DOMINATED SPLITTING FROM CONSTANT PERIODIC DATA AND GLOBAL RIGIDITY OF ANOSOV AUTOMORPHISMS

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Abstract. We show that an $\mathrm{SL}(d, \mathbb{R})$ cocycle over a hyperbolic system with constant periodic data has a dominated splitting whenever the periodic data indicates it should. This implies global periodic data rigidity of generic Anosov automorphisms of $\mathbb{T}^d$. Further, our approach also works when the periodic data is narrow, that is, sufficiently close to constant. We can show global periodic data rigidity for certain non-linear Anosov diffeomorphisms in a neighborhood of an irreducible Anosov automorphism with simple spectrum.

1. Introduction

Anosov diffeomorphisms are a well-studied class of examples in the field of dynamical systems. By definition an Anosov diffeomorphism is a diffeomorphism of a Riemannian manifold M such that TM has a continuous splitting into a complementary pair of Df-invariant bundles E^s and E^u, the vectors of which are uniformly contracted or expanded, respectively, by Df. We refer to E^s and E^u as the stable and unstable bundles of f, respectively. There is a simple algebraic construction of Anosov diffeomorphisms: take the action of a matrix $L \in \mathrm{GL}(d, \mathbb{Z})$ that has no eigenvalues of modulus 1 on the torus $\mathbb{R}^d/\mathbb{Z}^d$. Diffeomorphisms thus obtained are called *hyperbolic toral automorphisms* or *Anosov automorphisms*.

Anosov automorphisms have strong rigidity properties. One of the most basic, due to Franks and Manning, is that if f is an Anosov diffeomorphism that is in the same free homotopy class as an Anosov automorphism L, then there is a homeomorphism h such that $hfh^{-1} = L$, see e.g. [KH95, Sec. 2.3]. We call h a *conjugacy.* If h were a C^1 diffeomorphism, then it would imply that for every f-periodic point p of period n the derivative D_pf^n is conjugate as a linear map to L^n with the conjugacy given by D_ph. Consequently, if there is a periodic point where D_pf^n is not conjugate to L^n, then any conjugacy h cannot be a C^1 diffeomorphism. Hence we may view each periodic point p as coming with a possible obstruction to the differentiability of h. If all of these obstructions vanish, then we say that f has the same *periodic data* as L. We say that L is C^∞ *periodic data rigid* if for any C^∞ Anosov diffeomorphism f in the same homotopy class as L, if f has the same periodic data as L, then f is C^∞ conjugate to L. One of our main results is the following:

Theorem 1.1. *Suppose that L is an Anosov automorphism of $\mathbb{T}^3$. Then L is C^∞ periodic data rigid.*

In the case where L automorphism is conformal on its stable and unstable bundles, this result is due to Kalinin and Sadovskaya [KS09].

Date: September 27, 2023.

We remark that Theorem 1.1 is not perturbative: we do not consider only those f that are C^1 close to L. That said, this result builds upon and relies heavily on the large amount of work for the local, perturbative version of the problem, which asks whether any perturbation of an Anosov automorphism that has the same periodic data is C^1 conjugate back to the original automorphism. The main obstacle to proving global versions of existing local results is the construction of a particular type of Df-invariant splitting of the unstable bundle.

The main contribution of this article to the periodic data rigidity program is overcoming the problems that arise when not working perturbatively. We explain what the issue is in the simplest setting where it occurs, namely, in dimension 3. Let $L\colon \mathbb{T}^3 \to \mathbb{T}^3$ be an Anosov automorphism with real eigenvalues of distinct absolute values. If one has an Anosov diffeomorphism on $\mathbb{T}^3$ with a 2-dimensional unstable bundle E^u homotopic to L, then, a priori, there is no reason to expect that E^u has a splitting into two Df-invariant subbundles. However, note that the unstable bundle of L does split into two subbundles corresponding to the two unstable eigenvalues. If f is an Anosov diffeomorphism with the same periodic data as L, then the main remaining difficulty for establishing periodic data rigidity is showing that the unstable bundle of f has a splitting with the same properties as the splitting of the unstable bundle of L.

In this paper, we show show how to construct a Df-invariant splitting of E^u from just the periodic data of Df. The type of splitting of E^u that we seek is called a *dominated splitting*, which we now define. Suppose that $\sigma\colon X \to X$ is a homeomorphism of a compact metric space. If $A\colon X \to \mathrm{GL}(d,\mathbb{R})$ is a continuous map, then we may consider the products of the matrices $A(x)$ along a trajectory of σ. We write $A^n(x) = A(\sigma^{n-1}(x))\cdots A(x)$. We say that a continuous splitting $E \oplus F$ of the bundle $X \times \mathbb{R}^d$ is A-invariant if $A(x)E(x) = E(\sigma(x))$ and $A(x)F(x) = F(\sigma(x))$. A continuous splitting $E\oplus F$ is called a *dominated splitting* if this splitting is A-invariant and there exist constants $C > 0$ and $0 < \tau < 1$ such that for every $x \in X$ and $n \in \mathbb{N}$,

$$\|A^n(x)|F_x\| < C\tau^n m(A^n(x)|E_x),$$

where $m(A) = \|A^{-1}\|^{-1}$ denotes the conorm of the matrix A. We say that the dominated splitting *has index* k if $\dim E = k$. See [BG09] for a useful discussion of this and related notions.

A closely related notion is that of uniform hyperbolicity. We say that that a linear cocycle $A\colon X \to \mathrm{GL}(d,\mathbb{R})$ over a homeomorphism (X,σ) as above is *uniformly hyperbolic* if there exists a non-trivial continuous A-invariant splitting $E^s \oplus E^u$, a continuous metric $\|\cdot\|_x$ on the vector bundle $X \times \mathbb{R}^d$, and $\lambda > 1$ such that for each $x \in X$,

$$\|A(x)|_{E^s}\| < \lambda^{-1} < 1 < \lambda < m(A(x)|_{E^u}). \tag{1}$$

In words, vectors in E^s are uniformly contracted and those in E^u are uniformly expanded. Formally, an *Anosov diffeomorphism* $f\colon M \to M$ is a diffeomorphism of a Riemannian manifold for which the cocycle $Df\colon TM \to TM$ is uniformly hyperbolic. One defines cocycles and dominated splittings completely analogously if the bundle is not a trivial bundle.

As outlined above, the obstacle to making progress in the global rigidity problem for Anosov automorphisms is using the periodic data to construct a dominated splitting. In the local problem, such a splitting always exists for perturbative reasons, see, for example, [DeW21, Thm. 2.3].

Our main technical result is the following, which produces such a splitting. We work in the more abstract setting of cocycles over subshifts of finite type because the arguments in this context are more transparent. We say that a cocycle $A\colon \Sigma \to \mathrm{SL}(d,\mathbb{R})$ over a subshift of finite type Σ has *constant* periodic data if there is a list of numbers $\lambda_1 \geq \cdots \geq \lambda_d$ such that for each periodic point p of period n the moduli of the eigenvalues of $A^n(p)$ are $e^{n\lambda_1}, \dots, e^{n\lambda_d}$. Our main theorem is the following.

Theorem 1.2. *Suppose that Σ is a transitive, invertible, subshift of finite type and that $A\colon \Sigma \to \mathrm{SL}(d,\mathbb{R})$ is a Hölder continuous cocycle with constant periodic data associated to exponents $\lambda_1 \geq \cdots \geq \lambda_d$. If $\lambda_k > \lambda_{k+1}$, then A has a dominated splitting of index k.*

In fact, we are able to prove an even stronger result, which allows us to produce dominated splittings even when the periodic data is not constant, but instead is "narrow," in the sense that it lies sufficiently close to constant periodic data. We formally define δ-narrow periodic data in Definition 2.4, but we remark here that perturbations of Anosov automorphisms have narrow periodic data.

