

A COUNTEREXAMPLE TO THE SHUB ENTROPY CONJECTURE

ABSTRACT. This is an expository note. For every finite $r \geq 1$, we construct a non-invertible C^r self-map of $S^1 \times \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$ with zero topological entropy and eigenvalue 2 for f_* on the second homology group. Every orbit either converges to a fixed point or is asymptotic to an orbit of period 20. The idea is to start with $z \mapsto z^2$ on the Riemann sphere (the first $\mathbb{S}^2$ factor) and then destroy the entropy of the doubling map on the equator by pushing all points either to the north pole or to the south pole in a non-uniform fashion via a computer program like algorithm. This algorithm is using the “clock” variable $t \in S^1$ which plans how to push z to the north and auxiliary “memory and amplification” variables $(w_1, w_2) \in \mathbb{S}^2 \times \mathbb{S}^2$ which execute this push. If the clock “stalls” then the push never happens and the orbits sink to the south. This algorithm can be realized as a single explicit C^r map. The low regularity of the map is due to roughness of perturbations which are used to set the pushing algorithm in motion. One can replace $\mathbb{S}^2 \times \mathbb{S}^2$ by any 4-manifold as auxiliary variables have local nature. With some extra care one can reduce the dimension of auxiliary variable to 3 and obtain a similar map on $S^1 \times \mathbb{S}^2 \times \mathbb{S}^3$.

The example was constructed by undisclosed model by OpenAI [O26] and released on the historic slop-drop day, October 6, 2026. I claim no credit for the construction, I used GPT Sol 6.1 to simplify and help me understand OpenAI proof. It is completely elementary and could be understood by anyone with basic calculus knowledge.

Andrey Gogolev.

1. INTRODUCTION

Michael Shub asked whether

$$h_{\text{top}}(f) \geq \log \rho(f_*)$$

for every C^1 self-map of a closed smooth manifold, where f_* is the induced action on homology [S74, S06]. Celebrated work of Yosef Yomdin establishes this conjecture for C^∞ maps [Y87]. Instead of surveying the history of the problem let me refer the reader to the beautiful expository survey by Anatole Katok [K86] “A Conjecture about Entropy”, which captures perfectly the state of the art pre-Yomdin. Yosef

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Yomdin recently passed away and didn't see his old work confirmed to be optimal. I take this opportunity to translate an excerpt from his reminiscences [Y26]:

“The most remarkable thing about the Max Planck Institute was that it was undoubtedly a place of pilgrimage for all the world's best mathematicians. I spoke with M. Gromov there several times, and these conversations helped me greatly in my work. A. Katok also visited once. I told him about my work, and he explained several problems in dynamics that he thought I might try to approach. He particularly emphasized the Entropy Conjecture, which I began to think about more and more from then on. Three years later, by then at IHES, I managed to solve this problem.”

From now on let's fix the target regularity $r \geq 1$ and proceed to construct the posited C^r map $f: M \rightarrow M$, where $M = S^1 \times \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$. We will use (t, z, w_1, w_2) as coordinates on M and view $t \in S^1 = \mathbb{R}/20\mathbb{Z}$ as the clock coordinate. We identify $\mathbb{S}^2$ with the extended complex plane $\widehat{\mathbb{C}}$ via the standard stereographic projection, so that $|z| = 1$ is the equator, where the entropy is concentrated. We denote by N for the north ∞ pole and by S the south pole $0 \in \mathbb{C}$. We also do the same identification for remaining $\mathbb{S}^2$ factors and view $w_1, w_2 \in \mathbb{C}$, these are auxiliary coordinates which will always settle at a fixed point (and to avoid confusion we won't call 0 the south pole when auxiliary coordinates are considered).

Once the map is defined, it will be clear that it is homotopic to $(t, z, w_1, w_2) \mapsto (t, z^2, 0, 0)$ hence $\rho(f_*) = 2$.

