RIGIDITY LECTURE NOTES

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Lecture I

What is rigidity? Weak form of equivalence or coincidence of some simple invariants implies a strong form of equivalence. Today we will discuss several examples of rigidity of rank one, *i.e.,* for maps, diffeomorphisms and flows.[1]

1. **Shub-Sullivan.** *Let $f, g\colon S^1 \to S^1$ be C^r, $r \geq 2$, expanding maps. Assume that there exists a absolutely continuous conjugacy: $h \circ f \circ h^{-1} = g$. Then f and g are C^r conjugate.*

Proof. To make the proof very simple also *assume that h is a homeomorphism* and recall the following.

Theorem (Krzyżewski-Sacksteder)**.** *If $f\colon S^1 \to S^1$ is a C^r expanding map. Then there exists a C^{r-1} smooth f-invariant measure $\mu_f = \rho_f(x)dx$.*

By ergodicity of expanding maps we have $h^*\mu_g = \mu_f$. We can assume that $h(0) = 0$. Then

$$I_f(x) = \int_0^x \rho_f(x)dx = \int_0^{h(x)} \rho_g(x)dx = I_g(h(x))$$

Functions I_f and I_g are C^r. Hence, by the implicit function theorem, h is also C^r. □

2. **Expanding maps rigidity.** *Let $E_d : x \mapsto dx$. Let $f\colon S^1 \to S^1$ be a degree d expanding map, $h \circ f = E_d \circ h$. Let λ_f the be Lyapunov exponent of f with respect to μ_f. Assume $\lambda_f = \log d$ then f is smoothly conjugate to E_d.*

Proof.

$$\lambda_f = h_{\mu_f}(f) = h_{h_*\mu_f}(L)$$

But $h_{Leb}(E_d) = \log d$. Hence by uniqueness of the measure of maximal entropy $h_*\mu_f = Leb$. Hence, by Shub-Sullivan, h is smooth. □

3. **Katok entropy rigidity.** *Let (S, g) be a negatively curved surface and let $X^t\colon T^1S \to T^1S$ be its geodesic flow. Denote by λ the Liouville measure. Then*

$$h_\lambda(X^t) = h_{top}(X^t)$$

if and only of g is a hyperbolic metric (constant curvature).

4. **Avez rigidity.** *If $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ is a C^r, $r \geq 2$, Anosov diffeomorphism which has C^2 stable and unstable foliations then f is C^r conjugate to a linear automorphism.*

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[1]Disclaimer: The rich subject of rigidity of higher rank actions will be ignored.

5. **de la Llave-Marco-Moriyon rigidity.** *Let $f, g\colon \mathbb{T}^2 \to \mathbb{T}^2$ be conjugate C^r-smooth Anosov diffeomorphisms, $h \circ f = g \circ h$. Assume*

$$\forall p = f^k p \quad \exists C: \ Df^k(p) = CDg^k(h(p))C^{-1} \tag{$\star$}$$

Then h is $C^{r-\varepsilon}$ smooth.

Proof.

1. *Equilibrium states.* Given an Anosov diffeo f and a Hölder potential φ there exists a unique f-invariant measure $\mu_{f,\varphi}$, called *equilibrium state* which maximizes metric pressure $h_\mu(f) + \int \varphi d\mu$. Note that by uniqueness if ψ is *cohomologous* to φ, $\varphi = \psi + u \circ f - u$, then $\mu_{f,\varphi} = \mu_{f,\psi}$. The equilibrium state for $-\log Jac^u(f)$ is the SRB measure which has absolutely continuous conditional measure on unstable leaves of f. Denote this equilibrium state by m_f.
2. *Functoriality.* If $f = h^{-1} \circ g \circ h$ then $h^*\mu_{g,\varphi} = \mu_{f,\varphi\circ h}$. Follows directly from functoriality of metric entropy.
3. *Smoothness along foliations.*

   Theorem (Livshits)**.** *If f is a transitive Anosov diffeomorphism and φ and ψ are Hölder potentials such that*

   $$\forall p = f^k p: \qquad \sum_{x \in \mathcal{O}(p)} \varphi(x) = \sum_{x\in\mathcal{O}(p)} \psi(x)$$

   then φ is cohomologous to ψ.