Theorem 1.3. *Suppose that (Σ, σ) is a transitive, invertible, subshift of finite type and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$ with $\lambda_k > \lambda_{k+1}$. For each $\beta \in (0,1)$ there exists $\delta > 0$ such that if A is a β-Hölder $\mathrm{SL}(d,\mathbb{R})$ cocycle with δ-narrow periodic data centered at $(\lambda_1, \dots, \lambda_d)$ then A has a dominated splitting of index k.*

As far as the authors are aware, the actual construction of a dominated splitting from periodic data has not been carried out except in systems satisfying particular bunching conditions. One of the only examples the authors are familiar with is due to Butler [But17] in the context of the rigidity of the geodesic flow on complex hyperbolic space. Butler's argument makes use of the 1:2 ratio between Lyapunov exponents in this setting and involves delicate control of the regularity of the bundles he is considering [But17, Lem. 3.10]. In contrast, as we shall see, for constant periodic data our approach works for bundles that are merely Hölder with no control at all needed on the Hölder exponent. Further results constructing dominated splittings are known with other assumptions. For example, with fiber bunching, an additional constraint on the cocycle requiring $\|A\|/m(A)$ to be small, Velozo Ruiz constructed a dominated splitting for $\mathrm{SL}(2,\mathbb{R})$ cocycles over subshifts of finite type under an assumption which is equivalent to the following: there exists $\varepsilon > 0$ such that for each periodic point p^n of period n, $A^n(p^n)$ has an eigenvalue of modulus at least $e^{\varepsilon n}$ [VR20]. However, as Velozo Ruiz demonstrates in that same paper, that without fiber bunching the same assumption on the eigenvalues of periodic points is insufficient to imply the existence of a dominated splitting. (For a more geometric example in the smooth setting see [Gog10, Thm. 1].) In [CCZ23], those authors obtain that for cocycles that are twisting and admit holonomies, e.g. are fiber bunched, that there is uniform growth of the norm of the cocycle if one assumes that the largest exponent of every periodic point is the same. The construction of dominated splittings is also of interest in the field of Anosov representations. For example, in order to study Anosov representations, Kassel and Potrie construct a dominated splitting for locally constant cocycles that have a uniform gap in their periodic data [KP22].

1.1. **Applications to non-constant periodic data.** Perhaps the most interesting application of our approach is that we are able to prove global periodic data rigidity for systems that are C^1 close to Anosov automorphisms. When we talk about periodic data rigidity of Anosov diffeomorphisms that are not Anosov automorphisms, the problem is stated somewhat differently. In this case, if we have two Anosov diffeomorphisms on a torus in the same free homotopy class, then as before, there is a Hölder conjugacy h between them. If f is not an Anosov automorphism, then the periodic data may vary among periodic points of a given period. Thus the conjugacy h provides a "marking" of the points. Consequently, we say that f and g have the same *periodic data with respect to a conjugacy* h, if for each periodic point p of f of period k the linear maps D_pf^k and $D_{h(p)}g^k$ are conjugate. We say that an Anosov diffeomorphism is $C^{1+\text{Hölder}}$ rigid if whenever another C^2 Anosov diffeomorphism is conjugate to it by a conjugacy h and the diffeomorphism has the same periodic data with respect to the conjugacy, then h is a $C^{1+\text{Hölder}}$ diffeomorphism.

We say that an Anosov automorphism is *irreducible* if the characteristic polynomial of $L \in \mathrm{GL}(d,\mathbb{Z})$ is irreducible over $\mathbb{Z}$. This is a standard and often necessary assumption when studying rigidity in higher dimensions.

We deduce a global version of the main result of the second author's thesis [Gog08, Thm. A].

Theorem 1.4. *Let L be an irreducible hyperbolic automorphism of $\mathbb{T}^d$, $d \geq 3$, with simple Lyapunov spectrum. Then there exists a C^1-neighborhood $\mathcal{U} \subseteq \mathrm{Diff}^2(\mathbb{T}^d)$ of L such that any $f \in \mathcal{U}$ satisfying Property $\mathcal{A}$ and any C^2 Anosov diffeomorphism g with the same periodic data as f are $C^{1+\text{Hölder}}$ conjugate.*

A perturbation of L has the same intermediate-speed foliations that L does and Property $\mathcal{A}$, which conjecturally always holds, asserts that these foliations are transitive. Examples of open sets of diffeomorphisms which satisfy Property $\mathcal{A}$ in dimensions up to 5 were given in [Gog08, Sections 4.2 and 5]. In fact, in dimension 3, Property $\mathcal{A}$ always holds. So, we can actually conclude a stronger result, generalizing [GG08].

Theorem 1.5. *Fix exponents $\lambda_1 > \lambda_2 > 0 > \lambda_3$. Then there exists $\delta > 0$ such that if f is any C^2 Anosov diffeomorphism of $\mathbb{T}^3$ with δ-narrow spectrum centered at $\lambda_1, \lambda_2, \lambda_3$, then f is globally $C^{1+\text{Hölder}}$ periodic data rigid. In particular, this implies that there is a C^1 open neighborhood of any Anosov automorphism of $\mathbb{T}^3$ with simple Lyapunov spectrum such that every member of that neighborhood is periodic data rigid.*

The proof of the above theorems follows from applying our Theorem 1.3 to the approach in Theorem A of [GG08, Gog08]. Indeed, for Theorem 1.4, by choosing a sufficiently small $\mathcal{U}$ we have that f has a narrow dominated splitting, in particular, it has narrow periodic data. By the main assumption, diffeomorphism g also has narrow periodic data and, hence, Theorem 1.3 yields a dominated splitting for g. After this the proof in [Gog08] goes through even when $g \notin \mathcal{U}$.

1.2. **Applications to smooth rigidity of Anosov automorphisms.** The theorem formulated at the beginning of the introduction is a corollary to a more general higher dimensional result which continues the work on higher dimensional toral automorphisms by the second author, Guysinsky, Kalinin and Sadovskaya [GG08, Gog08, KS09, GKS11] and provides a global version of the higher dimensional result of [GKS11].

Corollary 1.6. *Let $L\colon \mathbb{T}^d \to \mathbb{T}^d$ be an irreducible Anosov automorphism such that no three of its eigenvalues have the same modulus. Then L is $C^{1+\text{Hölder}}$ rigid.*

Anosov automorphisms are also known to exist on nilmanifolds, which are compact quotients of a nilmanifold by a lattice. Work of the first author [DeW21] in this setting can be now globalized which we can briefly formulate as follows. The conditions of "irreducibility" and "sorted spectrum" are defined in [DeW21], but we note here that they are necessary for even local rigidity.

Corollary 1.7. *Suppose that $A\colon N/\Gamma \to N/\Gamma$ is an irreducible Anosov automorphism of a nilmanifold N/Γ with simple, sorted Lyapunov spectrum. Then A is globally $C^{1+\text{Hölder}}$ periodic data rigid. In particular, this shows that there exist non-toral nilmanifold Anosov automorphisms in arbitrarily high dimension which are $C^{1+\text{Hölder}}$ rigid.*

1.3. **Relationship to prior work.** Now we provide some commentary on how the current work fits into the rigidity program. Until very recently, the strongest local periodic data rigidity result available in higher dimensions was [GKS11], which requires that the Anosov automorphism has no more than three eigenvalues with the same modulus, as well as be irreducible. This builds on earlier work such as [GG08, Gog08], which studies the case where each eigenvalue of the automorphism has a distinct modulus. Those results produced a conjugacy that is $C^{1+\text{Hölder}}$, though not necessarily one that is C^∞. There is another line of work that shows that one may "bootstrap" a C^1 conjugacy to a C^∞ conjugacy. One of results in this direction is [Gog17], which takes advantage of strong Diophantime properties of linear invariant foliations and bootstraps a C^1 conjugacy between an Anosov automorphism of $\mathbb{T}^3$ and its C^1 small perturbation all the way to C^∞. Recently Kalinin, Sadovskaya, and Wang showed that it is possible to bootstrap a C^1 conjugacy between an Anosov automorphism and a C^k small perturbation all the way to a C^∞ conjugacy [KSW23]. Due to the KAM scheme used in that paper, k may be very large, however. Even more recently Zhenqi Wang announced that local rigidity holds for all toral automorphisms with irreducible characteristic polynomial in any dimension without any spectral restrictions. Similarly to [Gog17], this result also strongly utilizes Diophantine properties of invariant foliations and, similarly to [KSW23], requires that the perturbation be small in a high regularity norm in order to run a KAM scheme.

Some global rigidity results were already known due to work of Kalinin and Sadovskaya [KS09], which will be explained during the proof of Theorem 1.1. These results apply to hyperbolic toral automorphisms in dimension 3 that have a conjugate pair of complex eigenvalues, as well as hyperbolic toral automorphisms in dimension 4 for which the eigenvalues of the automorphism have only two distinct moduli. In each case, these assumptions are essentially that the eigenvalues of the automorphism have exactly two distinct moduli: one contracting, one expanding. There are certain other earlier results due to de la Llave that imply global rigidity under the assumption that the entire stable and unstable bundles of a non-linear Anosov diffeomorphism be conformal [dlL02]. The only dimension in which unrestricted, non-perturbative C^∞ periodic data rigidity of toral automorphisms was previously shown is dimension 2, which goes back to the late 1980s and is due to work of de la Llave, Marco, and Moriyón [dlL92, MM87, dlL87].

