SMOOTH RIGIDITY FOR VERY NON-ALGEBRAIC ANOSOV DIFFEOMORPHISMS OF CODIMENSION ONE

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Abstract. We study rigidity of Anosov diffeomorphisms in a sufficiently small C^1 neighborhood of a linear hyperbolic automorphisms of the 3-dimensional torus which has a pair of complex conjugate eigenvalues. In particular, we show that two very non-algebraic (an open and dense condition) Anosov diffeomorphisms from this neighborhood are smoothly conjugate if and only they have matching Jacobian periodic data.

1. Introduction

Recall that a diffeomorphism $f\colon M\to M$ is called *Anosov* if the tangent bundle admits a Df-invariant splitting $TM=E^s\oplus E^u$, where E^s is uniformly contracting and E^u is uniformly expanding under f. Basic examples of Anosov diffeomorphisms are *toral hyperbolic automorphisms* $L\colon\mathbb{T}^d\to\mathbb{T}^d$ which are given by *hyperbolic matrices* in $GL(d,\mathbb{Z})$, *i.e.,* matrices whose spectrum is disjoint with the unit circle in $\mathbb{C}$.

Let $f_1,f_2\colon M\to M$ be transitive Anosov diffeomorphisms which are conjugate via a homeomorphism h, $h\circ f_1=f_2\circ h$. We will say that f_1 and f_2 have matching *periodic data* if for every periodic point $p=f_1^k(p)$ the differentials $(Df_1^k)_p$ and $(Df_2^k)_{h(p)}$ are conjugate (in particular, they have the same spectrum). By differentiating the conjugacy relation one immediately sees that matching of periodic data is a necessary assumption for the conjugacy to be C^1. A weaker assumption which we will consider here is matching of Jacobian periodic data. Namely, we say that f_1 and f_2 have matching *Jacobian periodic data* if every periodic point $p=f_1^k(p)$

$$(J^sf_1^k)_p=(J^sf_2^k)_{h(p)}\quad\text{and}\quad(J^uf_1^k)_p=(J^uf_2^k)_{h(p)}$$

where J^sf_i and J^uf_i stand for Jacobians of the restrictions of Df_i, $i=1,2$, to the stable and unstable distributions, respectively. If the equality holds only for stable (or only for unstable) Jacobians then we will talk about matching of *stable (respectively, unstable) Jacobian periodic data*.

In dimension 2 matching of periodic data implies smoothness of the conjugacy by work of de la Llave, Marco and Moriyón [dlL87, MM87, dlL92]. In higher dimensions a lot of work was devoted to periodic data rigidity (characterization of smooth conjugacy class) of hyperbolic automorphisms, see *e.g.,* [dlL04, KS09, GKS11, DW21].

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In particular, in dimension 3 the problem was solved for automorphisms with a pair of complex eigenvalues by Kalinin and Sadovskaya [KS09] and for automorphisms with real spectrum by Gogolev and Guysinsky [GG08, G17]. Further, in proximity of automorphism with real spectrum matching of periodic data implies $C^{1+\text{hölder}}$ regularity of the conjugacy on an open set of Anosov diffeomorphisms in dimension 3 and higher [GG08, G08].

In this paper we transfer some of ideas of [GRH20a] from the setting of expanding maps to the setting of Anosov diffeomorphisms. In particular, we have open sets of Anosov diffeomorphisms where we obtain optimal smoothness of the conjugacy using less data, such as Jacobian periodic data, or stable Jacobian periodic data as opposed to full periodic data which was commonly used before. To the best of our knowledge this is the first instance when smooth (not just $C^{1+\text{hölder}}$) conjugacy classes were characterized on an open set of diffeomorphisms in dimension > 2.

1.1. **Results in dimension 3.** We present the following results for Anosov diffeomorphisms in dimension 3.

Theorem 1.1. *Let $L\colon \mathbb{T}^3 \to \mathbb{T}^3$ be an Anosov automorphism with a pair of complex conjugate eigenvalues. Then there exists a C^1 neighborhood $\mathcal{U}$ of L such that all C^r, $r \geqslant 2$, Anosov diffeomorphisms $f_1, f_2 \in \mathcal{U}$ with matching Jacobian periodic data are either $C^{r}*$ conjugate or the SRB measure coincides with the measure of maximal entropy for f_1.*

Above $r_* = r$ if r is not integer and $r_* = (r-1) + Lip$ otherwise. Note that there is a unique topological conjugacy h which is C^0 close to $id_{\mathbb{T}^3}$, $h \circ f_1 = f_2 \circ h$, coming from structural stability. The condition on matching of Jacobian periodic data is imposed relative to this conjugacy h.

Remarks.
1. Recall that that SRB and MME measures do not coincide if and only $-\log J^u f_1$ is not cohomologous to a constant. This can be detected from two periodic points with different sums of unstable Lyapunov exponents. Hence the above theorem solves the smooth rigidity problem in a C^1 neighborhood of L on a C^1-open and C^∞-dense subset. The obvious remaining problem is to handle the case when $\log J^u f_1$ is cohomologous to a constant. It is not hard to see by perturbing L along unstable foliation that the conjugacy is not necessarily smooth if we only assuming matching of Jacobian periodic data. However, the problem that remains in this case is establishing smooth of the conjugacy under assumption of matching of (full) periodic data.
2. We can replace the assumption on C^1-closeness to L by an appropriate bunching assumption at periodic points.
3. If both f_1 and f_2 are volume preserving then it is enough to assume matching of unstable Jacobian periodic data because the stable Jacobian periodic data are given by reciprocals and, hence, match automatically.

Given two conjugate Anosov diffeomorphisms f_1 and f_2, $h \circ f_1 = f_2 \circ h$, and Hölder continuous functions $\varphi_1, \varphi_2 : M \to \mathbb{R}$, we say that (f_1, φ_1) is *equivalent* to (f_2, φ_2) and write

$$(f_1, \varphi_1) \sim (f_2, \varphi_2)$$

if there exists a continuous function $u : M \to \mathbb{R}$ such that

$$\varphi_1 - \varphi_2 \circ h = u - u \circ f_1$$

Then, by the Livshits theorem [L72], $(f_1, \varphi_1) \sim (f_2, \varphi_2)$ if and only if for every periodic point $x \in Fix(f_1^n)$

$$\sum_{k=0}^{n-1} \varphi_1(f_1^k(x)) = \sum_{k=0}^{n-1} \varphi_2(f_2^k(h(x)))$$

Also recall that a potential $\varphi \colon M \to \mathbb{R}$ is called an *almost coboundary* over $f \colon M \to M$ if φ is cohomologous to a constant, that is,

$$\varphi = u - u \circ f + c$$

for some function u and a constant c.

In fact, when f_1 and f_2 are C^3, Theorem 1.1 is a consequence of the following more general result.

Theorem 1.2. *Let $L \colon \mathbb{T}^3 \to \mathbb{T}^3$ be an Anosov automorphism with a pair of complex conjugate eigenvalues whose absolute value is greater than 1. Fix a number $\kappa \in (\frac{1}{2}, 1]$. Then there exists a C^1 neighborhood $\mathcal{U}$ of L such that if $(f_1, \varphi_1) \sim (f_2, \varphi_2)$, where $f_1, f_2 \in \mathcal{U}$ are C^r, $r \geqslant 2 + \kappa$, and $\varphi_1, \varphi_2 \in C^{1+\kappa}(\mathbb{T}^3)$, then either h is uniformly C^r along unstable leaves or φ_1 is an almost coboundary over f_1.*

Corollary 1.3. *Let $L \colon \mathbb{T}^3 \to \mathbb{T}^3$ be an Anosov automorphism with a pair of complex conjugate eigenvalues of absolute value > 1. Then there exists a C^1 neighborhood $\mathcal{U}$ of L such that all C^r, $r \geqslant 3$, Anosov diffeomorphisms $f_1, f_2 \in \mathcal{U}$ with matching stable Jacobian periodic data are either $C^{r}*$ conjugate or the SRB measure coincides with the measure of maximal entropy for f_1^{-1}.*

The next corollary established smooth conjugacy only assuming matching of full Jacobian periodic data in the dissipative setting.

