Spectrum rigidity and integrability for codimension-one Anosov diffeomorphisms

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A Hyperbolic Day Online
Anosov Automorphisms on Torus

Let $A \in \text{GL}(n, \mathbb{Z})$, then it induces a diffeomorphism on $\mathbb{T}^n$.

Denote

$$\mathcal{T}\mathbb{T}^n = L_k^s \oplus \cdots \oplus L_1^s \oplus L^c \oplus L_1^u \oplus \cdots \oplus L_l^u$$

the finest dominated splitting of $A$.

i.e. each $L_i^{s/c/u}$ is generated by an eigenspace of $A$ with eigenvalues

$$\mu_1 = e^{\alpha+i\beta_1}, \ldots, \mu_k = e^{\alpha+i\beta_k},$$

where $k = \text{dim} L_i^{s/c/u}$.

**Rk.** Every $Df$-invariant bundle is integrable to a linear foliation on $\mathbb{T}^n$. 

Assume $f \in \text{Diff}(\mathbb{T}^n)$ is $C^1$-close to $A$ satisfying

- $f$ is partially hyperbolic;
- $f$ admits the finest dominated splitting

$$
\mathbb{T}^n = \bigoplus E_k^s \oplus \cdots \oplus E_1^s \oplus E^c \oplus E_1^u \oplus \cdots \oplus E_l^u
$$

with the same indices of $A$.

**Question.** How about the integrability of these invariant bundles of $f$?
Strong Foliations

\[ \mathcal{T}^{n} = (E_{k}^{s} \oplus \cdots \oplus E_{i}^{s}) \oplus \cdots \oplus E^{c} \oplus \cdots \oplus (E_{j}^{u} \oplus \cdots \oplus E_{l}^{u}). \]

The stable manifold theorem shows:

- \( \forall 1 \leq i \leq k, E_{k}^{s} \oplus \cdots \oplus E_{i}^{s} \) is integrable;
- \( \forall 1 \leq j \leq l, E_{j}^{u} \oplus \cdots \oplus E_{l}^{u} \) is integrable.
Center Foliations

\[ \mathbb{T}^n = E_k \oplus \cdots \oplus (E_i \oplus \cdots \oplus E^c \oplus \cdots \oplus E^u_j) \oplus \cdots \oplus E^u_l. \]

The invariant manifold theory of Hirsch-Pugh-Shub implies:

\[ \forall 1 \leq i \leq k, \forall 1 \leq j \leq l, \]

\[ E^s_i \oplus \cdots \oplus E^s_1 \oplus E^c \oplus E^u_1 \oplus \cdots \oplus E^u_j \]

is integrable.
The existence of strong and center foliations implies:

\begin{itemize}
  \item \( \forall 1 \leq i \leq j \leq k, \ E_j^s \oplus \cdots \oplus E_i^s \) is integrable;
  \item \( \forall 1 \leq i \leq j \leq l, \ E_i^u \oplus \cdots \oplus E_j^u \) is integrable.
\end{itemize}

**Question.** For \( \forall 1 \leq i \leq k \) and \( \forall 1 < j \leq l \), when \( E_i^s \oplus E_j^u \) is integrable?
Definition. We say $f$ has spectrum rigidity in $E_i^{s/c/u}$ if its Lyapunov exponents

$$
\lambda_{E_i^{s/c/u}}(\mu, f) \equiv \lambda_{L_i^{s/c/u}}(A), \quad \forall \mu \in \mathcal{M}_{\text{erg}}(f).
$$
Let $A \in \text{GL}(n, \mathbb{Z})$ be **pseudo-Anosov** (i.e. $A$ is ergodic, its characteristic polynomial $p_A$ is irreducible over $\mathbb{Z}$ and not a polynomial of $t^k$ for $k \geq 2$) and $\dim L^c = 2$.

**Theorem [F. R. Hertz]** For every $f$ $C^r$-close to $A$, if $E^s_f \oplus E^u_f$ is integrable, then $f$ conjugates to $A$ by a homeomorphism $h$. Moreover, $h$ is $C^1$-smooth along center foliation, which implies

$$\lambda^c(f, \mu) \equiv 0 = \lambda^c(A), \quad \forall \mu \in \mathcal{M}_{\text{erg}}(f).$$

**Rk.** Here $r = 22$ if $n = 4$, and $r = 5$ if $n \geq 6$. Using strong center bunching condition and KAM theory.
We say $f \in \text{Diff}(\mathbb{T}^3)$ is a DA-diffeomorphism, if

- $f$ is partially hyperbolic $TM = E^s \oplus E^c \oplus E^u$;
- $f_* : \pi_1(\mathbb{T}^3) = \mathbb{Z}^3 \to \mathbb{Z}^3$ satisfies $f_* = A \in \text{GL}(3, \mathbb{Z})$ is Anosov.

