Surgery for partially hyperbolic dynamical systems II. Blow-up of a complex curve.

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Abstract. In this paper we use the blow-up surgery introduced in [**Go16**] to produce new higher dimensional partially hyperbolic flows. The main contribution of the paper is the *slow-down construction* which accompanies the *blow-up construction*. This new ingredient allows to dispose of a rather strong *domination assumption* which was crucial for results in [**Go16**]. Consequently we gain more flexibility which allows to construct new *volume-preserving* partially hyperbolic flows as well as new examples which are not fiberwise Anosov. The latter are produced by starting with the geodesic flow on complex hyperbolic manifold which admits a totally geodesic complex curve. Then by performing the slow-down first and the blow-up second we obtain a new (volume-preserving) partially hyperbolic flows.

1. Introduction

This paper is a sequel to [**Go16**] and familiarity with [**Go16**] would help the reader. We keep the introduction brief, still we will recall all definitions which are needed.

A flow $\varphi^t\colon M \to M$ is *partially hyperbolic* if the tangent bundle TM splits into Df-invariant continuous subbundles $TM = E^s \oplus E^c \oplus E^u$ such that

$$\|D\varphi^t(v^s)\| < \lambda^t < \|D\varphi^t(v^c)\| < \mu^t < \|D\varphi^t(v^u)\|,\ t \geq 1, \tag{1.1}$$

for some Riemannian metric $\|\cdot\|$, some $\lambda < 1 < \mu$ and all unit vectors $v^s \in E^s$, $v^c \in E^c$ and $v^u \in E^u$. Then it is clear that the generating vector field $\dot\varphi$ lies in the center subbundle E^c.

An invariant submanifold $N \subset M$ is called an *Anosov submanifold* for φ^t if $TN = E^s \oplus \dot\varphi \oplus E^u$. Note that then the flow φ^t_N is given by the restriction $\varphi^t|_N$ is an Anosov flow. Further, the flow $\varphi^t\colon M \to M$ is called *locally fiberwise* at N if a neighborhood of N can be smoothly identified with $\mathbb{D}^k \times N$, where $\mathbb{D}^k = \{x \in \mathbb{R}^k : \|x\| < 1\}$, in such a way that the restriction $\varphi^t|_{\mathbb{D}^k \times N}$ has the product form

$$\varphi^t(x, y) = (a^t(x), \varphi^t_N(y)), \tag{1.2}$$

where a^t is a linear hyperbolic saddle flow.

Remark 1.1. Note that locally fiberwise assumption in this paper is weaker than the one in [**Go16**] as we no longer require $E^s \oplus E^u$ to be tangent to the N-fibers in the neighborhood $\mathbb{D}^k \times N$. Such weakening is crucial for the examples which we consider here. Also note that the locally fiberwise assumption implies that the normal bundle to N is trivial. This assumption is not crucial for our argument, but it does simplify notation and calculations a lot.

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Now we can blow-up M along $\{0\} \times N$ by replacing each point in $\{0\} \times N$ with the projective space of lines which pass through this point perpendicularly to N. The blown-up manifold $\hat{M}$ comes with a canonical *blow-down map* $\pi\colon \hat{M} \to M$ which collapses each projective space to its base point. The preimage $\pi^{-1}(\{0\} \times N) \simeq \mathbb{R}P^{k-1} \times N$ is called *the exceptional set.* In smooth category, $\hat{M}$ is the result of replacing $\mathbb{D}^k \times N$ with $(\mathbb{D}^k \# \mathbb{R}P^k) \times N$. We will write $\tilde{\mathbb{D}}^k$ for $(\mathbb{D}^k \# \mathbb{R}P^k)$. If the flow $\varphi^t\colon M \to M$ is locally fiberwise at N then it induces a flow $\hat{\varphi}^t\colon \hat{M} \to \hat{M}$ such that the diagram

$$\begin{array}{ccc} \hat{M} & \xrightarrow{\hat{\varphi}^t} & \hat{M} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow{\varphi^t} & M \end{array} \tag{1.3}$$

commutes. The induced flow $\hat{\varphi}^t\colon \hat{M} \to \hat{M}$ may or may not be partially hyperbolic.

Main Theorem. *Let $\varphi^t\colon M \to M$ be a partially hyperbolic flow with C^1 invariant splitting $E^s \oplus E^c \oplus E^u$ and let $N \subset M$ be an invariant Anosov submanifold of M. Assume that the dynamics is locally fiberwise in a neighborhood of N. Let $\hat{\varphi}^t\colon \hat{M} \to \hat{M}$ the induced flow on $\hat{M}$. Then there exists a partially hyperbolic flow $\tilde{\varphi}^t\colon \hat{M} \to \hat{M}$ which coincides with $\hat{\varphi}^t$ outside of a neighborhood of the exceptional set.*

The Main Theorem builds up on the earlier work [**Go16**]. However, strictly speaking, it is not a generalization of the results in [**Go16**]. Indeed in [**Go16**] the author showed that the blown-up flow $\hat{\varphi}^t$ is itself partially hyperbolic under more restrictive assumptions, most importantly the *domination assumption*, which assures that the Anosov submanifold is sufficiently fast compared to the center. In this paper we have fully disposed of the domination assumption and, most interestingly, the Main Theorem applies to examples when $\hat{\varphi}^t$ is not partially hyperbolic. The proof of the Main Theorem relies on some tools developed in [**Go16**] but also develops different technology for controlling the returns. The key basic ingredient of the proof is the *slow-down* construction in the neighborhood of the Anosov submanifold which provides a remedy for absence of domination. Consequently, unlike results of [**Go16**], the construction here can only be used for flows and not for diffeomorphisms. The benefit of the slow-down construction is that we can also produce volume preserving examples which was impossible with techniques of [**Go16**].

We proceed to describe an application of our theorem in the setting of geodesic flows on compact complex hyperbolic manifolds. Let M be a compact complex hyperbolic manifold of dimension n (real dimension $2n$). One can realize M as a quotient space of the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ by an action of a cocompact lattice in the group of biholomorphic isometries, $\Gamma \subset SU(n,1)$. Assume that there exists a compact totally geodesic complex curve $N \subset M$. Then, up to conjugating lattice Γ, the embedding $N \subset M$ is induced by the first coordinate embedding $\mathbb{H}^1_{\mathbb{C}} \subset \mathbb{H}^n_{\mathbb{C}}$. Now consider the geodesic flow on the unit tangent bundle $\varphi^t\colon T^1M \to T^1M$. We view φ^t as a partially hyperbolic flow with $\dim E^s = \dim E^u = 1$. Because N is totally geodesic, φ^t restricts to T^1N. We blow-up $T^1N \subset T^1M$. It is easy to see that the induced flow $\hat{\varphi}^t\colon \widehat{T^1M} \to \widehat{T^1M}$ is not partially hyperbolic because it has periodic orbits with dominated splittings of different dimension signatures. Further, we can check (see Section 4) that all other assumptions of Main Theorem are satisfied as well. Hence we obtain the following corollary.

Corollary 1.2. *Let M be a compact complex hyperbolic manifold and let $N \subset M$ be a totally geodesic complex curve. Then the blow up $\widehat{T^1M}$ of T^1M along T^1N supports a partially hyperbolic flow $\tilde{\varphi}^t\colon \hat{M} \to \hat{M}$. Moreover, the flow $\tilde{\varphi}^t\colon \hat{M} \to \hat{M}$ can be chosen to be an arbitrarily C^∞ small perturbation of $\hat{\varphi}^t$.*

Note that the if φ^t preserves a smooth volume m then $\hat{\varphi}^t$ preserves a smooth measure $\pi^*(m)$. However the density of $\pi^*(m)$ vanishes on the exceptional set. Nevertheless, following the idea of Katok and Lewis [**KL96**], we adapt our Main Theorem to the conservative setting.

ADDENDUM 1.3. *Let $N \subset M$ and $\varphi^t \colon M \to M$ be as in the Main Theorem. Assume that φ^t preserves a smooth volume m which has product form in the neighborhood $\mathbb{D}^k \times N$; that is, $m|_{\mathbb{D}^k \times N} = vol \otimes vol_N$, where vol is the standard Euclidean volume on $\mathbb{D}^k$ and vol_N is a smooth $\varphi^t|_N$-invariant volume on N. Then there exists a partially hyperbolic flow on $\hat{M}$ which preserves a smooth non-degenerate volume.*

The following is a non-trivial corollary.

COROLLARY 1.4. *Let M be a compact complex hyperbolic manifold and let $N \subset M$ be a totally geodesic complex curve. Then the blow up $\widehat{T^1M}$ of T^1M along T^1N supports a volume preserving partially hyperbolic flow $\tilde{\varphi}^t \colon \widehat{T^1M} \to \widehat{T^1M}$.*

Finally we remark that similarly to [**Go16**, Section 3] one can take multiple blow-ups as well as connected sums along Anosov submanifolds and produce partially hyperbolic diffeomorphisms on manifolds with even more complicated topology. Prior to proving the result we include a short section presenting a concrete evidence (higher homotopy) of altered topology of the manifold after the blow-up surgery.