Corollary 1.4. *Let $L \colon \mathbb{T}^3 \to \mathbb{T}^3$ be an Anosov automorphism with a pair of complex conjugate eigenvalues. Fix a number $\kappa \in (\frac{1}{2}, 1]$. Then there exists a C^1 neighborhood $\mathcal{U}$ of L such that if f_1 is not volume preserving and $(f_1, \log Jf_1) \sim (f_2, \log Jf_2)$, where $f_1, f_2 \in \mathcal{U}$ are C^r, $r \geqslant 2 + \kappa$, and Jf_i is full Jacobian with respect to a fixed volume form, $i = 1, 2$, then f_1 and f_2 are $C^{r}*$ conjugate.*

2. Preliminaries

We will denote by W^s and W^u the stable and unstable foliations which are tangent to the stable distribution E^s and the unstable distribution E^u of an Anosov diffeomorphosm f, respectively. When it is necessary to indicate the Anosov diffeomorphism which is being considered we will write E^s_f, W^s_f, etc. By $W^s_{loc}(x)$ and $W^u_{loc}(x)$ we will denote local invariant manifolds centered at x whose size is given by the local product structure constant.

2.1. **Regularity of the stable foliation.** We recall the definition of the stable bunching parameter $b^s(f)$ which controls regularity of the stable foliation, which defined in terms of exponential rates. Namely, for an Anosov diffeomorphism f there exist constants $\mu_+ > \mu_- > 1$ and $\lambda_+ > \lambda_- > 1$ and $C > 0$ such that

$$\frac{1}{C}\mu_+^{-n} \leqslant \| Df^n(v^s) \| \leqslant C\mu_-^{-n} \| v^s \| \quad \text{and}$$
$$\frac{1}{C}\lambda_-^n \| v^u \| \leqslant \| Df^n(v^u) \| \leqslant C\lambda_+^n \| v^u \|$$

for all $n \geqslant 0$ and all $v^s \in E^s$, $v^u \in E^u$. Then the *stable bunching parameter* is given by

$$b^s(f) = \frac{\log \lambda_-}{\log \lambda_+} + \frac{\log \mu_-}{\log \lambda_+}$$

If $b^s(f)$ is not an integer (which we can always assume) then the stable foliation W^s and the stable distribution E^s are $C^{b^s(f)}$ regular [HPS77, H97]. In particular, the stable holonomy maps are $C^{b^s(f)}$. (In fact, a better point-wise version of this result holds [H97].) Symmetrically, the unstable foliation W^u and the stable distribution E^u are $C^{b^u(f)}$ regular, where the *unstable bunching parameter* is given by

$$b^u(f) = \frac{\log \mu_-}{\log \mu_+} + \frac{\log \lambda_-}{\log \mu_+}$$

Note that if the Anosov automorphism $L\colon \mathbb{T}^3 \to \mathbb{T}^3$ has one dimensional stable subbundle corresponding to an eigenvalue μ^{-1}, $|\mu| > 1$, and a pair of complex conjugate eigenvalues $\lambda, \bar{\lambda}$, then for small perturbations f we have $\mu_-^{-1} \gtrsim |\mu|^{-1}$ or $\mu_- \lesssim |\mu| = |\lambda|^2$. Also we have $\lambda_- \lesssim |\lambda| \lesssim \lambda_+$. Hence

$$\frac{\log \lambda_-}{\log \lambda_+} \lesssim 1 \quad \text{and} \quad \frac{\log \mu_-}{\log \lambda_+} \lesssim 2$$

For sufficiently small perturbations f the above ratio will be close to 1 and 2, respectively, and, hence, the stable foliation is $C^{3-\varepsilon}$, where $\varepsilon > 0$ can be taken arbitrarily small by controlling the size of the perturbation. In fact, we will only need $C^{2+\varepsilon}$ regularity for W^s. Calculating $b^u(f)$ in this setting gives $C^{\frac{3}{2}-\varepsilon}$ regularity of W^u.

2.2. **Cohomological equation over Anosov diffeomorphisms and periodic cycle functionals.** Here we recall an alternative approach to solving the cohomological equation $\varphi = u - u \circ f + const$ over and Anosov diffeomorphisms f. This approach is due to Katok and Kononenko who introduced it to study the cohomological equation over partially hyperbolic diffeomorphisms [KK96]. For Anosov diffeomorphisms this approach is much easier because local accessibility property is automatic, however we will need to slightly refine the argument in order to rely on a sub-collection consisting of null-homologous periodic cycle functions only.

A piecewise smooth path $\gamma\colon [0,1] \to M$ is called a *us-adapted path* if each smooth leg is entirely contained in a stable or unstable leaf of f. If $\gamma(1) = \gamma(0)$ then we say that γ is a *us-adapted loop*.

Given a Hölder continuous function $\varphi\colon M \to \mathbb{R}$ the *periodic cycle functionals* are defined in the following way. If γ lies entirely in a stable leaf then let

$$PCF_\gamma(\varphi) = \sum_{n\geqslant 0} \varphi(f^n(\gamma(0))) - \varphi(f^n(\gamma(1)))$$

If γ lies entirely in a stable leaf then let

$$PCF_\gamma(\varphi) = \sum_{n<0} \varphi(f^n(\gamma(1))) - \varphi(f^n(\gamma(0)))$$

Given a us-adapted path $\gamma = \gamma_1 * \gamma_2 * \ldots * \gamma_m$, with each leg γ_i entirely contained in a stable or an unstable leaf let

$$PCF_\gamma(\varphi) = \sum_{i=1}^{m} PCF_{\gamma_i}(\varphi)$$

Note that the value $PCF_\gamma(\varphi)$ only depends on the sequence of endpoints of γ_i.

If $\varphi = u - u \circ f + \mathit{const}$ then, by a direct calculation $PCF_\gamma(\varphi) = u(\gamma(0)) - u(\gamma(1))$. Hence values of periodic cycle functionals on us-adapted loops provide obstructions to solving the cohomological equation. It turns out that vanishing of these obstructions is a sufficient condition for existence of a solution.

Proposition 2.1 (Katok-Kononenko)**.** *If $f\colon M \to M$ is an Anosov diffeomorphism and φ is a Hölder continuous function such that $PCF_\gamma(\varphi) = 0$ for every us-adapted loop γ, then φ is an almost coboundary; that is, there exists a constant c and a Hölder continuous function u such that $\varphi = u - u \circ f + c$.*

Proof. Let x_0 be a fixed point, $f(x_0) = x_0$ and let $c = \varphi(x_0)$. Given a point $x \in M$ consider a us-adapted path γ connecting x_0 to x and let $u(x) = PCF_\gamma(\varphi)$. Note that $u(x)$ does not depend on choice of γ because a different choice would adjust the value of $u(x)$ by a value of periodic cycle function on a loop, which is zero by our assumption.

By a direct calculation we have

$$u(x) - u(f(x)) = PCF_\gamma(\varphi) - PCF_{f\circ\gamma}(\varphi) = \varphi(x_0) - \varphi(x)$$

Finally, it is standard to check Hölder continuity of u by checking that restrictions to stable and unstable leaves are Hölder continuous. ☐

We will need a version of the above proposition which is concerned with null-homologous us-adapted loops, *i.e.,* loops whose homology class vanishes in $H_1(M, \mathbb{Z})$. The next proposition can be easily derived from the abelian Livshits Theorem for Anosov flows given in [GRH20b, Theorem 3.3], however, for Anosov diffeomorphisms the proof is more direct and we include it here.

