ANOMALOUS PARTIALLY HYPERBOLIC DIFFEOMORPHISMS II: STABLY ERGODIC EXAMPLES.

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ABSTRACT. We construct examples of robustly transitive and stably ergodic partially hyperbolic diffeomorphisms f on compact 3-manifolds with fundamental groups of exponential growth such that f^n is not homotopic to identity for all $n > 0$. These provide counterexamples to a classification conjecture of Pujals.

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1. INTRODUCTION

Let M be a Riemannian 3-manifold. In this paper, we say that a diffeomorphism $f\colon M \to M$ is *partially hyperbolic* if the tangent bundle TM splits into three one-dimensional Df-invariant continuous subbundles $TM = E^{ss} \oplus E^c \oplus E^{uu}$ such that for some $\ell > 0$ and for every $x \in M$

$$\|Df^\ell|_{E^{ss}(x)}\| < \min\{1, \|Df^\ell|_{E^c(x)}\|\} \le \max\{1, \|Df^\ell|_{E^c(x)}\|\} < \|Df^\ell|_{E^{uu}(x)}\|.$$

Sometimes, a stronger notion of *absolute partial hyperbolicity* is used. This means that f is partially hyperbolic and there exists $\lambda < 1 < \mu$ such that

$$\|Df^\ell|_{E^{ss}(x)}\| < \lambda < \|Df^\ell|_{E^c(x)}\| < \mu < \|Df^\ell|_{E^{uu}(x)}\|$$

The subbundles E^{ss}, E^c and E^{uu} depend on f and we will indicate this, when needed, using a subscript, *e.g.*, E^{ss}_f.

In 2001, it was informally conjectured in a talk by E. Pujals[1] that all examples of transitive partially hyperbolic diffeomorphisms, up to taking finite lifts and iterates, fall into one of the following classes:

1. deformations of linear Anosov automorphisms of the 3-torus $\mathbb{T}^3$;
2. deformations of skew products over a linear Anosov automorphism of the 2-torus $\mathbb{T}^2$;
3. deformations of time-one maps of Anosov flows.

In this paper we provide counterexamples to Pujals' conjecture.

Theorem 1.1. *There exist a closed orientable 3-manifold M and an absolutely partially hyperbolic diffeomorphism $f\colon M \to M$ which satisfies the following properties*

- *M admits an Anosov flow;*

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[1] The conjecture was formalized in [BW], see also [HaPe, Section 2.2] and [CHHU].

- *f^n is not homotopic to the identity map for all $n > 0$;*
- *f is volume preserving;*
- *f is robustly transitive and stably ergodic.*

Note that, because M admits an Anosov flow, the fundamental group of M has exponential growth. Therefore M and its finite covers do not admit Anosov automorphisms and partially hyperbolic skew products. Further, because iterates of f are not homotopic to identity, it follows that f, its iterates, and their finite lifts are not homotopic to the time-one map of an Anosov flow. We conclude that the diffeomorphism f given by Theorem 1.1, indeed, gives a counterexample to the Pujals' conjecture.

We present two classes of examples, both of which yield the statement of Theorem 1.1

- one on the unit tangent bundle of a surface of genus two or higher;
- the second class is based on a transitive Anosov flow which admits a transverse torus disjoint with a periodic orbit.

Remark 1.2. It is in fact plausible that the latter construction can be applied to any transitive Anosov flow with a transverse torus. However for the sake of simplicity and clarity, we will only present here an example based on the specific Anosov flow constructed in [BL].

1.1. **Overview of the constructions.** In both constructions we start with a Riemannian manifold with an Anosov flow and then perform a deformation (by changing the Riemannian metric in the first case and by considering finite lifts in the second case) which preserves the strength of the partial hyperbolicity of the Anosov flow. Both manifolds admit an incompressible torus (in the first case it contains two periodic orbits and in the second case it is transverse to the flow) and in both cases we consider a Dehn twist in a neighborhood of this torus which preserves the partially hyperbolic structure provided the deformation is sufficiently large. Let us make a more detailed outline.

The first construction is based on the time-one map $f\colon T^1S \to T^1S$ of the geodesic flow on a hyperbolic surface S. We fix a simple closed geodesic γ and consider a Dehn twist ρ along γ. Its differential induces diffeomorphism $D\rho\colon T^1S \to T^1S$. To find a partially hyperbolic diffeomorphism in the mapping class of $D\rho \circ f$ we deform the hyperbolic metric within the space of hyperbolic metrics in such a way that the length of γ goes to zero. Time-one map f deforms accordingly and stays partially hyperbolic. Because the "collar" of γ becomes a very thin tube, it is possible to deform the Dehn twist ρ so that it becomes an "almost isometry" of the surface. Hence, taking the composition $D\rho \circ f$ does not destroy partial hyperbolicity of f. It is possible to adjust this perturbation in order to make it volume preserving and still have partial hyperbolicity. We will also point out that the same constructions works starting with the time-one map of Handel-Thurston Anosov flow.

Our second construction is an adaptation of the one in [BPP]. We start with a conservative Anosov flow transverse to a torus T (see [BL]). We compose the time-N map of the flow, for some large N, with a Dehn twist along the torus, supported on a fundamental domain. In the non-transitive case considered in [BPP] the unstable and strong unstable (resp. stable and strong stable) foliations were kept unchanged in the negative (resp. positive) iterates of the fundamental domain; then the proof of partial hyperbolicity relied on the fact that, for N large enough, the Dehn twist preserves the transversality of the foliations. In the transitive case, N cannot be chosen larger than the smallest return time on the torus; moreover, none of the foliations are kept unchanged. However both difficulties would resolve if we could increase the return time without changing the dynamics, in particular the strength of the partial hyperbolicity. In the current paper we do it by considering the lift of the Anosov flow on a sufficiently large finite cyclic cover of the original manifold.

It is possible to construct examples without considering finite lifts by using a different mechanism, not preserving the strength of the partial hyperbolicity. This requires a different approach and will be delegated to a future paper.

1.2. **Pujals' conjecture revisited.** A partially hyperbolic diffeomorphism f is called *dynamically coherent* if the subbundles $E^{ss} \oplus E^c$ and $E^c \oplus E^{uu}$ are tangent to invariant 2-dimensional foliations, denoted by W^{cs}_f and W^{cu}_f, respectively. Then, these foliations intersect along an invariant 1-dimensional foliation W^c_f tangent to E^c. Under some technical assumptions, the pair (f, W^c_f) is known to be structurally stable [HPS]: for every diffeomorphism g, C^1-close to f, there is a homeomorphism h conjugating the foliations W^c_f and W^c_g and the points $h^{-1}gh(x)$ and $f(x)$ are uniformly bounded distance apart in the center leaf $W^c_f(x)$. We say that h is a *W^c-conjugacy.*

An example of non-dynamically coherent partially hyperbolic diffeomorphism on the torus $\mathbf{T}^3$ has been built by [HHU_3]. This example is not transitive and not absolutely partially hyperbolic.

Pujals' conjecture admits a stronger formulation, for dynamically coherent partially hyperbolic diffeomorphisms. It asserts that all such diffeomorphisms must be W^c-conjugated (up to finite iterates and lifts to finite covers) to one of the three models. Details can be found in [CHHU].

Remark 1.3. Counterexamples to this strong version of Pujals' conjecture were given recently in [BPP], though the examples are not transitive. We do not know if the transitive examples presented here are, in fact, dynamically coherent (see subsection 1.3).

Remark 1.4. Several positive classification results were established in [BW, BBI, BI, Pa, HP_1, HP_2] and certain families of 3-manifolds are now known only to admit partially hyperbolic diffeomorphisms which are on Pujals' list.

In our view, what Pujals was proposing is that it could be possible to reduce the classification of partially hyperbolic diffeomorphisms in dimension 3 to the classification of Anosov flows (even though the latter are far from being classified). The new examples greatly enrich the partially hyperbolic zoo in dimension 3. Still, the program of reducing the classification to that of Anosov flows, should not be abandoned.

The following questions arise naturally:

Question 1. *Assume that a manifold M with exponential growth of fundamental group admits a partially hyperbolic diffeomorphism. Does it also admit an Anosov flow?*

Notice that the main known obstruction for a manifold to admit an Anosov flow is the non-existence of Reebless foliations. This is also an obstruction for the existence of partially hyperbolic diffeomorphisms [BI].

Question 2. *Let $f: M \to M$ be a (dynamically coherent) partially hyperbolic diffeomorphism homotopic to the identity, is it W^c-conjugate to an Anosov flow*[2]*?*

The counterpart of the above question on *small manifolds* (*i.e.,* with fundamental group of polynomial growth) has been addressed and admits a complete answer [HP_1]. Among manifolds with fundamental group of exponential growth, only the case of solvable fundamental group is known [HP_2].

In fact, a natural (albeit somewhat vague) question which arises in view of our examples is the following:

Question 3. *Let $f: M \to M$ be a partially hyperbolic diffeomorphism on a 3-manifold M admitting an Anosov flow. What is the relationship between f and this flow?*

For instance, does M admit an Anosov flow X so that both f and X leave positively (resp. negatively) invariant the same strong unstable (resp. stable) cone-field ? Is the center-foliation of f (if it exists) equivalent to the center foliation of a topologically Anosov flow?

Note that a 3-manifold may admit many Anosov flows which are not topologically orbit equivalent (see [BBY] and references therein) and so the answer to the previous question may depend on the Anosov flow on M.

1.3. **Further properties and questions.** New examples are the source of new questions, but also a motivation to look again at previous ones. For example, in Section 3.8 we show that one of our examples possesses[3] periodic center leaves with new type of dynamical behavior which was not present in previous examples (see [BDV, Section 7.3]). Here by *periodic center leaf* we mean a complete curve tangent to E^c invariant by some power of f.

Other questions that must be tackled in view of the new examples pertain their dynamics, ergodicity in the volume preserving case, etc (see for example [CHHU, Wi]). Let us formulate some questions which we believe to be interesting:

Question 4. *Are the new examples presented in this paper dynamically coherent?*

Since the examples are constructed by composing with a perturbation (with relatively large support) we do not have much control on the dynamics or structure of the bundles after perturbation (we establish partial hyperbolicity using cone-field criteria). Notice that thanks to some criteria

[2] As remarked in [BW] it might be better to consider W^c-conjugacy to a topologically Anosov flow for technical reasons. See [BW, Conjecture 1].

[3] It is plausible that all our new examples have this property.

introduced in [BW] it is enough to show that the examples have a *complete* center-stable manifold (meaning that the saturation of a center-stable leaf by strong stable ones is complete in the metric induced by the Riemannian metric of the manifold). If one shows dynamical coherence, it seems natural to test other properties too:

Question 5. *Are the examples plaque-expansive?*

(See [HPS, Chapter 7] for definitions.) It is natural to study the W^c-conjugacy type of the examples, what is the dynamics of the center leaves, etc. Also note that all previously known examples have the property that they admit models with *smooth* center foliations.[4] It is unlikely that this will be the case for our examples, still we pose this as a question:

Question 6. *Is it possible to homotope any of the examples of the current paper to a partially hyperbolic diffeomorphism with a smooth center foliation?*

1.4. **Organization of the paper.** As we have already mentioned, the paper contains two families of examples which provide a proof of Theorem 1.1. The presentation of each of the examples is independent and can be read in any order. The only exception is the proof of robust transitivity and stable ergodicity which is the same proof and is carried out in Subsection 2.8.

In Section 2 we present the example on the unit tangent bundle of a hyperbolic surface and a related example on a certain graph manifold. In Section 3 we present the example starting with a transitive Anosov flow transverse to a torus which is not a suspension. Also, in Section 3.8 we discuss properties of periodic center leaves for the latter example.