   Then ($\star$) verifies the assumption of Livshits and implies that $\varphi = -\log Jac^u f$ is cohomologous to $\psi \circ h = -\log Jac^u g \circ h$. And hence, by functoriality,

   $$m_f = \mu_{f,\varphi} = \mu_{f,\psi\circ h} = h^*\mu_{g,\psi} = h^* m_g$$
4. *Calculus.* If ξ is a measurable partition subordinate to W^u_f then for m_f almost every x we have that $h|_{\xi(x)}\colon \xi(x) \to h(\xi(x))$ is absolutely continuous and, hence, smooth. It follows that $h \in C^r_u(\mathbb{T}^2)$, that is, h is smooth along unstable leaves. Applying the same argument to f^{-1} and g^{-1} which are conjugate via the same h, $h \circ f^{-1} = g^{-1} \circ h$, we also obtain $h \in C^r_s(\mathbb{T}^2)$. Then it remains to prove that

   $$C^{r-\varepsilon}(\mathbb{T}^2) \subset C^r_s(\mathbb{T}^2) \cap C^r_u(\mathbb{T}^2)$$

   which is an easy exercise when $r = \infty$.

□

6. **Otal-Croke marked length spectrum rigidity.** *Let (S, g_1) and (S, g_2) be negatively curved surfaces. Denote by $[\gamma]$ the free homotopy class loops $S^1 \to S$. For any non-trivial $[\gamma]$ consider the unique geodesic representatives $\gamma_1 \in [\gamma]$ and $\gamma_2 \in [\gamma]$ for g_1 and g_2, correspondingly and assume that*

$$\ell_{g_1}(\gamma_1) = \ell_{g_2}(\gamma_2) \tag{$\blacklozenge$}$$

Then there exists an isometry $\sigma\colon (S, g_1) \to (S, g_2)$.

Recently local marked length spectrum rigidity was established by Guillarmou-Lafeuvre for higher dimensional negatively curved manifolds M. Namely, if g_2 is sufficiently close to g_1 in C^N topology, $N = \frac{3}{2}\dim M + 8$, then ($\blacklozenge$) implies isometry.

To make the proof very simple assume that $g_2 = \rho^2 g_1$ (which is, after moding out by Diff(S), a finite codimension assumption in the case of surfaces). This proof is due to Katok and precedes Otal-Croke.

Proof. Denote by λ_i the Riemannian volume and by A_i the total area of g_i, $i = 1, 2$. Using Birkhoff ergodic theorem and Anosov closing lemma we can approximate λ_1 by a measure supported on a single periodic geodesic γ: $\lambda_1 \approx \frac{A_1}{\ell_{g_1}(\gamma)}\delta_\gamma$, where δ_γ is the uniform measure on γ. Then

$$A_2 = \int_S \rho^2 d\lambda_1 = \frac{\int_S \rho^2 d\lambda_1 \int_S d\lambda_1}{\int_S d\lambda_1} \geq \frac{\left(\int_S \rho d\lambda_1\right)^2}{A_1} \approx \left(\frac{A_1}{\ell_{g_1}(\gamma)}\right)^2 \frac{\left(\int_\gamma \rho d\delta_\gamma\right)^2}{A_1}$$
$$= A_1 \frac{\ell_{g_2}(\gamma)^2}{\ell_{g_1}(\gamma)^2} \geq A_1 \frac{\ell_{g_2}(\bar\gamma)^2}{\ell_{g_1}(\gamma)^2} = A_1,$$

where $\bar\gamma$ is the g_2-geodesic homotopic to γ. Hence we obtain that $A_2 \geq A_1$ and, using the symmetric argument we also have $A_1 \geq A_2$. It follows that the equality must be achieved in the Cauchy-Shwartz inequality above. Hence ρ is constant, and, hence, $\rho = 1$. □

7. **Rigidity of Anosov flows in dimension 3.** Anosov flows $X_1^t, X_2^t \colon M \to M$ are called *conjugate* via $h \colon M \to M$ if

$$\forall t \qquad h \circ X_1^t = X_2^t \circ h$$

If X_1^t and X_2^t are C^1 close transitive Anosov flows then Anosov structural stability yields an *orbit equivalence* $h \colon M \to M$ which send orbits of X_1 to orbits of X_2 preserving the time direction. However, typically a true conjugacy does not exists. Indeed, similarly to (♦), the periods of periodic orbits provide an obstruction to existence of the conjugacy. Applying the flow version of Livshits theorem to $D_{X_1}h - 1$ yields the following characterization.