We also mention that recently the second author and F. Rodriguez Hertz developed a new matching functions approach to rigidity of Anosov diffeomorphisms which disregards the dominated splitting structure, even when it is present, and yields global results [GRH]. This approach only applies to highly non-linear (very non-algebraic) Anosov diffeomorphisms with one dimensional stable foliation. Still there is some overlap between our Theorem 1.5 and [GRH, Theorem 1.5]. Namely, under certain further spectral assumptions on $L\colon \mathbb{T}^3 \to \mathbb{T}^3$ the method of [GRH] gives C^∞ periodic data rigidity for diffeomorphisms in an open and dense subset of a neighborhood of L, but not necessarily all diffeomorphisms in an open neighborhood of L as is obtained in Theorem 1.5.

1.4. **Remarks on the proof.** We now say a couple words about the ideas behind the proof of Theorem 1.2. In order to construct a dominated splitting, we use a characterization of domination that we may verify for each orbit individually. We will verify this criterion by using a shadowing argument. The characterization of uniform hyperbolicity is due to Yoccoz [Yoc04] in dimension two and was extended by Bochi and Gourmelon [BG09] to a characterization of domination in higher dimensions.

Theorem 1.8. [BG09, Thm. A] *Suppose that $f\colon X \to X$ is a homeomorphism of a compact metric space and that $\mathcal{E}$ is a continuous vector bundle over X. For a continuous cocycle $\mathcal{A}\colon \mathcal{E} \to \mathcal{E}$, the following are equivalent:*

(1) *$\mathcal{A}$ has a dominated splitting of index k,*
(2) *There exist $C > 0$ and $\tau < 1$ such that $\frac{\sigma_{k+1}(\mathcal{A}(n,x))}{\sigma_k(x)} < C\tau^n$ for all $x \in X$ and $n \geq 0$.*

This is essentially the statement that the singular values of $\mathcal{A}(n,x)$ separate exponentially fast. The proof of Theorem 1.2 will proceed by using a shadowing argument to check the criterion in Theorem 1.8. We will shadow an arbitrary trajectory by a periodic trajectory and then compare the two trajectories for a small amount of time near the point where they are closest. Due to the tight constraints on the periodic data, this yields usable estimates on the growth of singular values along that arbitrary trajectory.

Acknowledgements: The authors are grateful to Daniel Mitsutani for many helpful discussions. The authors are also grateful to Dmitry Dolgopyat and Amie Wilkinson for helpful comments on the manuscript. The first author was supported by the National Science Foundation under Award No. DMS-2202967. The second author was partially supported by NSF award DMS-2247747.

After the paper was finished, the authors learned from Jairo Bochi during a visit to Penn State that Misha Guysinsky announced a similar result some years ago for $\mathrm{SL}(2,\mathbb{R})$ cocycles [Sad13, Thm. 1.2]. We are grateful to Misha Guysinsky for subsequent discussion, which revealed that he has some different techniques that yield similar results and may be used to address a related question of Velozo Ruiz [VR20] about cocycles that are close to fiber bunched [Guy23].

2. Preliminaries

In this section we review some standard definitions and set notation that will be used through the rest of the paper.

2.1. **Subshifts of finite type.** We now give some definitions concerning subshifts of finite type. For a natural number m, we let Q be an $m \times m$ matrix with entries $\{0,1\}$. We may consider the space of all bi-infinite strings of symbols $\{1,\ldots,m\}^{\mathbb{Z}}$. Then we restrict to the subspace only allowing the substrings admitted by Q. For $\omega \in \{1,\ldots,m\}^{\mathbb{Z}}$, we write its elements as $(\ldots,\omega_{-1},\omega_0,\omega_1,\ldots)$. A *subshift of finite type* or SFT is the subspace:

$$\Sigma = \{\omega \in \{1,\ldots,m\}^{\mathbb{Z}} : Q_{\omega_i\omega_{i+1}} = 1\}.$$

The dynamics on this subspace is the left shift σ, defined by

$$(\sigma(\omega))_i = \omega_{i+1}.$$

We require that for every pair (i,j) there is some ℓ_{ij} with $Q_{ij}^{l_{ij}} > 0$. This is equivalent to (Σ,σ) being transitive.

There is also a natural metric on Σ. Write

$$N(\omega,\eta) = \min\{|n| : \omega_n \neq \eta_n\},$$

then for $\omega \neq \eta$ we define the metric:

$$d(\omega,\eta) = e^{-N(\omega,\eta)}.$$

Remark 2.1. The most important property of the shift that we will use is that there exists some uniform ℓ such that if $\omega = (\omega_0,\ldots,\omega_n)$ is any valid finite string, then there exists a finite string η of length at most ℓ such that $\omega\eta$ is valid, and, if repeated cyclically, defines a valid periodic word in Σ. This follows because transitivity of the SFT implies that there is a finite word going from any symbol to any other symbol.

In this paper, we consider cocycles over SFTs. In this context, a cocycle is defined by a map $A\colon \Sigma \to \mathrm{GL}(d,\mathbb{R})$. We always consider the case when this map is Hölder continuous. In a natural way, the map A defines a linear cocycle on the trivial bundle $\mathcal{A}\colon \Sigma\times\mathbb{R}^d \to \Sigma\times\mathbb{R}^d$. Calling both of these a "cocycle" is a standard abuse of terminology. One of the main properties that we use is summarized in the following remark.

Remark 2.2. If A is a β-Hölder cocycle, then there exists C such that if $\omega,\eta \in \Sigma$, and then if for all $|i| \le n$, $\omega_i = \eta_i$, then

$$\|A(\omega) - A(\eta)\| < Ce^{-\beta n},$$

which follows from the definition of Hölder continuity because $d(\omega,\eta) \le e^{-n}$.

Below, we will also consider more general cocycles that are not trivial. Suppose that (X,d) is a compact metric space and that $\mathcal{E}$ is a vector bundle over X. If $\sigma\colon X \to X$ is a homeomorphism of X, the we say that a *linear cocycle* over σ is a vector bundle map $\mathcal{A}\colon \mathcal{E} \to \mathcal{E}$ that fibers over σ. We write

$$\mathcal{A}(n,x)\colon \mathcal{E}_x \to \mathcal{E}_{\sigma^n(x)},$$

for the associated linear map induced by the cocycle. We will also need to consider "Hölder continuous" vector bundles. Before we formulate this, we note that for the applications in this paper, we only ever need to consider trivial bundles.

While one could formulate the theory in extreme generality, one can also formulate the theory in a more basic way. For a trivial bundle $X \times \mathbb{R}^d$, it is easy to see what a Hölder section of this bundle is. Hence it is natural to define a *Hölder* *vector bundle* over a metric space X to be a vector bundle that embeds as a Hölder subbundle of $X \times \mathbb{R}^d$. More generally, one can define a Hölder bundle using Hölder transition functions, although in this case it is more subtle to determine precisely the meaning of the bundle being β-Hölder because the composition of β-Hölder functions need not be β-Hölder. For a detailed discussion of Hölder vector bundles over metric spaces see [BG19, Sec. 2.2].

2.2. **Periodic data.** For a cocycle $\mathcal{A}$ over a homeomorphism $\sigma\colon X \to X$ on a vector bundle $\mathcal{E}$, one may consider the return map $\mathcal{A}(n,p)$ over any periodic point of period n. Typically, when one speaks of the "periodic data" of this cocycle, one is referring to the map that associates each periodic point p of period n with the conjugacy class of the matrix $\mathcal{A}(n,p)$ in $\mathrm{GL}(d,\mathbb{R})$. In our case, we consider a slightly more general notion.

Definition 2.3. *Given a list of real numbers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$, and we say that a cocycle $\mathcal{A}$ has periodic data with* constant exponents *of type $(\lambda_1, \dots, \lambda_d)$, if the following holds for each periodic point p^n of period n:*

If we order the eigenvalues α_i of $\mathcal{A}(n,p^n)$ according to their moduli and write them according to their algebraic multiplicity (so that there are d of them), then for each $1 \leq i \leq d$ we have $|\alpha_i| = e^{n\lambda_i}$.

Note that this condition is more general than the condition of having all periodic data conjugate to the power of a single fixed matrix even in the case where the eigenspaces are 1-dimensional.

We will also consider periodic data that is tightly clustered around particular values.