Finally, we made an effort to make this paper (topologically) self-contained and gave elementary proofs of certain known results on 3-manifolds relying on explicit description of the 3-manifolds at hand, see Section 3.6 and also Remark 2.16.

2. An example on the unit tangent bundle of a surface

2.1. **A sequence of hyperbolic surfaces.** Let S be an orientable closed surface of genus 2 or higher. A Riemannian metric on S is hyperbolic if the curvature is constant -1. Any closed surface S endowed with a hyperbolic metric g is called a hyperbolic surface. Any closed hyperbolic surface (S, g) has the hyperbolic plane $\mathbb{H}^2$ as its universal cover. In other words, (S, g) is isometric to the quotient of $\mathbb{H}^2$ by a discrete co-compact subgroup of isometries of $\mathbb{H}^2$ (Fuchsian group) acting freely on $\mathbb{H}^2$; we denote by $\Pi_g \colon \mathbb{H}^2 \to (S, g)$ this universal cover, which is a local isometry. The group of deck transformations of the cover Π_g is identified with the fundamental group $\pi_1(S)$.

[4][FG] presents special examples of a higher dimensional dynamically coherent partially hyperbolic diffeomorphisms with non-smooth center foliation. It is plausible that these examples cannot be homotoped to ones with smooth center foliation. However they are W^c conjugate to partially hyperbolic diffeomorphisms with smooth center foliation.

A marked surface is a surface endowed with a set of generators of $\pi_1(S)$. The space $\mathrm{Teich}(S)$ of equivalence classes of marked constant -1 curvature Riemannian metrics on S is called the Teichmüller space, and we refer to [FLP] for its properties.

Let γ be an essential (*i.e.,* $[\gamma] \neq 0$ in $\pi_1(S)$) simple closed curve in S. It is easy to see that there exists a sequence of hyperbolic metrics $\{[g_n] \in \mathrm{Teich}(S); n \geq 1\}$ such that γ is a geodesic for g_n and its length ℓ_n with respect to g_n monotonically decreases to 0 as $n \to \infty$. This can be seen, for instance, by using Fenchel-Nielsen coordinates on $\mathrm{Teich}(S)$. More precisely, there is a decomposition of the surface S in *pair of pants* (a *pair of pants* is topologically the sphere S^2 minus the interior of 3 disjoint discs), so that γ is one of the boundary component of a pair of pants. Then (see [FLP]) the length of the boundary components of the pair of pants (and hence the length of γ) can be chosen arbitrary.

We fix such a sequence of hyperbolic metrics $\{g_n; n \geq 1\}$. We denote $\Pi_n = \Pi_{g_n} : \mathbb{H}^2 \to (S, g_n)$ the corresponding universal covers.

2.2. **Geodesic flows.** Let TS be the tangent bundle of the surface S. We denote by T^1S the *unit tangent bundle* of S: given a Riemannian metric g on S, the unit tangent bundle is the level set

$$\{v \in TS : \|v\|_g = 1\}.$$

It is a smooth circle bundle over S. As we will endow S with a family of metrics, we can also define the unit tangent bundle without using a specific metric: T^1S is the circle bundle over S , defined as being the the quotient of the bundle $TS \backslash S$, (where S is the zero section) by identifying $v \in T_xS \setminus \{0_x\}$ and $u \in T_xS \setminus \{0_x\}$ if and only if $v = cu$ for some $c > 0$.

Note that, for hyperbolic metrics g_n introduced earlier, the geodesic flow on the tangent bundle TS restricts to the level set $\{v : \|v\|_{g_n} = 1\}$. Hence, via the above canonical identifications of all unit tangent bundles with T^1S, each metric g_n gives rise to its geodesic flow $\mathcal{G}_n$ on T^1S.

Let $f_n \colon T^1S \to T^1S$ be the time-one map of the geodesic flow $\mathcal{G}_n$ of g_n, $n \geq 1$, and let $f \colon T^1\mathbb{H}^2 \to T^1\mathbb{H}^2$ be the time-one map of the geodesic flow on the hyperbolic plane $\mathbb{H}^2$.

Then, because each (S, g_n) is covered by $\mathbb{H}^2$, we have the following commutative diagram

$$\begin{array}{ccc} T^1\mathbb{H}^2 & \xrightarrow{\ f\ } & T^1\mathbb{H}^2 \\ \downarrow & & \downarrow \\ T^1S & \xrightarrow{\ f_n\ } & T^1S \end{array} \tag{2.1}$$

where the vertical arrows are induced by the derivative of the covering map Π_n.

The hyperbolic metric g on $\mathbb{H}^2$ (resp. g_n on S) induces the Sasaki metric $\hat{g}$ on $T^1\mathbb{H}^2$ (resp. $\hat{g}_n$ on T^1S). We will only use the following properties of the Sasaki metrics:

- The metric $\hat{g}$ is invariant under derivatives of the isometries of $\mathbb{H}^2$,

- The derivative of the projection Π_n is a local isometry from $(T^1\mathbb{H}^2, \hat{g})$ to $(T^1S, \hat{g}_n)$.
- The Anosov splitting for the geodesic flow on $T^1\mathbb{H}^2$ is orthogonal with respect to the Sasaki metric.

Thus we have the following commutative diagram

$$\begin{array}{ccc} (T^1\mathbb{H}^2, \hat{g}) & \xrightarrow{f} & (T^1\mathbb{H}^2, \hat{g}) \\ \downarrow & & \downarrow \\ (T^1S, \hat{g}_n) & \xrightarrow{f_n} & (T^1S, \hat{g}_n) \end{array} \tag{2.2}$$

Here the vertical arrows are local isometries. This observation will be crucial for the proof of partial hyperbolicity of the example which we are about to construct.

2.3. **Collar neighborhoods of the geodesic γ for g_n.** Denote S^1 the circle $\mathbb{R}/\mathbb{Z}$ and by C_n, $n \geq 1$, the cylinder $[0,1]\times S^1$ equipped with hyperbolic metric

$$dx^2 + \ell_n^2 \cosh^2(x) dy^2, (x, y) \in [0, 1] \times S^1.$$

One can visualize C_n as follows: given an oriented geodesic σ of $\mathbb{H}^2$ there is a unique 1-parameter group $h_{\sigma,t}$ of isometries preserving σ. These isometries are called *translations of axis* σ and $h_{\sigma,t}$ acts on σ as a translation of length t. For $t \neq 0$ $h_{\sigma,t}$ acts freely and properly on $\mathbb{H}^2$: thus the quotient space $\mathbb{H}^2/h_t$, for $t > 0$, is a cylinder Σ_t (diffeomorphic to $\mathbb{R} \times S^1$) and, as $h_{\sigma,t}$ is an isometry, the hyperbolic metric g goes down on an hyperbolic metric g^t on Σ_t. The cylinder Γ_t supports a unique closed geodesic σ_t, which is the projection of σ; its length is t. We denote by $\Gamma_{t,+}$ and $\Gamma_{t,-}$ the two closed half cylinders obtained by cutting Γ_t along σ_t. We denote by $C_{t,1}$ the subset of $\Gamma_{t,+}$ of point whose (hyperbolic) distance from σ_t is less than or equal to 1. This provides an equivalent description

$$C_n \quad \text{is isometric to} \quad C_{l_n,1}$$

Let us denote by γ_n the curve γ considered as a geodesic of (S, g_n). If we choose an orientation on the "left" boundary component of C_n and on the geodesic γ_n then there exists a unique (up to the obvious S^1 action on C_n) locally isometric immersion

$$\varphi_n : C_n \to (S, g_n),$$

which sends the "left" boundary component to γ_n in orientation preserving manner as shown on Figure 1.

Clearly we can assume that $\ell_n < 2\sinh^{-1}(1/\sinh(1))$ for all $n \geq 1$. By the Collar Lemma [FM, p. 402], this condition implies that φ_n, $n \geq 1$, are, in fact, isometric embeddings. From now on we will identify the cylinder C_n with its image under φ_n and refer to it as the *neck*.

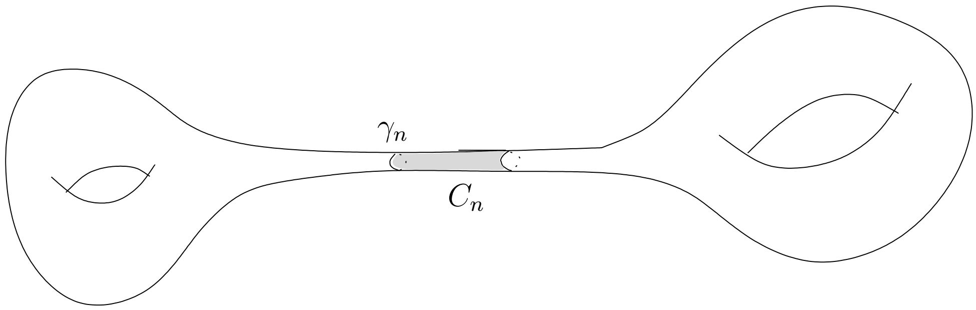

FIGURE 1

2.4. **Dehn twists.** Now define the Dehn twist $\rho_n\colon (S, g_n) \to (S, g_n)$ by

$$\rho_n(p) = \begin{cases} p, \text{if } p \notin C_n \\ (x, y + \rho(x)), \text{if } p = (x, y) \in C_n \end{cases} \tag{2.3}$$

Here $\rho\colon [0, 1] \to S^1$ is a C^∞ "twist function"; *i.e.*, ρ has the following properties

- ρ is C^∞ flat at 0 and at 1;
- $\rho(0) = \rho(1)$;
- ρ is increasing (we picked an orientation on S^1).

Next Proposition 2.3 asserts that the Dehn twists tend to the identity maps in the C^∞ topology as n tends to infinity. This statement does not have an obvious meaning, as each diffeomorphism ρ_n is considered on S endowed with the metric g_n: thus the C^∞ distance we consider on S depends on n. The idea is to consider lifts on $\mathbb{H}^2$ where the metric is fixed. Let us explain that precisely.

Definition 2.1. Consider the upper half plane $\mathbb{H}^2 \subset \mathbb{R}^2$. Let h_n be a sequence of diffeomorphisms on $\mathbb{H}^2$ and let $X_n \subset \mathbb{H}^2$ be a sequence of sets. We say that $d_{C^\infty}(h_n|_{X_n}, id) \to 0$ if for any $\varepsilon > 0$ and any $m > 0$ there is n_0 so that, for any $n \geq n_0$ we have the following property:
for any $x \in X_n$ there is an isometry ϕ_x of $\mathbb{H}^2$ so that $\phi_x(x) = (0, 1)$ and so that all the derivatives of $\phi_x \circ h_n \circ \phi_x^{-1}$ of order less than or equal to m at the point $(0, 1)$ are less than ε.

Definition 2.2. Let h_n be a sequence of diffeomorphisms on S supported on the neck C_n, $n > 0$ (that is, $h_n(x) = x$ for $x \notin C_n$). Let $\tilde{\gamma}_n$ and $\tilde{C}_n$ be the lifts of γ_n and C_n to $\mathbb{H}^2$ so that $\tilde{\gamma}$ is a boundary component of the strip $\tilde{C}_n$. Let $\tilde{h}_n$ be a lift of h_n on $\mathbb{H}^2$ such that $\tilde{h}_n$ is the identity on $\tilde{\gamma}_n$.

We say that the diffeomorphisms h_n tend to the identity with respect to the C^∞ distance on (S, g_n), and we write $d_{C^\infty,n}(h_n, id) \to 0$ if the restrictions of $\tilde{h}_n$ to $\tilde{C}_n$ tends to the identity map in the C^∞ topology.

We are now ready to state the key property of Dehn twists.