2. ASYMPTOTIC BEHAVIOUR AND VANISHING ENTROPY

We will choose a sufficiently large number $R > 2$ (4.1) and construct f so that every orbit either converges to a point of A_S or is asymptotic to a periodic orbit in A_N , where

$$A_N = S^1 \times \{N\} \times \{0\} \times \{0\}$$

and

$$\begin{aligned} A_S = & \{(10, S, 0, 0)\} \\ & \cup \{(10, S, w_1, 0) : |w_1| = 2R\} \\ & \cup \{(10, S, 0, w_2) : |w_2| = 2R\} \\ & \cup \{(10, S, w_1, w_2) : |w_1| = |w_2| = 2R\}. \end{aligned}$$

We call these the north and south attractors. The north attractor A_N will be a periodic circle, with $f(t, N, 0, 0) = (t + 1, N, 0, 0)$. The south attractor A_S will be fixed pointwise and consists of one isolated point, two circles, and a torus. It is not an attractor in the usual sense — points arbitrarily close to A_S can escape to A_N .

It already implies vanishing entropy. Every f -invariant probability measure μ is supported on $A_S \cup A_N$. Otherwise Poincaré recurrence gives a recurrent point $x \in \text{supp } \mu \setminus (A_S \cup A_N)$. The variational principle [M76] therefore gives

$$h_{\text{top}}(f) = \sup_{\mu} h_{\mu}(f) = 0.$$

Hence our goal is to show that for any $x \in M$ we have $f^n(x) \rightarrow A_S \cup A_N$, $n \rightarrow \infty$.

3. CLOCKWORK

The clock t -coordinate will always increase on $S^1 = \mathbb{R}/20\mathbb{Z}$ unless $t = 10$. For $|z| \geq 3/4$, the clock dynamics on S^1 is uniform rotation $t \mapsto t + 1$. For $|z| \leq 1/2$, it is $t \mapsto t + b(t)$, where we choose a smooth function $b : S^1 \rightarrow [0, 1]$ satisfying the following conditions,

$$(3.1) \quad \begin{aligned} b(t) = 0 &\iff t = 10, & b(t) = 1 &\text{ if } |t - 10| \geq 2, \\ t + b(t) &\leq 10 & (8 \leq t \leq 10). \end{aligned}$$

Here we identify S^1 with the interval $[0, 20]$. The last condition means that 10 is a semi-stable fixed point and the clock $t \mapsto t + b(t)$ will “stall” at 10.

Between these two regions we interpolate smoothly:

$$h(t, z) = b(t) + (1 - b(t))e(z),$$

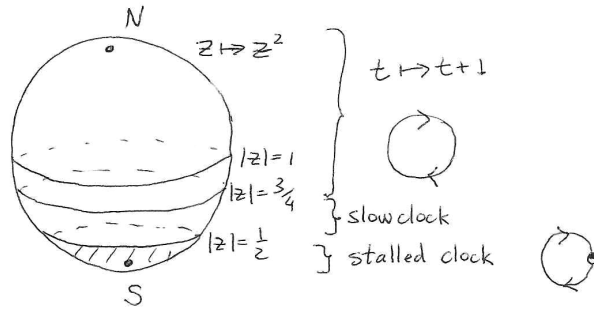


FIGURE 1

where $e: \mathbb{S}^2 \rightarrow [0, 1]$ is a smooth radial function which vanishes for $|z| \leq 1/2$ and equals one for $|z| \geq 3/4$. The dynamics on the first coordinate is then defined by

$$t_{n+1} = t_n + h(t_n, z_n) \pmod{20}.$$

Thus the clock advances by one outside $(8, 12)$, and also whenever the main coordinate lies on or outside the circle of radius $3/4$. It can slow down only near 10 when $|z| < 3/4$ and could stall at 10 if $|z| < 1/2$.

We use $\bar{t}_n$ for a real lift of the clock coordinate t_n . That is, if $\bar{b}: \mathbb{R} \rightarrow [0, 1]$ be the 20-periodic lift of b , then

$$(3.2) \quad \begin{aligned} \bar{h}(t, z) &= \bar{b}(t) + (1 - \bar{b}(t))e(z), \\ \bar{t}_{n+1} &= \bar{t}_n + \bar{h}(\bar{t}_n, z_n). \end{aligned}$$

Every orbit has one of the following two behaviors.

Stalled clock. The sequence $\bar{t}_n$ is bounded above. In this case $\bar{t}_n \rightarrow \bar{t}_* = 20k + 10$ for some $k \in \mathbb{Z}$. We will see that in this case orbits converge to A_S .