Definition 2.4. *Given a list of numbers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$ and $\delta > 0$, we say that the periodic data of a cocycle is δ*-narrow *if the following holds. For each periodic point p^n of period n, if we order the eigenvalues α_i of $\mathcal{A}(n,p^n)$ as above, then for each i we have*

$$e^{n(\lambda_i - \delta)} \leq |\alpha_i| \leq e^{n(\lambda_i + \delta)}.$$

In particular, note that a cocycle with constant exponents is a cocycle that is 0-narrow. This condition is similar to the condition of having "narrow band spectrum" that appears in normal forms theory. See, for example, [Kal20]. Unlike in the normal forms theory however, we do not have any constraints due to resonances between the different λ_i.

2.3. **Singular values.** Recall that the singular values of a map $A\colon \mathbb{R}^d \to \mathbb{R}^d$ are the eigenvalues of the positive square-root of A^*A. We write the singular values of a linear operator $A\colon \mathbb{R}^d \to \mathbb{R}^d$ as

$$\sigma_1 \geq \cdots \geq \sigma_d.$$

If $A\colon \mathbb{R}^d \to \mathbb{R}^d$ is a linear map, then we denote by A_k the induced map $A_k\colon \Lambda^k\mathbb{R}^d \to \Lambda^k\mathbb{R}^d$. A helpful fact that we use below is that

$$\|A_k\| = \sigma_1(A) \cdots \sigma_k(A). \tag{2}$$

We will use the following standard estimate on the perturbation of singular values.

Proposition 2.5. [Ste79] *Fix $d \geq 1$ then there exists $C > 0$ such that if $A\colon \mathbb{R}^d \to \mathbb{R}^d$ is a linear map and $\sigma_1 \geq \cdots \geq \sigma_d$ are the singular values of A, then if $E\colon \mathbb{R}^d \to$*

$\mathbb{R}^d$ *is a perturbation and we write the singular values of* $A+E$ *as* $\sigma_1' \geq \cdots \geq \sigma_d'$, *then for* $1 \leq i \leq d$,

$$|\sigma_i - \sigma_i'| \leq C\|E\|.$$

2.4. **Foliations.** In this paper we use the standard terminology for foliations. We say that a topological foliation of a smooth manifold M has *uniformly* C^r *leaves* if the leaves of the foliation are C^r and the r-jet of the foliation varies continuously. The same definition applies analogously for foliations that have uniformly $C^{1+\text{Hölder}}$ leaves. For more details, see [PSW97]. If we have two foliations $\mathcal{F}$ and $\mathcal{G}$ of a manifold M, then we say that a homeomorphism h *intertwines* $\mathcal{F}$ and $\mathcal{G}$ if it carries each leaf of $\mathcal{F}$ homeomorphically onto a leaf of $\mathcal{G}$. In other words, it is an isomorphism of topologically foliated manifolds.

2.5. **Anosov diffeomorphisms.** We say that a diffeomorphism $f\colon M \to M$ is Anosov if the tangent TM has a hyperbolic Df-invariant splitting. In this paper, we consider even finer splittings of TM. For an Anosov diffeomorphism f, we write its finest dominated splitting as:

$$E_l^{s,f} \oplus \cdots \oplus E_1^{s,f} \oplus E_1^{u,f} \oplus \cdots \oplus E_k^{u,f},$$

where the strength of the hyperbolicity increases from left to right. For an Anosov automorphism L, this finest dominated splitting has terms corresponding to each modulus of each eigenvalue of L. Under a C^1-small perturbation this splitting will persist, see [DeW21, Sec. 2.1.1], so that every Anosov diffeomorphism in a C^1 neighborhood of L will also have a splitting into the same number of bundles and these bundles will be uniformly Hölder. In this case, the bundles $E_i^{u,f}$ each integrate to a foliation. This is because due to the perturbative theory of Hirsch-Pugh-Shub [HPS77], the weak bundle bundle $E_i^{wu,f} = E_1^{u,f} \oplus \cdots \oplus E_i^{u,f}$ integrates to a uniformly $C^{1+\text{Hölder}}$ foliation, which we denote by $\mathcal{W}_i^{wu,f}$. Further, as every strong distribution integrates, $E_i^{u,f} \oplus \cdots \oplus E_k^{u,f}$ also integrates to a foliation. We call this foliation $\mathcal{W}_i^{uu,f}$. By intersecting these two foliations, one obtains that $E_i^{u,f}$ integrates to a foliation with uniformly $C^{1+\text{Hölder}}$ leaves which we denote by $\mathcal{W}_i^{u,f}$. Later we will see that periodic data assumption is enough to conclude that these foliations exist even for maps that are not perturbations of linear maps.

In order to state our results in higher dimension, we need to define the Property $\mathcal{A}$ that appears in [Gog08, p. 647]. Given a set B and a foliation $\mathcal{F}$, we define:

$$\mathcal{F}(B) = \bigcup_{x \in B} \mathcal{F}(x). \tag{3}$$

We say that a foliation is *transitive* if it has a dense leaf. We say that a foliation is *minimal* if every leaf is dense. We say that a foliation $\mathcal{F}$ of a manifold M is *tubularly minimal* if for each open set B, $\overline{\mathcal{F}(B)} = M$. It is shown by Gogolev [Gog08, Prop. 4] that transitivity and tubular minimality for foliations of compact manifolds are equivalent. Consequently, the following definition is equivalent to the definition of Property $\mathcal{A}$ in [GG08].

Definition 2.6. *Let* $\mathcal{U}$ *be a sufficiently* C^1 *small neighborhood of an irreducible Anosov automorphism* $L\colon \mathbb{T}^d \to \mathbb{T}^d$ *such that each* $f \in \mathcal{U}$ *the finest dominated splitting of* L *persists. Then we say that an Anosov diffeomorphism* f *has* Property $\mathcal{A}$ *for any* $1 \leq i \leq l-1$ *and* $1 \leq j \leq k-1$ *the foliations* $\mathcal{W}_i^{s,f}$ *and* $\mathcal{W}_j^{u,f}$ *are transitive.*

While it is not known how ubiquitous Property $\mathcal{A}$ is, there are many diffeomorphisms that satisfy it. For instance, any irreducible hyperbolic toral automorphism satisfies Property $\mathcal{A}$ [Gog08, Prop. 6].

3. Proof of the existence of a splitting

In this section we prove Theorem 1.2. We will use the following proposition, which is due to Kalinin [Kal11, Thm. 1.3].

Proposition 3.1. *Suppose that A is an $\mathrm{SL}(d,\mathbb{R})$ cocycle over a transitive, invertible, subshift of finite type that has constant periodic data with exponents $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$. Then for all $\varepsilon > 0$ there exists C_ε such that for all $\omega \in \Sigma, n \in \mathbb{N}$,*

$$\|A^n(x)\| \leq C_\varepsilon e^{n(\lambda_1+\varepsilon)}.$$

The first step in the proof of Theorem 1.2 is the following proposition. Note that the argument for the following proposition does not work if it is merely the case that the Lyapunov exponents of every measure are bounded away from zero rather than being close to constant.

Proposition 3.2. *Suppose that A is a Hölder continuous $\mathrm{SL}(d,\mathbb{R})$ cocycle with constant periodic data over a transitive shift with exponents*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{d-1} \geq \lambda_d.$$

For every sufficiently small $\gamma > 0$ and $\varepsilon > 0$, there exists $C > 0$ such that if p is a periodic point of period n and $\gamma n \leq i$, then

$$\|A^i(p)\| \geq C e^{\gamma(\lambda_1-\varepsilon)n}.$$

Proof. Suppose that p is a periodic point of period n. Then with i as in the statement of the proposition we may write:

$$A^n(p) = A^{n-i}(\sigma^i(p))A^i(p). \tag{4}$$

Because the periodic data are constant we have that $\|A^n(p)\| \geq e^{\lambda_1 n}$. Because $i \geq \gamma n$, we also have from Proposition 3.1 that for all $\varepsilon > 0$ there exists C_ε such that

$$\|A^{n-i}(\sigma^i(p))\| \leq C_\varepsilon e^{(1-\gamma)(\lambda_1+\varepsilon)n}.$$

Thus applying these estimates to (4) we see that:

$$e^{\lambda_1 n} \leq \|A^i(p)\| C_\varepsilon e^{(1-\gamma)(\lambda_1+\varepsilon)n}.$$

Thus

$$\|A^i(p)\| \geq C_\varepsilon^{-1} e^{n\lambda_1-(1-\gamma)(\lambda_1+\varepsilon)n} \tag{5}$$

$$\geq C_\varepsilon^{-1} e^{(\gamma\lambda_1-(1-\gamma)\varepsilon)n}. \tag{6}$$

Up to this point, we have no relationship between ε and γ, hence replacing ε with $\gamma\varepsilon/(1-\gamma)$, we obtain

$$\|A^i(p)\| \geq C_\varepsilon e^{\gamma(\lambda_1-\varepsilon)n},$$

as desired. □

Next, by considering the action of the cocycle on exterior powers, we are able to use the previous proposition to gain information about other singular values.