Let X_1^t and X_2^t be orbit equivalent transitive Anosov flows. Assume that

$$\forall p \in Per(X_1): \quad \mathrm{per}_{X_1}(p) = \mathrm{per}_{X_2}(h(p))$$

Then X_1^t is conjugate to X_2^t. Further, following in the footsteps of the proof for 2-dimensional Anosov diffeos one can obtain the following.

Theorem (de la Llave-Moriyon, Pollicott)**.** *Assume that X_1^t and X_2^t are conjugate transitive Anosov flows. Also assume, analogously to ($\star$), that the differentials of Poincaré return maps for all periodic points are conjugate. Then X_1^t and X_2^t are smoothly conjugate.*

Note that in the setting of geodesic flows any conjugacy is automatically smooth (one has that the conjugacy is volume preserving as an intermediate step in Otal's MLS rigidity proof and, hence, is smooth by following the de la Llave-Moriyon argument). That is, the assumption on differentials of return maps at periodic orbits is redundant. More generally, Feldman-Ornstein showed that the same holds for transitive contact Anosov flows. Hamenstädt generalized this result to higher dimensions assuming additionally C^1 stable and unstable foliations. We offer the following generalization.

Theorem (A.G. – F. Rodriguez Hertz)**.** *Let X_1^t and X_2^t be conjugate 3-dimensional transitive Anosov flows. Then either the conjugacy is smooth or X_1^t is a constant roof suspension of an Anosov diffeomorphism of $\mathbb{T}^2$.*

Lecture II

Today we will discuss rigidity of toral automorphisms similarly to rigidity of expanding maps and Anosov diffeos discussed last time. Any matrix $L \in SL(d, \mathbb{Z})$ induces a torus automorphism $L\colon \mathbb{T}^d \to \mathbb{T}^d$. If L is hyperbolic and $f\colon \mathbb{T}^d \to \mathbb{T}^d$ is an Anosov diffeo which is homotopic to L then, by work of Franks and Manning, f is conjugate to L, $h \circ f = L \circ h$. We are interested in higher regularity of h. Recall that the obstructions are carried by periodic orbits

$$\forall p = f^k p \quad \exists C: \quad Df^k(p) = CL^kC^{-1} \tag{$\star$}$$

Recall that if $d = 2$ then vanishing of obstructions implies that h is as regular as f by work of de la Llave-Marco-Moriyon.

8. Periodic data rigidity in dimension 3.

There are two cases to consider for 3-dimensional automorphisms with 2-dimensional unstable subbundle: the comformal case of a pair complex conjugate eigenvalues and the case when unstable subbundle admits a dominated splitting.

Theorem (Kalinin-Sadovskaya)**.** *Assume that* $L\colon \mathbb{T}^3 \to \mathbb{T}^3$ *has a pair of complex eigenvalue with* $0 < \lambda_1 < 1 < |\lambda_2| = |\lambda_3|$ *and that* C^r *Anosov diffeomorphism* f *is conjugate to* L, $h \circ f = L \circ h$. *If* $(\star)$ *then* h *is* $C^{r-\varepsilon}$.

Theorem (A.G. – Guysinsky)**.** *Assume that* $L\colon \mathbb{T}^3 \to \mathbb{T}^3$ *has real spectrum* $0 < \lambda_1 < 1 < \lambda_2 < \lambda_3$. *Consider* C^r *smooth Anosov diffeomorphism* f, *which is conjugate to* L; $h \circ f = L \circ h$. *If* $(\star)$ *then* h *is* $C^{r-\varepsilon}$.[2]

The bootsrap of regularity of h is carried out in several steps: to Lipschitz, to C^1 and then to $C^{r-\varepsilon}$. In the conformal case Kalinin and Sadovskaya built an invariant conformal structure on the unstable subbundle of f. Then presence of the conformal structure allows to employ apply one-dimensional techniques to the 2-dimensional problem and obtain regularity along the unstable foliation.