Ticking clock. We have $\bar{t}_n \rightarrow +\infty$, so the clock keeps going around the circle. A full circle takes at least 20 iterates and could take much longer due to a slow down at $10 \pmod{20}$. Now we roughly describe the purpose of different time intervals on the clock which are used to interact with the auxiliary variables (w_1, w_2) to execute the push to N after which the orbit converges to the periodic circle A_N .

- (1) *Reset:* $[0, 3]$. Both auxiliary coordinates are set to zero, clearing the “memory.”
- (2) *Preparation:* $[4, 7]$. Tiny “signals” s_i are added to the auxiliary variables w_i , $i = 1, 2$. Each signal records the direction of the square of the z -coordinate at its predicted future push time. The signal is scaled down according to the predicted delay before activation. The signal function is merely C^r due to its sensitive dependence on the initial point near the stalled clock regime.
- (3) *Amplification:* $[4, 15]$. Amplification of the signals occur in the auxiliary variables through the possibly long wait near 10. It increases a small signal without changing its direction. If the clock stalls, the signal is never used to correct the z -coordinate.
- (4) *Activation and Correction:* $[15, 18]$. Each auxiliary variable is read at most once and used to push the main variable z to N . The stored correction in the auxiliary variables has same direction as the square of z , so it increases the absolute value of z . On the subinterval $[16, 17]$ at least one auxiliary variable is read and is strong enough to push z into the basin of N .

- (5) *Dynamics in the basin of A_N .* After the push the z -coordinate tends to the north pole, the clock advances by one at every step, and no new signals are prepared. The next reset sets both auxiliary variables permanently to 0, and the orbit becomes asymptotic to the periodic circle A_N .

4. THE SIGNAL PREPARATION

Here we explain the construction of the signals s_i sent to the auxiliary variables during preparation. To determine their direction and magnitude, we first follow the dynamics without corrections to the main coordinate up to the predicted activation time.

We start with the initial dynamics $G: \mathbb{R} \times \mathbb{S}^2 \rightarrow \mathbb{R} \times \mathbb{S}^2$ given by

$$G(t, z) = (t + \bar{h}(t, z), z^2), \quad G^m(t, z) = (T_m(t, z), z_m).$$

Choose R so that

$$(4.1) \quad R > \|DG\|^r.$$

We write $\|\cdot\|$ for the uniform norm.

Remark 4.1. We note that $\|DG\| \geq 2$, so $R > 2^r \geq 2$. If we use the product of the standard metric on $\mathbb{R}$ and a standard round metric on $\mathbb{S}^2$ then one can arrange $\|DG\| < 4$.

Choose smooth $\beta_1, \beta_2: S^1 \rightarrow [0, 1]$ and intervals J_1, J_2 such that

$$\begin{aligned} \text{supp } \beta_i &\subset J_i \subset (15, 18), & |J_i| < 1, & i = 1, 2, \\ \max\{\beta_1(t), \beta_2(t)\} &= 1, & 16 \leq t \leq 17. \end{aligned}$$

These functions will be responsible for activating the auxiliary variables and providing the push toward N . Also define the non-periodic functions $\beta_i^0: \mathbb{R} \rightarrow [0, 1]$ by

$$\beta_i^0(t) = \begin{cases} \beta_i(t \bmod 20), & 0 < t < 20, \\ 0, & t \notin (0, 20). \end{cases}$$

Choose a smooth function $\alpha: \mathbb{R} \rightarrow [0, 1]$ such that

$$\text{supp } \alpha \subset (4, 7), \quad \alpha = 1 \quad \text{on } [5, 6].$$

Choose a smooth cut-off $\chi: [0, \infty) \rightarrow [0, 1]$ such that

$$\chi(|z|) = 1 \quad (1/2 \leq |z| \leq 2), \quad \chi(|z|) = 0 \quad (|z| \leq 1/3 \text{ or } |z| \geq 3).$$

Since $|J_i| < 1$, for each fixed (t, z) , there is at most one integer $m \geq 1$ with $\beta_i^0(T_m(t, z)) > 0$. We define the *signal functions* $s_i: S^1 \times \mathbb{S}^2 \rightarrow \mathbb{C}$

using the representative $0 \leq t < 20$ of the clock coordinate.

$$s_i(t, z) = \begin{cases} \alpha(t)\chi(|z|)R^{1-m}\beta_i^0(T_m(t, z))\left(\frac{z}{|z|}\right)^{2^{m+1}}, & \text{if } m \text{ exists,} \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently, since at most one summand is non-zero at each point,

$$(4.2) \quad s_i(t, z) = \alpha(t)\chi(|z|) \sum_{m \geq 1} R^{1-m}\beta_i^0(T_m(t, z))\left(\frac{z}{|z|}\right)^{2^{m+1}}.$$

This series representation will be used to prove C^r regularity. The non-periodic cut-off β_i^0 ensures that only the predicted visit in the current clock turn contributes to the signal.

