \end{cases}$$



$\xrightarrow{\varphi}$



$$\boxed{(P_T)_* \curvearrowright \pi_1(T, \gamma_0) \simeq (P_\Sigma)_* \curvearrowright \pi_1(\Sigma, \gamma)}$$

Thanks

CONSTRUCTION OF HYPERBOLIC MODELS