Proposition 3.3. *Suppose that A is a Hölder continuous* $\mathrm{SL}(d,\mathbb{R})$ *cocycle with constant periodic data over a transitive shift with exponents*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{d-1} \geq \lambda_d.$$

Then for every sufficiently small $\gamma, \varepsilon > 0$ and a fixed $\ell \in \mathbb{N}$, there exists $C_{\gamma,\varepsilon} > 0$ such that if p is a periodic point of period n and $\gamma n \leq i \leq \gamma n + \ell$, then

$$C_{\gamma,\varepsilon}^{-1} e^{\gamma n(\lambda_k - \varepsilon)} \leq \sigma_k(A^i(p)) \leq C_{\gamma,\varepsilon} e^{\gamma n(\lambda_k+\varepsilon)} \tag{7}$$

and hence in particular:

$$\sigma_k(A^i(p)) \geq C_{\gamma,\varepsilon} e^{\gamma n(\lambda_k - \lambda_{k+1} - \varepsilon)} \sigma_{k+1}(A^i(p)).$$

Proof. We proceed by induction on k. The base case when $k = 1$ is immediate from Proposition 3.1 and Proposition 3.2. Now suppose that the result holds for $k-1$, then we show it holds for k.

We first check the upper bound in equation (7). We apply Proposition 3.2 to the cocycle A_{k-1} induced by A on $\Lambda^{k-1}\mathbb{R}^d$. Hence for whatever sufficiently small $\gamma, \varepsilon > 0$ we choose there exists $C_{\gamma,\varepsilon} > 0$ such that for any periodic point p of period n and any $i \geq \gamma n$, $i \leq \gamma n + \ell$, we have that

$$\|A_{k-1}^i(p)\| \geq C_{\gamma,\varepsilon} e^{\gamma n(\lambda_1 + \cdots + \lambda_{k-1} - \varepsilon)}. \tag{8}$$

At the same time, by Proposition 3.1, we have that for all $i \geq \gamma n$

$$\|A_k^i(p)\| \leq C_\varepsilon e^{i(\lambda_1 + \cdots + \lambda_k + \varepsilon)}. \tag{9}$$

In particular, for our fixed finite $\ell > 0$ because $|\gamma n - i| < \ell$, there exists $C'_\varepsilon > 0$ such that for any $\gamma n \leq i \leq \gamma n + \ell$, we have

$$\|A_k^i(p)\| \leq C'_\varepsilon e^{\gamma n(\lambda_1 + \cdots + \lambda_k + \varepsilon)}. \tag{10}$$

Recalling now that the norm of a linear map on the k-th exterior power is the product of the first k singular values, we see that by dividing equation (10) by equation (8), we see that there exists $C > 0$ such that

$$\sigma_k(A^i(p)) \leq C e^{\gamma n(\lambda_k + 2\varepsilon)}.$$

We now check the lower bound in Equation (7). First, we apply Proposition 3.4 to get that

$$\sigma_1 \cdots \sigma_k = \|A_k^i(p)\| \geq e^{\gamma n(\lambda_1 + \cdots + \lambda_k - \varepsilon)}. \tag{11}$$

But by the inductive hypothesis, we know that for $1 \leq j \leq k-1$ that

$$\sigma_j(A^i(p)) \leq C e^{\gamma n(\lambda_j + \varepsilon)}. \tag{12}$$

Thus we obtain from equation (11), that for all $i \geq \gamma n$

$$\sigma_k(A^i(p)) \geq C e^{\gamma n(\lambda_k - (k-1)\varepsilon)}.$$

This gives the second required bound and hence we have finished the induction. Since this is a finite induction we can redefine the initial ε so that all posited inequalities of Proposition 3.3 are satisfied. $\square$

We now use a shadowing argument to upgrade the above estimate for periodic points to an estimate along every trajectory.

Proposition 3.4. *Suppose that A is a β-Hölder* $\mathrm{SL}(d,\mathbb{R})$ *cocycle with constant periodic data over a transitive subshift of finite type Σ. Suppose the periodic data has associated exponents*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{d-1} \geq \lambda_d.$$

If $\lambda_k > \lambda_{k+1}$, then there exist $c, \tau > 0$ such that for every $\omega \in \Sigma, n \in \mathbb{N}$,

$$\sigma_k(A^n(\omega)) \geq ce^{\tau n}\sigma_{k+1}(A^n(\omega)).$$

Proof. Let $\omega \in \Sigma$. By Remark 2.1, we then form a sequence of shadowing orbits p^n of period $2n + O(1)$ that we define by setting $p_i^n = \omega_i$ for $|i| \leq n$ adding a uniformly bounded size padding so that we may then repeat the sequence of symbols periodically. Fix some $0 < \gamma < 1$. The constants appearing in the following estimates are now uniform independent of ω, γ and n, and depend only on the cocycle A. The periodic orbit p^n shadows ω hence if we only consider iterates of p^n when p^n is closest to ω, $A(p^n)$ and $A(\omega)$ will be very close: this is what will make the argument work.

We now consider iterating the cocycle γn times starting from ω. Then we may write:

$$\prod_{i=0}^{\gamma n} A(\sigma^i(\omega)) = \prod_{i=0}^{\gamma n}\left[A(\sigma^i(p^n)) + O(e^{-\beta(1-\gamma)n})\right], \tag{13}$$

where, as in Remark 2.2, the big O term hides the Hölder constant of A and is due to p^n exponentially shadowing ω on a trajectory of length n.

We begin by estimating the difference between the two sides of equation (13). For any fixed $\kappa > 0$ that there exists $C_\kappa > 0$ such that $\binom{n}{k} \leq C_\kappa e^{nk\kappa}$.[1] Aside from the main term in the product, there are $\gamma n - 1$ groups of terms which we number from $1 \leq k \leq \gamma n$, where k is the number of copies of the term $O(e^{-\beta(1-\gamma)n})$ appearing. Each of these groups contains $\binom{\gamma n}{k}$ members, each being of size at most $e^{\eta(\gamma n - k)}e^{-\beta(1-\gamma)nk}$. Thus by the binomial formula, the sum of all the terms involving one of the error terms is:

$$\begin{aligned}\left\|\prod_{i=0}^{\gamma n}\left[A(\sigma^i(p^n)) + O(e^{-\beta(1-\gamma)n})\right] - \prod_{i=0}^{\gamma n} A(\sigma^i(p^n))\right\| &= \sum_{k=1}^{\gamma n} e^{\eta(\gamma n-k)}e^{-\beta(1-\gamma)nk}C_\kappa e^{\kappa nk}\\ &= C_\kappa e^{\eta\gamma n}\sum_{k=1}^{n} e^{-\eta k+(\kappa-\beta(1-\gamma)nk}.\end{aligned}$$

Hence as this series is decaying exponentially fast it is dominated by its first term, hence the sum of all the error terms is a matrix of size at most:

$$C'_\kappa e^{(\eta\gamma+\kappa-\beta(1-\gamma))n}. \tag{14}$$

There exists a choice, only depending on β and η, of $\gamma, \kappa > 0$ sufficiently small, such that the bound (14) is at most $O(e^{-\delta n})$ for some small $\delta > 0$. Hence, for this

[1] To see this, use the standard estimate that $\binom{n}{k} \leq n^k/k!$. Then to see that $n^k/k! \leq C_\kappa \exp(\kappa nk)$ take logarithms. Thus we are checking when $k\ln(n) - \ln(k!) \leq \kappa nk + \ln(C_\kappa)$. As long as $\kappa n > \ln(n)$ this holds with $\ln(C_\kappa) = 0$. Hence we need only consider n less than some fixed N_κ. But once n is fixed, the $\ln(k!)$ grows the fastest, so there are only finitely many remaining exceptions, hence the conclusion holds.

choice of γ we have

$$\prod_{i=0}^{\gamma n} A(\sigma^i(\omega)) = O(e^{-\delta n}) + \prod_{i=0}^{\gamma n} A(\sigma^i(p^n)).$$

But for fixed γ, by Proposition 3.3, there exist c, τ such that if $\sigma_k(n)$ and $\sigma_{k+1}(n)$ are the k and $(k+1)-$th singular values of the second term then $\sigma_k(n) \geq ce^{\gamma n \tau}\sigma_{k+1}(n)$. Thus by applying Proposition 2.5, we have found that, independent of ω, there exist $c_1, c_2, \delta, \gamma, \tau > 0$ such that for all n,

$$\sigma_k(A^{\gamma n}(\omega)) \geq \left\| \prod_{i=0}^{\gamma n} A(\sigma^i(\omega)) \right\| \geq c_1 e^{\gamma n \tau} - c_2 e^{-\delta n}.$$

From this the conclusion of the proposition is immediate. □

We can now deduce the main theorem of this paper.