9. De la Llave counterexample for dimensions ≥ 4.

In general, rigidity of automorphism in dimensions ≥ 4 is false. The following example is due to de la Llave.

Let A and B be hyperbolic automorphisms of $\mathbb{T}^2$ with $Av = \lambda v$ and $Bu = \mu u$ where $\mu > \lambda > 1$. Consider the product automorphism $L(x, y) = (Ax, By)$ and its perturbation

$$f(x, y) = (Ax + \varphi(y)v, By),$$

where $\varphi(y) = \sin(2\pi y_1)$. Then the conjugacy h has the form

$$h(x, y) = (x + \psi(y)v, y)$$

and can be calculated explicitly in the form of series. Let $r_0 = \log \lambda / \log \mu$. Note that $r_0 < 1$. One can then check that ψ is C^{r_0}, but no more regular.

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[2] To obtain the global version one needs to use, additionally, more recent results of Wang-Sun on approximation of Lyapunov exponents, Velozo on $SL(2, \mathbb{R})$ cocycles and also thesis of Potrie.

10. **Lyapunov spectrum rigidity.** Let $L\colon \mathbb{T}^d \to \mathbb{T}^d$ be an automorphism and let $f\colon \mathbb{T}^d \to \mathbb{T}^d$ be a volume preserving C^1 small perturbation of L, $h \circ f = L \circ h$. Instead of assuming that the Lyapunov exponents at periodic match we can make the same assumption for volume Lyapunov exponents, that is,

$$\chi_i^f = \chi_i^L,\ i = 1, \dots d \qquad (\star)$$

If f is not ergodic we can take average Lyapunov exponent.

10.1. *Dimension 2.* If $d = 2$ then

$$h_{vol}(f) = \chi_f = \chi_L = h_{vol}(L) = h_{h^*vol}(f)$$

Hence, by uniqueness of the measure of maximal entropy we have $h^*vol = vol$. Then one concludes that h is smooth following the proof in Lecture 1.

10.2. *Dimension 3.*

Theorem (Saghin-Yang)**.** *Assume that $L\colon \mathbb{T}^3 \to \mathbb{T}^3$ has real spectrum $0 < \lambda_1 < 1 < \lambda_2 < \lambda_3$. Assume that f is a C^1 small perturbation of L which satisfies $(\star)$. Then f is smoothly conjugate to L.*

Proof. Similarly to the 2-dimensional case one can use Pesin's formula to obtain $h^*vol = vol$. Then, just as in the 2-dimensional case, $h \in C_s^\infty(\mathbb{T}^3)$, that is, h is smooth along the stable foliation. By, comparing the rates at which points diverge in the universal cover, one has $h(W_f^{wu}) = W_L^{wu}$. However, one cannot conclude smoothness along W_f^{wu} because the conditional measures of volume typically fail to be absolutely continuous. This is the main issue and we need the following lemma to overcome it.

Lemma 0.1 (Ledrappier)**.** *Let W be a uniformly expanding foliation for a preserving diffeomorphism $f\colon M \to M$. Let m be an ergodic invariant measure. Let ξ be a measurable, Markov partition subordinate to W. Denote by $m_{\xi(x)}$ the conditional measures on $\xi(x)$. Conditional entropy $H(f^{-1}\xi|\xi)$ is defined by*

$$H(f^{-1}\xi|\xi) = \int_M -\log m_{\xi(x)}(f^{-1}(\xi(fx))dm$$

Then the conditional measures m_ξ are absolutely continuous if and only if

$$H(f^{-1}\xi|\xi) = \int_M \log Jac(f|_W)dm$$

We can apply the lemma to W_f^{wu} and vol. Indeed,

$$H(f^{-1}\xi|\xi) = H(L^{-1}(h(\xi))|h(\xi)) = \log \lambda_2$$

and

$$\int_{\mathbb{T}^3} \log Jac^{wu}(f)dvol = \chi_f^{wu} = \log \lambda_2$$

hence W_f^{wu} is absolutely continuous and the rest of the proof proceeds in the same way as for periodic data rigidity. □

Theorem (AG-Kalinin-Sadovskaya)**.** *Assume that $L\colon \mathbb{T}^3 \to \mathbb{T}^3$ has a pair of complex eigenvalues $0 < \lambda_1 < 1 < |\lambda_2| = |\lambda_3|$. Assume that f is a C^1 small perturbation of L which satisfies $(\star)$. Then f is smoothly conjugate to L.*

Proof. The challenge here is to establish regularity of h along the two dimensional unstable foliation. To do that we need to control two pieces of data for f: jacobian and quasi-conformal distortion.