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Higher homotopy of $\widehat{T^1M}$. To see that the examples given by Corollaries 1.2 and 1.4 live on new manifolds one can look at higher homotopy groups. Namely, we will prove that new examples are not homotopic to time-1 maps of geodesic flows on negatively curved manifolds. To do that we will additionally assume that complex dimension of M is $n \geq 3$. Then the real dimension of M is $2n$ and the dimension of T^1M is $4n-1$.

We begin by observing that since M is aspherical the long exact sequence in higher homotopy groups for the sphere bundle T^1M yields the isomorphism $\pi_{2n-1}(T^1M) \simeq \pi_{2n-1}(\mathbb{S}^{2n-1}) \simeq \mathbb{Z}$. Hence, to see that the new flow $\tilde{\varphi}^t$ is not derived from a geodesic flow (possibly on a different manifold), it is sufficient to verify that $\pi_{2n-1}(\widehat{T^1M})$ is not isomorphic to $\mathbb{Z}$.

To do this calculation we will first construct a special 2-fold cover $\hat{Q} \to \widehat{T^1M}$. Recall that $\widehat{T^1M}$ is obtained by blowing-up $T^1N \subset T^1M$ in the normal direction. Let's consider the same construction but by using the "spherical blow-up" instead of the projective one. Namely, let's replace each point in T^1N with the sphere of rays which are based at this point and are perpendicular to N. In this way we obtain a manifold Q with boundary $T^1N \times \mathbb{S}^{4n-5}$. Let $a \colon T^1N \times \mathbb{S}^{4n-5} \to T^1N \times \mathbb{S}^{4n-5}$ be an involution given by $a(x,y) = (x,-y)$. Then, clearly,

$$\widehat{T^1M} = Q/a$$

Now consider another copy of Q which we denote by Q' and form the double $\hat{Q} = Q \sqcup Q'$ by identifying the boundaries via the identity map. The double $\hat{Q}$ is a closed manifold and $T^1N \times \mathbb{S}^{4n-5} \subset \hat{Q}$ is a separating codimension 1 submanifold. Consider its two-sided thickening

$$\{t : -1/2 \leq t \leq 1/2\} \times T^1N \times \mathbb{S}^{4n-5}$$

where $t = 0$ corresponds to $T^1N \times \mathbb{S}^{4n-5}$. Recall that the odd dimensional sphere $\mathbb{S}^{4n-5} \subset \mathbb{C}^{n-2}$ admits the S^1-action given by complex multiplication, which we denote by ξ^t, $\xi^1 = id_{\mathbb{S}^{4n-5}}$.

Notice that $\xi^{1/2}$ is precisely the antipodal map $y \mapsto -y$. Hence we can extend the involution a to the thickening as follows

$$a(t, x, y) = (-t, x, \xi^{t+1/2}(y))$$

It is straightforward to verify that a is still an involution and that $a(-1/2, x, y) = (1/2, x, y)$. Hence a can be extended to the rest of $\hat{Q}$ outside the thickening by using the identity map $Q \to Q'$. Hence we obtain a fixed point free involution $a\colon \hat{Q} \to \hat{Q}$ which gives a 2-folded covering map

$$\hat{Q} \to \hat{Q}/a = Q/a = \widehat{T^1M}$$

Hence we have $\pi_{2n-1}(\widehat{T^1M}) = \pi_{2n-1}(\hat{Q})$ which now can be calculated. Namely, consider $\hat{Q}$ as the union of two open set U and V (corresponding to Q and Q') so that $U \cap V = (-1/2, 1/2) \times T^1N \times \mathbb{S}^{4n-5}$. Now we would like to apply the Seifert- van Kampen argument to U and V. First notice that U is homotopy equivalent to Q which, in turn, is homotopy equivalent to $T^1M \backslash T^1N$. Because T^1N has codimension $4n-4$ it does not affect $(2n-1)$-th homotopy group and we have $\pi_{2n-1}(U) = \pi_{2n-1}(V) = \mathbb{Z}$. Now it remains to notice that $T^1N \times \mathbb{S}^{4n-5}$ have vanishing homotopy in the range from 2 to $4n-6$. Because $4n - 6 \geq 2n$ the Seifert-van Kampen argument works easily to produces the isomorphism $\pi_{2n-1}(\hat{Q}) \simeq \pi_{2n-1}(U) \oplus \pi_{2n-1}(V) = \mathbb{Z}^2$, as desired.

2. The proof of the Main Theorem

2.1. Outline of the proof. The partially hyperbolic splitting $TM = E^s \oplus E^c \oplus E^u$ for $\varphi^t\colon M \to M$ induces a splitting $T\hat{M} = \hat{E}^s \oplus \hat{E}^c \oplus \hat{E}^u$ which is invariant under $D\hat{\varphi}^t\colon T\hat{M} \to T\hat{M}$. It can be checked in local coordinates that, because the partially hyperbolic splitting is C^1, the induced splitting $\hat{E}^s \oplus \hat{E}^c \oplus \hat{E}^u$ is continuous. Under and additional domination assumption on φ^t at N (and also a stronger locally fiberwise assumption) the latter splitting is partially hyperbolic and this situation was examined in [**Go16**]. However, in general, this splitting is not partially hyperbolic. To recover partial hyperbolicity we modify $\hat{\varphi}^t$ in the neighborhood of the exceptional set. Recall that by the locally fiberwise assumption, in the neighborhood of N, the generator of the flow is given by

$$\frac{\partial \varphi^t}{\partial t}(x, y) = X(x) + Y(y),$$

where X is the vector field on $\mathbb{D}^k$ which generates the hyperbolic saddle a^t and Y is the generator of φ^t_N. We consider a smooth bump function $\rho\colon \mathbb{D}^k \to \mathbb{R}$ which is radially symmetric, that is, $\rho(x) = \bar{\rho}(\|x\|)$ where smooth function $\bar{\rho}$ verifies

1. $\bar{\rho}(s) = \rho_0 < 1$, for $s \leq \delta$;
2. $\bar{\rho}$ is strictly increasing on $(\delta, 2\delta)$ and $|\bar{\rho}'(s)| < 1/\delta$ for $s \in (\delta, 2\delta)$;
3. $\bar{\rho}(s) = 1$ for $s \geq 2\delta$

Here the constant ρ_0 only depends on the contraction and expansion rates of $D\varphi^t$ along invariant subbundles. Constant δ will need to be chosen sufficiently small.

Given such a bump function ρ we replace the flow $\varphi^t|_{\mathbb{D}^k \times N}$ with a new flow φ^t_ρ whose generator is given by a *slow-down of the saddle* X

$$\frac{\partial \varphi^t_\rho}{\partial t}(x, y) = \rho X(x) + Y(y) \tag{2.4}$$

Because $\rho = 1$ on the boundary of $\mathbb{D}^k$ the flow φ^t_ρ extends to the rest of M as φ^t and then the blown-up flow $\hat{\varphi}^t_\rho$ is the posited partially hyperbolic flow.

Now we briefly outline the proof of partial hyperbolicity before proceeding with a more detailed argument. First note that on the δ-neighborhood of N the flow φ^t_ρ is a direct product of

the slow saddle $a^{\rho_0 t}$ and φ_N^t. Therefore, by choosing ρ_0 small enough, the domination condition of [**Go16**] holds on the δ-neighborhood and the estimates provided in [**Go16**] yield partial hyperbolicity of $\hat{\varphi}_\rho^t$ with respect to the splitting $T\hat{M} = \hat{E}^s \oplus \hat{E}^c \oplus \hat{E}^u$ on the δ-neighborhood of the exceptional set. Also, by construction, $\hat{\varphi}_\rho^t$ coincides with $\hat{\varphi}^t$ outside the 2δ-neighborhood of the exceptional set. The main technical difficulty is that the splitting $\hat{E}^s \oplus \hat{E}^c \oplus \hat{E}^u$ does not remain invariant as orbits cross the transition region ($\delta \le s \le 2\delta$). However, one can still consider cones centered at these non-invariant distributions and verify the Cone Criterion for partial hyperbolicity.

In what follows we will only establish *the splitting into unstable and center-stable subbundles.* Roughly speaking, this follows from the fact that the damage done to the cones in the transition region ($\delta \le s \le 2\delta$) is controlled uniformly (in δ) thanks to the second property of $\bar{\rho}$ and the fact that orbits spend a bounded time of order δ in the transition region. Because all our constructions are time-symmetric, repetitions of the arguments also yields a splitting into center-unstable and stable subbundles and hence full partial hyperbolicity.

