HIGHER-ORDER ASYMPTOTIC FORMULAE FOR PERIODIC ORBITS OF 3-DIMENSIONAL VOLUME-PRESERVING ANOSOV FLOWS AND THE FOULON CONJECTURE

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ABSTRACT. Let X^t be a volume-preserving Anosov flow of class C^7 on a closed 3-manifold. We consider a sequence of periodic orbits which shadow a homoclinic loop of a fixed periodic orbit γ, and derive higher-order asymptotic expansions for their periods T_n and for the logarithms of their unstable multipliers $\log \lambda_n$. We give geometric interpretations of the coefficients in these expansions, in particular in terms of the Anosov cocycle (à la Hurder-Katok [HK91]) and the longitudinal Anosov cocycle (à la Foulon-Hasselblatt [FH03]), evaluated on γ. Following the idea of De Simoi-Leguil-Vinhage-Yang [DSLVY20], we compare these asymptotics when volume is the measure of maximal entropy. In this case, the equilibrium-state characterizations of volume and the measure of maximal entropy synchronize periods and exponents: $\log \lambda_n = hT_n$. Matching the coefficients of the two asymptotic expansions forces the cohomology classes of both cocycles to vanish. Then the work of Foulon-Hasselblatt on the longitudinal Anosov cocycle [FH03], together with Foulon's work on the contact case [Fou01], yields the Foulon Conjecture for flows of class C^7: if volume is the measure of maximal entropy, then, up to a finite cover, X^t is smoothly conjugate either to a constant-roof suspension of a hyperbolic automorphism of the 2-torus or to a finite cover of the constant-speed geodesic flow on a closed hyperbolic surface.

1. INTRODUCTION

Statement of the results. Let M be a closed 3-dimensional manifold and let $X^t\colon M \to M$ be an Anosov flow generated by a vector field X. That is, there is a DX^t-invariant splitting $TM = E^s \oplus \mathbb{R}X \oplus E^u$, where E^s is uniformly contracted and E^u is uniformly expanded. We will write $E^{0s} = E^s \oplus \mathbb{R}X$ and $E^{0u} = E^u \oplus \mathbb{R}X$ for the weak stable and weak unstable distributions, respectively.

Theorem 1.1. *Let $X^t\colon M \to M$ be a C^r, $r \geq 7$, volume-preserving Anosov flow on a closed connected 3-dimensional manifold. Assume that the invariant volume m is the measure of maximal entropy. Then there exists a finite cover such that the lifted flow is C^{r_*} conjugate[1] to one of the following algebraic flows:*

1. *a constant-roof suspension of a hyperbolic automorphism of $\mathbb{T}^2$;*
2. *a finite cover of the constant-speed geodesic flow on a closed hyperbolic surface.*

The first result in this direction is probably the entropy rigidity theorem of Katok: if the metric entropy of a geodesic flow on a surface of negative curvature equals

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[1]Here $r_* = r$ if $r \notin \mathbb{N}$, and $r_* = (r-1) + \mathrm{Lip}$ if $r \in \mathbb{N}$. We note that the proof still works if X^t is $C^{6+\mathrm{Lip}}$, but the author doesn't know how to lower the regularity assumption any lower.

to its topological entropy then the metric has constant negative curvature [Kat88]. Foulon handled the more general case of contact flows and posed the even more general case of 3-dimensional volume-preserving Anosov flows as a conjecture [Fou01]. De Simoi-Leguil-Vinhage-Yang developed an approach to this conjecture via asymptotic formulae [DSLVY20]. However the expansion for periods in [DSLVY20, Lemma 3.4] missed a term which lead the authors to conclude that the Anosov cocycle is trivial based only on the first order formulae. Here we carefully derive the higher order formula and prove the above theorem by further pursuing the beautiful approach of De Simoi-Leguil-Vinhage-Yang. The need for higher order terms leads to a stronger regularity assumption C^7 (compared to original C^5 assumption in [DSLVY20]).

Specifying to the setting of suspension flows we obtain

Corollary 1.2. *Let $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ be a C^r, $r \geq 7$, area-preserving Anosov diffeomorphism and let $\varphi = -\log J^u f$. If φ is cohomologous to a C^7 function, then f is C^{r_*} conjugate to a hyperbolic linear automorphism of $\mathbb{T}^2$.*

By "cohomologous to a C^7 function" we precisely mean that $\varphi = \bar{\varphi} + u \circ f - u$, where $\bar{\varphi}$ is C^7 and u is continuous.

