Spectrum rigidity and integrability for codimension-one Anosov diffeomorphisms

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A Hyperbolic Day Online

Anosov Automorphisms on Torus

Let $A \in \mathrm{GL}(n, \mathbb{Z})$, then it induces a diffeomorphism on $\mathbb{T}^n$.

Denote

$$T\mathbb{T}^n = L_k^s \oplus_\prec \cdots \oplus_\prec L_1^s \oplus L^c \oplus L_1^u \oplus_\prec \cdots \oplus_\prec L_l^u$$

the finest dominated splitting of A.

i. e. each $L_i^{s/c/u}$ is generated by an eigenspace of A with eigenvalues

$$\mu_1 = e^{\alpha + i\beta_1}, \cdots\cdots, \mu_k = e^{\alpha + i\beta_k},$$

where $k =$dim$L_i^{s/c/u}$.

Rk. Every Df-invariant bundle is integrable to a linear foliation on $\mathbb{T}^n$.

Dominated Splitting

Assume $f \in \textbf{Diff}(\mathbb{T}^n)$ is C^1-close to A satisfying

- f is partially hyperbolic;
- f admits the finest dominated splitting

$$T\mathbb{T}^n = E^s_k \oplus_{\prec} \cdots \oplus_{\prec} E^s_1 \oplus E^c \oplus E^u_1 \oplus_{\prec} \cdots \oplus_{\prec} E^u_l$$

with the same indices of A.

Question. How about the integrability of these invariant bundles of f?

Strong Foliations

$$T\mathbb{T}^n = \left(E_k^s \oplus \cdots \oplus E_i^s\right) \oplus \cdots \oplus E^c \oplus \cdots \oplus \left(E_j^u \oplus \cdots \oplus E_l^u\right).$$

The stable manifold theorem shows:

- $\forall 1 \leq i \leq k$, $E_k^s \oplus \cdots \oplus E_i^s$ is integrable;
- $\forall 1 \leq j \leq l$, $E_j^u \oplus \cdots \oplus E_l^u$ is integrable.

Center Foliations

$$T\mathbb{T}^n = E^s_k \oplus \cdots \oplus \left(E^s_i \oplus \cdots \oplus E^c \oplus \cdots \oplus E^u_j\right) \oplus \cdots \oplus E^u_l.$$

The invariant manifold theory of Hirsch-Pugh-Shub implies:

- $\forall 1 \leq i \leq k,\ \forall 1 \leq j \leq l,$

  $$E^s_i \oplus \cdots \oplus E^s_1 \oplus E^c \oplus E^u_1 \oplus \cdots \oplus E^u_j$$

  is integrable.

Intermediate Foliations

$$T\mathbb{T}^n = E^s_k \oplus \cdots \oplus E^c \oplus \cdots \oplus \left(E^u_i \oplus \cdots \oplus E^u_j\right) \oplus \cdots \oplus E^u_l.$$

The existence of strong and center foliations implies:

- $\forall 1 \leq i \leq j \leq k,\ E^s_j \oplus \cdots \oplus E^s_i$ is integrable;
- $\forall 1 \leq i \leq j \leq l,\ E^u_i \oplus \cdots \oplus E^u_j$ is integrable.

Question. **For** $\forall 1 \leq i \leq k$ **and** $\forall 1 < j \leq l$, **when** $E^s_i \oplus E^u_j$ **is integrable?**

Spectrum Rigidity

Definition. We say f has **spectrum rigidity** in $E_i^{s/c/u}$ if its Lyapunov exponents

$$\lambda_{E_i^{s/c/u}}(\mu, f) \equiv \lambda_{L_i^{s/c/u}}(A), \qquad \forall \mu \in \mathcal{M}_{\mathrm{erg}}(f).$$

Theorem of F. R. Hertz

Let $A \in \mathrm{GL}(n, \mathbb{Z})$ be **pseudo-Anosov** (i.e. A is ergodic, its characteristic polynomial p_A is irreducible over $\mathbb{Z}$ and not a polynomial of t^k for $k \geq 2$) and $\dim L^c = 2$.

Theorem [F. R. Hertz] For every f C^r-close to A , if $E^s_f \oplus E^u_f$ is integrable, then f conjugates to A by a homeomorphism h. Moreover, h is C^1-smooth along center foliation, which implies

$$\lambda^c(f, \mu) \equiv 0 = \lambda^c(A), \qquad \forall \mu \in \mathcal{M}_{\mathrm{erg}}(f).$$

Rk. Here $r = 22$ if $n = 4$, and $r = 5$ if $n \geq 6$. Using strong center bunching condition and KAM theory.

Dimension 3

We say $f \in$ **Diff**$(\mathbb{T}^3)$ is a **DA-diffeomorphism**, if

- f is partially hyperbolic $TM = E^s \oplus E^c \oplus E^u$;
- $f_* : \pi_1(\mathbb{T}^3) = \mathbb{Z}^3 \to \mathbb{Z}^3$ satisfies $f_* = A \in \mathrm{GL}(3, \mathbb{Z})$ is Anosov.

$\Longrightarrow A$ is irreducible, and A (or A^{-1}) has three eigenvalues satisfying

$$0 < |\mu^{ss}| < |\mu^{cs}| < 1 < |\mu^{u}|.$$

Dimension 3

Theorem. [Gan-S.] Let $f \in \mathbf{Diff}^2(\mathbb{T}^3)$ be a DA-diffeo, then the following are equivalent:

- f admits a minimal foliation $\mathcal{F}^{su}$ tangent to $E^s \oplus E^u$.
- f is Anosov and has spectrum rigidity in E^c:

$$\lambda^c(f) \equiv \log|\mu^{cs}|.$$

Theorem. [Hammerlindl-S.]
Let $f \in \mathbf{Diff}^2(\mathbb{T}^3)$ be a DA-diffeo. If f admits a lamination Λ^{su} tangent to $E^s \oplus E^u$, then f is Anosov and has spectrum rigidity in E^c.