Proposition 2.3. *The sequence of Dehn twists $\rho_n\colon (S, g_n) \to (S, g_n)$ constructed above has the following property*

$$d_{C^\infty,n}(\rho_n, id_S) \to 0, n \to \infty.$$

Proof. Clearly we only need to pay attention to the neck $C_n \subset S$. After rescaling the coordinates $(x, y) \mapsto (x, \ell_n y) = (\bar{x}, \bar{y})$ the expression for g_n becomes independent of n

$$g_n = d\bar{x}^2 + \cosh^2(\bar{x})d\bar{y}^2; \tag{2.4}$$

The formula for ρ_n becomes

$$\rho_n(\bar{x}, \bar{y}) = (\bar{x}, \bar{y} + \ell_n \rho(\bar{x})).$$

The proposition follows because ρ' and it's higher derivatives are uniformly bounded and $\ell_n \to 0$ as $n \to \infty$. (Here we slightly abused notation by writing $\ell_n \rho$ for the same "twist function" $[0, 1] \to (S^1, d\bar{y})$ into the rescaled circle.) □

The following is an immediate corollary.

Corollary 2.4. *Let $\rho_n^* g_n$ denote the pull-back metric of g_n using ρ_n. Then*

$$d_{C^\infty}(\rho_n^* g_n, g_n) \to 0, n \to \infty.$$

Let $D\rho_n\colon TS \to TS$, $n \geq 1$, be the differential map. We abuse notation and also write $D\rho_n\colon T^1S \to T^1S$ for the induced diffeomorphism of T^1S given by $[v] \mapsto [D\rho_n(v)]$. Because Sasaki construction only uses the first derivatives of the metric we also have the following.

Corollary 2.5. *The sequence of Dehn twists $\rho_n\colon (S, g_n) \to (S, g_n)$ constructed above has the following property*

$$d_{C^\infty}(D\rho_n, id_{T^1S}) \to 0, n \to \infty.$$

Corollary 2.6. *For Sasaki metrics $\hat{g}_n$, $n \geq 1$, we have*

$$d_{C^\infty}((D\rho_n)^* \hat{g}_n, \hat{g}_n) \to 0, n \to \infty.$$

2.5. **Large perturbations preserving a partially hyperbolic structure.** We now introduce a new definition in order to understand under what condition a large perturbation can preserve the (absolute) partially hyperbolic structure of a diffeomorphism.

Let (M, g) be a (not necessarily compact) complete Riemannian 3-manifold let $f\colon M \to M$ be an absolutely partially hyperbolic diffeomorphism for which the splitting $E^{ss} \oplus E^c \oplus E^{uu}$ is orthogonal and the following inequalities hold,

$$\|Df|_{E^{ss}(x)}\|_g < \lambda < \lambda' < \|Df|_{E^c(x)}\|_g < \mu' < \mu < \|Df|_{E^{uu}(x)}\|_g \tag{2.5}$$

where $\lambda' < \lambda < 1 < \mu' < \mu$ are constants. We say that a sequence of diffeomorphisms $\{h_n\colon M \to M; n \geq 1\}$ is *ph-respectful* relative to (f, g) if

$$\sup_{x \in M} \angle_g(Dh_n E_f^\sigma(x), E_f^\sigma(h_n(x)) \to 0, n \to \infty, \;\; \sigma = ss, c, uu \tag{2.6}$$

and

$$d_{C^0}(h_n^* g, g) \to 0, n \to \infty \tag{2.7}$$

Note that Equation (2.7) implies that both $\|Dh_n(x)\|$ and $\mathcal{M}(Dh_n(x)) = \|Dh_n^{-1}(h(x))\|^{-1}$ uniformly converge to 1 as $n \to +\infty$.

Proposition 2.7. *Let (M, g), f and a ph-respectful sequence $\{h_n\colon M \to M; n \geq 1\}$ be as above. Then for all sufficiently large n the diffeomorphism $h_n \circ f$ is absolutely partially hyperbolic with respect to g.*

Proof of Proposition 2.7. For $\alpha \in (0, \pi/2)$ define the following unstable cone field on (M, g)

$$\mathcal{C}^{uu}_\alpha = \{v \in T_xM : \angle(v, E^{uu}) < \alpha\}.$$

Then, by assumption (2.5) there exists $\alpha \in (0, \pi/4)$ such that

$$Df(\mathcal{C}^{uu}_{\pi/4}) \subset \mathcal{C}^{uu}_\alpha \subset \mathcal{C}^{uu}_{\pi/4} \tag{2.8}$$

Now pick any non-zero vector $v \in Df(\mathcal{C}^{uu}_{\pi/4})$. Using (2.7) and (2.6), we have

$$\angle(Dh_n v, E^{uu}) \leq \angle(Dh_n v, Dh_n(E^{uu})) + \angle(Dh_n(E^{uu}), E^{uu}) \to \angle(v, E^{uu}), n \to \infty$$

uniformly in $v \in Df(\mathcal{C}^{uu}_{\pi/4})$. It follows that for all sufficiently large n

$$Dh_n(\mathcal{C}^{uu}_\alpha) \subset \mathcal{C}^{uu}_{\pi/4}$$

and, by combining with (2.8) we obtain

$$D(h_n \circ f)(\mathcal{C}^{uu}_{\pi/4}) \subset \mathcal{C}^{uu}_{\pi/4}.$$

Also using (2.5) and (2.7) on can check that there exist constants $\nu > \nu' > 1$ such that for all sufficiently large n

$$\|D(h_n \circ f)v\|_g \geq \nu \|v\|_g \quad \text{if } \; v \in Df(\mathcal{C}^{uu}_{\pi/4})$$

and

$$\|D(h_n \circ f)v\|_g \leq \nu' \|v\|_g \quad \text{if } \; v \notin \mathcal{C}^{uu}_{\pi/4}.$$

Because we assume that the partially hyperbolic splitting of f is orthogonal, we can check that Df satisfies such inequalities. For large n, diffeomorphism h_n almost does not affect the norms of vectors and we obtain the posited inequalities for $D(h_n \circ f)$.

By reversing the time we can obtain analogous properties of the (analogously defined) stable cone field $\mathcal{C}^{ss}_{\pi/4}$ hold with respect to $(h_n \circ f)^{-1}$. It is well-known (see, *e.g.*, [HaPe]) that existence of such cone fields imply absolute partial hyperbolicity. □

2.6. **The basic example.**

We are ready to present the basic version of our example.

Theorem 2.8. *Let S be a surface of genus 2 or higher, let $f_n\colon T^1S \to T^1S$, $n \geq 1$ be the time-one maps of the geodesic flows and let $D\rho_n\colon T^1S \to T^1S$, $n \geq 1$, be the diffeomorphism induced by the Dehn twists ρ_n as described above. Then for all sufficiently large n the diffeomorphisms $D\rho_n \circ f_n$ are absolutely partially hyperbolic. Furthermore, these diffeomorphisms and their finite iterates are not homotopic to identity.*

Recall that the Anosov splitting for the geodesic flow on $T^1\mathbb{H}^2$ is orthogonal with respect to the Sasaki metric.

Proposition 2.9. *Let $f\colon T^1\mathbb{H}^2 \to T^1\mathbb{H}^2$ be the time-one map of the geodesic flow on the hyperbolic plane and let $\hat{g}$ be the Sasaki metric on $\mathbb{H}^2$. For each $n \geq 1$ denote by $D\tilde{\rho}_n\colon T^1\mathbb{H}^2 \to T^1\mathbb{H}^2$ a lift of $D\rho_n\colon T^1S \to T^1S$ with respect the locally isometric cover (2.2). Then the sequence $\{D\tilde{\rho}_n\colon T^1\mathbb{H}^2 \to T^1\mathbb{H}^2; n \geq 1\}$ is ph-respectful relative to $(f, \hat{g})$.*

We proceed with the proof of Theorem 2.8 assuming the above proposition.

Proof of Theorem 2.8. The lift of $D\rho_n \circ f_n$ with respect to the locally isometric cover of (2.2) is $D\tilde{\rho}_n \circ f$. Therefore it suffices to check absolute partial hyperbolicity of $D\tilde{\rho}_n \circ f$ with respect to $\hat{g}$. For sufficiently large n this follows by combining Propositions 2.9 and 2.7.

Notice that diffeomorphism f_n is clearly isotopic to the identity while $\rho_n \colon S \to S$ is well known to be of infinite order in the mapping class group of S (see *e.g.,* [FLP]). Because the horizontal homomorphisms in the commutative diagram

$$\begin{array}{ccc} \pi_1(T^1S) & \longrightarrow & \pi_1(S) \\ \Big\downarrow{\scriptstyle \pi_1(D\rho_n)} & & \Big\downarrow{\scriptstyle \pi_1(\rho_n)} \\ \pi_1(T^1S) & \longrightarrow & \pi_1(S) \end{array}$$

are epimorphisms we also have that $D\rho_n \colon S \to S$ has infinite order in the mapping class group. Therefore $D\rho_n \circ f_n$ is of infinite order in the mapping class group of T^1S. □

Proof of Proposition 2.9. Our strategy is to first establish properties (2.6) and (2.7) for the sequence $\{D\rho_n \colon (T^1S, \hat{g}_n) \to (T^1S, \hat{g}_n); n \geq 1\}$ and then deduce that (2.6) and (2.7) also hold for the lifts to $T^1\mathbb{H}^2$.

Recall that by Corollary 2.6 we already have

$$d_{C^\infty}((D\rho_n)^*\hat{g}_n, \hat{g}_n) \to 0, n \to \infty.$$

Because we take the lifts with respect to locally isometric covers $(T^1\mathbb{H}^2, \hat{g}) \to (T^1S, \hat{g}_n)$, it follows that

$$d_{C^\infty}((D\tilde{\rho}_n)^*\hat{g}, \hat{g}) \to 0, n \to \infty.$$

On a hyperbolic surface, the Anosov splitting can be read off locally from the metric. Therefore $D\rho_n$ actually preserves the Df_n-invariant splitting $TT^1S = E^{ss}_{f_n} \oplus E^c_{f_n} \oplus E^{uu}_{f_n}$ outside the neck. Further, on the neck, because all surfaces have the same universal cover, the splittings $TT^1S = E^{ss}_{f_n} \oplus E^c_{f_n} \oplus E^{uu}_{f_n}$ are uniformly continuous in n. Hence, Corollary 2.5 yields

$$\sup_{x \in M} \angle(D(D\rho_n)E^\sigma_{f_n}(x), E^\sigma_{f_n}(D\rho_n(x)) \to 0, n \to \infty, \ \ \sigma = ss, c, uu.$$

And because the splittings for f_n lift to the splitting for f we obtain

$$\sup_{x \in M} \angle(D(D\tilde{\rho}_n)E^\sigma_{f}(x), E^\sigma_{f}(D\tilde{\rho}_n(x)) \to 0, n \to \infty, \ \ \sigma = ss, c, uu,$$

which concludes the proof. □

2.7. **The volume preserving modification.** Denote by m_n the Liouville volume on (T^1S, g_n).

Theorem 2.10. *Let S be a surface of genus 2 or higher, let $f_n \colon T^1S \to T^1S$, $n \geq 1$ be the time-one maps of the geodesic flows and let $\rho_n \colon S \to S$, $n \geq 1$, be the Dehn twists as described earlier. Then there exists a sequence of diffeomorphisms $\{h_n \colon T^1S \to T^1S; n \geq 1\}$ (which fiber over ρ_n) such that diffeomorphisms $h_n \circ f_n$ preserve m_n and for all sufficiently large n*

diffeomorphisms $h_n \circ f_n$ are absolutely partially hyperbolic. Furthermore, these diffeomorphisms and their finite iterates are not homotopic to identity.