Proof of Theorem 1.2. From Proposition 3.4, the cocycle satisfies the criterion of Bochi–Gourmelon (Theorem 1.8), and hence has a dominated splitting, so we are done. □

3.1. **Narrow periodic data.** In this section, we show that our result that produces dominated splittings works even when the periodic data is only close to being constant. For sufficiently narrow periodic data, one can still obtain a dominated splitting.

We will not give a full proof of this result as it follows along the same lines as the proof of Theorem 1.2. Instead we sketch the proof in the case of $\mathrm{SL}(2,\mathbb{R})$ cocycles, which will show how one may obtain an explicit estimate on δ, which we find interesting for comparison with other situations where narrow band spectrum appears.

In fact, we study a more precise assumption on the spectrum than we have mentioned previously. We will assume that that for the $\mathrm{SL}(2,\mathbb{R})$ cocycle A that for each periodic point the upper exponent lies in the interval $(\lambda_1 - \delta_-, \lambda_1 + \delta_+)$. We will now consider what constraints on δ_-, δ_+ and γ are required for the argument to go through. There are two that appear in the course of the proof.

The first is the constraint showing that periodic orbits are hyperbolic on γ-proportion of the orbit. Just as we showed in Proposition 3.2, we may obtain an estimate so that for all $\varepsilon, \gamma > 0$ there exist $C, \ell > 0$ such that for any $\gamma n \leq i \leq \gamma n + \ell$ and any periodic point p of period n,

$$\|A^i(p)\| \geq C_{\gamma,\varepsilon} e^{((\lambda_1 - \delta_-) - (1-\gamma)(\lambda_1 + \delta_+ + \varepsilon))n}.$$

Hence to proceed with the shadowing part of the proof and establish that there is norm growth along a given trajectory, we must have that

$$\gamma\lambda_1 + \gamma\delta_+ - \delta_- - \delta_+ > 0. \tag{15}$$

The other constraint on γ and δ_-, δ_+ appears in shadowing part of the argument (Proposition 3.4). We must choose $0 < \gamma, \kappa < 1$ so that in equation (14), we have

$$\eta\gamma + \kappa - \beta(1-\gamma) < 0. \tag{16}$$

As η was any number satisfying $\|A\| \leq e^{\eta}$, we could have instead picked a much better number by using work of Kalinin [Kal11, Thm. 1.3] to choose a metric that is adapted to the upper bound on the periodic data, i.e. we can arrange that

$\eta < \lambda_1 + \delta_+ + \varepsilon_1$ for any $\varepsilon_1 > 0$. Thus we see that in order for γ to satisfy this inequality, we must have that:

$$\gamma < \frac{\beta - \kappa}{\beta + \kappa + \lambda_1 + \delta_+ + \varepsilon_1}. \tag{17}$$

In particular, as the inequalities are strict and ε_1 and κ may be arbitrarily small, for equations (15) and (17) to be satisfied simultaneously, we need that $0 < \gamma < 1$ satisfies:

$$\frac{\delta_- + \delta_+}{\lambda_1 + \delta_+} < \gamma < \frac{\beta}{\beta + \lambda_1 + \delta_+}. \tag{18}$$

It is certainly possible to choose such a γ when $\delta_- + \delta_+ > 0$ is sufficiently small. Interestingly, however, we see is that when λ_1 is large, that we may take $\delta_- + \delta_+$ to be relatively large: in fact, we may take their sum to be arbitrarily close to β. It is useful to note that the case where λ_1 is close to 0 is not particularly interesting because these cocycles are actually fiber bunched and were already analyzed by the work of Velozo Ruiz [VR20]. Note that by following along the lines above one may similarly compute constraints on the narrowness of the periodic data in higher dimensions once the values of various constants are fixed.

It is interesting to relate the prior discussion to a question due to Katok that has been studied by Travis Fisher and the second author [Fis06, Gog10]. One way to formulate it is as follows.

Question 3.5. *Let L be an Anosov automorphism with only two Lyapunov exponents $-\lambda < \lambda$. Assume that f is a volume preserving diffeomorphism that is Hölder conjugate to L. Is f also Anosov?*

Hence the question is asking for existence of a dominated splitting for Df, which can naturally be considered as a Hölder continuous cocycle over f, which is a uniformly hyperbolic homeomorphism. The Hölder continuity of the conjugacy and its inverse is essentially equivalent to a narrowness condition on the periodic data. In particular, by studying the rate at which points approach a periodic point, we can deduce that if h is C^θ and h^{-1} is C^ω, then the positive exponent of the periodic points of Df lies in $(\lambda - \delta_-, \lambda + \delta_+)$ for some δ_-, δ_+ satisfying

$$\theta = \frac{\lambda - \delta_-}{\lambda}, \quad \omega = \frac{\lambda}{\lambda + \delta_+}. \tag{19}$$

In [Gog10, Thm. 1], examples of diffeomorphisms conjugate to Anosov diffeomorphisms but are not Anosov are constructed. In fact, quantitative bounds are obtained: for each $\varepsilon > 0$, there exists a diffeomorphism f Hölder conjugate to Anosov such that $\theta\omega = 1/8 - \varepsilon$, Df does not have a dominated splitting, and f is not Anosov. Necessarily, such an f cannot have narrow spectrum. On the other hand Fisher's result [Fis06, Thm. 5.1], [Gog10, Thm. 2], says that if $\theta\omega > 1/2$ then f is Anosov.

As regularity of the conjugacy implies a particular narrow spectrum condition, it is interesting to see how regular the conjugacy needs to be for us to be able to conclude that Df has a hyperbolic splitting by pulling Df back to a cocycle over L and using our theorem. Using (19), we may solve for δ_+ and δ_- in terms of θ, ω and λ. Using this information, we may rewrite the constraint in equation (18) in terms of θ and ω. There is one minor complication: the precise inequality in (18) depends on the strength of the hyperbolicity of the base transformation. So, we now explain what this equation should be when the base is an Anosov automorphism of $\mathbb{T}^2$ with exponents $\pm\lambda$. The left hand side is the same as before because it does not depend on the strength of the hyperbolicity of the base dynamics. The right hand side, however, takes a slightly different form because it does depend on the strength of the hyperbolicity of the base dynamics. In this case, the strength of the shadowing has an extra factor of λ, so instead of $O(e^{-\beta(1-\gamma)n})$ in equation (13) we get $O(e^{-\lambda\beta(1-\gamma)n})$ (see Definition 3.7). Thus we find that the analog of equation (16) should be

$$\eta\gamma + \kappa - \lambda\beta(1-\gamma) < 0.$$

Thus in order to conclude there is a dominated splitting we need to find γ satisfying

$$\frac{\delta_- + \delta_+}{\lambda_1 + \delta_+} < \gamma < \frac{\lambda\beta}{\lambda + \delta_+ + \lambda\beta}.$$

As we have pulled back the cocycle Df by h to be a cocycle over L, $\beta = \theta$. Thus simplifying using θ, ω, we obtain the constraint

$$1 - \theta\omega < \gamma < \frac{\theta\omega}{\theta\omega + 1}. \tag{20}$$

Thus there is a dominated splitting as long as $\omega\theta$ is larger than the positive root of $x^2 + x - 1$, which is $(\sqrt{5}-1)/2 > .618$. This is a weaker conclusion than Fisher's $\omega\theta > 1/2$. We note, however, that our approach is through general results on cocycles while Fisher's proof is very geometric and relies on Mañé's characterization of hyperbolicity, so it's not surprising that our result is a bit weaker.

The advantage of our approach is that it can also be applied to recovering dominated splittings.

Corollary 3.6. *If an Anosov automorphism L has a fine dominated splitting and a $C^{1+Hölder}$ diffeomorphism f is Hölder conjugate to L with θ and ω sufficiently close to 1, then f is also Anosov and has a fine dominated splitting.*

3.2. **Cocycles over general hyperbolic systems.** In the case of cocycles over general hyperbolic systems the same results hold. In principle, there are two ways to obtain such results. One way is to code the system by using the machinery of Markov partitions and then formally deduce the result in the general setting from the symbolic one. There will be some quantitative loss due to passing through a Hölder coding map. Since the argument only relies on shadowing one may also redo the entire argument presented above. In our view, this is a cleaner approach. As the argument is so similar, we omit it but we now describe the setting. The only advantage that symbolic setting has provided is expositional as shadowing takes a particularly simple form for SFTs. The properties we require in the general setting are the same as those considered in [Kal11] because the results of that paper are used in the proof. The first is essentially the definition of exponential shadowing.