Organization. In Section 2 we explain some needed preliminaries. We recall how the entropy assumption synchronizes the periods and the Lyapunov exponents of periodic points. We introduce an elaborate version of Hurder-Katok adapted charts for Anosov flows, the Anosov cocycle and the longitudinal Anosov cocycle (or the Foulon-Hasselblatt cocycle) and recall their rigidity properties. Finally we also introduce a finite regularity version of the Birkhoff normal form at a periodic point.

In Section 3 we prove the higher-order asymptotics for the periods of shadowing orbits. And in Section 4 we do the same for their Lyapunov exponents. Finally, in Section 5 we deduce our main result and the corollary.

Comments. The author is very grateful to Martin Leguil from whom he learnt about asymptotic formulae for the periods in July 2022. At the same time the author concluded that the main decreasing term of the order $n\mu^n$ was missing in the formula for the periods and communicated this error to M. Leguil. It was clear that pursuing higher order asymptotics is a plausible route for obtaining Foulon conjecture. The author began actively thinking about it in November 2025 after learning about [EVY] since it provided such a clean normal form at a periodic orbit.[2] By May 2026 the author was convinced that derivation of higher order asymptotics is possible and it was the matter of getting precise value of the coefficients in the formula which appeared to be a very tedious calculus task. At this point AI help (first Claude Opus 4.8 and then ChatGPT Sol 5.6) became invaluable for doing calculations. The most challenging one was the polynomial drift expansion (Lemma 3.2) which the author could derive, but not with the needed C^1-control. In August 2026 ChatGPT pointed out a paper of M. Gonchenko [Gon13] which had a lemma very close to what was needed and the task was finished soon after. The author also used Claude and ChatGPT to generate initial drafts on most of the proofs in this paper, all of which were substantially, but not completely, rewritten

[2]While psychologically the normal form of [EVY] was very important, the proof here only relies on the Birkhoff normal form.

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2. Preliminaries

We pass to the connected finite cover which orients E^s and E^u. The lifted volume is still the measure of maximal entropy, hence, it is enough to work with the lifted flow. Also note that for the lifted flow all periodic orbits have positive multipliers.

2.1. **Synchronization.** We begin with a well-known consequence of the assumption that volume is the measure of maximal entropy. Fix a background Riemannian metric on M. Let

$$\varphi^u(x) = \left.\frac{d}{dt}\right|_{t=0} \log J^u X^t(x)$$

be the infinitesimal logarithmic unstable Jacobian. The volume m is the SRB measure and, hence, is the equilibrium state of the potential $-\varphi^u$. The pressure of this potential is zero [BR75, Theorem 3.3 and Proposition 4.4]. By assumption, the same measure is the equilibrium state of the zero potential, whose pressure is the topological entropy $h = h_{\mathrm{top}}(X^1)$. It follows that the potentials are cohomologous, that is,

$$\varphi^u = h + Xu,$$

where u is a Hölder function differentiable in the flow direction; see, e.g., [Sh26, Lemma 2.4]. Integrating over a periodic orbit gives the following lemma.

Lemma 2.1. *If $p = X^T(p)$, then*

$$\log J^u X^T(p) = hT.$$

2.2. **Non-stationary Birkhoff normal form and Anosov cocycle.** Let X^t be a C^r, $r \geq 7$, volume-preserving Anosov flow on a closed 3-dimensional manifold, and let Ω be the invariant volume form. Here we elaborate on the construction of adapted transverse coordinates by Hurder and Katok [HK91, Sections 4]. We write the transition maps through cubic order in non-stationary Birkhoff normal form, so that the Anosov cocycle can be read directly from this normal form.

Proposition 2.2. *There exist $\varepsilon > 0$ and a family of embeddings*

$$\Psi_q \colon (-\varepsilon, \varepsilon)^2 \longrightarrow M, \qquad \Psi_q(0,0) = q, \quad q \in M,$$

with the following properties.