Proof. Recall that the neck $C_n \subset (S, g_n)$ can be equipped with coordinates $(\bar{x}, \bar{y}) \in [0,1] \times S^1$ so that the expression for g_n (2.4) is independent of n. We identify T^1C_n with $[0,1] \times S^1 \times S^1$ and will use α for the last (angular) coordinate with the agreement that vectors with $\alpha = 0$ are tangent to geodesics $y = \text{const}$.

Locally the Liouville volume m_n is the product of the Riemannian volume on (S, g_n) and the angular measure on the tangent circle. A direct calculation in $(\bar{x}, \bar{y}, \alpha)$ coordinates yields the following formula for m_n on T^1C_n

$$dm_n = \cosh(\bar{x})(\cosh^{-1}(\bar{x})\cos^2(\alpha) + \cosh(\bar{x})\sin^2(\alpha))d\bar{x}d\bar{y}d\alpha \tag{2.9}$$

Now define $h_n \colon (T^1S, \hat{g}_n) \to (T^1S, \hat{g}_n)$ by

$$h_n(v) = \begin{cases} v, \text{if } v \notin T^1C_n \\ (\bar{x}, \bar{y} + \ell_n\rho(\bar{x}), \alpha), \text{if } v = (\bar{x}, \bar{y}, \alpha) \in T^1C_n \end{cases}$$

Note that diffeomorphisms h_n, indeed, fiber over the Dehn twists ρ_n, and hence, $h_n \circ f_n$ and its finite iterates are not homotopic to identity for all $n \geq 1$.

The geodesic flows leave corresponding Liouville measures invariant. Thus, to show that $h_n \circ f_n$ preserves m_n we have to check that h_n preserves m_n. But h_n preserves $\bar{x}$ and α coordinates and the expression for the density of m_n (2.9) does not depend on $\bar{y}$. Hence, indeed, $h_n^* m_n = m_n$.

It remains to establish absolute partial hyperbolicity of $h_n \circ f_n$ for sufficiently large n. Just as in the proof of Theorem 2.8, we will check that a sequence of lifts $\{\tilde{h}_n \colon T^1\mathbb{H}^2 \to T^1\mathbb{H}^2; n \geq 1\}$ (taken with respect to locally isometric covers) is a ph-respectful sequence relative to $(f, \hat{g})$.

The restriction of the diffeomorphism $D\rho_n \colon T^1S \to T^1S$ to T^1C_n is given by the formula

$$D\rho_n(\bar{x}, \bar{y}, \alpha) = (\bar{x}, \bar{y} + \ell_n\rho(\bar{x}), D_{\bar{x}}(\alpha)),$$

where $D_{\bar{x}} \colon S^1 \to S^1$ is induced by the matrix

$$\begin{pmatrix} 1 & 0 \\ \ell_n\rho'(\bar{x}) & 1 \end{pmatrix}.$$

Therefore we can decompose h_n as

$$h_n = h_n' \circ D\rho_n,$$

where the restriction of h_n' to T^1C_n is given by

$$h_n'(\bar{x}, \bar{y}, \alpha) = (\bar{x}, \bar{y}, D_{\bar{x}}^{-1}(\alpha)).$$

Since $\ell_n \to 0$ as $n \to \infty$ we have $d_{C^\infty}(h_n', id_{T^1S}) \to 0$ as $n \to \infty$. Now we consider lifts $\tilde{h}_n' \colon T^1\mathbb{H}^2 \to T^1\mathbb{H}^2$ which fiber over $id_{\mathbb{H}^2}$ so that we also have $d_{C^\infty}(\tilde{h}_n', id_{T^1\mathbb{H}^2}) \to 0$. It is easy to see, that this last conclusion together with ph-respectful properties of $\{D\tilde{\rho}_n; n \geq 1\}$ implies the sequence $\{\tilde{h}_n = \tilde{h}_n' \circ D\tilde{\rho}_n; n \geq 1\}$ is also ph-respectful relative to $(f, \hat{g})$. $\square$

2.8. **Stable ergodicity and robust transitivity.** Here we establish an addendum to Theorem 2.10, which finally completes our first proof of Theorem 1.1.

Addendum 2.11. *The volume preserving absolutely partially hyperbolic diffeomorphism F_0 constructed in the proof of Theorem 2.10 admits a stably ergodic and robustly transitive perturbation F.*

We do not know whether F_0 is ergodic or not with respect to the volume m but it is possible to consider a small C^1 volume preserving perturbation F_1 of F_0 to make it stably ergodic (see [BMVW], in this context it is also possible to make a C^∞-small perturbation to obtain ergodicity, [HHU_2]).

This means that there exists a C^1-neighborhood $\mathcal{U}_1$ of F_1 such that for every $F \in \mathcal{U}_1$ which preserves m and is of class C^2 we have that F is ergodic[5]. Moreover, every $F \in \mathcal{U}_1$ is accessible (see [BMVW]). Therefore, if F preserves m then it is transitive (even if it is not C^2).

However, in principle, it is possible that a dissipative perturbation of F_1 is not transitive. To obtain robust transitivity we shall perform yet another (volume preserving) perturbation of F_1 within $\mathcal{U}_1$ to obtain both properties at the same time.

We have the following.

Proposition 2.12. *Let $f\colon M \to M$ be a volume preserving partially hyperbolic diffeomorphism of a 3-dimensional manifold such that it has a normally hyperbolic circle leaf whose dynamics is conjugate to a rotation. Then, there exists an arbitrarily C^1-small volume preserving perturbation of f which makes it stably ergodic and robustly transitive.*

Proof. The proof compiles several well known results. Let us fix $\varepsilon > 0$. As explained above, one can make a $\varepsilon/3$-small C^1-volume preserving perturbation f_1 of f such that f_1 is ergodic (in particular it is transitive). If ε is sufficiently small, by normal hyperbolicity of the circle, the circle persists and it is at C^1-distance smaller than $\varepsilon/3$ of a rotation.

We would like to put ourselves in the hypothesis of [BDV, Proposition 7.4] which requires the construction of blenders. Notice first, that since f is partially hyperbolic, the first hypothesis of the proposition is verified.

Notice that it was proved originally in [BD] (and later in [HHTU] in the conservative setting) that it is always possible to make a $\varepsilon/3$-small C^1 (volume preserving) perturbation which creates a blender.

Using the transitivity and the normally hyperbolic circle leaf we know that the center-stable and center-unstable manifolds of the circle are robustly dense (see the proof of [BDV, Proposition 7.4]). Making a perturbation along the circle (of C^1-size less than $\varepsilon/3$) one gets a Morse-Smale dynamics on the circle with only two periodic orbits (since it is close to a rotation) and by chosing their position one can guarantee that their strong manifolds intersect the activating region of the blender. This provides the second and third hypothesis of [BDV, Proposition 7.4] which provide the desired robust transitivity.

A further arbitrarily small C^1-perturbation gives stable ergodicity in addition to robust transitivity. □

[5]The fact that F has to be C^2 is for technical reasons which we shall not explain here.

The diffeomorphism F_0 is in the hypothesis of the Proposition since there is at least one center circle leaf disjoint from all perturbations and since the dynamics on the circle is the time-1 map of a flow (and therefore conjugate to a rotation). This completes the proof of Theorem 1.1 for this type of manifolds.

2.9. **Further remarks.**

2.9.1. *Multiple Dehn twists.* Note that given a collection of disjoint simple closed geodesics on a hyperbolic surface we can pick a sequence of hyperbolic metrics so that the lengths of all these geodesics go to zero. By considering disjoint collars we can perform Dehn twists along these geodesics simultaneously and, thus, obtain partially hyperbolic representative in corresponding mapping class.

2.9.2. *Higher dimensions.* By Mostow rigidity, the mapping class group of a negatively curved manifold of dimension 3 or higher is finite. Therefore our scheme cannot be applied to a geodesic flow on such manifold to produce partially hyperbolic diffeomorphisms whose finite iterates are not homotopic to identity.

2.10. **Modification on graph manifolds: examples based on Handel-Thurston Anosov flows.** The purpose of this section is to explain that our construction can be modified to yield partially hyperbolic diffeomorphism on graph manifold which admit Handel-Thurston Anosov flows [HT]. We merely observe that our mechanism for hyperbolicity is very well compatible with Handel-Thurston mechanism and, hence, they can be applied at the same time.

Let us briefly recall the Handel-Thurston construction. Let S be a hyperbolic surface of genus two or higher and let γ be a simple closed geodesic on S. Cutting along γ creates two boundary components for T^1S which are 2-tori $\mathbb{T}^2 = S^1 \times S^1$, where the first S^1 corresponds to γ and the second S^1 to the fiber circle. To a obtain the graph manifold M reglue these boundary components with a shearing map $F\colon \mathbb{T}^2 \to \mathbb{T}^2$ given by

$$(x, \alpha) \mapsto (x + a\alpha, \alpha),$$

where $a \in \mathbb{Z}\backslash\{0\}$. Because the angular coordinate stays unchanged, the differential DF matches the Anosov vector field on the boundary components. Hence the geodesic flow on T^1S induces a flow on M. The construction is summarized on the following figure taken from [HT]. Among other things, Handel and Thurston showed that if one makes an appropriate choice of a, then this flow is a volume preserving Anosov flow.

Now, as before, we fix a sequence of hyperbolic metrics $\{g_n; n \geq 1\}$ on S such that the length of γ tends to zero. Each of these metrics yields a Handel-Thurston flow on the graph manifold M whose time-one map is denoted by $f_n\colon M \to M$. Note that our construction of the Dehn twist occurs in the "one-sided collar" of γ. Hence the Dehn twists ρ_n, $n \geq 1$, induce diffeomorphism $D\rho_n\colon M \to M$ (which are no longer differentials, but rather "glued differentials"; however we keep the same notation for consistency).

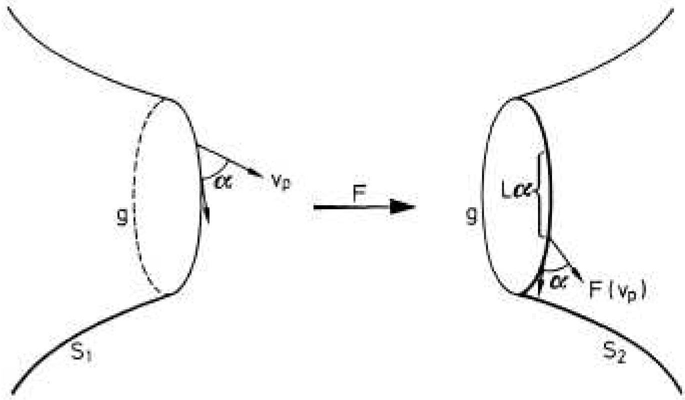

FIGURE 2. Shear with $a = 1$

Theorem 2.13. *Let M, f_n and $D\rho_n$, $n \geq 1$, be all as described above. Then for all sufficiently large n the diffeomorphisms $D\rho_n \circ f_n$ are absolutely partially hyperbolic. Furthermore, these diffeomorphisms and their finite iterates are not homotopic to identity.*

Sketch of the proof of Theorem 2.13. The mechanism of hyperbolicity of Handel and Thurston is summarized on Figures 3 and 5 taken from [HT].