- *The unstable jacobian $Jac^u(f)$ is continuously cohomologous to constant*

$$Jac^u(f) = \frac{\rho(fx)}{\rho(x)}|\lambda_2|^2$$

The proof uses absolute continuity of W^u and then the measurable Livshits theorem for scalar cocycles.

- *There exists a Hölder continuous Riemannian metric on E^u such that $Df|_{E^u}$ is conformal*

$$\|Dfv_x\| = a(x)\|v_x\|, \quad \forall v_x \in E^u$$

The tool for proving this is the *trichotomy of Kalinin-Sadovskaya.* Namely, if 2×2 matrix cocycle over f has only one volume exponent then:

1. the cocycle is conformal;
2. the cocycle admits a continuous invariant line bundle;
3. the cocycle admits an invariant pair of transverse line bundles;

Note that in our setting we have that at the fixed point p, $Df|_{E^u}(p)$ is close to an irrational rotation which eliminates possibilities 2 and 3.

- *The conjugacy h is Lipschitz along W^u.*

We approximate h by an h_0 which is C^1 along W^u. Then define

$$h_n = L^n \circ h \circ f^{-n}$$

By using conformality and the fact that $a(x)$ is cohomologous to a constant it is easy to check that h_n are uniformly bounded in C^1 topology and, in fact converge to h.

- It follows from the Rademacher theorem that h is differentiable almost everywhere along W^u, that is,

$$D^u h \circ D^u f = D^u L \circ D^u h$$

has a measurable solution $D^u h$. Then by a result of Sadovskaya it must have a continuous version and hence, $h \in C^{1+}(W^u)$. After that the classical de la Llave bootsrap argument kicks in.

□

Combining the above techniques yields a higher dimensional result.

Theorem (Saghin-Yang/ AG-Kalinin-Sadovskaya)**.** *Let f be a C^1 small perturbation of $L\colon \mathbb{T}^d \to \mathbb{T}^d$, where L^4 is a hyperbolic irreducible automorphism such that no three eigenvalues have the same absolute value. Then the conjugacy $h \in C^{1+Holder}$.*

Remark 0.2. In contrast to periodic data rigidity, the Lyapunov spectrum rigidity is an "extremal" property of the automorphism. It is easy to produce non-linear Anosov diffeomorphisms with the same Lyapunov spectrum which are not C^1 conjugate.

11. **Rigidity of partially hyperbolic automorphisms.**

11.1. *Dimension 3.*

Theorem (Saghin-Yang). *Let $L\colon \mathbb{T}^3 \to \mathbb{T}^3$ be the product partially hyperbolic automorphism*

$$L(x, y, z) = (A(x, y), z)$$

and let f be a volume preserving perturbation with the same (average) Lyapunov exponents ($\star$). Then f is smoothly conjugate to

$$L'(x, y, z) = (A(x, y), z + \alpha(x, y))$$

Proof. Consider the semi-conjugacy $h\colon \mathbb{T}^3 \to \mathbb{T}^2$, $h \circ f = A \circ h$. Let ξ be a measurable, Markov partition partition subordinate to W^u_f (by pulling-back the Markov partition $h(\xi)$ for A).
Invariance Principle. (Avila-Viana, Tahzibi-Yang)

$$H_{vol}(f^{-1}(\xi)|\xi) \leq H_{h_* vol}(A^{-1}(h(\xi))|h(\xi))$$

and the equality holds if and only if the conditional measures are invariant under the center holonomy. We have

$$\chi^{uu}_f = H_{vol}(f^{-1}\xi|\xi) \leq H_{h_* vol}(A^{-1}(h(\xi))|h(\xi)) \leq \chi^{uu}_L = \chi^{uu}_A$$