$\implies A$ is irreducible, and $A$ (or $A^{-1}$) has three eigenvalues satisfying

$$0 < |\mu^{ss}| < |\mu^{cs}| < 1 < |\mu^u|.$$
Theorem. [Gan-S.] Let $f \in \text{Diff}^2(T^3)$ be a DA-diffeo, then the following are equivalent:

- $f$ admits a minimal foliation $\mathcal{F}^{su}$ tangent to $E^s \oplus E^u$.
- $f$ is Anosov and has spectrum rigidity in $E^c$:

$$\lambda^c(f) \equiv \log |\mu^{cs}|.$$ 

Theorem. [Hammerlindl-S.] Let $f \in \text{Diff}^2(T^3)$ be a DA-diffeo. If $f$ admits a lamination $\Lambda^{su}$ tangent to $E^s \oplus E^u$, then $f$ is Anosov and has spectrum rigidity in $E^c$. 
Codimension-One

Let $A \in \text{GL}(n, \mathbb{Z})$ be hyperbolic with finest dominated splitting

$$\mathcal{T} \mathcal{T}^n = L_k^s \oplus \cdots \oplus L_1^s \oplus L^u,$$

where $\dim L^u = 1$. ($A$ is totally irreducible.)

Assume $f \in \text{Diff}^2(T^n)$ is $C^1$-close to $A$ satisfying

- $f$ is Anosov;
- $f$ admits finest dominated splitting

$$\mathcal{T} \mathcal{T}^n = E_k^s \oplus \cdots \oplus E_1^s \oplus E^u$$

with same indices of $A$. 
Theorem. [Gogolev-S.] For every $1 < j \leq k$, if $E^s_j \oplus E^u$ is integrable, then $f$ has spectrum rigidity in $E^s_i$:

$$\lambda(E^s_i, f) \equiv \lambda(L^s_i, A), \quad \forall i = 1, \cdots, j - 1.$$ 

Rk. The reverse is true if $\dim L^s_i \leq 2$ for $1 \leq i < j$, see [Gogolev-Kalinin-Sadovskaya].
Corollary. [Gogolev-S.]
Let $A \in \text{GL}(n, \mathbb{Z})$ be codimension-one with real spectrum. For $\forall f \in \text{Diff}_m^2(\mathbb{T}^n) \ C^1$-close to $A$ with splitting

$$\mathcal{T}\mathbb{T}^n = E^s_{n-1} \oplus \cdots \oplus E^s_1 \oplus E^u,$$

if

- $E^s_{n-1} \oplus E^u$ is integrable;
- the metric entropy $h_m(f) = h_m(A)$;

then $f$ is smoothly conjugate to $A$. 
Main Idea in Dimension 3

Figure 1: Spectrum rigidity: $\lambda^c(p) \equiv \lambda^c(q)$ for $\forall p, q \in \text{Per}(f)$
Lemma. If $\mathcal{F}$ is a foliation on $\mathbb{T}^n$ sub-foliated by a linear foliation $\mathcal{L}$ with dense leaves, i.e.

$$\mathcal{L}(x) \subset \mathcal{F}(x), \quad \overline{\mathcal{L}(x)} = \mathbb{T}^n, \quad \forall x \in \mathbb{T}^n,$$

then $\mathcal{F}$ is a linear foliation.

\[ \Downarrow \]

If $E_i^s \oplus E_u$ is integrable, then its foliation is linear after conjugacy, thus $E_{i-1}^s \oplus E_u$ is integrable. **Play Induction!**
Let $A \in \text{GL}(n, \mathbb{Z})$ be hyperbolic and irreducible. Let
\[
\TT^n = L^s_k \oplus \cdots \oplus L^s_1 \oplus L^u_1 \oplus \cdots \oplus L^u_l
\]
be the finest dominated splitting of $A$ with $k > 1$.

Assume $f \in \text{Diff}^2(\TT^n)$ be $C^1$-close to $A$ with splitting
\[
\TT^n = E^s_k \oplus \cdots \oplus E^s_1 \oplus E^u_1 \oplus \cdots \oplus E^u_l.
\]
For every $1 < j \leq k$, denote
\[
E^{ss} = E^s_k \oplus \cdots \oplus E^s_j, \quad E^{cs} = E^u_{j-1} \oplus \cdots \oplus E^u_1,
\]
then
\[
\TT^n = E^{ss} \oplus E^{cs} \oplus E^u.
\]
Theorem. [Gogolev-S.]
If $E^{ss} \oplus E^u$ is integrable, then $f$ has spectrum rigidity in $E^{cs} = E^{s}_{j-1} \oplus \cdots \oplus E^{s}_1$, i. e.

$$\lambda(E^s_i, f) \equiv \lambda(L^s_i, A), \quad \forall i = 1, \cdots, j - 1.$$ 

Rk. The reverse is true if $\dim L^s_i \leq 2$ for $1 \leq i < j$. 

General Dimensions
Thank You!