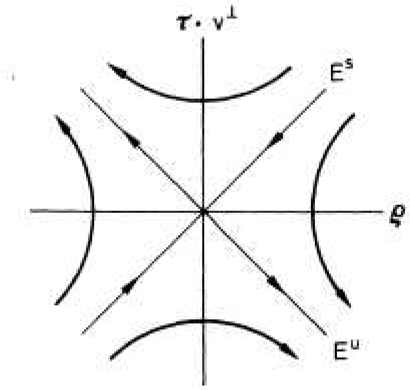

Fig. 1. Action of $D\psi$ on $\rho \times (\tau \cdot v^{\perp})$ plane

FIGURE 3

Figure 3 depicts the action of the geodesic flow on the plane transverse to the flow in certain special coordinates. Figure 5 depicts the action of the differential DF of the gluing map on the transverse plane in the same coordinates. Here the strength of shear depends on L — the length of γ. Hence, as length of γ tends to zero, DF becomes close to identity. It readily follows that the Anosov splitting of the Handel-Thurston flow gets arbitrarily

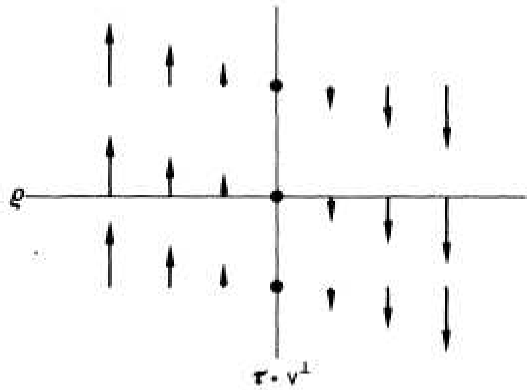

Fig. 2. Action of DF [resp. DF^{-1}] on the $\rho\times(\tau\cdot v^{\perp})$ plane for fixed v_p with $0<\sphericalangle(Tg,v)<\pi$ [resp. $\pi<\sphericalangle(Tg,v)<2\pi$]

FIGURE 4

close to the original Anosov splitting. Hence the former is almost orthogonal. Further, the expansion and contraction rates are also remain almost the same. Therefore, by direct inspection one can check that our proof of Theorem 2.8 goes through in this setting as well.

To finish the proof, one must show that the resulting diffeomorphism and its iterates are not homotopic to identity. For the sake of brevity, we will only indicate the needed results from 3-manifold topology:

- In [Wa] it is shown that for irreducible and sufficiently large 3-manifolds (the manifolds we are dealing here are irreducible since they admit an Anosov flow and sufficiently large since they contain an incompressible tori) homotopy and isotopy classes coincide, so it is enough to show that $D\rho_n$ is not isotopic to the identity.
- In [McC, Proposition 4.1.1] it is shown that in this situation (i.e. the manifold is not the mapping torus of a linear Anosov diffeomorphism of $\mathbb{T}^2$), if $D\rho_n$ were isotopic to the identity then one could assume that the isotopy fixes the torus on which we have cut the manifold all along the isotopy.
- In [Jo, Proposition 25.3] the mapping class group of the resulting pieces after cutting along the tori is studied. In particular, one can use this result and the previous remark to check that $D\rho_n$ is not isotopic to identity.

□

Remark 2.14. The construction of a volume preserving modification of this example is the same as the one in Subsection 2.7. Existence of stably ergodic and robustly transitive perturbations can be seen in the same way as in Subsection 2.8.

Remark 2.15. Similarly to our remark in 2.9.1, we can also pick two collections of disjoint simple closed geodesic on S (the collections may coincide) and perform Handel-Thurston surgery with respect to one set and the Dehn twist construction with respect to the second set.

Remark 2.16. We also would like to remark that in the case when at least one of the Dehn twists is done on a non-separating geodesic along which the Handel-Thurston surgery is not performed then one can directly check that the induced homology automorphism is of infinite order. Hence, in this case, one does not need to rely on 3-manifold theory to see that the constructed diffeomorphism and its iterates are not homotopic to identity. In contrast, if the Handel-Thurston surgery and the Dehn twist are being done on the same geodesic (separating or not) the homology of this geodesic vanishes in the resulting graph manifold; and hence, the Dehn twist is identity in homology.

3. An example on a graph manifold

3.1. **Anosov flows transverse to tori.** Let Y be an Anosov vector field on a 3-manifold M and let $T \subset M$ be a torus transverse to Y. It is well known that T must be incompressible [Br]. A systematic study of Anosov flows transverse to tori has been recently carried out in [BBY], yet some questions still remain open. Most parts of our construction work well for all examples of Anosov flows transverse to tori which are not suspensions, however, at some stages we shall rely on the specific example from [BL] (particularly, Lemma 3.8 below).

We will consistently use the same notation for vector fields and flows generated by them; *e.g.,* we write Y^t for the flow generated by the vector field Y. We begin our presentation with the following lifting construction.

Proposition 3.1. *For every $t_1 > 0$ there exists a finite connected covering $\hat{M} \to M$ such that if $\hat{Y}$ is the lift of Y and $\hat{T}$ is a (connected component of) lift of T then for all $t \in (0, t_1)$ one has $\hat{T} \cap \hat{Y}^t(\hat{T}) = \emptyset$. Moreover, with respect to the metric on $\hat{M}$ induced by the covering map, the C^1-norm of $\hat{Y}$ is the same as the one of Y and the time-one map $\hat{Y}^1$ is (ℓ, λ, μ)-partially hyperbolic whenever Y^1 is.*

Above we call a partially hyperbolic diffeomorphism $f : M \to M$ a (ℓ, λ, μ)*-partially hyperbolic* if

$$\|Df^\ell|_{E^{ss}(x)}\| < \lambda < \|Df^\ell|_{E^c(x)}\| < \mu < \|Df^\ell|_{E^{uu}(x)}\|$$

Proof. Consider the manifold M_0 obtained by cutting M along T. If M_0 is not connected, then this means that $Y^t(T)$ is disjoint from T for all $t \neq 0$ and therefore there is no need to consider a lift of M, *i.e.,* the posited property holds for Y.

If M_0 is connected, then it has two boundary components T_1 and T_2 such that Y points inwards on T_1 and outwards on T_2. Let $t_0 > 0$ be the minimal time for an orbit to go from T_1 to T_2 so that for every $x \in T_1$ we have $Y^t(x) \notin T_2$ for $0 \le t \le t_0$. Now glue $[t_1/t_0] + 1$ copies of M_0 by identifying the copies of T_2 with the copies of T_1 and closing-up the last copy to obtain a compact boundaryless manifold $\hat{M}$ which covers M. Clearly, if $\hat{T}_1$ is a lift of T_1 then $\hat{Y}^t(\hat{T}_1) \cap \hat{T}_1 = \emptyset$ for $0 < t \le t_1$.

The fact that the norm and the (ℓ, λ, μ)-partial hyperbolicity are not affected is direct from the fact that the differentiable and metric structures are obtained by lifting those from M. $\square$

Remark 3.2. It is also easy to see that the bundles $E^\sigma_{\hat{Y}}$ are the lifts of the bundles E^σ_Y as well as the $\hat{Y}_t$-invariant foliations ($\sigma = cs, cu, ss, uu$).

3.2. **Coordinates in flow boxes.** As before, we consider an Anosov vector field Y transverse to a torus T. Since Y is transverse to T we obtain that the foliations W^{cs}_Y and W^{cu}_Y induce (transverse) foliations $\mathcal{L}^{cs}_Y$ and $\mathcal{L}^{cu}_Y$ on T.

We can consider coordinates $\theta_T \colon T \to \mathbb{T}^2$ where $\mathbb{T}^2 = \mathbb{R}^2/_{\mathbb{Z}^2}$ with the usual (x, y)-coordinates (mod 1). (The choice of θ_T is not canonical, and will be specified later in Lemma 3.8.)

We denote by F^s and F^u the foliations $\theta_T(\mathcal{L}^{cs}_Y)$ and $\theta_T(\mathcal{L}^{cu}_Y)$, respectively.

By Proposition 3.1, for each $N > 0$ and integer $K \geq 1$ there exist a finite covering $\hat{M}_{N,K} \to M$ such that $\hat{T} \cap \hat{Y}^t(\hat{T}) = \emptyset$ for $0 \leq t \leq NK$. Then the set

$$\mathcal{U}_N = \bigcup_{0 \leq t \leq N} \hat{Y}_t(\hat{T})$$

is injectively embedded in $\hat{M}_{N,K}$ and the first K-iterates by $\hat{Y}^N$ of $\mathcal{U}_N$ have mutually disjoint interiors. (We have slightly abused the notation by ignoring the dependence of $\mathcal{U}_N$ on K, but this will not cause any confusion.)

Consider the "straightening diffeomorphism" $H_N : \mathcal{U}_N \to [0, 1] \times \mathbb{T}^2$ given by

$$H_N(\hat{Y}_t(p)) = \left(\frac{t}{N}, \theta_T(p)\right), p \in \hat{T} \tag{3.1}$$

For fixed N and K we shall denote by $\hat{W}^\sigma_N$ and $\hat{E}^\sigma_N$ the corresponding foliations and invariant bundles for the lift $\hat{Y}$ ($\sigma = cs, cu, ss, uu$). (Again, we suppress dependance on K to avoid overloading the notation.)

We also denote by $\mathcal{F}^{uu}$ and $\mathcal{F}^{ss}$ the one-dimensional foliations of $[0, 1] \times \mathbb{T}^2$ which in $\{t\} \times \mathbb{T}^2$ coincide with the foliations $\{t\} \times F^u$ and $\{t\} \times F^s$ respectively.

Lemma 3.3. *Diffeomorphism H_N have the following properties*

- $H_N(\hat{W}^{cs}_N \cap \mathcal{U}_N) = [0, 1] \times F^s \stackrel{\text{def}}{=} \mathcal{F}^{cs}$
- $H_N(\hat{W}^{cu}_N \cap \mathcal{U}_N) = [0, 1] \times F^u \stackrel{\text{def}}{=} \mathcal{F}^{cu}$
- $DH_N(\hat{E}^{ss}_N)$ *converges to the tangent bundle of the foliation* $\mathcal{F}^{ss}$ *as* $N \to \infty$.
- $DH_N(\hat{E}^{uu}_N)$ *converges to the tangent bundle of the foliation* $\mathcal{F}^{uu}$ *as* $N \to \infty$.

It is important to remark that the above convergence is with respect to the standard metric in $[0, 1] \times \mathbb{T}^2$ and not with respect to the push forward metric from the manifold via H_N.

Proof. Because the differential of H_N maps the vector field $\hat{Y}$ to the vector field $\frac{1}{N}\frac{\partial}{\partial t}$ the first two properties follow. Note that the component of $\hat{E}^\sigma_N$ along $\hat{Y}$ is uniformly bounded ($\sigma = ss, uu$). Therefore, contraction by a factor $\frac{1}{N}$ implies the posited limit behavior in the latter properties. $\square$

3.3. **A diffeomorphism in a flow box which preserves transversalities.** Assume that there exists a smooth path $\{\varphi_s\}_{s\in[0,1]}$ of diffeomorphisms of $\mathbb{T}^2$ such that

- $\varphi_s = Id$ for s in neighborhoods of 0 and 1,
- the closed path $s \mapsto \varphi_s$ is not homotopically trivial in $\mathrm{Diff}(\mathbb{T}^2)$,
- for every $s \in [0,1]$:
$$\varphi_s(F^u) \pitchfork F^s.$$

We use the coordinate chart $H_N\colon \mathcal{U}_N \to [0,1]\times\mathbb{T}^2$ to define diffeomorphism $\mathcal{G}_N : \mathcal{U}_N \to \mathcal{U}_N$ by
$$(s,x,y) \mapsto (s, \varphi_s(x,y)).$$

The following lemma is immediate from our choice of $\{\varphi_s\}_{s\in[0,1]}$.