2.2. Cones near the exceptional set. We will need to introduce more notation in order to proceed with the precise description of the cones and the estimates. Denote by $\tilde{\mathbb{D}}^k_{<\delta} \times N$ the δ-neighborhood of the exceptional set, that is, the preimage

$$\pi^{-1}(\{x \in \mathbb{D}^k : \|x\| < \delta\} \times N)$$

Denote by $TN = E^s_N \oplus E^c_N \oplus E^u_N$ the Anosov splitting of the restriction φ_N^t (*i.e.,* $E^c_N = \dot{\varphi}^t_N$) and by $(E^s_N \oplus E^c_N \oplus E^u_N) \oplus H$ the product splitting on $\tilde{\mathbb{D}}^k_{<\delta} \times N$. Given a small number $\omega > 0$ define the cones on $\tilde{\mathbb{D}}^k_{<\delta} \times N$

$$\begin{aligned} \mathcal{C}^u_\omega(x,y) &= \{v \in T_{(x,y)}(\tilde{\mathbb{D}}^k_{<\delta} \times N) : \measuredangle(v, E^u_N) < \omega\} \\ \mathcal{C}^{cs}_\omega(x,y) &= \{v \in T_{(x,y)}(\tilde{\mathbb{D}}^k_{<\delta} \times N) : \measuredangle(v, E^s_N \oplus E^c_N \oplus H) < \omega\} \end{aligned} \tag{2.5}$$

Remark 2.1. The splitting $E^s_N \oplus (E^c_N \oplus H) \oplus E^u_N$ coincides with the splitting $\hat{E}^s \oplus \hat{E}^c \oplus \hat{E}^u$ on the exceptional set only.

Recall that $\lambda < 1 < \mu$ are the constants from the definition of partial hyperbolicity (1.1). Also let $\lambda' \in (\lambda, 1]$ and $\mu' \in [1, \mu)$ be the some constants for which we have

$$c^{-1}(\lambda')^t \le \|Da^t(v)\|/\|v\| \le c(\mu')^t,$$

where $c > 0$.[1] Here a^t is the hyperbolic saddle given by the locally fiberwise structure (1.2) and $v \in T\mathbb{D}^k$. Now we pick a constant $\rho_0 > 0$ which enters the definition of the function ρ in the previous subsection such that we have the following inequality

$$\left(\frac{\lambda'}{\mu'}\right)^{\rho_0} > \max(\lambda, \mu^{-1}) \tag{2.6}$$

This is the *domination condition* [**Go16**, (2.3)] on the flow φ^t_ρ. This condition yields required estimates on the cones on $\tilde{\mathbb{D}}^k_{<\delta} \times N$ for the blown-up flow. We pick $\rho_0 < 1$. (If domination condition holds with $\rho_0 = 1$, our Main Theorem was already established in [**Go16**]. In either case we can always choose $\rho_0 < 1$.) Precisely, we have the following lemma.

Lemma 2.2. *There exist $\omega > 0$, $c > 0$, $\kappa > 1$ and $\delta_0 > 0$ such that for all $\delta < \delta_0$ there exists a Riemannian metric $\|\cdot\|_\delta$ on $\hat{M}$, which coincides with the metric $\|\cdot\|$ coming from M outside the δ-neighborhood of the exceptional set, such that the cone fields $\mathcal{C}^u_\omega$ and $\mathcal{C}^{cs}_\omega$ defined above*

[1] Constant μ' and λ' can be chosen to be arbitrarily close to the "outer" and "inner" spectral radii of a^t by choosing large $c > 0$.

are eventually (forward and backward) invariant under $D\varphi_\rho^t$ and verify the following hyperbolic properties:

1. *for all finite orbits segments $\{\varphi_\rho^s(x,y), 0 \le s \le t\}$, which are entirely contained in the δ-neighborhood of the exceptional set and for all $v \in \mathcal{C}_\omega^u(x,y)$*

$$\|D\varphi_\rho^t(v)\|_\delta > \mu^t \|v\|_\delta,\ t \ge 0$$

2. *for all finite orbits segments $\{\varphi_\rho^s(x,y), 0 \le s \le t\}$, which are entirely contained in the δ-neighborhood of the exceptional set, for all $v \in \mathcal{C}_\omega^u(x,y)$ and for all $w \in \mathcal{C}_\omega^{cs}(x,y)$ with $D\varphi^t w \in \mathcal{C}_\omega^{cs}(\varphi^t(x,y))$*

$$\frac{\|D\varphi_\rho^t(v)\|_\delta}{\|v\|_\delta} > c\kappa^t \frac{\|D\varphi_\rho^t(w)\|_\delta}{\|w\|_\delta},\ t \ge 0$$

The proof of this lemma is the basic technical ingredient of the prequel paper [**Go16**]. More precisely, the construction of appropriate Riemannian metric $\|\cdot\|_\delta$ is given in Section 5.1 of [**Go16**]. (For this construction we need to assume that the Riemannian metric $\|\cdot\|$ from the definition of partial hyperbolicity (1.1) on $\mathbb{D}^k \times N$ is a direct sum of the canonical flat metric and a metric on N. It was explained in Section 5.3.2 of [**Go16**] that such assumption can be made without loss of generality.) Then Lemma 5.1 of [**Go16**] gives partial hyperbolicity of the splitting $E_N^s \oplus (E_N^c \oplus H) \oplus E_N^u$. Finally, the fact that the estimates hold for the vectors in the cones (with proper choice of ω) is proved in Section 5.3.4 of [**Go16**].

2.3. Control along the center in the transition domain. Consider the transition domain $A_\delta \times N$, where $A_\delta = \tilde{\mathbb{D}}^k_{<2\delta} \cap \tilde{\mathbb{D}}^k_{>\delta}$. Recall that the Riemannian metric $\|\cdot\|_\delta$ restricted to this domain is the direct sum of the flat metric $\|\cdot\|$ and a metric on N. Also recall that the flow φ_ρ^t is generated by $\rho(x)X(x) + Y(y)$, $(x,y) \in A_\delta \times N$. It follows that, even though ρ is not constant, the splitting $E_N^s \oplus E_N^c \oplus E_N^u \oplus H$ stays invariant within this domain. Note that because of the nature of the dynamics of the hyperbolic saddle (invariance under rescaling) and because $\rho \ge \rho_0$ with ρ_0 independent of δ, there exists a uniform upper bound on time T which an orbit can spend in $A_\delta \times N$

$$T \le C_1, \tag{2.7}$$

where C_1 is a constant which depends on a^t and ρ_0, but does not depend on δ and ρ.

We proceed to explain how to control extra distortion which occurs along the "horizontal" distribution H. Hence we focus on the dynamics of reparametrized saddle flow a_ρ^t generated by ρX. The extra distortion which occurs along H is due to ρ-driven shear, however we will see that such shear is controlled uniformly in δ. We will perform all calculations using the canonical Euclidean structure on $A_\delta \subset \mathbb{D}^k$. Let $v_0 \in T_x A_\delta$ be a unit tangent vector and let $v_0^t = Da_\rho^t v_0$. We would like to obtain uniform control on derivative of the norm of v_0^t. Clearly it is sufficient to estimate the derivative at $t = 0$. Denote by V the vector field on A_δ obtained by translating v_0 to every point. Then by the definition of derivative a vector field we have

$$\frac{Da_\rho^t V - V \circ a_\rho^t}{t} = -\mathcal{L}_{\rho X} V = [V, \rho X]$$

Hence

$$v_0^t = V \circ a_\rho^t + [V, \rho X]t + h.o.t.$$

We proceed to estimate the derivative.

$$\frac{d\|v_0^t\|}{dt}\Big|_{t=0} = \frac{1}{2\|v_0\|}\frac{d\|v_0^t\|^2}{dt}\Big|_{t=0} = \frac{1}{2\|v_0\|}\frac{d\langle V\circ a_\rho^t + [V,\rho X]t, V\circ a_\rho^t + [V,\rho X]t\rangle}{dt} =$$
$$\frac{\langle [V,\rho X], v_0\rangle}{\|v_0\|} \le \frac{\|[V,\rho X]\|\,\|v_0\|}{\|v_0\|} = \|\rho[V,X] + V(\rho)X\| \le$$
$$\|[V,X]\| + |V(\rho)|\|X\| \le C_2 + \|\nabla\rho\|C_3\delta \le C_4$$

In the last inequality we used the fact that $\|X\| \le C_3\|\delta\|$ on A_δ and that $\|\nabla\rho\| \le 1/\delta$ by the construction of ρ.

Now, using the above bound and the time control (2.7), we immediately obtain.

LEMMA 2.3. *Assume that an orbit segment $\{a_\rho^s(x),\, 0 \le s \le T\}$ is entirely contained in A_δ, then for all $v \in T_xA_\delta$, $x \in A_\delta$*

$$\frac{\|Da_\rho^T v\|}{\|v\|} \le C_5, \quad \textit{and} \quad \frac{\|Da_\rho^T v\|}{\|v\|} \ge \frac{1}{C_5},$$

where C_5 is a constant which does not depend on ρ and δ.