Proposition 2.2. *Assume that $f\colon M \to M$ is an Anosov diffeomorphism such that $f_* n \neq n$ for all non-zero $n \in H_1(M, \mathbb{Z})$. Assume that φ is a Hölder continuous function such that $PCF_\gamma(\varphi) = 0$ for every null-homologous us-adapted loop γ. Then φ is an almost coboundary.*

Proof. Let $\tilde{M}$ be the universal abelian cover of M, that is, the cover which corresponds to the commutator subgroup $[\pi_1 M, \pi_1 M]$; its group of Deck transformations is given by $H_1(M, \mathbb{Z})$. Let x_0 be a fixed point for f and let $\tilde{x}_0$ be a fixed point for a lift $\tilde{f}\colon \tilde{M} \to \tilde{M}$ of f. Also let $\tilde{\varphi}$ be the lift of φ.

Note that homologically trivial us-adapted loops on M are precisely those loops which correspond to elements in the commutator subgroup of $\pi_1(M)$ and, hence, lift to loops on $\tilde{M}$. Therefore, the preceding proof can be repeated verbatim on $\tilde{M}$. Namely, if $c = \tilde{\varphi}(\tilde{x}_0)$ and $\tilde{u}(x) = PCF_\gamma(\tilde{\varphi})$, where γ connects x_0 to x, then we have a solution to the cohomological equation on $\tilde{M}$:

$$\tilde{\varphi} = \tilde{u} - \tilde{u} \circ \tilde{f} + c$$

Let $w\colon H_1(M, \mathbb{Z}) \to \mathbb{R}$ be given by $w(T) = \tilde{u}(T(\tilde{x}_0))$. Then, because periodic cycle functionals are invariant under Deck transformations: $PCF_{T\circ\gamma}(\tilde{\varphi}) = PCF_\gamma(\tilde{\varphi})$, $T \in H_1(M, \mathbb{Z})$, one can easily verify that w is a homomorphism. And also similarly, $\tilde{u} \circ T - \tilde{u} = w(T)$.

Now for any $T \in H_1(M, \mathbb{Z})$ we have

$$\tilde{\varphi} \circ T = \tilde{u} \circ T - \tilde{u} \circ \tilde{f} \circ T + c = \tilde{u} + w(T) - \tilde{u} \circ \tilde{f} - w(f_*(T)) + c = \tilde{\varphi} + w(T) - w(f_*(T))$$

Hence, because $\tilde{\varphi} \circ T = \tilde{\varphi}$, we obtain that $w(T) = w(f_*(T))$ or $w((id - f_*)T) = 0$ for all T. By assumption $id - f_*$ has trivial kernel, hence, we conclude that $w \equiv 0$. This means that $\tilde{u}$ is also equivariant under the Deck group and, thus, descends to a function $u\colon M \to \mathbb{R}$ and gives a solution to the cohomological equation on M: $\varphi = u - u \circ f + c$. □

2.3. **Regularity of simple periodic cycle functionals.** The simplest PCF is given by a quadruple of points. Here we show that under assumptions of Theorem 1.2 such PCF are C^1 along unstable leaves.

Let $a \in W^s(b)$. Then there is a canonical holonomy map $Hol_{a,b}\colon W^u_{loc}(a) \to W^u_{loc}(b)$ which takes a to b and which is given by sliding along stable leaves. If stable and unstable foliations have global product structure (as is the case for Anosov diffeomorphisms on tori), then the holonomy map can be continuously extended to a map $Hol_{a,b}\colon W^u(a) \to W^u(b)$ in a unique way.

Let $\gamma(a, b, x)$ be a us-adapted loop connecting a to b, b to $Hol_{a,b}(x)$, to x and then back to a. Given a potential φ define $\rho^\varphi_{a,b}\colon W^u(a) \to \mathbb{R}$ via the periodic cycle functional of $\gamma(a, b, x)$

$$\rho^\varphi_{a,b}(x) = PCF_{\gamma(a,b,x)}(\varphi)$$

Lemma 2.3. *Let $L\colon \mathbb{T}^3 \to \mathbb{T}^3$ be an Anosov automorphism with a pair of complex conjugate eigenvalues. Fix a number $\kappa > \frac{1}{2}$ and assume let $\varphi\colon M \to \mathbb{R}$ be a Hölder* *function which is uniformly $C^{1+\kappa}$. Let f be a sufficiently small perturbation of the automorphism L such that $\lambda_+ < \mu_-^\kappa$. Then $\rho^\varphi_{a,b}\colon W^u(a)\to\mathbb{R}$ is C^1 regular.*

Proof. Recall that by definition

$$\rho^\varphi_{a,b}(x) = \sum_{n\geqslant 0}\varphi(f^n(b)) - \varphi(f^n(a)) + \sum_{n\geqslant 0}\varphi(f^n(x)) - \varphi(f^n(Hol_{a,b}(x))) \\ + \sum_{n<0}\varphi(f^n(x)) - \varphi(f^n(a)) + \sum_{n<0}\varphi(f^n(b)) - \varphi(f^n(Hol_{a,b}(x)))$$

Note that the first series term is just a constant. The third series term can be easily seen to be C^1 along W^u by calculating the formal derivative and observing that the resulting series converge uniformly and hence, by Weierstrass M-test, give a bona fide derivative of the series. The last series term is of the same nature as the third one, but precomposed with the holonomy map. Since $Hol_{a,b}$ is C^1, we conclude that the last term is also C^1 along W^u. Hence, it remains to analyze the second series term.

We denote by D_u the restriction of derivative to E^u and calculate the formal derivative of the second series term:

$$\sum_{n\geqslant 0} D_u\varphi(f^n(x))D_uf^n(x) - D_u\varphi(f^n(Hol_{a,b}(x)))D_uf^n(Hol_{a,b}(x))D_uHol_{a,b}(x)$$

We proceed with an estimate using the triangle inequality by splitting the above series into a sum of two series. Note that the points $f^n(Hol_{a,b}(x))$ and $f^n(x)$ are close and we can identify unstable subspaces at these points using a finite number of smooth charts. In this way the compositions of differentials which appear below make sense.

$$\sum_{n\geqslant 0} D_u\varphi(f^n(x))D_uf^n(x) - D_u\varphi(f^n(Hol_{a,b}(x)))D_uf^n(Hol_{a,b}(x))D_uHol_{a,b}(x) \\ = \sum_{n\geqslant 0}\big(D_u\varphi(f^n(x)) - D_u\varphi(f^n(Hol_{a,b}(x)))\big)D_uf^n(x) \\ + \sum_{n\geqslant 0} D_u\varphi(f^n(Hol_{a,b}(x)))\big(D_uf^n(x) - D_uf^n(Hol_{a,b}(x))D_uHol_{a,b}(x)\big)$$

We will see that both series above converge uniformly. This then implies that $\rho^\varphi_{a,b}$ is indeed D_u-differentiable with a continuous derivative given by the above series.

For estimating the first series we use the fact that $D^u\varphi$ is Hölder with exponent $\kappa > 1/2$. (Indeed, recall that E^u is C^1 and φ is $C^{1+\kappa}$.)

$$\|\big(D_u\varphi(f^n(x)) - D_u\varphi(f^n(Hol_{a,b}(x)))\big)D_uf^n(x)\| \\ \leqslant C\,dist(f^n(x), f^n(Hol_{a,b}(x)))^\kappa\lambda_+^n \leqslant C\mu_-^{-n\kappa}\lambda_+^n$$

Hence, because $\mu_-^{-\kappa}\lambda_+ < 1$, the series converge uniformly.