1. *The embeddings are C^{r-1} and $q \mapsto \Psi_q$ is Hölder continuous in the C^{r-1} topology. Their images $\mathcal{T}_q$ are transverse to the flow.*
2. *The curves $\Psi_q(\cdot, 0)$ and $\Psi_q(0, \cdot)$ parametrize the local strong stable and local strong unstable manifolds of q, respectively.*
3. *Adapted coordinates respect area froms*

$$\Psi_q^*(\iota_X \Omega) = dx \wedge dy.$$

4. *Let $F_{q,t} \colon \mathcal{T}_q \to \mathcal{T}_{X^t(q)}$ be the local Poincaré map and consider its coordinate representation*

$$\mathcal{F}_{q,t} = \Psi_{X^t(q)}^{-1} \circ F_{q,t} \circ \Psi_q.$$

Then,

$$\mathcal{F}_{q,t}(x,y) = \big(\mu(q,t)x\big(1 - A(q,t)xy\big), \mu(q,t)^{-1}y\big(1 + A(q,t)xy\big)\big) + R_{q,t}(x,y), \tag{2.1}$$

where $\mu(q,t) > 0$. *Moreover,* $R_{q,t}$ *vanishes to order three at the origin:*

$$D^j R_{q,t}(0,0) = 0, \qquad 0 \le j \le 3.$$

Definition 2.1. *The coefficient* $A(q,t)$ *in* (2.1) *is the* Anosov cocycle. *Equivalently,*

$$A(q,t) = -\frac{1}{2\mu(q,t)}\partial_x^2\partial_y(\mathcal{F}_{q,t})_1(0,0) = \frac{\mu(q,t)}{2}\partial_x\partial_y^2(\mathcal{F}_{q,t})_2(0,0).$$

It is routine to check that A is a Hölder continuous additive cocycle. Hurder and Katok [HK91, Theorem 3.4] classified volume-preserving Anosov flows with smooth weak unstable distribution as precisely those for which $[A] = 0$. However, we will not use this result in this paper.

Remark 2.3. If $p = X^{T_0}(p)$, then $\mathcal{F}_{p,T_0}$ is an area-preserving germ at a hyperbolic fixed point. Recall the standard Birkhoff normal form

$$(x,y) \longmapsto \big(\mu x B(xy), \mu^{-1} y B(xy)^{-1}\big), \qquad B(0) = 1,$$

formula (2.1) gives

$$B'(0) = -A(p,T_0).$$

Thus $A(p,T_0)$ is the hyperbolic twist. See Birkhoff [Bir27, Chapter VIII, Section 1, pp. 211–213] and Hurder-Katok [HK91, Section 5].

2.3. **Longitudinal Anosov cocycle.** For $q \in M$ and $T > 0$, consider the adapted transversals based at the points q and $X^T(q)$. We write the transition time between them as $T + \tau_{q,T}$; thus $\tau_{q,T}$ is the unique C^{r-1} function such that

$$X^{T+\tau_{q,T}(x,y)}\Psi_q(x,y) = \Psi_{X^T(q)}\big(\mathcal{F}_{q,T}(x,y)\big), \qquad \tau_{q,T}(0,0) = 0.$$

Since the coordinate axes parametrize the strong stable and strong unstable manifolds, we have

$$\tau_{q,T}(x,0) = \tau_{q,T}(0,y) = 0.$$

Definition 2.2. *The* longitudinal Anosov cocycle*, or the* Foulon-Hasselblatt cocycle*, is defined as*

$$K(q,T) = \partial_x\partial_y\tau_{q,T}(0,0).$$

Then K is a cocycle whose cohomology class $[K]$ is independent of the choice of adapted charts [FH03].

Theorem 2.4 (Foulon-Hasselblatt [FH03, Theorem 3])**.** *Let* X^t *be a* C^r*,* $r \ge 2$*, volume-preserving Anosov flow on a closed* 3*-manifold. Then*

$$\begin{aligned} [K] = 0 \quad &\iff \quad E^s \oplus E^u \in C^{r-1} \\ &\iff \quad X^t \text{ is a contact flow or a constant-roof suspension.} \end{aligned}$$

2.4. **A finite-order Birkhoff normal form.** Fix a periodic orbit γ, a point $p \in \gamma$, and let T_0 be its period. The adapted chart of Proposition 2.2 gives C^{r-1} area-preserving coordinates (ξ, η) on $\mathcal{T}_p$ relative to which the return map has the form

$$F(\xi,\eta) = \big(\mu\xi(1-a\xi\eta) + R_1(\xi,\eta), \mu^{-1}\eta(1+a\xi\eta) + R_2(\xi,\eta)\big), \tag{2.2}$$

where $\mu = \mu(p, T_0) \in (0,1)$, $a = A(p, T_0)$, and $D^j R_1(0,0) = D^j R_2(0,0) = 0$ for $0 \le j \le 3$. We will need more refined coordinates at the periodic point p.