2.4. Cones away from the exceptional set. To define the cones on $M\backslash(\tilde{\mathbb{D}}^k_{>2\delta}\times N)$ we use the same ω given by Lemma 2.2 and let

$$\mathcal{C}^u_\omega(p) = \{v \in T_p(M\backslash(\tilde{\mathbb{D}}^k_{>2\delta}\times N)) : \measuredangle(v,\hat{E}^u) < \omega\}$$
$$\mathcal{C}^{cs}_\omega(p) = \{v \in T_p(M\backslash(\tilde{\mathbb{D}}^k_{>2\delta}\times N)) : \measuredangle(v,\hat{E}^c\oplus\hat{E}^s) < \omega\}$$

Because $\varphi_\rho^t = \varphi^t$ and $\|\cdot\|_\delta = \|\cdot\|$ on $M\backslash(\tilde{\mathbb{D}}^k_{>2\delta}\times N)$ we then have invariance and hyperbolicity properties of these cones for orbit segments which stay in $M\backslash(\tilde{\mathbb{D}}^k_{>2\delta}\times N)$ by partial hyperbolicity of the flow φ^t.

2.5. Proof of partial hyperbolicity. To obtain partially hyperbolic splitting $E^u_\rho \oplus E^{cs}_\rho$ for φ^t_ρ we use the cone criterion applied to $\mathcal{C}^u_\omega$ and $\mathcal{C}^{cs}_\omega$. We recall that on $\tilde{\mathbb{D}}^k_{<\delta}\times N$ the cone families are centered at E^u_N and $E^s_N\oplus E^c_N\oplus H$ while on $M\backslash(\tilde{\mathbb{D}}^k_{>2\delta}\times N)$ the cone families are centered at $\hat{E}^u$ and $\hat{E}^c\oplus\hat{E}^s$. Note also that our cone families are not defined in the transition domain $A_\delta\times N$. However, we don't need to extend cones there because orbits spend a uniformly bounded time in $A_\delta\times N$.

By preceding discussion the cones are eventually invariant and and possess hyperbolic properties required by the Cone Criterion as long as the orbit stays disjoint with $A_\delta\times N$. Hence we are left to analyze the case when $\varphi^s(p)\in A_\delta\times N$, $0<s<T$, with p and $\varphi^T(p)$ in the boundary of $A_\delta\times N$. For the sake of concreteness we can focus on the case when $p\in\partial(\tilde{\mathbb{D}}^k_{<\delta}\times N)$ and $\varphi^T(p)\in\partial(\tilde{\mathbb{D}}^k_{>2\delta}\times N)$. (The other two cases $p\in\partial(\tilde{\mathbb{D}}^k_{>2\delta}\times N)$, $\varphi^T(p)\in\partial(\tilde{\mathbb{D}}^k_{<\delta}\times N)$ and $p\in\partial(\tilde{\mathbb{D}}^k_{>2\delta}\times N)$, $\varphi^T(p)\in\partial(\tilde{\mathbb{D}}^k_{>2\delta}\times N)$ can be treated completely analogously.) Recall that cone aperture ω is a fixed number given by Lemma 2.2 and is independent of δ. Also recall that $\hat{E}^s$, $\hat{E}^c$ and $\hat{E}^u$ are continuous distributions[2] which coincide with E^s_N, $E^c_N\oplus H$ and E^u_N, respectively, on the exceptional set. Hence for all sufficiently small δ we have

$$dist(E^s_N\oplus E^c_N\oplus H(q), \hat{E}^s\oplus\hat{E}^c(q)) < \frac{\omega}{10}$$

and

$$dist(E^u_N(q), \hat{E}^u(q)) < \frac{\omega}{10}$$

[2]Here we rely on the smoothness assumption for the partially hyperbolic splitting of φ^t in an essential way.

for all $q \in \tilde{\mathbb{D}}^k_{<3\delta} \times N$. Because, locally in the neighborhood of the exceptional set, the flow φ^t_ρ preserves both splittings $E^u_N \oplus (E^c_N \oplus H) \oplus E^s_N$ and $\hat{E}^s \oplus \hat{E}^c \oplus \hat{E}^u$ it follows that

$$D\varphi^T_\rho(E^u_N(p)) \subset \mathcal{C}^u_\omega(\varphi^T_\rho(p)),$$
$$D\varphi^{-T}_\rho(\hat{E}^c \oplus \hat{E}^s(\varphi^T_\rho(p))) \subset \mathcal{C}^{cs}_\omega(p)$$

Combining this observation with control provided by Lemma 2.3 one can easily verify the following statement.

LEMMA 2.4. *There exist constants $C_6 > 0$ and $C_7 > 0$ such that for all sufficiently small $\delta > 0$ and for all points $\{p, \varphi^T(p)\} \subset \partial(A_\delta \times N)$ we have*

$$D\varphi^T_\rho(\mathcal{C}^u_\omega(p)) \subset \mathcal{C}^u_{C_6\omega}(\varphi^T_\rho(p)),$$
$$D\varphi^{-T}_\rho(\mathcal{C}^{cs}_\omega(\varphi^T_\rho(p))) \subset \mathcal{C}^{cs}_{C_6\omega}(p),$$
$$\|D\varphi^T_\rho v\|_\delta \geq C_7\|v\|_\delta,\ v \in \mathcal{C}^u_\omega(p),$$
$$\|D\varphi^{-T}_\rho v\|_\delta \geq C_7\|v\|_\delta,\ v \in \mathcal{C}^{cs}_\omega(\varphi^T_\rho(p))$$

Now note that by decreasing δ we can increase the return time to the 2δ-neighborhood of the exceptional set, $\tilde{\mathbb{D}}^k_{<2\delta} \times N$, as much as we wish. This observation combined with Lemma 2.4 implies that $\mathcal{C}^u_\omega$ is eventually forward invariant and $\mathcal{C}^{cs}_\omega$ is eventually backward invariant for all sufficiently small δ. Finally the exponential expansion of vectors in $\mathcal{C}^u_\omega$ and domination of $\mathcal{C}^u_\omega$ over $\mathcal{C}^{cs}_\omega$ can be checked by using a standard argument: subdividing the orbit into segments and pasting together the estimates given by Lemmas 2.2, 2.4 as well as hyperbolicity of cone families outside $\tilde{\mathbb{D}}^k_{<2\delta} \times N$. This arguments takes an advantage of the long return time to $\tilde{\mathbb{D}}^k_{<2\delta} \times N$ one more time. We suppress detailed estimates as they are very standard.

3. Volume preserving modification via Katok-Lewis trick

We first formulate a standard lemma.

LEMMA 3.1. *Let M be a smooth manifold equipped with a smooth non-degenerate volume form m. Assume that a flow generated by a smooth vector field preserves m. Consider a smooth function $\rho\colon M \to \mathbb{R}$, $\rho > 0$. Then the flow generated by ρX preserves m/ρ.*

PROOF. By Cartan's formula

$$0 = \mathcal{L}_X m = \iota_X dm + d\iota_X m = d\iota_X m$$

and similarly $\mathcal{L}_X(m/\rho) = d\iota_X(m/\rho)$. We calculate

$$\mathcal{L}_{\rho X}(m/\rho) = \rho\mathcal{L}_X(m/\rho) + d\rho \wedge \iota_X(m/\rho) = \rho d\iota_X(m/\rho) + \frac{1}{\rho}d\rho \wedge \iota_X m =$$
$$\rho d(\frac{1}{\rho}\iota_X m) + \frac{1}{\rho}d\rho \wedge \iota_X m = \rho\left(-\frac{1}{\rho^2}d\rho \wedge \iota_X m + \frac{1}{\rho}d\iota_X m\right) + \frac{1}{\rho}d\rho \wedge \iota_X m = d\iota_X m = 0$$

□

The goal of this section is to prove the Addendum 1.3. Recall that we assume that $\varphi^t\colon M \to M$ preserves a smooth volume m and $m|_{\mathbb{D}^k \times N} = vol \otimes vol_N$. Recall that φ^t_ρ is a slow down of φ^t along $\mathbb{D}^k$. By Lemma 3.1, the flow φ^t_ρ also locally preserves the smooth volume $m_\rho|_{\mathbb{D}^k \times N} = \frac{1}{\rho}vol \otimes vol_N$. Note that $m_\rho = m$ near the boundary and hence extend to a smooth φ^t_ρ-invariant volume on the whole of M which we still denote by m_ρ. Because $\rho = \rho_0$ is a constant on $\mathbb{D}^k_{<\delta}$, we see that m_ρ still have a product form $\frac{1}{\rho_0}vol \otimes vol_N$ on $\mathbb{D}^k_{<\delta} \times N$.

3.1. Replacing the smooth structure. If we equip $\mathbb{D}^k$ with the standard Euclidean coordinates $(x_1, x_2, \dots, x_k)$ then

$$vol = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_k. \tag{3.8}$$

By commutativity of (1.3) the flow $\hat{\varphi}_\rho^t$ preserves $\pi^* m_\rho$, which is a smooth measure away from the exceptional set.