For the second series note that $D_u\varphi$ is uniformly bounded and hence, we need to estimate $D_uf^n(x) - D_uf^n(Hol_{a,b}(x))D_uHol_{a,b}(x)$. To do that notice that $f^n\circ$

$Hol_{a,b} = Hol_{f^n(a),f^n(b)} \circ f^n$. Hence

$$\begin{aligned}\| D_u f^n(x) - D_u f^n(Hol_{a,b}(x)) D_u Hol_{a,b}(x) \| \\ = \| D_u f^n(x) - D_u Hol_{f^n(a),f^n(b)}(f^n(x)) D_u f^n(x) \| \\ \leqslant \| Id - D_u Hol_{f^n(a),f^n(b)}(f^n(x)) \| \cdot \| D_u f^n(x) \| \leqslant C\mu_-^{-n}\lambda_+^n\end{aligned}$$

where for the bound

$$\| Id - D_u Hol_{f^n(a),f^n(b)}(f^n(x)) \| \leqslant C dist(f^n(x), f^n(Hol_{a,b}(x))) \leqslant C\mu_-^{-n}$$

we used the fact W^s is a C^2 foliation and, hence, $D_u Hol_{z,y}$ is uniformly Lipschitz in $y \in W^s_{loc}(z)$, $z \in \mathbb{T}^3$, and $D_u Hol_{z,z} = Id$. Therefore, because $\lambda_+ < \mu_-$, the second series also converge uniformly. □

2.4. **Relation between stable holonomy and simple periodic cycle functionals.** Consider a quadruple of points a, $b \in W^s(a)$, $x \in W^u(a)$ and $Hol_{a,b}(x)$ as in the preceding section. The Jacobian of $Hol_{a,b}$ can be calculated using the relationship $f^n \circ Hol_{a,b} = Hol_{f^n(a),f^n(b)} \circ f^n$. Indeed, taking Jacobians of both sides yields

$$J\,Hol_{a,b}(x) = \frac{J^u f^n(x) J\,Hol_{f^n(a),f^n(b)}(f^n(x))}{J^u f^n(Hol_{a,b}(x))}$$

Recall that $J\,Hol_{f^n(a),f^n(b)} \to 1$ as $n \to +\infty$ because holonomy is uniformly C^1 and $Hol_{z,z} = Id$, $z \in \mathbb{T}^3$. Hence, by taking logarithms and passing to the limit as $n \to +\infty$ we obtain the following expression for the Jacobian of the holonomy

$$\log J\,Hol_{a,b}(x) = \sum_{n \geqslant 0} \log J^u f(f^n(x)) - \log J^u f(f^n(Hol_{a,b}(x)))$$

This formula give the relationship of the Jacobian of the holonomy to the simple periodic functional. Namely, if $\varphi = \log J^u f$ then we have

$$\begin{aligned}\rho^{\varphi}_{a,b}(x) = \log J\,Hol_{a,b}(x) - \log J\,Hol_{a,b}(a) \\ + \sum_{n<0} \varphi(f^n(x)) - \varphi(f^n(a)) + \sum_{n<0} \varphi(f^n(b)) - \varphi(f^n(Hol_{a,b}(x)))\end{aligned}$$

Remark 2.4. The formula above becomes much nicer if one considers the Jacobian of holonomy relative to the conditional measures of the SRB measure of f. Recall that the density of such conditional measure on $W^u(a)$ normalized to be equal to 1 at a, is given by

$$\theta_a(x) = \prod_{n<0} \frac{J^u f(f^n(a))}{J^u f(f^n(x))}$$

Then the Jacobian of holonomy relative to the SRB conditional measures on $W^u(a)$ and $W^u(b)$ is given by

$$J^{SRB} Hol_{a,b}(x) = J\,Hol_{a,b}(x) \frac{\theta_b(Hol_{a,b}(x))}{\theta_a(x)}$$

Taking logarithms and using the formula for $\rho^{\varphi}_{a,b}$ we have

$$\log J^{SRB} Hol_{a,b}(x) = \rho^{\varphi}_{a,b}(x) + \log J^{SRB} Hol_{a,b}(a)$$

Hence, up to an additive constant simple PFC is the same as logarithmic Jacobian of the holonomy relative to the conditionals of the the SRB measure.

This expression lets us establish regularity of $\rho^{\varphi}_{a,b}$ when f is merely C^2. Note that for Lemma 2.3 to apply when $\varphi = \log J^u f$, we must have that φ is $C^{1+\kappa}$, which means that f has to be $C^{2+\kappa}$ regular.

Lemma 2.5. *Let $L\colon \mathbb{T}^3 \to \mathbb{T}^3$ be an Anosov automorphism with a pair of complex conjugate eigenvalues. Let f be a sufficiently small perturbation of the automorphism L and let $\varphi = \log J^u f$. Then $\rho^{\varphi}_{a,b}\colon W^u(a) \to \mathbb{R}$ is C^1 regular.*

Proof. The preceding formula expresses $\rho^{\varphi}_{a,b}$ as a sum of four terms. Recall that according to the discussion in Section 2.1 the map $Hol_{a,b}$ is C^2 (and this is why we need f to be at least C^2). Hence the first term $J\,Hol_{a,b}$ is C^1 regular. The second term is just a constant. Then, C^1 regularity of of the third and, similarly, the last series term, are easy to see by observing that φ is C^1, differentiating the series formally with respect to x and observing exponential convergence of the resulting series. ☐

2.5. **Non-stationary linearization for expanding foliations.** Let $f\colon M \to M$ be a C^r, $r \geqslant 2$, diffeomorphism which leaves invariant a continuous foliation W with uniformly C^r leaves. Assume that W is an *expanding foliation*, that is $\| Df(v) \| > 1$, for all $v \in E$ where $E \subset TM$ is the distribution tangent to W. The following proposition on non-stationary linearization is a particular instance of the normal form theory developed by Guysinsky and Katok [GK98] and further refined by Kalinin and Sadovskaya [S05, KS09].

Proposition 2.6. *Let f be a C^r, $r \geqslant 2$, diffeomorphism and let W be an expanding foliation as described above, $E = TW$. Assume that there exist $\nu \in [0,1]$ and $\gamma \in (0,1)$ such that*

$$\| (Df^n|_E)^{-1} \| \cdot \| Df^n|_E \|^{1+\nu} \leqslant C\gamma^n$$

for all $n \geqslant 1$. Then for all $x \in M$ there exists $\mathcal{H}_x : E(x) \to W^u(x)$ such that

1. *$\mathcal{H}_x$ is a C^r diffeomorphism for all $x \in M$;*
2. *$\mathcal{H}_x(0) = x$;*
3. *$D_0\mathcal{H}_x = id$;*
4. *$\mathcal{H}_{fx} \circ D_x f = f \circ \mathcal{H}_x$;*
5. *$D\mathcal{H}_x$ is Lipschitz along W;*
6. *such family $\mathcal{H}_x$, $x \in M$, is unique among linearizations satisfying the above properties; moreover, uniqueness still holds among linearizations which do not necessarily obey item 5 above, but with ν-Hölder dependence of $D\mathcal{H}_x$ along W;*
7. *if $y \in W(x)$ then $\mathcal{H}_y^{-1} \circ \mathcal{H}_x : E(x) \to E(y)$ is affine;*
8. *the map $x \to \mathcal{H}_x$ from M to $Imm^r(E(x), M)$ is Hölder, in particular, the map $\hat{\mathcal{H}} : E \to M$, given by $\hat{H}(x,v) = H_x(v)$ is continuous;*

Such family $\{\mathcal{H}_x, x \in M\}$ is called *non-stationary linearization/normal form or affine structure* along W.