The following proposition is essentially the same as a more general higher order normal form [Gon13, Lemma 1.1]. However, we save one extra derivative (compared to C^{r-4}) and we have a better control on the error terms. We include a proof in Appendix A.

Proposition 2.5. *There exist C^{r-3} area-preserving coordinates (x, y) on $\mathcal{T}_p$ in which*

$$\begin{aligned} F(x,y) = \big(&\mu x\big(1 - axy + b(xy)^2\big) + r_1(x,y),\\ &\mu^{-1} y\big(1 + axy + (a^2 - b)(xy)^2\big) + r_2(x,y)\big), \end{aligned} \tag{2.3}$$

where $0 < \mu < 1$ is the stable multiplier of p, $a = A(p, T_0)$, and the error terms have the form

$$\begin{aligned} r_1(x,y) &= x^3 y^2 \rho_1(x,y),\\ r_2(x,y) &= x^2 y^3 \rho_2(x,y), \end{aligned} \tag{2.4}$$

where $\rho_1, \rho_2 \in C^{r-6}$ and $\rho_1(0,0) = \rho_2(0,0) = 0$.

Let $T_0 + \tau$ be the return time in these coordinates. Then there exists $v \in C^{r-5}$, vanishing on both coordinate axes, such that $\widehat{\tau} = \tau + v \circ F - v$ has the expansion

$$\widehat{\tau}(x,y) = K(p, T_0)xy + O((xy)^2). \tag{2.5}$$

3. Asymptotic formula for the periods

The formula below refines the first-order expansion derived in [GRH22]. We fix a periodic orbit γ and the Birkhoff coordinates given by Proposition 2.5.

Let $\mathcal{O}$ be an orbit homoclinic to γ. Let $p_{\text{out}} \in \mathcal{O} \cap W^u_{\text{loc}}(p) \cap \mathcal{T}_p$ be its last intersection with the transversal $\mathcal{T}_p$ before the excursion and let $p_{\text{in}} \in \mathcal{O} \cap W^s_{\text{loc}}(p) \cap \mathcal{T}_p$ be its first intersection after the excursion. After replacing these points by $F^{-j}(p_{\text{out}})$ and $F^j(p_{\text{in}})$ for some $j \ge 0$, and shrinking $\mathcal{T}_p$ if needed, we may assume that the open excursion from $p_{\text{out}} = (0, y_{\text{out}})$ to $p_{\text{in}} = (x_{\text{in}}, 0)$ is disjoint from $\mathcal{T}_p$. The excursion defines a Poincaré map G from a neighborhood of p_{out} in $\mathcal{T}_p$ to a neighborhood of p_{in} in $\mathcal{T}_p$ with $G(p_{\text{out}}) = p_{\text{in}}$, and an excursion-time function T_{exc} on its domain, which is characterized by

$$X^{T_{\text{exc}}(q)}(q) = G(q).$$

In the refined Birkhoff coordinates, G and T_{exc} are C^{r-3}.

For all sufficiently large n, the associated pseudo-orbit is shadowed by a periodic orbit γ_n which intersects $\mathcal{T}_p$ $(n+1)$ times. Label its intersections $p_i = (x_i, y_i) \in \mathcal{T}_p$, $0 \le i \le n$, ordered by time so that p_0 is close to p_{in} and p_n is close to p_{out}. Then p_n belongs to the domain of G for all sufficiently large n. We have

$$p_i = F^i(p_0), \qquad p_0 = G(p_n).$$

We denote the period of γ_n by T_n.

Throughout all local calculations that follow we work in the first quadrant: $x_i > 0$ and $y_i > 0$ for $0 \le i \le n$. Indeed, this is just a matter of properly orienting the local coordinates. (Recall that p has positive multipliers after passing to a finite cover.)