Codimension-One

Let $A \in \mathrm{GL}(n, \mathbb{Z})$ be hyperbolic with finest dominated splitting

$$T\mathbb{T}^n = L^s_k \oplus_\prec \cdots \oplus_\prec L^s_1 \ \oplus \ L^u,$$

where $\dim L^u = 1$. (A is totally irreducible.)

Assume $f \in \mathrm{Diff}^2(\mathbb{T}^n)$ is C^1-close to A satisfying

- f is Anosov;
- f admits finest dominated splitting

$$T\mathbb{T}^n = E^s_k \oplus_\prec \cdots \oplus_\prec E^s_1 \ \oplus \ E^u$$

  with same indices of A.

Codimension-One

Theorem. [Gogolev-S.] For every $1 < j \leq k$, if $E^s_j \oplus E^u$ is integrable, then f has spectrum rigidity in E^s_i:

$$\lambda(E^s_i, f) \equiv \lambda(L^s_i, A), \qquad \forall i = 1, \cdots, j-1.$$

Rk. The reverse is true if $\dim L^s_i \leq 2$ for $1 \leq i < j$, see [Gogolev-Kalinin-Sadovskaya].

Applications

Corollary. [Gogolev-S.]

Let $A \in \mathrm{GL}(n, \mathbb{Z})$ be codimension-one with real spectrum. For $\forall f \in \mathbf{Diff}^2_m(\mathbb{T}^n)$ C^1-close to A with splitting

$$T\mathbb{T}^n = E^s_{n-1} \oplus \cdots \oplus E^s_1 \oplus E^u,$$

if

- $E^s_{n-1} \oplus E^u$ is integrable;
- the metric entropy $h_m(f) = h_m(A)$;

then f is smoothly conjugate to A.

Main Idea in Dimension 3

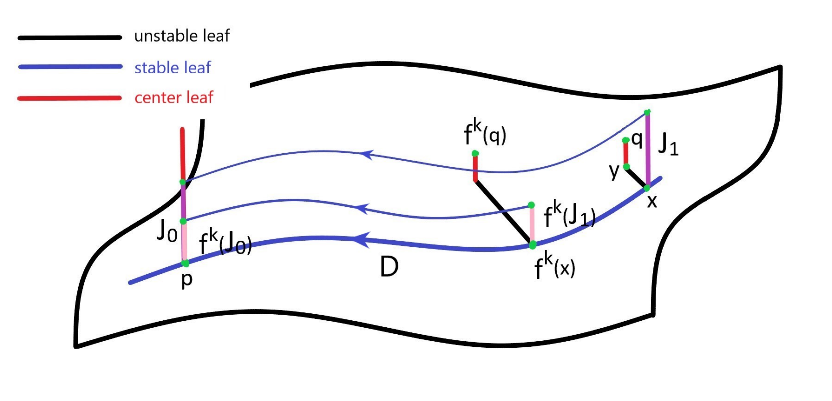

Figure 1: Spectrum rigidity: $\lambda^c(p) \equiv \lambda^c(q)$ for $\forall p, q \in \mathrm{Per}(f)$

Main Idea

Lemma. If $\mathcal{F}$ is a foliation on $\mathbb{T}^n$ sub-foliated by a linear foliation $\mathcal{L}$ with dense leaves, i. e.

$$\mathcal{L}(x) \subset \mathcal{F}(x), \qquad \overline{\mathcal{L}(x)} = \mathbb{T}^n, \qquad \forall x \in \mathbb{T}^n,$$

then $\mathcal{F}$ is a linear foliation.

$$\Downarrow$$

If $E_i^s \oplus E^u$ is integrable, then its foliation is linear after conjugacy, thus $E_{i-1}^s \oplus E^u$ is integrable. **Play Induction!**

General Dimensions

Let $A \in \mathrm{GL}(n, \mathbb{Z})$ be hyperbolic and irreducible. Let

$$T\mathbb{T}^n = L^s_k \oplus_\prec \cdots \oplus_\prec L^s_1 \ \oplus \ L^u_1 \oplus_\prec \cdots \oplus_\prec L^u_l$$

be the finest dominated splitting of A with $k > 1$.

Assume $f \in \mathrm{Diff}^2(\mathbb{T}^n)$ be C^1-close to A with splitting

$$T\mathbb{T}^n = E^s_k \oplus_\prec \cdots \oplus_\prec E^s_1 \ \oplus \ E^u_1 \oplus_\prec \cdots \oplus_\prec E^u_l.$$

For every $1 < j \leq k$, denote

$$E^{ss} = E^s_k \oplus \cdots \oplus E^s_j, \qquad E^{cs} = E^u_{j-1} \oplus \cdots \oplus E^u_1,$$

then

$$T\mathbb{T}^n = E^{ss} \oplus E^{cs} \oplus E^u.$$

General Dimensions

Theorem. [Gogolev-S.]
If $E^{ss} \oplus E^u$ is integrable, then f has spectrum rigidity in $E^{cs} = E^s_{j-1} \oplus \cdots \oplus E^s_1$, i. e.

$$\lambda(E^s_i, f) \equiv \lambda(L^s_i, A), \qquad \forall i = 1, \cdots, j-1.$$

Rk. The reverse is true if $\dim L^s_i \leq 2$ for $1 \leq i < j$.

Thank You !