By the assumption on the Lyapunov spectrum ($\star$) we have that both inequalities above are, in fact, equalities. Hence, the center holonomy is absolutely continuous, hence, smooth, both within W^{cu} and W^{cs}. Therefore W^c is a smooth circle fibration. Straightening this fibration we can conjugate f to a diffeomorphism of the form

$$(x, y, z) \mapsto (g(x, y), \alpha_{(x,y)}(z))$$

Then applying the earlier 2-dimensional argument to A and g we obtain that g is smoothly conjugate to A. □

11.2. *Dimension 4.*

Theorem. *Let $L\colon \mathbb{T}^4 \to \mathbb{T}^4$ be an irreducible partially hyperbolic diffeomorphism, $\lambda_1 < |\lambda_2| = |\lambda_3| = 1 < \lambda_4$. And let f be a volume preserving C^{infty} small perturbation with the same Lyapunov exponents ($\star$). Then f is smoothly conjugate to L.*

This relies on two big reults:

F. Rodriguez Hertz dichotomy: Either

1. f is conjugate to L (and conjugacy is smooth along W^c via a KAM argument); or
2. f is accessible.

Avila-Viana: If f is accessible then f has at least one non-zero center exponent.

Combining this with ($\star$) we have that f is conjugate to L. We only need to check smoothness along stable and unstable foliations. Berg proved that volume is the unique measure of maximal entropy for L. As before we have:

$$h_{h_* vol}(L) = h_{vol}(f) = \chi^{uu}_f = \chi^{uu}_L = h_{vol}(L)$$

Hence, the same argument as in dimension 2, we obtain smoothness along W^{uu}, and, similarly, along W^{ss}.

1. Lecture III

This lecture is entirely based on a joint work with F. Rodriguez Hertz, under preparation.

1. **Otal-Croke marked length spectrum rigidity.** *Let* (S, g_1) *and* (S, g_2) *be negatively curved surfaces. Denote by* $[\gamma]$ *the free homotopy class loops* $S^1 \to S$. *For any non-trivial* $[\gamma]$ *consider the unique geodesic representatives* $\gamma_1 \in [\gamma]$ *and* $\gamma_2 \in [\gamma]$ *for* g_1 *and* g_2*, correspondingly and assume that*

$$\ell_{g_1}(\gamma_1) = \ell_{g_2}(\gamma_2) \qquad (\blacklozenge)$$

Then there exists an isometry $\sigma\colon (S, g_1) \to (S, g_2)$.

2. **Khalil-Lafont question.** *Consider additional data: two positive smooth functions* $\varphi_1, \varphi_2\colon S \to \mathbb{R}$. *Instead of* $(\blacklozenge)$ *assume that*

$$\forall[\gamma] \quad \int_{\gamma_1} \varphi_1(\gamma(t))dt = \int_{\gamma_2} \varphi_2(\gamma(t))dt \qquad (\heartsuit)$$

Does it follow that (S, g_1) *and* (S, g_2) *are homothetic? That is, does there exists a constant* $c > 0$ *and an isometry* $\sigma\colon (S, g_1) \to (S, c^2 g_2)$.
Note that if $\varphi_1 = \varphi_2 = 1$ then this is precisely MLS rigidity. Also note that if $g_1 = c^2 g_2$ then $(\varphi_1, \varphi_2) = (\varphi, c\varphi)$ verify $(\heartsuit)$.

3. **Sharpened MLS rigidity.** *Let* (S, g_1) *and* (S, g_2) *be negatively curved surfaces. Let* $\varphi_i : T^1S \to \mathbb{R}$ *be smooth functions such that* φ_1 *is not an abelian coboundary. Assume that for every homologically trivial homotopy class of loops* $[\gamma] \in \pi_1(S)$, $[\gamma] \neq 0$*, we have*

$$\int_{\gamma_1} \varphi_1 = \int_{\gamma_2} \varphi_2$$

Then there exists a constant $c > 0$ *and an isometry* $\sigma\colon (S, g_1) \to (S, c^2 g_2)$.