Proposition 3.1. *There exist constants C and $c \neq 0$ such that*

$$\begin{aligned} T_n = nT_0 + T_{\mathrm{exc}}(p_{\mathrm{out}}) + C\mu^n + cK(p,T_0)n\mu^n \\ - c^2A(p,T_0)K(p,T_0)n^2\mu^{2n} + O(n\mu^{2n}). \end{aligned}$$

For the sake of doing asymptotics we subdivide γ_n into the *excursion part* from p_n to p_0 which is governed by G and return time T_{exc} and the *local passage* from p_0 to p_n which is governed by the local dynamics F and the roof function $T_0 + \tau$.

$$T_n = \sum_{i=1}^{n}(T_0 + \tau(x_i, y_i)) + T_{\mathrm{exc}}(p_n) \tag{3.1}$$

We will first expand the product $I_n = x_0y_0$ in terms of μ^n and then use it to expand T_n.

It will be useful to consider the local passage $(p_i)_{i=0}^n$ as parametrized via the x-coordinate of the first point $x_0 \approx x_{\mathrm{in}}$ and the y-coordinate of the last point $y_n \approx y_{\mathrm{out}}$. In particular, we will view x_n and y_0 as functions of (x_0, y_n). The following lemma concerns only the local dynamics F and gives the (x_0, y_n)-parametrization and asymptotic expansion of x_n and y_0.[3] Then Lemma 3.3 incorporates the excursion map G and gives asymptotics of I_n.

Remark. The next lemma is almost the same as a lemma by M. Gonchenko [Gon13, Lemma 1.2]. We follow contracting summation-operator proof from [Gon13, Section 4.2.2]. It is stated in [Gon13, Lemma 1.2] that C^1 control can be achieved, but the details were not explicitly given. We carefully calculate the coefficients the polynomial Π_n and establish C^1 control of the remainder.

Lemma 3.2. *For all sufficiently large n, we have*

$$\begin{aligned} x_n &= \mu^n x_0 \Pi_n(s) + \Delta_{1,n}(x_0, y_n), \\ y_0 &= \mu^n y_n \Pi_n(s) + \Delta_{2,n}(x_0, y_n), \end{aligned} \tag{3.2}$$

where $s = \mu^n x_0 y_n$ and the main term is given by the polynomial

$$\Pi_n(s) = 1 - nas + \left(\frac{3}{2}a^2n^2 + \left(b - \frac{1}{2}a^2\right)n\right)s^2, \tag{3.3}$$

and the errors are given by functions $\Delta_{1,n}$ and $\Delta_{2,n}$ which are defined on a neighborhood of $(x_{\mathrm{in}}, y_{\mathrm{out}})$ and are controlled in C^1 norm

$$\|\Delta_{1,n}\|_{C^1} + \|\Delta_{2,n}\|_{C^1} \le C\mu^{3n}.$$

Remark. The last summand of the main term $\mu^n x_0 \Pi_n(s)$ is of order μ^{3n} in the C^0-norm, but of order μ^{2n} in the C^1-norm.

Proof.

Step 1: *Local passage as a fixed point of a summation operator.*

Let

$$B(u) = 1 - au + bu^2,$$

[3]Note that while not explicitly indicated in the notation y_0 also depends on n as well as the whole local passage $(p_i)_{i=0}^n$.

where a and b are the constants coming from the Birkhoff normal form (2.3). Define the integrable map

$$F_0(x,y) = \big(\mu x B(xy), \mu^{-1} y B(xy)^{-1}\big).$$

Then (2.3) can be written as

$$F = F_0 + \mathcal{R}, \qquad \mathcal{R}(x,y) = \big(x^3y^2\rho_1(x,y), x^2y^3\rho_2(x,y)\big), \tag{3.4}$$

where the two functions ρ_j are C^1 and vanish at the origin. Here we used

$$B(u)^{-1} = 1 + au + (a^2 - b)u^2 + O(u^3)$$

and absorbed the $O(u^3)$ term into the second remainder.