5. THE COUNTEREXAMPLE

Choose a smooth $p : [0, \infty) \rightarrow [0, 2]$ satisfying

$$\begin{aligned} p(u) &= u \quad (0 \leq u \leq 1), & p(u) &= 2 \quad (2 \leq u \leq 3R), \\ p &\text{ is increasing on } [1, 2], & p(u) &= 0 \quad (u \geq 4R). \end{aligned}$$

For $w \in \mathbb{C} \setminus \{0\}$ let

$$P(w) = p(|w|)\frac{w}{|w|}, \quad P(\infty) = P(0) = 0.$$

Thus we have a smooth map $P : \mathbb{S}^2 \rightarrow \mathbb{C} \subset \mathbb{S}^2$ with $|P| \leq 2$, and $P(w) = w$ on the closed unit disk. We will use

$$(5.1) \quad p(u) \geq \min(u, 1), \quad 0 \leq u \leq 3R.$$

Choose a smooth $\gamma : S^1 \rightarrow [0, R]$ such that

$$\gamma = 0 \quad \text{on } [0, 3] \cup [19, 20], \quad \gamma = R \quad \text{on } [4, 18].$$

This function is responsible for setting the auxiliary variables to zero after each clock turn and for amplification of the signals on $[4, 18]$.

Define the *correction function*

$$C(t, w_1, w_2) = 2\beta_1(t)P(w_1) + 2\beta_2(t)P(w_2), \quad |C| \leq 8.$$

Our map $f : (t, z, w_1, w_2) \mapsto (t', z', w'_1, w'_2)$ is finally defined by

$$(5.2) \quad \boxed{\begin{aligned} t' &= t + h(t, z), \\ z' &= z^2 + C(t, w_1, w_2), \\ w'_i &= \gamma(t)P(w_i) + s_i(t, z), \quad i = 1, 2. \end{aligned}}$$

At $z = N$, set $z' = N$. Note that $|w'_i| \leq 2R + 1 < 3R$. Thus every auxiliary variable lies in the disk after the first iteration: $|w| < 3R$.

A visit to $[0, 3]$ resets both auxiliary coordinates to 0. Signals are added during $(4, 7)$, and corrections to z occur only during $(15, 18)$.

6. REGULARITY

The main component of (5.2) is smooth at N . In the local chart $\zeta = 1/z$, it is given by

$$\zeta' = \frac{\zeta^2}{1 + C(t, w_1, w_2)\zeta^2}.$$

Since $|C| \leq 8$, the denominator is non-zero for $|\zeta| < 1/4$.

Hence all ingredients in the definition of f are smooth except the signal functions s_i , $i = 1, 2$.

The chain rule gives

$$(6.1) \quad \|D^j G^m\| \leq C_j \|DG\|^{jm}, \quad j \geq 1.$$

This is standard. Indeed, let $K = \|DG\| \geq 2$. For $j = 1$, the chain rule expresses $D(G^m)$ as a composition of m derivatives of G . Thus the estimate holds with $C_1 = 1$.

Now induct on j . Assume the estimate holds for all derivative orders below j , and let $H_m = \|D^j G^m\|$. Differentiating $G^{m+1} = G \circ G^m$ j times gives one term $DG(G^m)D^j G^m$, whose norm is at most KH_m . Each remaining term contains derivatives of G^m of orders below j whose orders add up to j . The induction hypothesis bounds their product by a constant times K^{jm} . Consequently,

$$H_{m+1} \leq KH_m + B_j K^{jm}.$$

Here B_j depends on j , the derivative bounds of G up to order j , and the previously obtained constants $C_1, \dots, C_{j-1}$, but is independent of m . Choosing C_j sufficiently large and inducting on m therefore proves (6.1). Since T_m is the first coordinate of G^m , it satisfies the same derivative bounds. By the chain rule, these bounds also control the derivatives of $\beta_i^0 \circ T_m$ appearing in the signals s_i .