Definition 3.7. *We say that two orbit segments $x, f(x), \dots f^n(x)$ and $p, f(p), \dots f^n(p)$ are* exponentially δ-close with exponent $\lambda > 0$ *if for every $i = 0, \dots, n$, we have*

$$d(f^i(x), f^i(p)) \le \delta \exp(-\lambda \min\{i, n-i\}). \tag{21}$$

Definition 3.8. *(Closing Property) We say that a homeomorphism $f\colon (X,d) \to (X,d)$ of a metric space satisfies the* closing property *if there exist $c, \lambda, \ell > 0$ such that for any $x \in X$ and $n > 0$ there exists $0 \le j \le \ell$ and a point $p \in X$ such that $f^{n+j}(p) = p$ and the orbit segments $x, f(x), \dots, f^n(x)$ and $p, f(p), \dots, f^n(p)$*

are exponentially $\delta = cd(x, f^n(x))$ close with exponent λ. Further, we assume that there exists a point $y \in X$ such that for every $i = 0, \ldots, n$

$$d(f^i(p), f^i(y)) \leq \delta e^{-\lambda i} \text{ and } d(f^i(y), f^i(x)) \leq \delta e^{-\delta(n-i)}.$$

It is well-known that locally maximal hyperbolic sets satisfy these properties. As a special case, the above apply to Anosov diffeomorphisms. See for instance [KH95, Ch. 18].

Using these definitions, we formulate a more general version of our main technical result which should be useful for future applications. We remark, however, that this result is not needed for the applications in this paper because the unstable and stable bundles of an Anosov diffeomorphism of a torus are always trivial.

Theorem 3.9. *Suppose that $f\colon X \to X$ is a homeomorphism of a compact metric space satisfying the closing property. Let $(\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d$ be a non-increasing list of real numbers with $\lambda_k > \lambda_{k+1}$ for some k, and let $\mathcal{E}$ be a Hölder vector bundle over X. Then for any uniform bound on the Hölder exponent of a Hölder linear cocycle $\mathcal{A}\colon \mathcal{E} \to \mathcal{E}$ there exists $\delta > 0$ such that any cocycle that satisfies those regularity bounds and has δ-narrow periodic data has a dominated splitting of index k.*

4. Application to Anosov Diffeomorphisms

From the main technical result we can now deduce our periodic data rigidity theorems. We remark that this section is mostly an explanation of how to combine our Theorem 1.2 with existing results. That said there are other novelties such as Observation 4.4, which turns local C^1 results into global C^1 results.

4.1. **Rigidity of non-linear Anosov diffeomorphisms.** We now turn to the proof of our non-linear rigidity theorem, Theorem 1.4. We organize the proof into steps with Lemma 4.1 being the main novel step. The remaining steps are already present in [GG08, Gog08]. Nonetheless, we discuss them to show how they fit together. To begin, we must recover the dominated splitting.

Lemma 4.1. *Suppose that L is an Anosov automorphism of $\mathbb{T}^d$. Let $\varepsilon > 0$ be a sufficiently small number, then there is a C^1 neighborhood $\mathcal{U}$ of L such that if $f \in \mathcal{U}$, and g is any C^2 Anosov diffeomorphism with the same periodic data as f, then $T\mathbb{T}^d$ has a continuous Dg-invariant splitting into bundles $E_i^{u,g}$, $E_j^{s,g}$ corresponding to the bundles of L. Further, there is a Hölder metric on $E^{u,g}$ such that if $\lambda_1 \geq \cdots \geq \lambda_m > 0$ are the unstable Lyapunov exponents of L, for $v \in E_i^{u,g}$, we have*

$$e^{\lambda_i - \varepsilon}\|v\| \leq \|Dgv\| \leq e^{\lambda_i + \varepsilon}\|v\|. \tag{22}$$

The analogous statement holds for $E^{s,g}$ as well.

Proof. Fix a small $\delta' > 0$ and let $\mathcal{U}'$ be a δ'-small C^1 neighborhood of L. Let $f \in \mathcal{U}'$ and assume that g is an Anosov diffeomorphism with the same periodic data as f.

Lemma 4.2. *Then any diffeomorphism g with the same periodic data as f has the Hölder exponent of its stable and unstable bundles, and these constants depend only on L and δ' and hence is uniform in g.*

Proof. As is standard (see [DeW21, Thm 2.3]), for all $f \in \mathcal{U}'$, being δ'-close to L, must satisfy the same exponential bounds

$$C^{-1}\mu_1^n \leq \|D^s f^n\| \leq C\mu_2^n, \quad C^{-1}\nu_1^n \leq \|D^u f^{-n}\| \leq C\nu_2^n, \quad n \geq 1,$$

where $\mu_i \in (0,1)$ and $\nu_i \in (0,1)$ only depend on L and δ'. Since f and g have the same periodic data we have the same bounds for g over all periodic points $p = g^n p$:

$$C^{-1}\mu_1^n \leq \|D_p^s g^n\| \leq C\mu_2^n, \quad C^{-1}\nu_1^n \leq \|D_p^u g^{-n}\| \leq C\nu_2^n.$$

Now we can apply [Kal11, Theorem 1.3] to conclude that all g also satisfy exponential bounds for all points, and uniformly in g. Namely for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$C_\varepsilon^{-1}(\mu_1 - \varepsilon)n \leq \|D^s g^n\| \leq C_\varepsilon(\mu_2 + \varepsilon)^n, \quad C_\varepsilon^{-1}(\nu_1 - \varepsilon)^n \leq \|D^u g^{-n}\| \leq C_\varepsilon(\nu_2 + \varepsilon)^n.$$

It is well known that the unstable subbundle of such g is Hölder with a uniform constant and uniform Hölder exponent given by

$$\min\left\{1, \frac{\log(\mu_2 + \varepsilon) + \log(\nu_2 + \varepsilon)}{\log(\mu_1 - \varepsilon)}\right\}.$$

This Hölder property of subbundles is due to Anosov and the explicit expression for the exponent was given by Hirsch and Pugh [HP70]. □

Now we note that the unstable subbundle of g is trivial because g is conjugate to L [DeW22, Prop. 35]. Hence the unstable differential $D^u g$ defines a cocycle over g which is Hölder uniformly in g. We note that the above lower bound on the Hölder exponent could only improve if δ' is chosen to be even smaller. We are in the position to apply Theorem 1.3 to conclude that there exists a δ such that if the periodic data of g is δ-narrow than g admits finest dominated splitting. Recall that $\delta > 0$ depends only on spectrum of L and on (the lower bound on) the Hölder exponent of $D^u g$.

So now we can choose an even smaller C^1 neighborhood $\mathcal{U}$ of L such that periodic data of $f \in \mathcal{U}$ (and hence of g as well) is δ-narrow. Then Theorem 1.3 indeed applies to such g and we obtain a dominated splitting for g which matches the dominated splitting for f. □

Next, we will use the following lemma to obtain that weak foliations exist even globally.

Lemma 4.3. *Let L be an Anosov automorphism of $\mathbb{T}^d$. Then there exists a C^1 open neighborhood of L in $\mathrm{Diff}^{1+}(\mathbb{T}^d)$, such that for any $f \in \mathcal{U}$, the following holds. If g is any C^2 Anosov diffeomorphism with the same periodic data as f, then $E^{u,g}$ has a $D^u g$ invariant splitting into bundles $E_i^{u,g}$ corresponding to the bundles of L. Further, the weak flag of bundles $E_1^{u,g} \oplus \cdots \oplus E_i^{u,g}$ is uniquely integrable to a weak foliation with uniformly $C^{1+Hölder}$ leaves, which we denote by $\mathcal{W}_i^{wu,g}$. Further, h intertwines the $\mathcal{W}_i^{wu,g}$ and $\mathcal{W}_i^{wu,f}$ foliations.*

Proof. The argument follows along the same lines as in [DeW21], which works globally as well as locally. The point is that if γ is a curve tangent to $E_i^{wu,f}$, then points in γ separate at most at a particular exponential rate under the iteration of f due to the estimate in equation (22). As the conjugacy h intertwines the unstable foliations of f and g, we may then compare how fast points within a fixed unstable leaf separate as we iterate the maps f and g. When we look in the universal cover, we see that they are separating at a particular exponential rate. Hence as restricted to unstable leaves h is a quasi-isometry and carries $\mathcal{W}_i^{uu,f}$ to $\mathcal{W}_i^{uu,L}$, this implies that $h^{-1}(\gamma)$ lies within a single leaf of $\mathcal{W}_i^{wu,f}$ and hence the curves tangent to $E_i^{wu,g}$

lie tangent to a single topological foliation, which is intertwined by h with $\mathcal{W}_i^{wu,f}$. We refer to [DeW21, Section 4] for complete details of this argument. $\square$

We are now ready for the proof of Theorem 1.4. The main technical point in concluding is ensuring that strong unstable foliations are carried to strong unstable foliations by the conjugacy.