We rescale the coordinates (x_i, y_i), $0 \le i \le n$, to be of order one:

$$\widehat{x}_i = \mu^{-i} x_i, \qquad \widehat{y}_i = \mu^{i-n} y_i.$$

Then $x_i y_i = \mu^n \widehat{x}_i \widehat{y}_i$. The decomposition (3.4) then reads

$$\widehat{x}_{i+1} = \widehat{x}_i B(\mu^n \widehat{x}_i \widehat{y}_i) + \mathcal{E}_{i,n}(\widehat{x}_i, \widehat{y}_i), \tag{3.5}$$

$$\widehat{y}_i = \widehat{y}_{i+1} B(\mu^n \widehat{x}_i \widehat{y}_i) + \mathcal{H}_{i,n}(\widehat{x}_i, \widehat{y}_i), \tag{3.6}$$

where

$$\mathcal{E}_{i,n}(x,y) = \mu^{2n-1} x^3 y^2 \rho_1(\mu^i x, \mu^{n-i} y),$$
$$\mathcal{H}_{i,n}(x,y) = -\mu^{2n+1} B(\mu^n xy) x^2 y^3 \rho_2(\mu^i x, \mu^{n-i} y).$$

Recall that ρ_1 and ρ_2 are at least C^1. Vanishing of ρ_1 and ρ_2 at the origin implies

$$\|\mathcal{E}_{i,n}\|_{C^1} + \|\mathcal{H}_{i,n}\|_{C^1} \le C\mu^{2n}(\mu^i + \mu^{n-i}).$$

Summing gives

$$\sum_{i=0}^{n-1} \big(\|\mathcal{E}_{i,n}\|_{C^1} + \|\mathcal{H}_{i,n}\|_{C^1}\big) \le C\mu^{2n}. \tag{3.7}$$

We view (3.5) and (3.6) as a system of equations with $\widehat{x}_0 = x_0$ and $\widehat{y}_n = y_n$ given. We now reinterpret this system as a fixed point problem for a certain summation operator. For a sequence $(\xi, \eta) = ((\xi_i, \eta_i))_{i=0}^n$, let

$$\|(\xi,\eta)\|_\infty = \max_{0\le i\le n} (|\xi_i| + |\eta_i|), \qquad \mathcal{B}_n(R) = \{(\xi,\eta) : \|(\xi,\eta)\|_\infty \le R\}.$$

We equip $\mathcal{B}_n(R)$ with the uniform metric

$$d_\infty((\xi,\eta),(\widetilde{\xi},\widetilde{\eta})) = \|(\xi,\eta) - (\widetilde{\xi},\widetilde{\eta})\|_\infty.$$

For each n, $\mathcal{B}_n(R)$ is a closed ball in $(\mathbb{R}^2)^{n+1}$. Choose R, independently of n, so that $|x_0| + |y_n| < R/2$. Writing (3.5) and (3.6) in difference form we have

$$\widehat{x}_{i+1} - \widehat{x}_i = \widehat{x}_i(B(\mu^n \widehat{x}_i \widehat{y}_i) - 1) + \mathcal{E}_{i,n}(\widehat{x}_i, \widehat{y}_i),$$
$$\widehat{y}_i - \widehat{y}_{i+1} = \widehat{y}_{i+1}(B(\mu^n \widehat{x}_i \widehat{y}_i) - 1) + \mathcal{H}_{i,n}(\widehat{x}_i, \widehat{y}_i).$$

Summing the first equality from 0 to $i-1$ and the second from i to $n-1$ defines the summation operator $\mathcal{S}_n(\xi,\eta)=(\xi',\eta')$ by

$$\xi_i' = \widehat{x}_0 + \sum_{j=0}^{i-1} \big(\xi_j(B(\mu^n\xi_j\eta_j)-1) + \mathcal{E}_{j,n}(\xi_j,\eta_j)\big),$$
$$\eta_i' = \widehat{y}_n + \sum_{j=i}^{n-1} \big(\eta_{j+1}(B(\mu^n\xi_j\eta_j)-1) + \mathcal{H}_{j,n}(\xi_j,\eta_j)\big),$$

whose fixed points are precisely the solutions of (3.5)–(3.6) with $\widehat{x}_0 = x_0$ and $\widehat{y}_n = y_n$.