4. **Examples of non abelian coboundaries.** Let X be an Anosov vector field on a closed manifold M. A Hölder continuous function $\varphi\colon M \to \mathbb{R}$ is called an *abelian coboundary* if there exists a closed 1-form ω and a Hölder function (differentiable along X) such that

$$\varphi = Xu + \omega(X)$$

Notice that the decomposition $\varphi = Xu+\omega(X)$ is highly non-unique because we can change ω by any an exact 1-form. Indeed given any smooth function $v\colon M \to \mathbb{R}$ we can write a different decomposition

$$\varphi = (\omega + dv)(X) + X(u - v)$$

If X is the geodesic flow on (S, g_1) then there two classes of non abelian coboundaries (to which Sharpened MLS rigidity would apply).

1. Function $\varphi\colon T^1S \to \mathbb{R}$ is not an abelian coboundary if it is non-negative and takes at least one positive value.
2. Function $\varphi\colon T^1S \to \mathbb{R}$ is not an abelian coboundary if it is a pullback of a non-zero function on the surface, in other words, $\varphi(v, x) = \varphi(x)$.

5. **An example without rigidity.** Let S be a surface equipped with a cohomologically non-trivial closed $1-$form $\omega\colon TS \to \mathbb{R}$. Consider two non-isometric Riemannian metrics g_1 and g_2 on S. Then the corresponding unit tangent bundles are naturally embedded in the full tangent bundle $T^1_{g_i}S \subset TS$ and we can define $\varphi_i : T^1_{g_i}S \to \mathbb{R}$ by $\varphi_i(x,v) = \omega_x(v)$. Let $\gamma_i \subset T^1_{g_i}S$ be homotopic unit-speed closed g_i-geodesics. Then

$$\int_{\gamma_1} \varphi_1(\gamma_1(s))ds = \int_{\gamma_1} \omega(\gamma_1(s))ds = \langle [\omega], [\gamma] \rangle = \int_{\gamma_2} \omega(\gamma_2(s))ds = \int_{\gamma_2} \varphi_2(\gamma_2(s))ds$$

where $[\omega]$ is the cohomology class of ω and $[\gamma]$ is the homology class of γ_i, $i = 1, 2$.

6. **Abelian Livshits theorem.** We follow Sharp and say that a transitive Anosov flow $X^t\colon M \to M$ is *homologically full* if every integral homology class contains a closed orbit of X^t.

Theorem. *Assume that $X^t\colon M \to M$ is a homologically full transitive Anosov flow and let $\varphi \in C^r(M)$, $r > 0$, $\varphi : M \to \mathbb{R}$ such that*

$$\int_\gamma \varphi = 0$$

for all homologically trivial closed orbits γ. Then there is a C^∞ smooth closed 1-form ω on M and a function $u \in C^{r-\epsilon}(M)$, where $\epsilon > 0$ is arbitrarily small, such that

$$\varphi = Xu + \omega(X)$$

Proof. By work of Sharp homologically trivial orbits equidistribute according to a certain equilibrium state. In particular, it follows that the homologically trivial orbits are dense. However, one can avoid using Sharp's machinery and give a simpler proof by using shadowing.

Let $\hat{M}$ be *the universal abelian cover* of M, that is, the cover which corresponds to the commutator subgroup $[\pi_1 M, \pi_1 M]$. Note that homologically trivial periodic orbits in M lift to periodic orbits in $\hat{M}$. Hence periodic orbits of the lifted flow are dense in $\hat{M}$ and, by applying the standard Smale argument we conclude that $X^t\colon \hat{M} \to \hat{M}$ is a transitive flow. Hence we can carry out the standard proof of Livshits theorem on $\hat{M}$. The conclusion is that the lift $\hat{\varphi}\colon \hat{M} \to \mathbb{R}$ is a coboundary (in the usual sense), which translates into φ being an abelian coboundary. □

7. **Matching rigidity for Anosov flows.**

Theorem. *Let $X_i^t\colon M \to M$, $i = 1, 2$ be $C^{1+\alpha}$ 3-dimensional transitive Anosov flows. Assume they are orbit equivalent via $H\colon M \to M$. Let $\varphi_i : M \to \mathbb{R}$ be C^1 functions. If $\int_{\gamma_1} \varphi_1 = \int_{H_*\gamma_1} \varphi_2$ for every X_1-closed orbit γ_1, then one of the following holds:*

1. *φ_i are X_i abelian coboundary;*
2. *H is C^1 after adjusting it through a time change.*

8. **Another application to 3-dimensional Anosov flows.** Recall from the first lecture.