Let's examine the form of $\pi^* m_\rho$ at the exceptional set. Because π is a product, we only need to look at the pullback of vol to $\tilde{\mathbb{D}}^k$ under $\tilde{\mathbb{D}}^k \to \mathbb{D}^k$. Recall that

$$\tilde{\mathbb{D}}^k = \{(x_1, x_2, \dots x_k, \ell) : (x_1, x_2, \dots x_k) \in \ell,\, \ell \in \mathbb{R}P^{k-1}\}$$

and that the standard smooth charts for $\tilde{\mathbb{D}}^k$ are given by extending the standard charts for the projective space $\mathbb{R}P^{k-1}$. Namely the i-th chart is given by

$$\begin{aligned}\Psi_i(u_1, u_2, \dots u_k) = \\ (u_1 u_i, u_2 u_i, \dots u_{i-1} u_i, u_i, u_{i+1} u_i, \dots u_k u_i, [u_1 : \ldots : u_{i-1} : 1 : u_{i+1} : \ldots : u_k])\end{aligned} \tag{3.9}$$

We can calculate the pull-back of vol

$$d(u_1 u_i) \wedge d(u_2 u_i) \wedge \ldots \wedge du_i \wedge \ldots \wedge d(u_k u_i) = u_i^{k-1} du_1 \wedge du_2 \wedge \ldots \wedge du_k.$$

Hence, when $k > 1$ the pull-back vanishes on the projective space. To remedy the situation we follow the idea of Katok-Lewis (which they used to construct non-standard higher rank volume preserving group actions.) Namely we replace the smooth structure on $\mathbb{D}^k$ by declaring that

$$\Phi \colon \vec{u} \mapsto \|\vec{u}\|^\alpha \vec{u},\, \alpha < 0$$

is a smooth chart near the origin (*i.e.,* by changing the smooth atlas). With respect to this chart the Euclidean norm of a vector $\vec{u}$ is given by

$$\|\vec{u}\|_{\text{new}} = \|\vec{u}\|^{1+\alpha} \tag{3.10}$$

Accordingly we change the smooth structure on M by declaring that $\Phi \times id_N \colon \mathbb{D}^k \times N \to M$ is a smooth chart at N. Note that M equipped with the new smooth atlas, which we denote by M^{new}, is obviously diffeomorphic to the original M. However, it is easy to check that $a_\rho^t \colon \mathbb{D}^k \to \mathbb{D}^k$ and, hence, $\varphi_\rho^t \colon M^{\text{new}} \to M^{\text{new}}$ fail to be smooth.

Accordingly we replace we replace charts (3.9) for $\tilde{\mathbb{D}}^k$ by composing Ψ_i and Φ, that is,

$$\begin{aligned}\Psi_i^{\text{new}}(u_1, u_2, \dots u_k) = \\ \big(f_\alpha(u_1, \dots, u_{i-1}, u_{i+1}, \dots u_k)\|u_i\|^\alpha (u_1 u_i, u_2 u_i, \dots u_i, \dots u_k u_i), [u_1 : \ldots : 1 : \ldots : u_k]\big),\end{aligned}$$

where

$$f_\alpha(u_1, \dots, u_{i-1}, u_{i+1}, \dots u_k) = (u_1^2 + u_2^2 + \ldots + u_{i-1}^2 + 1 + u_{i+1}^2 + \ldots + u_k^2)^{\alpha/2}$$

Because the new smooth structure amounts to mere reparametrization in the radial direction the projective dynamics remains exactly the same. A direct calculation in chart shows that $\hat{a}_\rho^t \colon \tilde{\mathbb{D}}^k \to \tilde{\mathbb{D}}^k$ is smooth with respect to the new smooth structure. Hence $\hat{\varphi}_\rho^t \colon \hat{M}^{\text{new}} \to \hat{M}^{\text{new}}$ is also smooth. Further, by appropriate choice of α we can now guarantee that $\pi^* m$ is a non-degenerate volume on $\hat{M}^{\text{new}}$. We present the chart calculation which determines the "right" value of α. In order to simplify notation we perform this calculation in the first chart Ψ_1^{new}. We also abbreviate $f_\alpha(u_2, u_3, \dots u_k)$ to simply f_α. Note that

$$df_\alpha \wedge du_2 \wedge du_3 \wedge \ldots \wedge du_k = 0$$

This is very helpful for the calculation:

$$d(f_\alpha\|u_1\|^\alpha u_1)\wedge d(f_\alpha\|u_1\|^\alpha u_1u_2)\wedge\ldots\wedge d(f_\alpha\|u_1\|^\alpha u_1u_k)=$$
$$d(f_\alpha\|u_1\|^\alpha u_1)\wedge(u_2d(f_\alpha\|u_1\|^\alpha u_1)+f_\alpha\|u_1\|^\alpha u_1du_2)\wedge\ldots\wedge(u_kd(f_\alpha\|u_1\|^\alpha u_1)+f_\alpha\|u_1\|^\alpha u_1du_k)=$$
$$(f_\alpha\|u_1\|^\alpha u_1)^{k-1}d(f_\alpha\|u_1\|^\alpha u_1)\wedge du_2\wedge\ldots\wedge du_k=$$
$$(f_\alpha\|u_1\|^\alpha u_1)^{k-1}\big(f_\alpha d(\|u_1\|^\alpha u_1)\wedge du_2\wedge\ldots\wedge du_k+\|u_1\|^\alpha u_1df_\alpha\wedge du_2\wedge du_3\wedge\ldots\wedge du_k\big)=$$
$$(f_\alpha\|u_1\|^\alpha u_1)^{k-1}(\alpha+1)f_\alpha\|u_1\|^\alpha du_1\wedge du_2\wedge\ldots\wedge du_k)=(\alpha+1)f_\alpha^k\|u_1\|^{k\alpha}u_1^{k-1}$$

Notice that f_α is a smooth function. Hence the pull-back of *vol* is a smooth and non-degenerate on M^{new} when $k\alpha+k-1=0$, *i.e.,*

$$\alpha=-\frac{k-1}{k}$$

REMARK 3.2. It is crucial for this construction that the initial volume on $\mathbb{D}^k$ given by (3.8) has constant density. Indeed, if we allow for a non-trivial density $\beta(x_1,\ldots x_k)$ and begin with $\beta dx_1\wedge dx_2\wedge\ldots\wedge dx_k$ instead, then all computations go through in the same way. However the expression for the density after the blow-up in the chart Ψ_i^{new} will have an additional factor

$$\beta(f_\alpha(u_1,\ldots,u_{i-1},u_{i+1},\ldots u_k)\|u_i\|^\alpha(u_1u_i,u_2u_i,\ldots u_i,\ldots u_ku_i))$$

which is not C^1 at the exceptional set given by $u_i=0$ (unless the Taylor coefficients of β up to order k vanish). Hence we have a positive continuous density which is not C^1 on the exceptional set. This issue, in fact, gives us an additional difficulty to overcome in the proof of Corollary 1.4.

3.2. Partial hyperbolicity in volume preserving setting. We now have a volume preserving flow $\hat\varphi_\rho^t\colon M^{\text{new}}\to M^{\text{new}}$. Here we explain that this flow is also partially hyperbolic provided that constant ρ_0 (from the definition of ρ) is chosen to be sufficiently small. Namely, we amend the domination condition (2.6), as follows

$$\left(\frac{\lambda'}{\mu'}\right)^{\rho_0}>\max(\lambda,\mu^{-1}),\ \ \lambda<(\lambda')^{\rho_0/k},\ \ (\mu')^{\rho_0/k}<\mu \tag{3.11}$$

Clearly these inequalities are verified for a sufficiently small ρ_0.