Lemma 3.4. *The diffeomorphism $\mathcal{G}_N$ has the following properties:*

- *$\mathcal{G}_N(H_N^{-1}(\mathcal{F}^{uu}))$ is transverse to $\hat{W}^{cs}_{\hat{Y}} = H_N^{-1}(\mathcal{F}^{cs})$,*
- *$\mathcal{G}_N(\hat{W}^{cu}_{\hat{Y}}) = \mathcal{G}_N(H_N^{-1}(\mathcal{F}^{cu}))$ is transverse to $H_N^{-1}(\mathcal{F}^{ss})$.*

Let $h_N\colon \hat{M}_{N,K} \to \hat{M}_{N,K}$ be the diffeomorphism which coincides with $\mathcal{G}_N$ on $\mathcal{U}_N$ and is identity outside $\mathcal{U}_N$.

Corollary 3.5. *There exists $N_0 > 0$ such that for $N \geq N_0$:*

- *$Dh_N(E^{uu}_{\hat{Y}})$ is transverse to $E^{cs}_{\hat{Y}}$,*
- *$Dh_N(E^{cu}_{\hat{Y}})$ is transverse to $E^{ss}_{\hat{Y}}$.*

Proof. This follows by combining Lemma 3.4, Lemma 3.3 and the fact that outside $\mathcal{U}_N$ the diffeomorphism h_N is the identity. □

Remark 3.6. Notice that if there is a volume form ω on T such that φ_s preserves the form $(\theta_T)_*(\omega)$ for every s, then h_N preserves volume form $\omega \wedge dY$.

3.4. **Proof of partial hyperbolicity.** Let diffeomorphism $f_{N,K}\colon \hat{M}_{N,K} \to \hat{M}_{N,K}$ be the time-N map of the flow generated by the vector field $\hat{Y}$ on $\hat{M}_{N,K}$.

Proposition 3.7. *For any sufficiently large N there exists $K_0 = K_0(N)$ such that for all $K \geq K_0$ the diffeomorphism $h_N \circ f_{N,K}$ is absolutely partially hyperbolic.*

Proof. Let $F_{N,K} = h_N \circ f_{N,K}$. The proof uses the cone-field criteria.

Recall (see Proposition 3.1) that for large enough N and any K we have that $f_{N,K} : \hat{M}_{N,K} \to \hat{M}_{N,K}$ is $(1,\lambda,\mu)$-partially hyperbolic for some $\lambda < 1 < \mu$. We shall choose $1 < \mu_1 < \mu_0 < \mu$ (and $\lambda < \lambda_0 < \lambda_1 < 1$ for the symmetric argument).

First consider a fixed value of $N \geq N_0$ given by Corollary 3.5. Then, using partial hyperbolicity and Corollary 3.5, we can choose a cone-field $\mathcal{E}^{uu}$ about $E^{uu}_{\hat{Y}}$ such that

- cone-field $\mathcal{E}^{uu}$ is transverse to $E^{cs}_{\hat{Y}}$;
- $Df_{N,K}(\overline{\mathcal{E}^{uu}}) \subset \mathcal{E}^{uu}$;
- cone-field $Dh_N(\mathcal{E}^{uu})$ is also transverse to $E^{cs}_{\hat{Y}}$;

and

- for every $v \in \mathcal{E}^{uu} \setminus \{0\}$ one has that $\|Df_{N,K}v\| > \mu_0\|v\|$;
- for every $v \in T_x\hat{M}_{N,K}$ such that $Df_{N,K}v \notin \mathcal{E}^{uu}$ one has that $\|Df_{N,K}v\| \leq \mu_1\|v\|$.

We shall show that if K is sufficiently large then we can construct a cone-field $\mathcal{C}^{uu}$ such that for a sufficiently large iterate $n_K > 0$ we have

$$DF^{n_K}_{N,K}(\overline{\mathcal{C}^{uu}}) \subset \mathcal{C}^{uu}$$

and there exists $\hat{\mu} > 1$ such that

$$\|DF^{n_K}_{N,K}v\| > \hat{\mu}\|v\|, \text{ if } v \in \mathcal{C}^{uu} \setminus \{0\}.$$

and

$$\|DF^{n_K}_{N,K}v\| \leq \hat{\mu}\|v\|, \text{ if } DF^{n_K}_{N,K}v \notin \mathcal{C}^{uu}.$$

Consider the minimal n_0 such that $Df^{n_0}_{N,K}(Dh_N(\overline{\mathcal{E}^{uu}})) \subset \mathcal{E}^{uu}$. Because the angle between $Dh_N(\mathcal{E}^{uu})$ and $E^{cs}_{\hat{Y}}$ is uniformly bounded from below, the value of n_0 is independent of K. Define the cone-field $\mathcal{C}^{uu}$ as follows:

- $\mathcal{C}^{uu} = Dh_N(\mathcal{E}^{uu})$ on $\mathcal{U}_N$,
- $\mathcal{C}^{uu} = \mathcal{E}^{uu}$ outside $\mathcal{U}_N$.

Claim. *There exists $n > 0$ such that $DF^n_{N,K}(\overline{\mathcal{C}^{uu}}) \subset \mathcal{C}^{uu}$.*

Proof. Notice that $\mathcal{C}^{uu}$ is indeed smooth since h_N coincides with the identity in a neighborhood of the boundary of $\mathcal{U}_N$. Let us first show that if $n = 2\ell n_0$ with $\ell \geq 2$ and $K \gg 2n_0$ then

$$DF^n_{N,K}(\overline{\mathcal{C}^{uu}}) \subset \mathcal{C}^{uu}.$$

This is quite direct. Notice first that for points $x \notin \mathcal{U}_N \cup f^{-1}_{N,M}(\mathcal{U}_N)$ one has that $DF_{N,K}(\overline{\mathcal{C}^{uu}}(x)) \subset \mathcal{C}^{uu}(f_{N,M}(x))$ since this holds true for $\mathcal{E}^{uu}$ and $f_{N,M}$ (and $F_{N,K} = f_{N,M}$ on $f^{-1}_{N,M}(\mathcal{U}_N)$). For points $x \in f^{-1}_{N,M}(\mathcal{U}_N)$ we have that $DF_{N,K}(\overline{\mathcal{C}^{uu}(x)}) = Dh_N Df_{N,K}(\overline{\mathcal{C}^{uu}(x)}) \subset Dh_N(\mathcal{E}^{uu}(f_{N,K}(x))) = \mathcal{C}^{uu}(F_{N,K}(x))$. Finally, if $x \in \mathcal{U}_N$ then, by construction, we have that $DF^{n_0}_{N,K}(\overline{\mathcal{C}^{uu}}(x)) \subset \mathcal{C}^{uu}(F_{N,K}(x))$. This implies the inclusion

$$DF^{2n_0}_{N,K}(\overline{\mathcal{C}^{uu}}) \subset \mathcal{C}^{uu}.$$

□

To show absolute partial hyperbolicity (*i.e.,* the existence of $\hat{\mu}$ as stipulated earlier) it is enough to consider large enough ℓ as above and $n_K = 2\ell n_0$ (and $K \geq n_K$, recall that n_0 and ℓ are independent of K). Let $C = \max_x\{\|DF_{N,K}(x)\|, \|DF^{-1}_{N,K}(x)\|\}$.

Notice first that for any point $x \in M_{N,K}$ one has that $F^i_{N,K}(x) \in \mathcal{U}_N$ for at most one value of $0 \leq i \leq n_K$. This implies that:

- on the one hand, if $v \in \mathcal{C}^{uu}$, then $\|DF^{n_k}_{N,K}v\| \geq C^{-1}\mu_0^{n_k-1}\|v\|$;
- on the other hand, if $DF^{n_k}_{N,K}v \notin \mathcal{C}^{uu}$ then $\|DF^{n_k}_{N,K}v\| \leq C\mu_1^{n_k-1}$.

which for large enough n_k verifies the desired properties.

Finally, by reversing the time, a symmetric argument provides a cone-field $\mathcal{C}^{ss}$ with analogous properties and, therefore, yields absolute partial hyperbolicity of $F_{N,K}$ for a sufficiently large K. □

3.4.1. *Some remarks on this approach.* The approach presented here requires to consider large finite coverings of the initial manifold in order to ensure large return times to the transverse tori. This method provides uniform bounds on the constants of the partial hyperbolicity and can be compared with the mechanism used in section 2 to construct examples starting from geodesic flows in constant negative curvature.

After finishing the first draft of this paper, we discovered a different mechanism which guarantees partial hyperbolicity after composing with Dehn twists and allows one to avoid passing to finite covers. This mechanism is related to the one here yet involves some different ideas. The current construction may actually suit better when trying to understand the dynamics of new examples.

We will leave to it a future paper to explore this different mechanism which will also unify the mechanisms for both (families of) examples presented in this paper (the ones with an incompressible torus transverse to the flow and the ones with an incompressible torus not transverse to the flow).

3.5. **Bonatti-Langevin's example.** Notice that to this point we do not know if there is an Anosov flow transverse to a torus for which the posited family of diffeomorphisms $\{\varphi_s\}_{s\in[0,1]}$ exists. For this purpose we shall introduce a specific class of Anosov flow examples from [BL] which will also make easy the task of showing volume preservation. It is plausible that other examples, *e.g.,* those that can be found in [BBY], also can serve as the Anosov flow ingredient in our construction. However we haven't checked it.

A relevant remark is that Proposition 3.7 can be applied to the suspension of a linear Anosov diffeomorphism of $\mathbb{T}^2$ which gives rise to a manifold where every partially hyperbolic is leaf conjugate to an Anosov flow ([HP$_2$]). It is important to consider a manifold on which the diffeomorphism h_N is not isotopic to the identity in order to obtain a new example of partially hyperbolic dynamics.

A volume preserving transitive Anosov flow $X_0^t\colon M_0 \to M_0$ was built in [BL] which admits a torus T_0 transverse to the flow. At the same time, the flow admits a periodic orbit disjoint from T_0. We denote by X_0 the vector field generating this flow.

We will state in the following lemma the properties about this flow that we shall use to construct a family φ_s as in the previous subsection:

Lemma 3.8. *There exist coordinates $\theta : T_0 \to \mathbb{T}^2$ such that:*

- *In the coordinates (x, y) of $\mathbb{T}^2$ the foliation F^s is a foliation with two horizontal circles and such that every leaf is locally of the form $(x, g^s(x))$, where g^s is a function with derivative smaller than $\frac{1}{4}$.*
- *In the coordinates (x, y) of $\mathbb{T}^2$ the foliation F^u is a foliation with two vertical circles and such that every leaf is locally of the form $(g^u(y), y)$, where g^u is a function with derivative smaller than $\frac{1}{4}$.*

- *The flow is orthogonal to T_0 and if ω is the area form induced in T_0 by the invariant volume form one has that $\theta_*(\omega)$ is the standard area form $dx \wedge dy$ on $\mathbb{T}^2$.*

Proof. This follows by inspection of the construction in [BL].

The first two properties follow directly from the computation of the holonomy between the entry torus and the exit torus performed in page 639 of [BL]. The relation between h_t and the intersection of the center-stable and center-unstable foliations with T_0 can be inferred from the arguments at the end of the proof of the main theorem (see the end of page 642 and page 643 of [BL]).

Preservation of the volume form in this coordinates is provided by [BL, Lemma 3.1]. □

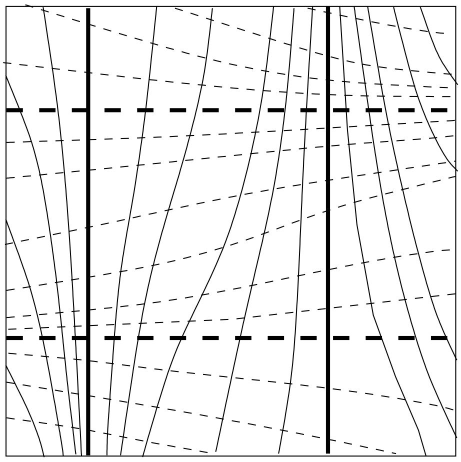

FIGURE 5. The foliations F^s and F^u in $\mathbb{T}^2$ for the Bonatti-Langevin's example.