Theorem (de la Llave-Moriyon, Pollicott)**.** *Assume that X_1^t and X_2^t are orbit equivalent 3-dimensional transitive Anosov flows. Assume*

$$\forall p \in Per(X_1) : \quad T_p = \mathrm{per}_{X_1}(p) = \mathrm{per}_{X_2}(h(p)) = T_{h(p)} \qquad \text{(A1)}$$

Also assume that the differentials of Poincaré return maps for all periodic points are conjugate:

$$\forall p \in Per(X_1)\ \exists C:\ \ DX_1^{T_p}(p) = C \circ DX_2^{T_{h(p)}}(h(p)) \circ C^{-1} \tag{A2}$$

Then X_1^t and X_2^t are smoothly conjugate.

In the first lecture we discussed what happens if (A2) is dropped.

If one drops the assumption (A1) and, keeps (A2) instead, then the matching theorem could be applied to infinitesimal stable and unstable jacobians yields the following.

Theorem. *Assume that X_1^t and X_2^t are orbit equivalent 3-dimensional transitive Anosov flows. Assume that the differentials of Poincaré return maps for all periodic points are conjugate (A2). Then X_1^t and X_2^t are smoothly orbit equivalent.*

9. **Outline of the proof of Sharpened MLS rigidity.**

9.1. *Reduction to a reparametrization.* Applying the Matching Theorem to the geodesic flows X_1 and X_2 we obtain a C^{1+} orbit equivalence H. Let

$$\tilde{X}_1 = DH(X_1)$$

Then obviously,

$$\tilde{X}_1 = \rho X_2, \rho > 0$$

that is $\tilde{X}_1$ is a C^1 flow which is reparametrization of X_2. Our goal is to prove that $\tilde{X}_1^t$ and X_2/c are conjugate.

9.2. *Matching of homologically trivial spectra.* Both flows $\tilde{X}_1$ and X_2 are contact. Denote by α and β the contact $1-$forms for X_2 and $\tilde{X}_1$, respectively. Then we have that $d\alpha$ is an exact X_2-invariant $2-$form. On the other hand, by using Cartan's formula, $d\beta$ is also X_2-invariant. Hamenstädt proved that such form is unique, that is $d\beta = cd\alpha$, $c > 0$. Hence the $1-$form

$$\mu = \beta - c\alpha$$

is closed. Plugging X_2 yields a formula for ρ in terms of μ

$$\rho = \frac{1}{c + \mu(X_2)}$$

9.3. *Sharpening the reparametrization.* Our goal now is to show that μ is exact. Then the periods of $Y_1 = c\tilde{X}_1$ and $Y_2 = X_2$ match and we would conclude that Y_1 and Y_2 are conjugate. To do this we rely on work of R. Sharp, which is based on earlier work of Katsuda-Sunada.

If $Y^t\colon M \to M$ is a homologically full Anosov flow then the functional $\beta\colon H^1(M, \mathbb{R}) \to \mathbb{R}$ given by

$$\beta([\theta]) = P_Y(\theta(Y)) = \sup_{\mu}\{h_\mu + \int_M \theta(Y)d\mu\}$$

attains a unique minimum at $\xi_Y \in H^1(M, \mathbb{R})$. Geodesic flows are special among homologically full flows:

Fact 1. If Y is a geodesic flow then $\xi_Y = 0$;

Using the minimizer property it is not hard to study the behavior of the minimizer under the reparametrizations.

Fact 2. If $Y_1 = Y_2/(1 + \omega(Y_2))$ then

$$\xi_1 = \xi_2 + \beta(\xi_2)[\omega]$$

Since both Y_1 and Y_2 come from geodesic flows this boils down to $\beta(\xi_2)[\omega] = 0$. But one can also check that $\beta(\xi_2) = h_{\mu_{\xi_2}}(Y_2) > 0$. Hence $\omega = dv$.

We conclude that Y_1 and Y_2 are conjugate and, hence, we can apply Otal-Croke theorem to obtain the posited isometry $\sigma\colon (S, g_1) \to (S, c^2 g_2)$.