The proof of partial hyperbolicity is the same as the one given in Section 2. The only difference which requires some commentary is the Lemma 2.2 for $\hat\varphi_\rho^t\colon M^{\text{new}}\to M^{\text{new}}$ under the condition (3.11). Recall that the proof of this lemma mostly rests on Lemma 5.1 of [**Go16**] and the proof of Lemma 5.1 is the only place which requires some adjustments. We indicate how (3.11) must be used in the proof of Lemma 5.1. Recall that on the small neighborhood of the projective space the dynamics of $\hat a_\rho^t$ is given by

$$\hat a_\rho^t(s,v)=(\hat{\hat a}_\rho^t(s),\bar a_s^t(v)),\ \ s\in\mathbb{R}P^{k-1},v\in\mathbb{R}_+$$

where $\hat{\hat a}_\rho^t\colon\mathbb{R}P^{k-1}\to\mathbb{R}P^{k-1}$ is the projectivization of a_ρ^t (which coincides with the restriction of $\hat a_\rho^t$ to $\mathbb{R}P^{k-1}$) and $\bar a_s^t$ is the cocycle over $\hat{\hat a}_\rho^t$ given by the action of a_ρ^t on lines (see the proof of Lemma 5.3 in [**Go16**]).[3]

The estimate on $\hat{\hat a}_\rho^t$ (Claim 5.4 of [**Go16**]) remains exactly the same as the alternation of the smooth structure did not change the projective dynamics. The place where (3.11) is needed is the inequality (5.16) of [**Go16**] (estimate on the cocycle $\bar a_s^t$). Indeed, given a small $\vec u$, according to (3.10), we have the local estimate

$$\|a_\rho^t(\vec u)\|_{\text{new}}=\|a_\rho^t(\vec u)\|^{1+\alpha}\le(c(\mu_0'^\rho)^t\|\vec u\|)^{1+\alpha}=c^{1/k}(\mu')^{\rho_0t/k}\|\vec u\|_{\text{new}}$$

[3]One difference which appears is that even though, with respect to the new smooth chart Ψ^{new}, a_ρ^t still sends lines to line the cocycle $\bar a_s^t$ is no longer linear. This, however, does not present any additional difficulty.

and similarly

$$\|a_\rho^t(\vec{u})\|_{\text{new}} \geq c^{-1/k}(\lambda')^{\rho_0 t/k}\|\vec{u}\|_{\text{new}}$$

This effects the last inequality in the proof of Lemma 5.3 of [**Go16**]. Namely, we obtain an exponential upper bound in

$$\max\left(\left(\frac{\lambda'}{\mu'}\right)^{\rho_0}, (\mu')^{\rho_0/k}\right)$$

(and, analogously, a lower bound with $(\lambda')^{\rho_0/k}$) Hence, in order for the rest of the proof to work we need to use (3.11) instead of (2.6).

4. The example

In this section we introduce geodesic flows on complex hyperbolic manifolds in detail and then prove Corollaries 1.2 and 1.4.

4.1. Complex hyperbolic manifolds. First recall that 1-dimensional complex hyperbolic space can be identified with 2-dimensional real hyperbolic space with metric equal to one quarter of the standard Poincaré metric. The linear fractional transformations form the group of holomorphic isometries (to generate the full group of isometries one also needs the anti-holomorphic transformation) and can be identified with $PSU(1,1) = \pm Id\backslash SU(1,1)$. Because of the $\frac{1}{4}$ multiple in the expression for the metric the curvature is -4 and the contraction and expansion rates of the geodesic flow on the complex hyperbolic space are twice bigger. It follows that the full stable and unstable horocycles of geodesic flows on higher dimensional complex hyperbolic manifolds contain one dimensional "fast" horocycles which correspond to the complex lines in the tangent bundle. This yields a partially hyperbolic splitting which is different from the Anosov one and makes the geodesic flow on complex hyperbolic manifold suitable for the blow-up surgery.

We begin by summarizing some standard material on complex hyperbolic manifolds. We mostly follow the lucid exposition by D.B.A. Epstein [**E84**]. Consider the following Hermitian quadratic forms on $\mathbb{C}^{n+1}$ of signature $(n,1)$.

$$Q(x) = \sum_{i=1}^{n} z_i\bar{z}_i - z_{n+1}\bar{z}_{n+1}$$

$$\hat{Q}(x) = \sum_{i=1}^{n-1} z_i\bar{z}_i + z_n\bar{z}_{n+1} + \bar{z}_n z_{n+1}$$

These forms have the following associated matrices

$$J = diag(1,1,\dots 1,-1)$$

$$\hat{J} = \begin{pmatrix} Id & 0 \\ 0 & J_0 \end{pmatrix}$$

respectively. Here $J_0 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Let $SU(n,1;Q)$ and $SU(n,1;\hat{Q})$ be the groups of $(n+1)\times(n+1)$ complex matrices which have determinant 1 and preserve corresponding form. These groups are conjugate in $GL(n+1)$ by

$$T = \begin{pmatrix} Id & 0 \\ 0 & T_0 \end{pmatrix}$$

where $T_0 = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$.

Recall that the complex hyperbolic n-space $\mathbb{H}^n_{\mathbb{C}}$ can be defined as

$$\mathbb{H}^n_{\mathbb{C}} = \{[x] \in \mathbb{C}P^n : Q(x) < 0\}$$

Clearly the action of $SU(n,1;Q)$ on $\mathbb{C}^{n+1}$ induces an action on $\mathbb{H}^n_{\mathbb{C}}$ and, in fact, $SU(n,1)$ coincides with the group of biholomorphic isometries of $\mathbb{H}^n_{\mathbb{C}}$. If Γ is a discrete cocompact subgroup of $SU(n,1)$ acting on the right then the orbit space

$$M = \mathbb{H}^n_{\mathbb{C}}/\Gamma$$

is a closed complex hyperbolic manifold. Moreover, every closed complex hyperbolic manifold arises in this way.

4.2. The geodesic flow as a homogenous flow. We describe M and its unit tangent bundle as homogeneous spaces. The group $SU(n,1;Q)$ acts transitively on $\mathbb{H}^n_{\mathbb{C}}$ and the stabilizer of $[(0,0,\dots 0,1)]$ is

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & \overline{\det A} \end{pmatrix} : A\bar{A}^t = Id \right\} \simeq U(n).$$

The stabilizer of a tangent vector is the group $W(n-1)$ given by[4]

$$W(n-1) = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & \bar{\lambda} & 0 \\ 0 & 0 & \bar{\lambda} \end{pmatrix} : A\bar{A}^t = Id, \lambda^2 = \det A \right\}$$

Hence we have

$$M = U(n)\backslash SU(n,1;Q)/\Gamma \qquad T^1M = W(n-1)\backslash SU(n,1;Q)/\Gamma.$$

The same descriptions work using $SU(n,1;\hat{Q})$ as the underlying Lie group with embeddings of $W(n-1)$ and $U(n)$ are conjugated by T. Also note that $W(0) = \{\pm Id\}$ and we will write $PSU(1,1)$ instead of $W(0)\backslash SU(1,1)$.

From now on it would be more convenient to only use the form $\hat{Q}$ and we abbreviate $SU(n,1;\hat{Q})$ to $SU(n,1)$.

Now recall the Lie algebras

$$\mathfrak{u}(n-1) = \{A \in M_{n-1} : \bar{A}^\intercal = -A\}$$

and

$$\mathfrak{su}(n,1) = \mathfrak{su}(n,1,\hat{Q}) = \{B \in M_{n+1} : Tr(B) = 0, \;\; \bar{B}^\intercal \hat{J} + \hat{J}B = 0\} \tag{4.12}$$

If we write a traceless matrix $B \in \mathfrak{su}(n,1)$ in block form, then $B \in \mathfrak{su}(n,1)$ if and only if

$$B = \begin{pmatrix} A & v \\ -J_0\bar{v}^\intercal & D \end{pmatrix}$$

where $A \in \mathfrak{o}(n-1)$ and $D = \left(\begin{smallmatrix} a & ib \\ ic & -\bar{a} \end{smallmatrix}\right)$, $a \in \mathbb{C}$, $b, c \in \mathbb{R}$.

The geodesic flow $d_t \colon T^1M \to T^1M$ is given by $W(n-1)g\Gamma \mapsto d_tW(n-1)g\Gamma = W(n-1)d_tg\Gamma$, where

$$d_t = \begin{pmatrix} Id & 0 \\ 0 & d_t^0 \end{pmatrix}, \;\; \text{with} \;\; d_t^0 = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

The strong stable and strong unstable horocycle subgroups are

$$h_t^{s/u} = \begin{pmatrix} Id & 0 \\ 0 & h_t^{s0/u0} \end{pmatrix}, \;\; \text{with} \;\; h_t^{s0} = \begin{pmatrix} 1 & it \\ 0 & 1 \end{pmatrix}, \;\; h_t^{u0} = \begin{pmatrix} 1 & 0 \\ it & 1 \end{pmatrix}.$$

We refer to [**FK01**] for a more detailed exposition on the geodesic flow as a homogeneous flow.

[4]Notice that, by mapping to the $(n-1)\times(n-1)$ upper diagonal matrix A, the group $W(n-1)$ is a double cover of $U(n-1)$. It is curious to notice that, unlike in the real case, $W(n-1)$ is not isomorphic to $U(n-1)$. However using the fact that $\det \colon U(n) \to U(1)$ is a trivial principal fiber bundle one can check that $W(n-1)$ is diffeomorphic to $U(n-1)$.