Recall the series representation (4.2), which we can write as

$$s_i(t, z) = \sum_{m \geq 1} s_{i,m}(t, z),$$

$$s_{i,m}(t, z) = \alpha(t) \chi(|z|) R^{1-m} \beta_i^0(T_m(t, z)) \left(\frac{z}{|z|} \right)^{2^{m+1}}.$$

The summands are smooth. We will show that their C^r norms are summable.

Recall that χ is supported on $1/3 \leq |z| \leq 3$. Repeated application of the product rule gives for $k \geq 0$,

$$\left\| \chi(|z|) \left(\frac{z}{|z|} \right)^k \right\|_{C^j} \leq c_j (1+k)^j,$$

where c_j is independent of k . Indeed, view the power as a product of k copies of $z/|z|$. Applying the product rule j times produces at most $(1+k)^j$ terms. Each term contains at most j differentiated factors, whose derivatives are bounded, while the undifferentiated direction factors have absolute value one.

Applying the product rule distributes at most r differentiations among the factors defining $s_{i,m}$. Combining the preceding estimate with (6.1) therefore gives

$$\|s_{i,m}\|_{C^r} \leq M_r R^{1-m} \|DG\|^{rm}.$$

These norms are summable by the choice of R (4.1). The series therefore converges in C^r on $[0, 20] \times \mathbb{S}^2$. The summands vanish near both poles and near the endpoints of this interval, so s_1, s_2 are C^r on $S^1 \times \mathbb{S}^2$.

All remaining terms other than s_i are smooth. Hence f is C^r .

7. ASYMPTOTIC BEHAVIOR

Here we will use the notation $(t_n, z_n, w_{1,n}, w_{2,n})$ for an orbit of f . We also use the clock lift $\bar{t}_n \in \mathbb{R}$ defined by (3.2).

7.1. Stalled clock. If $\bar{t}_n \rightarrow \bar{t}_* = 10 + 20k$ then eventually $C = 0$. Also $|z_n| < 3/4$ for all sufficiently large n , since otherwise $\bar{h}(\bar{t}_n, z_n) = 1$. In this region $z_{n+1} = z_n^2$, so $z_n \rightarrow S$.

Also, eventually $\gamma(t_n) = R$ and $s_1 = s_2 = 0$, and each auxiliary variable follows $w_i \mapsto RP(w_i)$. According to the definition of P each w_i is either 0 or will become constant with absolute value $2R$. Hence orbits with bounded $\bar{t}_n$ converge to a fixed point of f .

7.2. Ticking clock. First observe that $|C| \leq 8$ gives

$$|z'| \geq |z|^2 - 8 \geq 2|z|$$

whenever $|z| \geq 4$. Thus an orbit entering the region $|z| \geq 4$ stays there and tends to N . Here $h = 1$ and $s_1 = s_2 = 0$, so the clock advances by one at every step and a later reset makes both auxiliary variables zero permanently. Also $\bar{t}_{n+1} = \bar{t}_n + 1$ and our orbit is asymptotic to an orbit of exact period 20 in A_N .

It remains to show that if the clock doesn't stall then the orbit must enter the region $|z| \geq 4$. We argue by contradiction and suppose that $\bar{t}_n \rightarrow +\infty$ and $|z_n| < 4$ for every n . Choose a complete clock turn,

which we identify with $[0, 20)$. Visits to $[0, 3]$ resets both auxiliary variables before preparation.

Let I be the set of preparation indices, namely those with $\alpha(\bar{t}_k) > 0$, in the chosen turn. Recall that α is supported inside $(4, 7)$ and equals to 1 on $[5, 6]$, so I has at most three indices. Let k_0 be the first index where α is positive, that is, $k_0 = \min I$.

For every $k \in I$ we claim that

$$1/2 < |z_k| < 2.$$

Indeed, the region $4 \leq t \leq 10$, $|z| \leq 1/2$ is forward invariant: there $C = 0$, $e = 0$, and (3.1), prevents the clock from passing 10. Entering this region would contradict $\bar{t}_n \rightarrow \infty$. On the other hand, $|z_k| \geq 2$ would give $|z_{k+1}| = |z_k|^2 \geq 4$, since no correction is applied during preparation. Hence $\chi(|z_k|) = 1$ for every $k \in I$.