Notice that for X_0^t one can easily construct a family of diffeomorphisms $\{\varphi_s\}_{s\in[0,1]}$ consisting of translations in $\mathbb{T}^2$ (which preserve the Lebesgue measure and transversality between the bundles) and verifies the properties of the previous section.

Corollary 3.9. *There exists a finite cover M of M_0 and a number $N > 0$ such that if we denote by f the time-N map of the flow X^t (the lift of X_0^t to M) and a volume preserving diffeomorphism $h_N \colon M \to M$ which is a Dehn twist along a lift T of T_0 such that the diffeomorphism $F = h_N \circ f$ is a conservative absolutely partially hyperbolic diffeomorphism.*

Proof. It follows directly from the previous lemma that there exists a family $\{\varphi_s\}_{s\in[0,1]}$ with the properties required by Proposition 3.7. This gives the desired statement (see Remark 3.6 for the volume preservation). □

3.6. **The iterates of the Dehn twist are not isotopic to the identity.** According to [Br], every torus transverse to an Anosov flow is incompressible (*i.e.,* its fundamental group is injected in the fundamental group of the ambient manifold). Furthermore, in the case of the [BL]-example, the manifold obtained by cutting along the torus is a circle bundle, and the gluing map along the torus does not preserve the homology class (in the torus) of the fibers: in the terminology of the topology of 3-manifolds, this means that the torus belongs to the Jaco-Shalen-Johannson (JSJ) family of the ambient manifold. A result of Johannson [Jo, Proposition 25.3] (see also [McC, Proposition 4.1.1]) provides a general criterion for proving that a Dehn twist along a given torus of the JSJ family, and its iterates, are not isotopic to the identity. This criterium indeed applies to the Dehn twist along the transverse torus of the example of [BL] (see also [Ba]).

Nevertheless, there is a very elementary proof for the specific example given in Section 3.5. The goal of this section is to present this simple proof.

In Section 3.1, for any manifold M endowed with a transitive Anosov flow X and a torus T transverse to X, for any integer $n > 0$, we build a covering obtained by considering n copies of the manifold M_0 with boundary obtained by cutting M along T. Let us denote by $\Pi_n\colon \tilde{M}_n \to M$ this n-cyclic covering. Note that $\Pi_n^{-1}(T)$ is a family of n disjoint tori $\hat{T}_0, \dots, \hat{T}_{n-1}$. In this section we will denote by V the underlying manifold of [BL]-example and by $\Pi_n\colon V_n \to V$ the n-cyclic covering defined above.

Proposition 3.10. *Let n be an integer divisible by 4 and let the manifold V, the transverse torus T and the covering $\Pi_n\colon V_n \to V$ be as described above. Let $\hat{T}_0, \dots, \hat{T}_{n-1}$ be the lifts of T to V_n. Then for any non-zero homology class $[\gamma]$ of $H_1(\hat{T}_0, \mathbb{Z})$, the Dehn twist along $\hat{T}_0$ in the direction of $[\gamma]$ and its positive iterates are not homotopic to the identity map on V_n.*

3.6.1. *Description of the ambient manifold V and its 2-cover.* Let V_0 be the compact manifold with boundary obtained by cutting V along T. By construction in [BL] we have the following facts:

- the boundary of V_0 consists of two tori T_0 and T_1;
- V_0 is the total space of a circle bundle $S^1 \to V_0 \overset{p}{\to} B$;
- the base B is the projective plane $\mathbb{R}P^2$ with the interiors of two disjoint disks removed, or equivalently the Mœbius band with the interior of a disk removed;
- the structure group of $p\colon V_0 \to B$ contains orientation-reversing diffeomorphisms and $p\colon V_0 \to B$ is the unique circle bundle over B for which the total space V_0 is oriented.

We denote by $\varphi\colon T_0 \to T_1$ the orientation reversing gluing diffeomorphism (that is, $V = V_0/\varphi$).

We proceed with an explicit description of $p\colon V_0 \to B$. Let

$$\tilde{B} = \mathbb{R}/\mathbb{Z} \times [-1,1] \setminus \left(D\left((0,0), \frac{1}{2}\right) \cup D\left((\frac{1}{2},0), \frac{1}{2}\right)\right)$$

where $D((t,0),\frac{1}{2})$ is the open disk of radius $\frac{1}{2}$ centered at the point $(t,0) \in \mathbb{R}/\mathbb{Z} \times [-1,1]$. Let

$$\tilde{V}_0 = \tilde{B} \times \mathbb{R}/\mathbb{Z}.$$

Note that B is diffeomorphic to the quotient of $\tilde{B}$ by the involution without fixed points $(r,s) \mapsto (r+\frac{1}{2}, -s)$ and V_0 is the quotient of $\tilde{V}_0$ by the free involution $(r,s,t) \mapsto (r+\frac{1}{2}, -s, -t)$. We denote by $\pi_0\colon \tilde{V}_0 \to V_0$ this 2-fold cover.

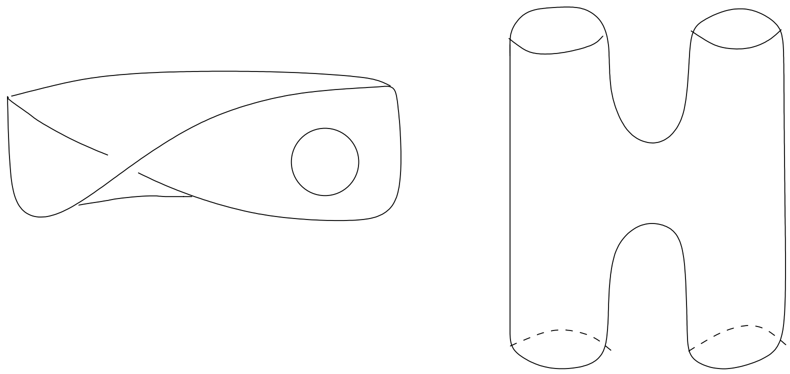

FIGURE 6. The base space B (to the left) and its 2-fold covering $\tilde{B}$ (to the right).

Note that π_0 restricts to a diffeomorphism on each connected component of $\partial\tilde{V}_0$ (to a connected component of ∂V_0). Let us denote

$$\pi_0^{-1}(T_0) = T_{0,0} \sqcup T_{0,1} \text{ and } \pi_0^{-1}(T_1) = T_{1,0} \sqcup T_{1,1}.$$

Let $\tilde{\varphi}_0\colon T_{0,0} \to T_{1,0}$ and $\tilde{\varphi}_1\colon T_{0,1} \to T_{1,1}$ be the unique diffeomorphisms which project by π_0 to φ. We denote by $\tilde{\varphi}\colon T_{0,0} \cup T_{1,0} \to T_{0,1} \cup T_{1,1}$ the assembled diffeomorphism which coincides with $\tilde{\varphi}_i$ on $T_{i,0}$, $i = 0, 1$.

Let

$$\tilde{V} = \tilde{V}_0/\tilde{\varphi}.$$

One can easily check that π_0 induces a covering map $\pi\colon \tilde{V} \to V$.

3.6.2. *Description of the gluing map φ.* The manifold $\tilde{V}_0$ is a product of $\tilde{B}$ and the circle S^1. The horizontal 2-foliation (whose leaves are the $\tilde{B} \times \{t\}$) and the vertical circle bundle passes to the quotient by the involution on V_0. They induce on each connected component $T_0, T_1, T_{0,0}, T_{0,1}, T_{1,0}, T_{1,1}$ two transverse foliations by circles called respectively meridians (induced by the horizontal 2-foliation) and parallels (or fibers).

According to [BL] the gluing map φ is chosen so that it maps the parallels and meridians of T_0 to meridians and parallels of T_1, respectively. Thus $\tilde{\varphi}$ exchanges parallels with meridians in the same way (for other constructions see [Ba]).

We will need the following remark.

Remark 3.11. Fix an orientation of V_0. It induces orientations of T_0 and of T_1 (as boundary orientation). We orient $\tilde{V}_0$ so that the projection π_0 preserves the orientation. Now $T_{i,j}$ inherit of the boundary orientation and $\pi_0\colon T_{i,0} \cup T_{i,1} \to T_i$ is orientation preserving.

Up to changing all orientations, the gluing map φ is a quarter of turn in the positive direction, that is, (in coordinates putting the meridians and the fibers in horizontal and vertical position) a rotation by $+\frac{\pi}{4}$.

Because π is orientation preserving, both restrictions of $\tilde{\varphi}$ to $T_{0,0}$ and $T_{0,1}$ (endowed with the boundary orientations) are rotations by $+\frac{\pi}{4}$.

3.6.3. *$4m$-cyclic covers of V and $\tilde{V}$; the proof of Proposition 3.10.* Recall that the cyclic covering $\Pi_n\colon V_n \to V$ is obtained by considering n copies $V_0 \times \{i\}$, $i \in \mathbb{Z}/n\mathbb{Z}$, of V_0 and by gluing $T_0 \times \{i\}$ with $T_1 \times \{i+1\}$ using φ. In the same way be define a cyclic covering $\tilde{\Pi}_n\colon \tilde{V}_n \to \tilde{V}$ obtained by considering n copies $\tilde{V}_0 \times \{i\}$, $i \in \mathbb{Z}/n\mathbb{Z}$, of $\tilde{V}_0$ and by gluing $(T_{0,0} \cup T_{0,1}) \times \{i\}$ with $(T_{1,0} \cup T_{1,1}) \times \{i+1\}$ using $\tilde{\varphi}$. It is easily to check that π_0 induces a covering map $\pi_n\colon \tilde{V}_n \to V_n$ which projects to π.

Proposition 3.10 is a corollary of the next lemma.

Lemma 3.12. *For any $n \in 4\mathbb{N} \setminus \{0\}$, the homomorphism $H_1(T_{i,j} \times \{k\}, \mathbb{Z}) \to H_1(\tilde{V}_n)$, $i,j \in \{0,1\}$, $k \in \mathbb{Z}/n\mathbb{Z}$ induced by the inclusion $T_{i,j} \times \{k\} \subset \tilde{V}_n$ is injective.*

Proof of Proposition 3.10. The Dehn twist h_γ on V_n along $T_0 \times \{i\}$ in the direction of $\gamma \in H_1(T_0, \mathbb{Z})$ can be lifted to a diffeomorphisms $\tilde{h}_\gamma\colon \tilde{V}_n \to \tilde{V}_n$ which is the composition of two Dehn twists with disjoint supports, one along $T_{0,0} \times \{i\}$ and one along $T_{0,1} \times \{i\}$ in the direction of the lifts of γ. For proving that h_γ^k, $k \in \mathbb{Z} \setminus \{0\}$, is not homotopic to the identity map of V_n, it is enough to prove that $\tilde{h}_\gamma^k$ is not homotopic to the identity on $\tilde{V}_n$.

Notice that, by construction, $T_{0,0} \times \{i\} \cup T_{0,1} \times \{i\}$ does not disconnect $\tilde{V}_n$. This implies that there is a closed path σ which intersects $T_{0,0} \times \{i\} \cup T_{0,1} \times \{i\}$ transversely at a single point contained in $T_{0,0} \times \{i\}$. In particular, the homology class of σ in $H_1(\tilde{V}_n, \mathbb{Z})$ is non-trivial. Let $[\gamma_0]$ be the homology class of the lift of γ to $T_{0,0} \times \{i\}$. Lemma 3.12 implies that the class $[k.\gamma_0]$ in $H_1(\tilde{V}_n, \mathbb{Z})$ does not vanish when $k \neq 0$. Clearly

$$\left(\tilde{h}_\gamma^k\right)_* ([\sigma]) = [\sigma] + k[\gamma_0] \neq [\sigma].$$

Thus the action of $\tilde{h}_\gamma^k$ on homology is non-trivial, completing the proof. □

3.6.4. *Oriented surfaces in $\tilde{V}_4$ and the proof of Lemma 3.12.*

Lemma 3.13. *There is a smooth oriented closed surface $\Sigma_0 \subset \tilde{V}_4$ which intersects transversally each circle fiber of $\tilde{V}_0 \times \{0\}$ at exactly one point.*

Proof. Let $S = \tilde{B} \times \{0\} \subset \tilde{V}_0$. It is a surface with boundary which intersects each circle fiber in exactly one point.