3. Proofs of results in dimension 3

For all the proofs in this section we will assume that L has one real eigenvalue of absolute value < 1 and a pair of complex eigenvalues of absolute value > 1. If the real eigenvalue has absolute value > 1 then one can consider inverses and conclude the same results.

3.1. **Theorem 1.2 implies Theorem 1.1 with a caveat.** The caveat is that we need to make an additional assumption that f_i are at least $C^{2+\kappa}$ regular. Under this assumption we explain that Theorem 1.2 applies in the setting of Theorem 1.1.

Fix a Riemannian metric on $\mathbb{T}^3$ and let $\varphi_i = \log J^u f_i$, $i = 1, 2$. Because unstable Jacobian periodic data match we have $(f_1, \varphi_1) \sim (f_2, \varphi_2)$. In order to apply Theorem 1.2 we also need to check that $\varphi_i \in C^{1+\kappa}(\mathbb{T}^3)$ with $\kappa > \frac{1}{2}$. Note that this is not immediate because the unstable subbundle is merely $C^{\frac{3}{2}-\varepsilon}$. However, because the stable foliation W_i^s is C^2 we can pick C^2- coordinate charts on $\mathbb{T}^3$ such that W_i^s is "horizontal" with respect to these charts. Then the differential Df_i has upper triangular form in these charts. Taking the determinant of Df_i yields the following relation

$$\varphi_i = \log J^u f_i = \log J f_i - \log J^s f_i$$

We have $\log J f_i \in C^{1+\kappa}(\mathbb{T}^3)$ because we have assumed that f_i are $C^{2+\kappa}$. And $\log J^s f_i$ is also $C^{1+\kappa}$ because the stable subbundle E_i^s is C^2. Therefore φ_i are indeed $C^{1+\kappa}$.

Applying Theorem 1.2 we obtain that $\log J^u f_i$ is cohomologous to a constant or the conjugacy h is uniformly C^r along the unstable foliation. In the former case we conclude that the equilibrium state for $-\log J^u f_i$ coincides with the equilibrium state for the constant, which precisely means that the SRB measure coincides with the measure of maximal entropy for f_i.

In the case when h is uniformly C^r along the unstable foliation we need to refer to classical arguments [dlL92] to conclude that h is C^{r*}. Indeed, from matching of stable Jacobian periodic data de la Llave concludes that h sends the SRB measure for f_1^{-1} to the SRB measure for f_2^{-1}. Same is true for conditional measures of these SRB measures along the stable leaves. Further, de la Llave argues that these conditionals are C^r smooth. And because the stable foliation is one dimensional we can conclude that h is uniformly C^{r+1} along stable foliation by integrating. Finally, given that h is uniformly C^r along both the stable and the unstable foliation, one employs the Journé's Lemma [J88], to conclude that h is a C^{r*} diffeomorphism.

3.2. **Proofs of Corollaries.** Here we explain how Corollaries 1.3 and 1.4 follow from Theorem 1.2.

Proof of Corollary 1.3. Let $\varphi_i = \log J^s f_i$, $i = 1, 2$. Then, from regularity of E^s we have $\varphi_i \in C^2(\mathbb{T}^3)$ and by the matching assumption $(f_1, \varphi_1) \sim (f_2, \varphi_2)$. Thus Theorem 1.2 applies and we have that either h is C^r along the unstable foliation, and we further get that h is C^r* as explained in Section 3.1, or φ_1 is cohomologous to a constant. In the latter case, the equilibrium state for $\varphi_1 = -\log J^u(f_1^{-1})$, which is the SRB measure for f_1^{-1} coincides with MME.

Proof of Corollary 1.4. Here we can apply Theorem 1.2 to full Jacobians $\varphi_i = \log Jf_i$. Diffeomorphism f_1 being dissipative implies that φ_1 is not cohomologous to a constant. Hence, Theorem 1.2 implies that h is C^r along unstable foliation, which in turn implies that h is C^r* is the same way as before.

3.3. **Outline of the proof of Theorem 1.2 (and Theorem 1.1).** By the assumption $(f_1, \varphi_1) \sim (f_2, \varphi_2)$ we have that φ_1 is cohomologous to $\varphi_2 \circ h$ over f_1. Because periodic cycle functionals vanish on coboundaries we have that

$$PCF_\gamma(\varphi_1) = PCF_\gamma(\varphi_2 \circ h)$$

for every us-adapted loop γ for f_1. Now we focus on simple PCFs given by four legs and associated functions $\rho^{\varphi}_{a,b}\colon W^u(a) \to \mathbb{R}$ as defined in the Section 2.3. The above equality of PFCs can be written in the following way

$$\rho^{\varphi_1}_{a,b} = \rho^{\varphi_2 \circ h}_{a,b} = \rho^{\varphi_2}_{h(a),h(b)} \circ h|_{W^u_{f_1}(a)}$$

We call such relation *matching of functions* $\rho^{\varphi_1}_{a,b}$ and $\rho^{\varphi_2}_{h(a),h(b)}$. This relation holds for all $a \in \mathbb{T}^3$ and $b \in W^s(a)$.

Now the proof splits into two cases. The first case is when all simple PCFs $\rho^{\varphi_1}_{a,b}$ are constant. In this case, we have, in fact, that $\rho^{\varphi_1}_{a,b} \equiv 0$ because we always have $\rho^{\varphi_1}_{a,b}(a) = 0$. We will deduce that such vanishing implies that all PCFs on null-homotopic us-adapted loops vanish. Then Proposition 2.2 allows us to conclude that φ_1 is an almost coboundary, which completes the proof in this case.

The second case in when $\rho^{\varphi_1}_{a,b}$ is non-constant for some a and b. Denote by p a fixed point of f_1 such that Df_1 has a pair of (non-real) complex conjugate eigenvalues. Such a point exists in proximity of $0 \in \mathbb{T}^3$ because we have assumed that f_1 is sufficiently close to L in C^1 topology and L has a pair of complex conjugate eigenvalues. Recall that by Lemma 2.3 $\rho^{\varphi_1}_{a,b}$ is C^1. Using minimality of the stable foliation, we can adjust the locations of the points a and b the stable manifold such that $a \in W^u_{f_1}(p)$ and $\rho^{\varphi_1}_{a,b}$ is has non-zero differential at p. Note that dynamics produces another matching relation as follows

$$\rho^{\varphi_1}_{a,b} \circ f_1|_{W^u_{f_1}(p)} = \rho^{\varphi_2}_{h(a),h(b)} \circ h|_{W^u_{f_1}(p)} \circ f_1|_{W^u_{f_1}(p)} = (\rho^{\varphi_2}_{h(a),h(b)} \circ f_2|_{W^u_{f_2}(h(p))}) \circ h|_{W^u_{f_1}(p)}$$

The differential $D(\rho^{\varphi_1}_{a,b} \circ f_1|_{W^u_{f_1}(p)})$ is also non-zero at p and has a kernel which is linearly independent from the kernel of $D\rho^{\varphi_1}_{a,b}$. This is because $Df_1(p)$ is a "expanding rotation" and doesn't have any real eigenvalues. Hence we have two independent matching relations on the neighborhood of of p in $W^u(p)$, which makes it possible to apply the Inverse Function Theorem to conclude that $h|_{W^u_{f_1}(p)}$ is a

C^1 diffeomorphism on a small neighborhood of p. Then we can use C^1 regularity of the stable holonomy to spread this regularity everywhere and conclude that h is uniformly C^1 along the unstable foliation $W^u_{f_1}$. The last step in the proof is to use uniqueness of the normal form for the unstable foliation to bootstrap regularity along $W^u_{f_1}$ from C^1 to C^r. Then concluding that h is a C^r* diffeomorphism was already explained at the end of Section 3.1.