Proof of Theorem 1.4. At this point from Lemmas 4.1 and 4.3, we see that there is a C^1 neighborhood of $L, \mathcal{U}$, in $\text{Diff}^2(\mathbb{T}^d)$ such that all of the structures and estimates described in those lemmas hold. Let $f \in \mathcal{U}$ and g be a C^2 Anosov diffeomorphism with the same periodic data as f. Then we must show that f and g are $C^{1+\text{Hölder}}$ conjugate. We already have from Lemma 4.3 that the weak flags of both f and g exist and are intertwined by the conjugacy.

We now use the periodic data assumption to show that the strong unstable foliations are also intertwined under the conjugacy. This step is an induction argument and carries over from [Gog08] without any changes. It relies on Property $\mathcal{A}$ and the orientability of f-invariant foliations. This argument at its core studies the holonomies of a strong unstable foliation between leaves of a weak unstable foliation. One deduces that these holonomies are necessarily isometric with respect to the affine parameters on the $\mathcal{W}_i^{u,f}$ leaves. From this, one obtains a contradiction if h does not carry the strong foliation of g to the strong foliation of f. In the process of induction one also obtains that the conjugacy is $C^{1+\text{Hölder}}$ along all one-dimensional expanding foliations $\mathcal{W}_i^{u,f}$.

The last step concludes global regularity of the conjugacy and its inverse. This step is standard and is an inductive application of the Journé's Lemma [Jou88]. First, Journé's Lemma needs to be applied inductively along the weak flag to obtain $C^{1+\text{Hölder}}$ smoothness of the conjugacy and its inverse along the unstable foliation. Then the whole argument has to be repeated to obtain $C^{1+\text{Hölder}}$ smoothness along the stable foliation. Finally, one more application of Journé's Lemma finishes the proof. $\square$

The proof of Theorem 1.5 follows along similar lines after a preparatory step dealing with the periodic data.

Proof of Thm. 1.5. This follows from [GG08, Thm. 2], which says that if two Anosov diffeomorphisms of $\mathbb{T}^3$ have the same periodic data, and each have a hyperbolic splitting of their 2-dimensional unstable bundle, then the two are $C^{1+\text{Hölder}}$ conjugate. Thus the theorem will follow from Lemma 4.1 and its sublemma 4.2, which gives uniform Hölder control on the stable and unstable bundles from just the periodic data. To conclude, one just needs to choose δ small enough that the splitting of such a diffeomorphism is uniformly Hölder, and then take δ even smaller so that having δ-narrow spectrum around $\lambda_1 > \lambda_2 > 0 > \lambda_3$ implies that there is a splitting of E^u as required by [GG08, Thm. 2]. $\square$

4.2. **Rigidity of Anosov automorphisms.** In the case of Anosov automorphisms, we can weaken the restriction on the dimension of the Lyapunov subspaces of the automorphisms. In addition, we can also obtain higher regularity of the conjugacy.

We begin with showing global C^∞ rigidity of Anosov automorphisms of $\mathbb{T}^3$. We now record the following observation.

Observation 4.4. If f is C^1 conjugate to L by a C^1 conjugacy h, then it is C^∞ conjugate to a map $\widetilde{f}$ that is C^1 close to L. To obtain this, conjugate f by $\widetilde{h}$, a C^∞ diffeomorphism that is C^1 close to h, so that f is C^∞ conjugated to $\widetilde{h}f\widetilde{h}^{-1}$, which is a C^∞ Anosov diffeomorphism that is C^1 close to L.

We can now use the above proposition to conclude our main theorem.

Proof of Theorem 1.1. It suffices to consider the case that L has a two dimensional unstable bundle. Then either L is conformal on its unstable bundle or its unstable bundle has hyperbolic splitting. We analyze each case separately. Let f denote an Anosov diffeomorphism with the same periodic data as L and h a conjugacy between f and L.

There are then two cases:

(1) If the unstable bundle is conformal then we may conclude by work of Kalinin and Sadovskaya. Let f denote the other Anosov diffeomorphism with the same periodic data as L. Then the restriction of Df to the unstable bundle of f preserves a conformal structure due to [KS10, Thm. 1.3]. Then as a conjugacy between f and L intertwines the stable and unstable foliations of these maps. By applying the argument in [GKS11, Prop. 2.3], this implies that h is $C^{1+\text{Hölder}}$ along the stable and unstable foliations of f and hence h is $C^{1+\text{Hölder}}$ by Journé's lemma [Jou88]. The result [KS09, Cor. 2.5] implies immediately that a C^1 conjugacy between L and a C^1 close Anosov diffeomorphism is C^∞. Thus we may now conclude by Observation 4.4.

(2) If the unstable bundle isn't conformal, then Theorem 1.5 implies that h is $C^{1+\text{Hölder}}$. To upgrade the conjugacy to C^∞ can be done using the main result of [Gog17], which says that a C^∞ Anosov diffeomorphism that is C^1 close to L and is C^1 conjugate to L is C^∞ conjugate to L. By applying Observation 4.4 to the result of the second author, we are done.

Having completed the analysis of the two cases, we may now conclude. □

We now turn to the proof of Corollary 1.6. We will not give a full proof as it follows the same lines as the proof of Theorem 1.4.

Sketch of proof of Cor. 1.6. The main difference from the setting of Theorem 1.4, is that in this setting, the intermediate foliations $\mathcal{W}_i^{u,L}$ may be two dimensional. That these foliations exist and have the desired properties follows from the same development in [DeW21] that we have mentioned before, which is indifferent to the dimensions of the leaves of the foliation. However, the argument for showing that strong foliations intertwine with strong foliations is slightly different, but follows the same lines as the argument in [GKS11, Prop. 2.3], which also deals with two dimensional foliations. The part of the argument that shows that the strong foliation is carried to the strong foliation follows by studying the holonomies of the strong foliation $\mathcal{W}_{i+1}^{uu,f}$ between leaves of the $\mathcal{W}_i^{u,f}$ foliation. By using that the differential of $Df|_{E_i^{u,f}}$ and $DL|_{E_i^{u,L}}$ can both be shown to be conformal, one may deduce that the holonomy is isometric. This argument does not use any global properties of f and hence may be applied in this case as well. □

We do not sketch the proof of Corollary 1.7 as the proof is *mutatis mutandis* the same as the one in [DeW21] and is quite similar to the two we have already described.

5. Remaining Questions

In view of the aforementioned results. Let us summarize some of the remaining questions in the study of periodic data rigidity of hyperbolic toral automorphisms. As these questions are central in the rigidity of hyperbolic systems and are the motivation for much research, we hope others in the field, and particularly newcomers, will find these precise statements useful. The first question is perhaps the most fundamental.

Question 5.1. *Suppose that $L \in \mathrm{GL}(d,\mathbb{Z})$ is an irreducible Anosov automorphism. If f is a C^1 small C^2 perturbation of L with the same periodic data as L are f and L $C^{1+Hölder}$ conjugate?*

As mentioned above, our results show that a positive answer to the above question should provide a positive answer to the global question as well. As was mentioned in the introduction, Zhenqi Wang has recently announced that the answer to this question is "Yes" if the perturbation is C^∞ small.

The next main question asks whether conjugacies are typically smooth. The following is referred to as the "bootstrapping problem."

Question 5.2. *Suppose that L is an irreducible Anosov automorphism and that f is a C^∞ Anosov diffeomorphism that is C^1 conjugate to L by a conjugacy h. Then is h C^∞?*

The result of [KSW23] shows that the answer is "Yes" when f is sufficiently C^∞ close to L.

In view of the disparity of regularity between what seems possible and what is known, it is interesting to ask whether all these results work for merely C^1 small perturbations or even globally. In fact, our Theorem 1.1 shows that in dimension 3 that perturbative restrictions are not necessary. However, it is entirely possible that in higher dimensions the situation is much more subtle. For example, the rigidity results of [KSW23] apply even to Anosov automorphisms that are *not* periodic data rigid, such as those containing Jordan blocks.

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