On $\mathcal{B}_n(R)$, the C^1 norms of the functions

$$(x,y)\mapsto x(B(\mu^n xy)-1) \quad \text{and} \quad (x,y,z)\mapsto z(B(\mu^n xy)-1)$$

are bounded by $C\mu^n$. Together with (3.7), this gives

$$\|\mathcal{S}_n(\xi,\eta)-\mathcal{S}_n(\widetilde{\xi},\widetilde{\eta})\|_\infty \le (Cn\mu^n + C\mu^{2n})\|(\xi,\eta)-(\widetilde{\xi},\widetilde{\eta})\|_\infty.$$

The same estimates give

$$\|\mathcal{S}_n(\xi,\eta)\|_\infty \le x_0 + y_n + Cn\mu^n + C\mu^{2n} < R$$

for all sufficiently large n. Thus $\mathcal{S}_n$ preserves $\mathcal{B}_n(R)$ and is a contraction towards the fixed point with Lipschitz constant $1/2$ with respect to $\|\cdot\|_\infty$ norm.

Step 2: *Truncated summation operator and estimates on the distance between fixed points.*

The rescaled dynamics of F_0 is

$$\widehat{x}^0_{i+1} = \widehat{x}^0_i B(\mu^n \widehat{x}^0_i \widehat{y}^0_i), \tag{3.8}$$
$$\widehat{y}^0_i = \widehat{y}^0_{i+1} B(\mu^n \widehat{x}^0_i \widehat{y}^0_i), \tag{3.9}$$

for $0 \le i \le n-1$, with $\widehat{x}^0_0 = x_0$ and $\widehat{y}^0_n = y_n$. Note that $\widehat{x}^0_{i+1}\widehat{y}^0_{i+1} = \widehat{x}^0_i\widehat{y}^0_i$. Thus the product $\widehat{x}^0_i\widehat{y}^0_i$ is independent of i. As before, writing (3.8) and (3.9) in difference form and summing up defines the F_0 summation operator $\mathcal{S}^0_n$. Explicitly, if $\mathcal{S}^0_n(\xi,\eta)=(\xi',\eta')$, then

$$\xi_i' = \widehat{x}_0 + \sum_{j=0}^{i-1} \xi_j(B(\mu^n\xi_j\eta_j)-1),$$
$$\eta_i' = \widehat{y}_n + \sum_{j=i}^{n-1} \eta_{j+1}(B(\mu^n\xi_j\eta_j)-1).$$

Thus the fixed point of $\mathcal{S}^0_n$ is precisely the rescaled local passage of F_0 satisfying the prescribed "boundary condition." The same argument shows that $\mathcal{S}^0_n$ preserves $\mathcal{B}_n(R)$ and is a contraction with Lipschitz constant $1/2$ and $(\widehat{x}^0_i,\widehat{y}^0_i)$ is its fixed point. By (3.7),

$$\sup_{\mathcal{B}_n(R)} \|\mathcal{S}_n - \mathcal{S}^0_n\|_\infty + \sup_{\mathcal{B}_n(R)} \|D\mathcal{S}_n - D\mathcal{S}^0_n\|_\infty \le C\mu^{2n}. \tag{3.10}$$

Here and below, $D\mathcal{S}_n$ and $D\mathcal{S}^0_n$ denote derivatives in the sequence variables.

Recall that for each fixed (x_0, y_n), the local passages $(\widehat{x}, \widehat{y})$ and $(\widehat{x}^0, \widehat{y}^0)$ are the fixed points of $\mathcal{S}_n$ and $\mathcal{S}_n^0$, respectively. Then

$$\begin{aligned}\|(\widehat{x}, \widehat{y}) - (\widehat{x}^0, \widehat{y}^0)\|_\infty &\le \|\mathcal{S}_n(\widehat{x}, \widehat{y}) - \mathcal{S}_n(\widehat{x}^0, \widehat{y}^0)\|_\infty + \|(\mathcal{S}_n - \mathcal{S}_n^0)(\widehat{x}^0, \widehat{y}^0)\|_\infty \\ &\le \frac{1}{2}\|(\widehat{x}, \widehat{y}) - (\widehat{x}^0, \widehat{y}^0)\|_\infty + \sum_{i=0}^{n-1} \big(\|\mathcal{E}_{i,n}\|_{C^0} + \|\mathcal{H}_{i,n}\|_{C^0}\big).\end{aligned}$$

Thus (3.10) gives

$$\|(\widehat{x}, \widehat{y}) - (\widehat{x}^0, \widehat{y}^0)\|_\infty = \max_{0\le i\le n} \big(|\widehat{x}_i - \widehat{x}_i^0| + |\widehat{y}_i - \widehat{y}_i^0|\big) \le 2C\mu^{2n}.$$