4.3. Totally geodesic holomorphic curve. The complex hyperbolic space $\mathbb{H}^1_{\mathbb{C}}$ can be identified with $\{z_1 = z_2 = \ldots = z_{n-1} = 0\} \cap \mathbb{H}^n_{\mathbb{C}}$. The group of holomorphic isometries $SU(1,1)$ of $\mathbb{H}^1_{\mathbb{C}}$ embeds into $SU(n,1)$ as lower diagonal block. Let Γ be a cocompact lattice in $SU(n,1)$ and let $\Gamma_1 = SU(1,1) \cap \Gamma$. We assume that Γ_1 is a cocompact subgroup of $SU(1,1)$. Hence the embedding $\mathbb{H}^1_{\mathbb{C}} \subset \mathbb{H}^n_{\mathbb{C}}$ yields the embeddings

$$N = U(1)\backslash SU(1,1)/\Gamma_1 \subset U(n)\backslash SU(n,1)/\Gamma = M, \text{ and}$$
$$T^1N = PSU(1,1)/\Gamma_1 \subset W(n-1)\backslash SU(n,1)/\Gamma = T^1M$$

where N is a totally geodesic one dimensional complex curve.

4.4. Parametrization of the neighborhood and the geodesic flow. We introduce a parametrization of a neighborhood $\mathcal{U}$ of $PSU(1,1)$ in $W(n-1)\backslash SU(n,1)$. This parametrization will be constructed to be Γ_1 equivariant and, hence, will descend to a parametrization of a neighborhood of T^1N in T^1M.

Pick a small $\varepsilon_0 > 0$ and take the following as a transversal to the Lie algebras of $SU(1,1)$ and $W(n-1)$. Using the block from (4.12) let

$$\mathbb{D}_{\varepsilon_0} = \left\{ \begin{pmatrix} 0 & v \\ -J_0\bar{v}^\intercal & 0 \end{pmatrix} \in \mathfrak{su}(n,1), \quad \text{where } \|v\| < \varepsilon_0 \right\}$$

This is a $(4n-4)$-dimensional transversal spanned by weak stable and unstable horocycles. Let $\Sigma = \Sigma_{\varepsilon_0} = \exp(\mathbb{D}_{\varepsilon_0})$.

Now we define a parametrization $p\colon \Sigma \times PSU(1,1) \to W(n-1)\backslash SU(n,1)$ of a neighborhood $\mathcal{U} = \mathcal{U}_{\varepsilon_0}$ of $PSU(1,1)$ in $W(n-1)\backslash SU(n,1)$ as follows

$$p(\sigma, u) = W(n-1)\sigma u. \tag{4.13}$$

To verify that this is a well-defined parametrization for a sufficiently small ε_0 it is sufficient to check that the map $P\colon W(n-1) \times \Sigma \times PSU(1,1) \to SU(n,1)$ given by $P(w,\sigma,u) = w\sigma u$ is a diffeomorphism on its image. And that the image contains a neighborhood of $W(n-1) \times PSU(1,1) \subset SU(n,1)$. To do this we consider a metric d on $SU(n,1)$ which is invariant under the right action of $PSU(1,1)$ and left action of $W(n-1)$. One can obtain such a metric by starting with a right invariant Riemannian metric and then averaging with respect to the left action of (compact group) $W(n-1)$.

Notice that $T_{id}\Sigma$, $\mathfrak{su}(1,1)$ and $\mathfrak{w}(n-1)$ span the full Lie algebra $\mathfrak{su}(n,1)$, and, hence, P is a local diffeomorphism on the neighborhood of $(0,0,0)$. More precisely, by choosing appropriately small $\varepsilon_0 > 0$ and $r > 0$ we have that the restriction of P to the neighborhood

$$\{w \in W(n-1) : d(w, id) < r\} \times \Sigma \times \{u \in PSU(1,1) : d(u, id) < r\}$$

is a local diffeomorhism on its image. Further, because $P(w'w, \sigma, uu') = w'P(w,\sigma,u)u'$ we obtain that each point $P(w', 0, u')$ has a neighborhood which has a uniform size (with respect to metric d) entirely contained in the image of P.

It remains to check that P is one-to-one. Let

$$\delta_0 = \sup_{\sigma \in \Sigma} d(id, \sigma)$$

Note that by choosing smaller ε_0 we can make $\delta_0 > 0$ as small as desired. Assume that $P(w_1, \sigma_1, u_1) = P(w_2, \sigma_2, u_2)$, *i.e.*,

$$w_2^{-1}w_1\sigma_1 = \sigma_2 u_2 u_1^{-1} \tag{4.14}$$

Then

$$d(w_2^{-1}w_1, u_2u_1^{-1}) \le d(w_2^{-1}w_1, w_2^{-1}w_1\sigma_1) + d(\sigma_2 u_2 u_1^{-1}, u_2 u_1^{-1}) = d(id, \sigma_1) + d(\sigma_2, id) \le 2\delta_0$$

Recall that $W(n-1)\times PSU(1,1)$ is (explicitly) properly embedded in $SU(n,1)$. Hence the last inequality implies that both $w_2^{-1}w_1$ and $u_2u_1^{-1}$ are close to id. On the other hand we have already shown that P is a local diffeomorphism on the neighborhood of id. Hence (4.14) implies that $w_2^{-1}w_1 = id$, $u_2u_1^{-1} = id$ and $\sigma_1 = \sigma_2$ proving that P is injective.

Finally, we let Γ_1 act on $\Sigma\times PSU(1,1)$ by $\gamma_1\colon (\sigma,u)\mapsto(\sigma,u\gamma_1)$. Our parametrization is equivariant with respect to the right action of Γ_1 and hence descends to a parametrization of a neighborhood of $T^1N\subset T^1M$ by $\Sigma\times PSU(1,1)/\Gamma_1\simeq\Sigma\times T^1N\simeq\mathbb{D}_{\varepsilon_0}\times T^1N$.[5]

4.5. Proof of Corollary 1.2. The Corollary 1.2 follows from the Main Theorem provided that we verify the locally fiberwise assumption with respect to our parametrization. We write v as a column vectors $v=(v_1,v_2)$ which parametrizes Σ. That is,

$$A(v_1,v_2)=\begin{pmatrix} 0 & v\\ -J_0\bar v^{\intercal} & 0\end{pmatrix}$$

and $\sigma(v_1,v_2)=\exp A(v_1,v_2)$.

Notice that

$$\begin{aligned} d_t\sigma(v_1,v_2)d_t^{-1} &= d_t\exp A(v_1,v_2)d_t^{-1}\\ &= \exp d_tA(v_1,v_2)d_t^{-1} = \exp A(e^{-t}v_1,e^tv_2)=\sigma(e^{-t}v_1,e^tv_2)\end{aligned}$$

Now we can deduce the formula for the geodesic flow using the coordinates $(v_1,v_2,u)\in\Sigma\times PSU(1,1)$

$$\begin{aligned} d_t(v_1,v_2,u) = W(n-1)d_t\sigma(v_1,v_2)u &= W(n-1)d_t\sigma(v_1,v_2)d_t^{-1}d_tu\\ &= (e^{-t}v_1,e^tv_2,d_tu)\end{aligned}$$

We conclude that with respect to the coordinates (v_1,v_2,u) the geodesic flow is the product of $(4n-4)$-dimensional hyperbolic saddle and the geodesic flow on a holomorphic curve. This verifies the assumption of the Main Theorem on locally fiberwise structure of d_t on $\mathcal{U}$.

Finally to see that the partially hyperbolic flow $\tilde\varphi^t$ could be chosen to be arbitrarily close to $\hat\varphi^t\colon\widehat{T^1M}\to\widehat{T^1M}$ in C^∞ topology recall that we obtain $\tilde\varphi^t$ by blowing up the reparametrized flow φ^t_ρ. The reparametrization is localized in the neighborhood of T^1N and is given by (2.4). Function ρ has to be chosen so that (2.6) holds:

$$\left(\frac{\lambda'}{\mu'}\right)^{\rho_0} > \max(\lambda,\mu^{-1})$$

In the current setting $\lambda'^{-1}=\mu'=e$ and $\lambda^{-1}=\mu=e^2$. Hence any value of $\rho_0<1$ would work. It follows that the function ρ can be chosen to be arbitrarily close to 1 in the C^∞ topology. Therefore φ^t_ρ can be arbitrarily C^∞ close to φ^t and, accordingly, $\tilde\varphi^t$ can be arbitrarily C^∞ close to $\hat\varphi^t$.

4.6. Proof of Corollary 1.4. Corollary 1.4 does not immediately follow from Addendum 1.3. The reason is that the pull-back of the Liouville volume form p^*vol under parametrization p has the form

$$\alpha(v_1,v_2)\omega_0\wedge vol_{T^1N},$$

[5]Notice that in particular we have shown that the normal bundle of T^1N in T^1M is trivial. This happens because $W(n-1)\cap PSU(1,1)=\{Id\}$. It was pointed out to us by Mike Davis that in general the normal bundle of N in M is twisted and the twisting is controlled by the Chern class.

where ω_0 is the standard volume form on $\mathbb{D}_{\varepsilon_0}$ and vol_{T^1N} is the Liouville volume form on T^1N. Indeed, because the Liouville measure comes from the Haar measure on $SU(n,1)$ and p is equivariant with respect to the right action of $PSU(1,1)$ the density α is independent of the u-coordinate. However, the dependence on v_1 and v_2 is non-trivial. Hence the Addendum 1.3 does not apply directly (cf. Remark 3.2). Our approach is to replace the flow φ_t with a different flow $\bar{\varphi}_t$ to which Addendum 1.3 can be applied. More precisely, on the neighborhood $\mathbb{D}_{\varepsilon_0} \times T^1N$ we will let

$$\bar{\varphi}^t = \bar{h} \circ \varphi^t \circ \bar{h}^{-1}$$

where $\bar{h} = (h, id_{T^1N})$ and h is C^1 small and tapers away to identity on the neighborhood of the boundary of $\mathbb{D}_{\varepsilon_0}$.