In the following three sections we fill in the details for the above outline.

Remark 3.1. The proof of Theorem 1.1 is exactly the same working with the specific potentials $\varphi_i = \log J^u f_i$, $i = 1, 2$, to conclude that h is C^r along the unstable foliation. The only difference is that due to possible lack of regularity of f_i one has to invoke Lemma 2.5 instead of Lemma 2.3.

3.4. **Case I: vanishing of simple PCFs.** We begin the proof of Theorem 1.2 by considering the case when simple PCFs $\rho^{\varphi_1}_{a,b} \equiv 0$ for all $a \in \mathbb{T}^3$ and $b \in W^s(a)$. We will prove, using induction, that $PCF_\gamma(\varphi_1) = 0$ for all null-homologous us-adapted loops γ, which handles this case by applying Proposition 2.2 and concluding that φ_1 is an almost coboundary.

Because γ null-homologous, it lifts to a loop $\tilde\gamma$ on the universal cover $\mathbb{R}^3$ and we have $PCF_{\tilde\gamma}(\tilde\varphi_1) = PCF_\gamma(\varphi_1)$, where $\tilde\varphi_1$ is the lift of φ_1 and the PCF on the universal cover is defined in the same way using the lifted dynamics. The advantage of working on the universal cover is that the lifted foliations $\tilde W^s$ and $\tilde W^u$ have global product structure because they are close to the linear foliations for L. In particular, the space of unstable leaves is homeomorphic to $\mathbb{R}$ and, hence, is linearly ordered. We will denote by $\mathcal{U}(x)$ the $\mathbb{R}$-coordinate of $\tilde W^u(x)$, $x \in \mathbb{R}^3$ (and similarly for $\mathbb{R}$-coordinates of paths which are entirely contained in unstable leaves).

We will write $\tilde\gamma = \gamma_1 * \gamma_2 * \ldots * \gamma_{2k}$ and we can assume that the legs γ_i are contained in unstable leaves for even i and contained in stable leaves for odd i; indeed, if there are two consecutive legs in the same stable (or unstable) leaf we can just combine them into a single leg. We will run induction on k. If $k = 2$ then the corresponding PCF is simple and vanishes by the assumption.

Now assume vanishing for all $\tilde\gamma$ with $2k-2$ legs or less. Pick a maximal leg γ_{2i}, that is, a leg such that $\mathcal{U}(\gamma_{2i}) \geqslant \mathcal{U}(\gamma_{2j})$ for all $j = 1, \ldots k$. We can cyclically relabel the legs if needed so that $2i \neq 2k$. By maximality we have $\mathcal{U}(\gamma_{2i-1}(0)) < \mathcal{U}(\gamma_{2i})$ and $\mathcal{U}(\gamma_{2i+1}(1)) < \mathcal{U}(\gamma_{2i})$. For concreteness, also assume that $\mathcal{U}(\gamma_{2i-1}(0)) \geqslant \mathcal{U}(\gamma_{2i+1}(1))$ (the other case is symmetric). Then, by global product structure the leaf $\tilde W^u(\gamma_{2i-1}(0))$ intersects the leaf $W^s(\gamma_{2i+1}(1))$ at a unique point q with $q \in \gamma_{2i+1}$. We use point q to subdivide γ_{2i+1} into two legs $\gamma_{2i+1} = \delta_1 * \delta_2$. Also consider a path $\varepsilon\colon [0,1] \to \tilde W^u(\gamma_{2i-1}(0))$ which connects $\gamma_{2i-1}(0)$ to q and let $\bar\varepsilon$ be the same path with reversed orientation which connects q to $\gamma_{2i-1}(0)$. By adding the legs ε and $\bar\varepsilon$ we can "decompose" $\tilde\gamma$ into two us-adapted loops

$$\alpha = \gamma_{2i-1} * \gamma_{2i} * \delta_1 * \bar\varepsilon$$

and

$$\beta = \gamma_1 * \ldots * (\gamma_{2i-2} * \varepsilon) * \delta_2 * \gamma_{2i+2} * \ldots \gamma_{2k}$$

Note that α has only 4 legs and β has $2k-2$ legs (or $2k-4$ if δ_2 is a point). Hence by the induction hypothesis $PCF_\alpha(\varphi_1) = PCF_\beta(\varphi_1) = 0$. Also recall that from the definition of periodic cycle functionals we have $PCF_{\bar{\varepsilon}}(\varphi_1) = -PCF_\varepsilon(\varphi_1)$. It follows that

$$PCF_{\tilde{\gamma}}(\varphi_1) = PCF_{\tilde{\gamma}}(\varphi_1) + PCF_\varepsilon(\varphi_1) + PCF_{\bar{\varepsilon}}(\varphi_1) = PCF_\alpha(\varphi_1) + PCF_\beta(\varphi_1) = 0$$

Remark 3.2. For convenience we made use of global product structure and one-dimensionality of W^s. However, this is not essential. A more tedious argument, which relies on local product structure only, can show that for any null-homologous γ the corresponding PCF can be written us a sum of simple $PCFs$ corresponding to loops of small diameter and, hence, vanishes.

3.5. **Case II: non-constant simple PCF.** Recall that the simple PCFs are C^1 by Lemma 2.3. We assume now that there exists $a \in \mathbb{T}^3$, $b \in W^s_{f_1}(a)$ and $x_0 \in W^u f_1(a)$ such that $D\rho^{\varphi_1}_{a,b}(x_0) \neq 0$. Then for any x in a sufficiently small neighborhood B of x_0 we also have $D\rho^{\varphi_1}_{a,b}(x) \neq 0$.

Let p be a fixed point of f_1 such that $Df_1|_{E^u_{f_1}(p)}$ has complex (non-real) eigenvalues. Such point exists for all f_1 which are sufficiently close to L in C^1 topology. By minimality property of $W^s_{f_1}$ we have $\mathbb{T}^3 = \cup_{x\in B} W^s_{f_1}(x)$. Hence, we can pick $x \in B$ such that $p \in W^s_{f_1}(x)$. The local stable holonomy $Hol_{x,p}\colon W^u_{loc}(x) \to W^u_{loc}(p)$ uniquely extends to a global holonomy $Hol_{x,p}\colon W^u_{f_1}(x) \to W^u_{f_1}(p)$ and we let $c = Hol_{x,p}(a)$. The following relation can be verified directly from definition of PCFs

$$\rho^{\varphi_1}_{c,b} - \rho^{\varphi_1}_{c,a} = \rho^{\varphi_1}_{a,b} \circ (Hol_{x,p})^{-1}$$

Recall that $D\rho^{\varphi_1}_{a,b}(x) \neq 0$. Because $Hol_{x,p}$ is a C^1 diffeomorphism and $p = Hol_{x,p}(x)$, the above relation implies that either $D\rho^{\varphi_1}_{c,b}(p) \neq 0$ or $D\rho^{\varphi_1}_{c,a}(p) \neq 0$ (or both). These two cases are fully analogous and, for concreteness, we assume that $D\rho^{\varphi_1}_{c,b}(p) \neq 0$.