For every $n \geq k_0$ up to clock entering $(15, 16)$, there is no correction ($C = 0$) and repeated squaring gives

$$(7.1) \quad \frac{z_n}{|z_n|} = \left(\frac{z_{k_0}}{|z_{k_0}|} \right)^{2^{n-k_0}}.$$

Starting at any preparation index k , the dynamics of first two variables $(\bar{t}_n, z_n)$ agrees with $G^{n-k}(\bar{t}_k, z_k)$, until the clock enters $(15, 16)$ (recall G from Section 4). During this clock turn, $\beta_i^0(\bar{t}_n) = \beta_i(\bar{t}_n \bmod 20)$. Each β_i^0 is positive at at most one reading $\bar{t}_{\ell_i}$, since $|J_i| < 1$. If this reading exists, (4.2) and (7.1) used for k give

$$(7.2) \quad s_i(\bar{t}_k \bmod 20, z_k) = \alpha(\bar{t}_k) R^{1+k-\ell_i} \beta_i^0(\bar{t}_{\ell_i}) \left(\frac{z_{k_0}}{|z_{k_0}|} \right)^{2^{\ell_i-k_0+1}}.$$

Otherwise every preparation input for that auxiliary is zero. Since $\max\{\beta_1(t), \beta_2(t)\} = 1$ on $[16, 17]$ at least one correction happens. The point is that while there could be several, up to 3, signals being sent

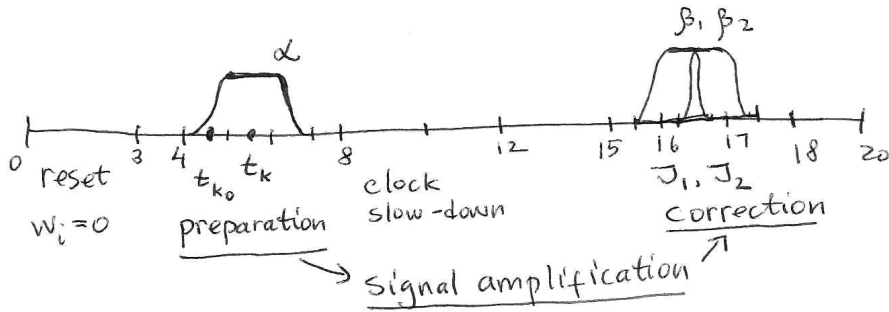


FIGURE 2

to the auxiliary variable during preparation they are all non-negative multiples of the same unit complex number and at least one of them is sufficiently strong where $\alpha = 1$. The auxiliary dynamics then amplifies the signal using P , which preserves this direction in $\mathbb{C}$, until correction happens at time ℓ_i .

Until correction, the variable z is being squared and the direction (argument) of z_n is the same as $(z_{k_0}/|z_{k_0}|)^{2^{n-k_0}}$. At $n = \ell_i$, the auxiliary therefore lies on the ray of the current square, giving

$$(7.3) \quad z_{n+1} = (|z_n|^2 + 2\beta_1(t_n)p(|w_{n,1}|) + 2\beta_2(t_n)p(|w_{n,2}|)) \left(\frac{z_n}{|z_n|} \right)^2.$$

Take $k \in I$ with $\alpha(\bar{t}_k) = 1$. Let m be the first index in this turn with $\bar{t}_m \geq 16$. Then $\bar{t}_m \in [16, 17)$, and each activation index ℓ_i that exists lies in $\{m-1, m, m+1\}$. Moreover, $\beta_i(t_m) = 1$ for at least one i ; choose such an i , so that $\ell_i = m$. Recalling $\alpha(\bar{t}_k) = 1$ and $\chi(|z_k|) = 1$, (7.2) gives $|s_i(\bar{t}_k \bmod 20, z_k)| = R^{k+1-m}$. Since $\gamma(t_k) = R$ and both terms lie on the same ray,

$$\begin{aligned} |w_{k+1,i}| &= |RP(w_{k,i}) + s_i(\bar{t}_k \bmod 20, z_k)| \\ &= Rp(|w_{k,i}|) + R^{k+1-m} \geq R^{k+1-m}. \end{aligned}$$