Let $\omega_1 = \alpha(v_1, v_2)\omega_0$. By rescaling ω_0 if needed, we may assume that $\alpha(0,0) = 1$. Denote by a^t the saddle flow, $a^t(v_1, v_2) = (e^{-t}v_1, e^t v_2)$. Note that, because α is continuous and a^t-invariant, we also have $\alpha(0, v_2) = \alpha(v_1, 0) = 1$.

LEMMA 4.1. *For all sufficiently small $\varepsilon_1 \in (0, \varepsilon_0)$ there exists a diffeomorphism $h\colon \mathbb{D}_{\varepsilon_1} \to h(\mathbb{D}_{\varepsilon_1}) \subset \mathbb{D}_{\varepsilon_0}$ such that $h_*\omega_1 = \omega_0$ and h commutes with the saddle flow, when defined:*

$$h \circ a_t = a_t \circ h$$

Before proving the lemma we first finish the proof of Corollary 1.4. First extend $h\colon \mathbb{D}_{\varepsilon_1} \to h(\mathbb{D}_{\varepsilon_1})$ to a diffeomorphism $h\colon \mathbb{D}_{\varepsilon_0} \to \mathbb{D}_{\varepsilon_0}$ which equals to identity near the boundary. Then replace the geodesic flow φ^t with the new flow $\bar{\varphi}^t$ by replacing the restriction $\varphi^t|_{\mathbb{D}_{\varepsilon_0} \times T^1N}$ with $(h \circ a^t \circ h^{-1}, \varphi^t_{T^1N})$. Clearly $\bar{\varphi}^t$ is smoothly conjugate to φ^t. Hence $\bar{\varphi}^t$ is partially hyperbolic with C^1 splitting. Further, T^1N remains $\bar{\varphi}^t$-invariant and, because h commutes with a^t on $\mathbb{D}_{\varepsilon_1}$ we have

$$\bar{\varphi}^t(v_1, v_2, u) = \varphi^t(v_1, v_2, u) = (a^t(v_1, v_2), \varphi^t_{T^1N}(u))$$

for $(v_1, v_2) \in \mathbb{D}_{\varepsilon_1}$. Hence the locally fiberwise assumption is also verified for $\bar{\varphi}^t$. On the neighborhood $\mathbb{D}_{\varepsilon_1} \times T^1N$ the $\bar{\varphi}^t$-invariant volume has the form $\bar{h}_*(\omega_1 \wedge vol_{T^1N}) = h_*\omega_1 \wedge vol_{T^1N} = \omega_0 \wedge vol_{T^1N}$ and hence the assumption of Addendum 1.3 is also verified. We conclude that Addendum 1.3 applies to $\bar{\varphi}^t$ and yields Corollary 1.4. □

Hence it only remains to prove the Lemma.

PROOF OF LEMMA 4.1. The idea of the proof is perform an a^t-equivariant Moser trick.[6] To obtain the diffeomorphism h such that $h_*\omega_1 = \omega_0$ consider the path $\omega_t = (1-s)\omega_0 + s\omega_1$, $s \in [0,1]$. Then, by the Poincaré Lemma, there exists η such that

$$d\eta = \omega_1 - \omega_0 = \gamma\omega_0, \;\; \gamma = \alpha - 1$$

Further, we can choose η to be a^t-invariant; that is, $\mathcal{L}_X\eta = 0$, where $X = \partial a^t/\partial t$. We proceed with the proof assuming this fact, which we will verify later via a direct calculation.

Because ω_s are non-degenerate forms the equation

$$\iota_{Y_s}\omega_s = \eta,$$

uniquely defines "time-dependent vector field" Y_s. Then, by Cartan's formula, we have for every $s \in [0,1]$

$$\mathcal{L}_{Y_s}\omega_s = (\iota_{Y_s} \circ d + d \circ \iota_{Y_s})\omega_s = d\beta$$

Hence by integrating Y_s we obtain a one-parameter family of diffeomorphisms h_s such that

$$(h_s)_*\omega_0 = \omega_s$$

[6]While such trick is standard in the context of equivariant cohomology, when the acting group is compact, see e.g. [**GS84**], we were unable to locate any prior work on "locally equivariant" Moser trick. While we only do it here for the saddle singularity, presumably it is much more general.

Recall that volume forms ω_s are invariant under X, *i.e.,* $\mathcal{L}_X\omega_s = 0$ Hence

$$0 = \mathcal{L}_X\beta = \mathcal{L}_X(\iota_{Y_s}\omega_s) = \iota_{Y_s}(\mathcal{L}_X\omega_s) + \iota_{\mathcal{L}_X Y_s}\omega_s = \iota_{\mathcal{L}_X Y_s}\omega_s,$$

which implies that $[X, Y_s] = \mathcal{L}_X Y_s = 0$ because ω_s in non-degenerate. It follows from the Frobenius Theorem that a^t commutes with h_s as posited. Note that $h_s(0,0) = (0,0)$. It remains to set $h = h_1$ and restrict to a sufficiently small disk $\mathbb{D}_{\varepsilon_1}$ such that $h(\mathbb{D}_{\varepsilon_1}) \subset \mathbb{D}_{\varepsilon_0}$.

Hence, to finish the proof of the Lemma it remains to show that the form η can be chosen to be a^t invariant. For the sake of notation we prove this fact only when $\dim \mathbb{D}_{\varepsilon_0} = 4$. The general case can be addressed in the same way.

We use coordinates (x_1, x_2, x_3, x_4). Then $\omega_0 = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$ and the generator of a^t is given by

$$X = -x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_4}$$

First let $\eta_0 = x_1 dx_2 \wedge dx_3 \wedge dx_4$. Then $d\eta_0 = \omega_0$ and, using Cartan formula $\mathcal{L}_X\eta_0 = \iota_X\omega_0 + d\iota_X\eta_0$ it is straightforward to verify that $\mathcal{L}_X\eta_0 = 0$, *i.e.,* η_0 is a^t-invariant.

Our goal now is to find an a^t-invariant function β such that $d(\beta\eta_0) = \gamma\omega_0$. We have

$$d(\beta\eta_0) = \beta\eta_0 + d\beta \wedge \eta_0 = \beta\omega_0 + x_1\frac{\partial\beta}{\partial x_1}\omega$$

Hence we need to solve the equation

$$\beta + x_1\frac{\partial\beta}{\partial x_1} = \frac{\partial}{\partial x_1}(x_1\beta) = \gamma$$

for β. Then

$$\beta(x_1, x_2, x_3, x_4) = \frac{1}{x_1}\int_0^{x_1}\gamma(q, x_2, x_3, x_4)dq$$

is a solution.

We check that β is a^t-invariant. Let $\Gamma = \Gamma(x_1, x_2, x_3, x_4) = \int_0^{x_1}\gamma(q, x_2, x_3, x_4)dq$. Because γ is a^t-invariant we have

$$\begin{aligned} 0 = \int_0^{x_1} X\gamma(q, x_2, x_3, x_4)dq &= -\int_0^{x_1} q\frac{\partial}{\partial q}\gamma(q, x_2, x_3, x_4)dq - x_2\frac{\partial}{\partial x_2}\Gamma + x_3\frac{\partial}{\partial x_3}\Gamma + x_4\frac{\partial}{\partial x_4}\Gamma \\ &= -x_1\gamma(x_1, x_2, x_3, x_4) + \Gamma(x_1, x_2, x_3, x_4) - x_2\frac{\partial}{\partial x_2}\Gamma + x_3\frac{\partial}{\partial x_3}\Gamma + x_4\frac{\partial}{\partial x_4}\Gamma = \Gamma + X\Gamma \end{aligned}$$

where we used integration by parts and the fundamental theorem of calculus. Now differentiating $x_1\beta = \Gamma$ with respect to X gives

$$X(x_1)\beta + x_1 X\beta = X\Gamma$$

which yields

$$x_1 X\beta = X\Gamma + x_1\beta = X\Gamma + \Gamma = 0.$$

Hence $X\beta = 0$. Finally by the product formula

$$\mathcal{L}_X\beta\eta_0 = X(\beta)\eta_0 + \beta\mathcal{L}_X\eta_0 = 0.$$

□

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