We consider $S \times \{0\} \subset \tilde{V}_0 \times \{0\} \subset \tilde{V}_4$ and $S \times \{2\} \subset \tilde{V}_4$. Consider the intersection of these surfaces with $\tilde{V}_0 \times \{1\}$:

- $S \times \{0\} \cap \tilde{V}_0 \times \{1\}$ is the image by $\tilde{\varphi}$ of $(S \cap (T_{0,0} \cup T_{0,1})) \times \{0\}$. In other words, it consists of exactly one circle fiber in $T_{1,0} \times \{1\}$ and one circle fiber in $T_{1,1} \times \{1\}$.
- in the same way $S \times \{1\} \cap \tilde{V}_0 \times \{1\}$ is the image by $\tilde{\varphi}^{-1}$ of $(S \cap (T_{1,0} \cup T_{1,1})) \times \{2\}$ that is, it consists of exactly one circle fiber in $T_{0,0} \times \{1\}$ and one circle fiber in $T_{0,1} \times \{1\}$.

There are two disjoint cylinders $C_0 \times \{1\}$ and $C_1 \times \{1\}$, each being a 1-parameter family of circle fibers in $\tilde{V}_0 \times \{1\}$ such that:

- the cylinder $C_0 \times \{1\}$ connects the fiber in $T_{1,0}$ with the one in $T_{0,0}$ and
- the cylinder $C_1 \times \{1\}$ connects the fiber in $T_{1,1}$ with the one in $T_{0,1}$.

The surface we obtain still has four boundary components which are circle fibers in $\tilde{V}_0 \times \{3\}$. And each boundary components of $\tilde{V}_0 \times \{3\}$ contains precisely one of these circles. We obtain the posited surface Σ_0 by gluing in the cylinders $C_0 \times \{3\}$ and $C_1 \times \{3\}$.

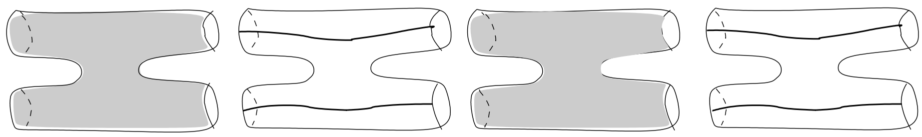

FIGURE 7. Construction of the surface Σ_0. It consists of the shaded surfaces (corresponding to leaves of the horizontal foliation) and the product of the curves connecting the boundaries multiplied by the fibers in the parts where the surfaces are not shaded.

To finish the proof it remains to see that Σ_0 is orientable. For this we need to pay attention to the orientations of the fibers glued by the cylinders. More precisely, orientability follows from our description of the gluing map and the following properties:

- the orientations of the circle fibers in $T_{1,0} \times \{1\}$ and in $T_{1,1} \times \{1\}$, (viewed as the image by $\tilde{\varphi}$ of boundary components of the surface $S \times \{0\}$) are the same (as circle fibers in $V_0 \times \{1\}$).
- the orientations of the circle fibers in $T_{0,0} \times \{1\}$ and in $T_{0,1} \times \{1\}$, (viewed as the image by $\tilde{\varphi}^{-1}$ of boundary components of the surface $S \times \{3\}$) are the same (as circle fibers in $V_0 \times \{s\}$).

But these follow from Remark 3.11, completing the proof of the lemma. □

Recall that $\tilde{V}_4$ is a cyclic 4-cover of $\tilde{V}_0$. Let ρ be a generator of the Deck transformation group (sending $\tilde{V}_0 \times \{i\}$ on $\tilde{V}_0 \times \{i+1\}$). Let $\Sigma_1 = \rho(\Sigma_0)$. We can now conclude the proof of Lemma 3.12 and therefore of Proposition 3.10.

Lemma 3.14. *Any non-trivial* $[\gamma] \in H_1(T_{i,j} \times \{k\}, \mathbb{Z})$ *has a non-zero intersection number with either* Σ_0 *or* Σ_1.

For the proof notice that $(T_{i,j} \times \{k\}) \cap (\Sigma_0 \cup \Sigma_1)$ is the union of exactly one meridian and one parallel.

3.7. **Stable ergodicity and robust transitivity.** The same argument as in Subsection 2.8 shows that in this setting one also obtains stable ergodicity and robust transitivity by a small C^1 volume preserving perturbation.

The argument only uses the existence of a circle leaf where the dynamics is a rotation; in this example, as Bonatti-Langevin's example has a periodic orbit which is disjoint from the transverse tori, this orbit is not altered by the modifications one has made and therefore, (any of the lifts) of this orbit remains a closed normally hyperbolic center curve whose dynamics is conjugate to a rotation and it can be used for the argument.

This completes the proof of Theorem 1.1 for this family of manifolds.

3.8. **Periodic center leaves.** In [BDV, Chapter 7] the classification of transitive partially hyperbolic systems in dimension 3 is also discussed. Problem 7.26 is related to the structure of the center leaves of a transitive partially hyperbolic diffeomorphisms. It asks whether it is possible to have periodic points both in circle leaves and in leaves homeomorphic to the line. It also asks whether it is possible that a transitive partially hyperbolic diffeomorphism has a periodic center leaf homeomorphic to $\mathbb{R}$ for which one end is contracting and the other one is expanding (such leaves are called *saddle node leaves*; see [BDV, Section 7.3.4]).

We shall show that some of our examples admit both periodic center leaves which are lines as well as periodic center leaves which are circles. The existence of complete curves tangent to E^c invariant under some iterate of f through periodic points is established in [BDU, HHU_1].

Proposition 3.15. *There exists a volume preserving, robustly transitive absolutely partially hyperbolic diffeomorphism F on a closed 3-manifold, which has periodic points on normally hyperbolic invariant circles and periodic points which do not belong to any periodic invariant circle tangent to the center bundle.*

Proof. The diffeomorphism is obtained following the procedure which we described in this section, starting with a transitive Anosov flow X with a transverse torus, then considering a large finite cyclic cover and composing the time N map (for some N large enough) and a Dehn twist h_N along one component of the lift of the transverse torus.

Specifically, recall notation of Section 3.6 : let V be the closed 3 manifold endowed with the of Anosov flow X which admits a transverse torus. The map $\pi\colon \tilde{V} \to V$ is the 2-fold cover defined in Section 3.6.1. Also recall that $\tilde{V}_4$ and, more generally, $\tilde{V}_{4m}$ are the cyclic covers of $\tilde{V}$ defined in Section 3.6.3. Let $\tilde{X}_{4m}$ denote the lift of X on $\tilde{V}_{4m}$. The Anosov flow $\tilde{X}_{4m}$ admits at least $8m$ non-isotopic transverse tori denoted by $T_{i,j} \times \{k\}$, $i, j \in \{0, 1\}$, $k \in \mathbb{Z}/4m\mathbb{Z}$ following the notation of Section 3.6.3. Here we will denote by $\tilde{T}_{4m}$ the torus $T_{0,0} \times \{0\} \subset \tilde{V}_{4m}$.

Our starting Anosov flow is $(\tilde{V}_4, \tilde{X}_4)$, with the transverse torus $\tilde{T}_0$. Notice that $(\tilde{V}_{4m}, \tilde{X}_{4m})$ is a cyclic cover of $(\tilde{V}_4, \tilde{X}_4)$ which respects dynamics. The torus $\tilde{T}_{4m}$ is one of the connected components of the lift of $\tilde{T}_0$ (the projection from $\tilde{T}_{4m}$ to $\tilde{T}_0$ is a diffeomorphism), and the return time of $\tilde{X}_{4m}$ to $\tilde{T}_{4m}$ tends to $+\infty$ as $m \to +\infty$. We can repeat the construction we have performed in Section 3.5: for a sufficiently large $N > 0$ there exist m such that the composition of the time N-map of the flow of $\tilde{X}_{4m}$ and the Dehn twist h_N along $\tilde{T}_{4m}$ (in the direction $[\gamma] \in H_1(\tilde{T}_{4m}, \mathbb{Z})$) is partially hyperbolic and volume preserving. Further, an extra small perturbation yields a robustly transitive stably ergodic diffeomorphisms f.

Now, as a consequence of Lemma 3.12 (arguing exactly as in the proof of Proposition 3.10) we obtain: given any closed curve $C \subset \tilde{V}_{4m}$ whose homological intersection number $i(C)$ with $\tilde{T}_{4m}$ does not vanish, and for any $n \neq 0$ we have

$$[f^n(C)] = [C] + i(C) \cdot [\gamma] \neq [C] \in H_1(\tilde{T}_{4m}, \mathbb{Z}).$$

Hence, if C is a periodic circle for f then its intersection number $i(C)$ vanishes.

Lemma 3.16. *For sufficiently large N and m, the center bundle of f is orientable and transverse to $\tilde{T}_{4m}$.*

Now we finish the proof of Proposition 3.15 assuming the above lemma. Because center bundle is oriented and transverse to $\tilde{T}_{4m}$, any closed center curve passing in a neighborhood of $\tilde{T}_{4m}$ has a non-trivial intersection number with $\tilde{T}_{4m}$, and therefore cannot be periodic. On the other hand, because f is robustly transitive, up to performing an arbitrarily small C^1-perturbation, it admits hyperbolic periodic points in the neighborhood of $\tilde{T}_{4m}$.

In [BDU] the existence of complete invariant center curves through these periodic points is shown (see also [HHU_1]). These curves are periodic and intersect $\tilde{T}_{4m}$, therefore, they cannot be circles.

Finally, the support of the Dehn twist h_N is disjoint from a hyperbolic basic set of the vector field $\tilde{X}_{4m}$ and this basic set contains periodic orbits of $\tilde{X}_{4m}$. The restriction of f to this basic set is a small perturbation of the time-N map of the flow X_{4m}. Hence these invariant circles persists as normally hyperbolic invariant circles. Up to performing an arbitrarily small perturbation, one may assume that each of these circles contains a hyperbolic periodic points. Hence, we have shown that f has both: periodic points on invariant center circles and periodic points not in invariant center circles, concluding the proof.

It remains to establish Lemma 3.16. □

Proof of Lemma 3.16. Let E^{cs}_X E^{cs}_f, E^{cu}_X E^{cu}_f be the center stable and unstable bundles of the vector field $\tilde{X}_{4m}$ and of f, respectively. When m tends to ∞ the time return of $\tilde{X}_{4m}$ to $\tilde{T}_{4m}$ also tends to infinity. As a consequence we obtain that, on the fundamental domain $\mathcal{U}_N$, the bundle E^{cs}_f tends to E^{cs}_X and E^{cu}_f tends to $Dh_N(E^{cu}_X)$. Therefore E^c_f tends to $E^{cs}_X \cap Dh_N(E^{cu}_X)$ which is everywhere transverse to the $\mathbb{T}^2$ fibers in the fundamental domain $\mathcal{U}_N$, concluding. □

We expect that our examples should posses saddle node center leaves, we state this as a question here.

Question 7. *Does f admit a periodic saddle node leaf?*

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