Note that the leaf $W^u_{f_1}(p)$ is fixed by f_1. Using the conjugacy relation we now have matching pairs

$$\rho^{\varphi_1}_{c,b} = \rho^{\varphi_2}_{h(c),h(b)} \circ h|_{W^u_{f_1}(p)}$$

and

$$\rho^{\varphi_1}_{c,b} \circ f_1|_{W^u_{f_1}(p)} = (\rho^{\varphi_2}_{h(c),h(b)} \circ f_2|_{W^u_{f_2}(h(p))}) \circ h|_{W^u_{f_1}(p)}$$

Further the differentials $D\rho^{\varphi_1}_{c,b}|_{E^u_{f_1}(p)}$ and $D(\rho^{\varphi_1}_{c,b} \circ f_1|_{W^u_{f_1}(p)})|_{E^u_{f_1}(p)}$ are linearly independent because $Df_1|_{E^u_{f_1}(p)}$ does not have real eigenvalues. It follows that the map

$$\mathcal{P}^{\varphi_1}_{c,b} = (\rho^{\varphi_1}_{c,b}, \rho^{\varphi_1}_{c,b} \circ f_1|_{W^u_{f_1}(p)})$$

has a full-rank differential at p and, hence, is a C^1 diffeomorphism when restricted to a sufficiently small neighborhood $\mathcal{U}$ of p. We also define

$$\mathcal{P}^{\varphi_2}_{h(c),h(b)} = (\rho^{\varphi_2}_{h(c),h(b)}, \rho^{\varphi_2}_{h(c),h(b)} \circ f_2|_{W^u_{f_2}(h(p))})$$

and by the matching relations we have

$$\mathcal{P}^{\varphi_1}_{c,b} = \mathcal{P}^{\varphi_2}_{h(c),h(b)} \circ h$$

Taking the inverse, we obtain the following formula for the restriction

$$h^{-1}|_{h(\mathcal{U})} = (\mathcal{P}^{\varphi_1}_{c,b})^{-1} \circ \mathcal{P}^{\varphi_2}_{h(c),h(b)}$$

By using a symmetric argument (and passing to an even smaller neighborhood $\mathcal{U}$ if needed) we also have that $h|_{\mathcal{U}}$ is C^1 and, hence, $h|_{\mathcal{U}}$ is a C^1 diffeomorphism.

Now let $q \in W^s_{f_1}(p)$ and let $\mathcal{U}_q$ be the image of $\mathcal{U}$ under the stable holonomy $Hol_{p,q}\colon W^u_{loc}(p) \to W^u_{loc}(q)$. Then, because h preserves the stable foliation we have that

$$h|_{\mathcal{U}_q} = Hol_{p,q} \circ h|_{\mathcal{U}} \circ (Hol_{p,q})^{-1}$$

and, hence, is also a C^1 diffeomorphism.

Finally, we will use minimality of the stable foliation to conclude that h is uniformly C^1 along $W^u_{f_1}$. Indeed, neighborhoods $\mathcal{U}_q$, $q \in W^s_{f_1}(p)$, sweep out the whole torus, so h is C^1 along $W^u_{f_1}$. To see uniformity, note that minimality actually implies that

$$\mathbb{T}^3 = \bigcup_{q \in W^s(p,R)} \mathcal{U}_q$$

where $W^s(p,R)$ is an segment of radius R in $W^s(p)$ relative to the intrinsic metric. Because the family of holonomies $Hol_{p,q}$, $q \in W^s(p,R)$, is uniformly C^1 we, indeed, can conclude that the same is true for $h|_{\mathcal{U}_q}$, $q \in W^s(p,R)$, yielding uniform C^1 smoothness along $W^u_{f_1}$ on the whole $\mathbb{T}^3$.

3.6. **A bootstrap argument.** Denote by $\mathcal{H}^i_x$, the affine structure for f_i along the unstable foliation $W^u_{f_i}$, $i = 1, 2$. The idea for bootstrap is to use uniqueness of the normal form. Indeed, since we have that h is uniformly C^1 along unstable foliation we can consider non-stationary linearization for f_1 given by $\mathcal{H}^{1\prime}_x = (h|_{W^u_{f_1}(x)})^{-1} \circ \mathcal{H}^2_x \circ Dh|_{E^u_{f_1}(x)}$. Then, if the normal form for $W^u_{f_1}$ is unique, we get $\mathcal{H}^{1\prime}_x = \mathcal{H}^1_x$ and conclude that $h|_{W_{f_1}(x)} = \mathcal{H}^2_x \circ Dh|_{E^u_{f_1}(x)} \circ (\mathcal{H}^1_x)^{-1}$ is C^r. This however, does not work so easily because the uniqueness guaranteed by item 6 of Proposition 2.6 requires $D\mathcal{H}^{1\prime}_x$ to be Hölder with respect to x along the unstable leaves. We do not have such regularity because h merely C^1 along unstable leaves. However, using this idea we can still establish smoothness along the leaf with conformal fixed point and then finish using denseness of this leaf.

We will begin by bootsrapping h from C^1 to C^r along the conformal leaf. We state the following proposition in somewhat more general context for the sake of future use and reference.

Proposition 3.3. *Let $f_i\colon M \to M$ be C^r, $r \geqslant 2$, diffeomorphisms which admit 2-dimensional expanding foliations W_i and satisfy assumptions of Proposition 2.6, $i = 1, 2$, $E_i = TW_i$. Assume that f_1 and f_2 are conjugate, $h \circ f_1 = f_2 \circ h$, and $h(W_1) = W_2$. Assume that p is a fixed point for f_1 such that $Df_1|_{E_1(p)}$ does not have real eigenvalues. Assume that $h|_{W_1(p)}$ is differentiable at p. Then $h|_{W_1(p)}$ is C^r.*

Proof. Denote by $\mathcal{H}^i_x$, $x \in M$, the affine structures for (f_i, W_i) given by Proposition 2.6. To prove the proposition we will show that $h|_{W_1(p)} = \mathcal{H}^2_{h(p)} \circ Dh|_{E_1(p)} \circ (\mathcal{H}^1_p)^{-1}$, which is clearly C^r. For that we need the following elementary lemma.

Lemma 3.4. *Let $H : \mathbb{C} \to \mathbb{C}$ be a continuous map of the complex plane $\mathbb{C}$. Assume that D_0H exists. Assume that there is $\lambda \in \mathbb{C}$, be such that $|\lambda| > 1$ and $H(\lambda z) = \lambda H(z)$ for every $z \in \mathbb{C}$. Then $H(z) = D_0H(z)$ for every $z \in \mathbb{C}$.*

Proof. First note that $H(0) = H(\lambda 0) = \lambda H(0)$, hence, $H(0) = 0$. Let $n_i \to +\infty$ such that $\lambda^{n_i}|\lambda|^{-n_i} \to \sigma$, $|\sigma| = 1$. Then, since H is differentiable at 0, for every $z \in \mathbb{C}$,

$$\frac{H(|\lambda|^{-n_i} z)}{|\lambda|^{-n_i}} \to D_0H(z)$$

as $n_i \to +\infty$. On the other hand,

$$\frac{H(|\lambda|^{-n_i} z)}{|\lambda|^{-n_i}} = \frac{\lambda^{-n_i}}{|\lambda|^{-n_i}} H\left(\lambda^{n_i}|\lambda|^{-n_i} z\right) \to \sigma^{-1}H(\sigma z),\ n_i \to +\infty$$

Hence $\sigma^{-1}H(\sigma z) = D_0H(z)$ for every $z \in \mathbb{C}$ and hence $H(z) = \sigma D_0H(\sigma^{-1}z)$ for every $z \in \mathbb{C}$ which implies that $H(z) = D_0H(z)$. □