Until time m , all inputs to this auxiliary lie on the same ray, $\gamma = R$, and $|w_{n,i}| < 3R$. Therefore (5.1) gives

$$|w_{n+1,i}| \geq R \min(|w_{n,i}|, 1), \quad k+1 \leq n < m.$$

Since $R^{n-m} \leq 1$ for $n \leq m$, induction yields

$$|w_{n,i}| \geq R^{n-m}, \quad k+1 \leq n \leq m.$$

In particular, $|w_{m,i}| \geq 1$ and $p(|w_{m,i}|) \geq 1$. And (7.3) now gives

$$|z_{m+1}| \geq |z_m|^2 + 2 > 2.$$

The next clock time $\bar{t}_{m+1}$ is in $[17, 18)$, so its correction $C(t_{m+1}, w_{1,m+1}, w_{2,m+1})$ is either zero or again aligned and positive. Hence $|z_{m+2}| \geq |z_{m+1}|^2 > 4$, contradicting our assumption. We conclude that $\text{id } \bar{t}_n \rightarrow \infty$ is therefore asymptotic to a periodic orbit in A_N .

Remark 7.1. Near the isolated fixed point in A_S the local dynamics is

$$(t, z, w_1, w_2) \mapsto (t + b(t), z^2, R w_1, R w_2).$$

8. FINAL REMARKS

Remark 8.1. The recurrent set is exactly $A_S \cup A_N$. The non-wandering set $\Omega(f)$ is larger: it contains the disk

$$\mathcal{D} = \{10\} \times \{|z| \leq 1/2\} \times \{0\} \times \{0\}.$$

Each point on $\mathcal{D}$ converges to the fixed point $\{10\} \times S \times \{0\} \times \{0\}$.

Take a point $x = (10, z_0, 0, 0) \in \mathcal{D}$, $z_0 \neq S$. Backward square roots and clock preimages give

$$G^d(\tau + 1, q) = (30, z_0), \quad \tau \in [16, 17), \quad 0 < |q| < 1.$$

Indeed, backward radii tend to 1, eventually making every backward clock step exactly one. Choose i with $\beta_i(\tau) = 1$. Set $s_0 = \tau$ and choose successive clock preimages satisfying

$$10 < s_n < s_{n-1}, \quad s_n + \bar{b}(s_n) = s_{n-1} \quad (n \geq 1).$$

Then $s_n \rightarrow 10$. Define the initial auxiliary pair $w_n = (w_{1,n}, w_{2,n}) \in \mathbb{C}^2$ by

$$w_{i,n} = \frac{q - z_0^{2^{n+1}}}{2R^n}, \quad w_{3-i,n} = 0.$$

For $y_n = (s_n, z_0, w_n)$, we have $|z| \leq 1/2$ until activation, and the selected auxiliary variable remains in the unit disk, since

$$|R^j w_{i,n}| \leq \frac{|q| + |z_0|^{2^{n+1}}}{2} < 1, \quad 0 \leq j \leq n.$$

Its unique activation at τ sends the main z -coordinate to q . The next reset clears both auxiliaries and the clock variable stalls at 30, so no new signals are added. Thus the clock and main coordinate follow G , giving

$$f^{n+1+d}(y_n) = x, \quad y_n \rightarrow x, n \rightarrow \infty$$

This proves that x is non-wandering.

Remark 8.2. With a modified signal one can replace the two auxiliary spheres by one $\mathbb{S}^3$, obtaining a similar C^r map on $S^1 \times \mathbb{S}^2 \times \mathbb{S}^3$.

Remark 8.3. Lipschitz counterexample in the invertible setting can be found in the report on the conversation among R. Bowen, M. Hirsch, A. Manning, C. Pugh, B. Sanderson, M. Shub and R. Williams [P75]. Likely finite smoothness counterexamples also exist in diffeomorphism setting.

Remark 8.4. Given the new example one can ask for examples with finite collection of periodic points of any given period. I expect that it should also be possible by replacing the periodic attractor by a quasi-periodic one. The remaining unresolved question is the Entropy Conjecture for aspherical manifolds. One can still hope for it to be true even in the C^0 setting.

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