Because eigenvalues $Df_1|_{E_1(p)}$ are complex, we can identify $E_1(p)$ with $\mathbb{C}$ so that $Df_1\colon E_1(p) \to E_1(p)$ becomes $z \mapsto \lambda z$. Note that $|\lambda| > 1$. Let $H\colon E_1(p) \to E_1(p)$ be given by $H = (\mathcal{H}^1_p)^{-1} \circ (h|_{W_1(p)})^{-1} \circ \mathcal{H}^2_{h(p)} \circ Dh|_{E_1(p)}$. We check the main assumption of the above lemma

$$\begin{aligned}
H \circ (z \mapsto \lambda z) &= (\mathcal{H}^1_p)^{-1} \circ (h|_{W_1(p)})^{-1} \circ \mathcal{H}^2_{h(p)} \circ (Dh|_{E_1(p)} \circ Df_1|_{E_1(p)}) \\
&= (\mathcal{H}^1_p)^{-1} \circ (h|_{W_1(p)})^{-1} \circ (\mathcal{H}^2_{h(p)} \circ Df_2|_{E_2(p)}) \circ Dh|_{E_1(p)} \\
&= (\mathcal{H}^1_p)^{-1} \circ ((h|_{W_1(p)})^{-1} \circ f_2|_{W_2(p)}) \circ \mathcal{H}^2_{h(p)} \circ Dh|_{E_1(p)} \\
&= ((\mathcal{H}^1_p)^{-1} \circ f_1|_{W_1(p)}) \circ (h|_{W_1(p)})^{-1} \circ \mathcal{H}^2_{h(p)} \circ Dh|_{E_1(p)} \\
&= Df_1|_{E_1(p)} \circ (\mathcal{H}^1_p)^{-1} \circ (h|_{W_1(p)})^{-1} \circ \mathcal{H}^2_{h(p)} \circ Dh|_{E_1(p)} = (z \mapsto \lambda z) \circ H
\end{aligned}$$

Also recall that $D\mathcal{H}^i_x = id_{E_i(x)}$ and hence $D_0H = id_{\mathbb{C}}$. Therefore, by the lemma $H = id_{\mathbb{C}}$, which precisely means that $h|_{W_1(p)} = \mathcal{H}^2_{h(p)} \circ Dh|_{E_1(p)} \circ (\mathcal{H}^1_p)^{-1}$. □

By applying Proposition 3.3 in our setting for unstable foliations, we have $h|_{W^u_{f_1}(p)} = \mathcal{H}^2_{h(p)} \circ Dh|_{E^u_{f_1}(p)} \circ (\mathcal{H}^1_p)^{-1}$. We would like to show a similar formula $h|_{W^u_{f_1}(x)} = \mathcal{H}^2_{h(x)} \circ C(x) \circ (\mathcal{H}^1_x)^{-1}$ for all $x \in \mathbb{T}^3$. To do that we can exploit denseness of $W^u_{f_1}(p)$. Given a point $x \in \mathbb{T}^3$ let $x_n \in W^u_{f_1}(p)$ be a sequence of points converging to x as $n \to \infty$. Then for $x \in W^u_{f_1}(p)$ we have

$$h|_{W^u_{f_1}(x)} = \mathcal{H}^2_{h(p)} \circ Dh|_{E^u_{f_1}(p)} \circ (\mathcal{H}^1_p)^{-1}$$
$$= \mathcal{H}^2_{h(x)} \circ ((\mathcal{H}^2_{h(x)})^{-1} \circ \mathcal{H}^2_{h(p)} \circ Dh|_{E^u_{f_1}(p)} \circ (\mathcal{H}^1_p)^{-1} \circ \mathcal{H}^1_x) \circ (\mathcal{H}^1_x)^{-1}$$

and letting

$$C(x) = (\mathcal{H}^2_{h(x)})^{-1} \circ \mathcal{H}^2_{h(p)} \circ Dh|_{E^u_{f_1}(p)} \circ (\mathcal{H}^1_p)^{-1} \circ \mathcal{H}^1_x$$

we obtain $h|_{W^u_{f_1}(x)} = \mathcal{H}^2_{h(x)} \circ C(x) \circ (\mathcal{H}^1_x)^{-1}$ for $x \in W^u_{f_1}(p)$. By item 7 of Proposition 2.6 maps $C(x)\colon E^u_{f_1}(x) \to E^u_{f_2}(h(x))$ are affine. One can easily check that $C(x)(0) = 0$ and hence $C(x)$, $x \in \mathbb{T}^3$, are, in fact, linear maps.

Now we would like to take a limit as $x_n \to x$ of

$$h|_{W^u_{f_1}(x_n)} = \mathcal{H}^2_{h(x_n)} \circ C(x_n) \circ (\mathcal{H}^1_{x_n})^{-1}$$

We left-hand-side converges to $h|_{W^u_{f_1}(x)}$, however, in order to be able to take the limit of the right-hand-side we also need the norm and conorm of $C(x_n)$ to be uniformly bounded. If that is the case, then from uniqueness of the limit we have that $C(x_n)$ converges to an invertible linear map $C(x)$, which has the same bounds on the norm and conorm, and

$$h|_{W^u_{f_1}(x)} = \mathcal{H}^2_{h(x)} \circ C(x) \circ (\mathcal{H}^1_x)^{-1}$$

Then from continuity property (item 8) of Proposition 2.6 and uniform bounds on $C(x)$, $x \in \mathbb{T}^3$, we can conclude that h is uniformly C^r along unstable foliation.

Thus, it remains to prove the following lemma.

Lemma 3.5. *There exists a constant $C > 0$ such that for every $x \in W^u_{f_1}(p)$, the linear maps $C(x) : E^u_{f_1}(x) \to E^u_{f_2}(h(x))$ defined above satisfy the following bounds*

$$\| C(x) \| \leqslant C, \;\; \| (C(x))^{-1} \| \leqslant C$$

Proof. Note that this lemma is not very obvious because the norm of $(\mathcal{H}^1_p)^{-1} \circ \mathcal{H}^1_x$ could explode as x goes to infinity inside the leaf $W^u_{f_1}(p)$.

We shall bound uniformly $\| C(x) \|$, $x \in \mathbb{T}^3$. The bound on the norm of $C(x)^{-1}$ follows from the same argument by interchanging the roles of f_1 and f_2 and working with h^{-1}.

Notice that $C(x) = (\mathcal{H}^2_{h(x)})^{-1} \circ h|_{W^u(x)} \circ \mathcal{H}^1_x$. If there is no uniform bound on $\| C(x) \|$, then there exist sequences $x_n \in W^u(p_1)$ and $v_n \in E^u_1(x_n)$ with $\| v_n \| \to 0$ and such that $\| C(x_n)v_n \| = 1$. Taking a subsequence if necessary, we have $x_\infty = \lim x_n \in \mathbb{T}^3$ and $w_\infty = \lim C(x_n)v_n \in E^u_2(h(x_\infty))$, $\| w_\infty \| = 1$. Let $z_n = \mathcal{H}^1_{x_n}(v_n)$, then $z_n \to \mathcal{H}^1_{x_\infty}(0) = x_\infty$. So we obtain that $h(z_n) \to h(x_\infty)$ and

$$\mathcal{H}^2_{h(x_n)}(C(x_n)v_n) \to \mathcal{H}^2_{h(x_\infty)}(w_\infty) \neq \mathcal{H}^2_{h(x_\infty)}(0) = h(x_\infty)$$

On the other hand,

$$\mathcal{H}^2_{h(x_n)}(C(x_n)v_n) = h\left(\mathcal{H}^1_{x_n}(v_n)\right) = h(z_n) \to h(x_\infty)$$

yielding a contradiction. □

Remark 3.6. Once Proposition 3.3 is established one could argue in a more ad hoc, but quicker way using holonomies along stables that

$$h|_{W^u(x)} = Hol_{h(x_p),h(x)} \circ h|_{W^u(p)} \circ Hol_{x,x_p}$$

where $x_p \in W^s(x) \cap W^u(p)$. Hence $h|_{W^u(x)}$ is uniformly C^2 for every $x \in \mathbb{T}^3$ due to C^2 regularity of holonomies. After that one can use uniqueness of normal forms as outlined at the beginning of this section to further bootstrap to C^r.

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