Having estimated the uniform norm, we now estimate the C^1 norms as functions of (x_0, y_n). The parameter-dependent contraction mapping theorem implies that the fixed points are C^1 functions of (x_0, y_n). Displaying the parameter dependence, the fixed point equation for $\mathcal{S}_n$ is

$$(\widehat{x}, \widehat{y}) = \mathcal{S}_n(\widehat{x}, \widehat{y}; x_0, y_n).$$

Differentiating using the chain rule gives

$$D_{(x_0,y_n)}(\widehat{x}, \widehat{y}) = D\mathcal{S}_n(\widehat{x}, \widehat{y}) D_{(x_0,y_n)}(\widehat{x}, \widehat{y}) + \partial_{(x_0,y_n)}\mathcal{S}_n,$$

and therefore

$$\big(I - D\mathcal{S}_n(\widehat{x}, \widehat{y})\big) D_{(x_0,y_n)}(\widehat{x}, \widehat{y}) = \partial_{(x_0,y_n)}\mathcal{S}_n.$$

Here $D\mathcal{S}_n$ denotes the derivative in the sequence variables, while $\partial_{(x_0,y_n)}\mathcal{S}_n$ is the differential with respect to (x_0, y_n). By definition of $\mathcal{S}_n$, it is simply the map

$$(h_1, h_2) \longmapsto \big((h_1, h_1, \dots, h_1), (h_2, h_2, \dots, h_2)\big),$$

which has norm one.

The same formulae apply to $\mathcal{S}_n^0$. Since both contractions have Lipschitz constant $1/2$, the operators on the left are invertible and their inverses have norm at most 2. The derivative of $\mathcal{S}_n^0$ is $Cn\mu^n$-Lipschitz on $\mathcal{B}_n(R)$. Therefore, by the preceding C^0 estimate and (3.10),

$$\begin{aligned}&\|D\mathcal{S}_n(\widehat{x}, \widehat{y}) - D\mathcal{S}_n^0(\widehat{x}^0, \widehat{y}^0)\| \\ &\qquad \le \|D\mathcal{S}_n(\widehat{x}, \widehat{y}) - D\mathcal{S}_n^0(\widehat{x}, \widehat{y})\| + Cn\mu^n \|(\widehat{x}, \widehat{y}) - (\widehat{x}^0, \widehat{y}^0)\|_\infty \le C\mu^{2n}.\end{aligned}$$

Note that $\partial_{(x_0,y_n)}\mathcal{S}_n = \partial_{(x_0,y_n)}\mathcal{S}_n^0$. Subtracting the differentiated fixed point equations gives

$$\begin{aligned}0 &= \big(I - D\mathcal{S}_n\big) D_{(x_0,y_n)}(\widehat{x}, \widehat{y}) - \big(I - D\mathcal{S}_n^0\big) D_{(x_0,y_n)}(\widehat{x}^0, \widehat{y}^0) \\ &= \big(I - D\mathcal{S}_n\big) D_{(x_0,y_n)}\big((\widehat{x}, \widehat{y}) - (\widehat{x}^0, \widehat{y}^0)\big) - \big(D\mathcal{S}_n - D\mathcal{S}_n^0\big) D_{(x_0,y_n)}(\widehat{x}^0, \widehat{y}^0) \\ &= \big(I - D\mathcal{S}_n\big)\Big[D_{(x_0,y_n)}\big((\widehat{x}, \widehat{y}) - (\widehat{x}^0, \widehat{y}^0)\big) \\ &\qquad - \big(I - D\mathcal{S}_n\big)^{-1}\big(D\mathcal{S}_n - D\mathcal{S}_n^0\big) D_{(x_0,y_n)}(\widehat{x}^0, \widehat{y}^0)\Big],\end{aligned}$$

where $D\mathcal{S}_n$ and $D\mathcal{S}_n^0$ are evaluated at $(\widehat{x}, \widehat{y})$ and $(\widehat{x}^0, \widehat{y}^0)$, respectively. Since $I - D\mathcal{S}_n$ is invertible, the expression in brackets vanishes. The second differentiated fixed point equation and the inverse bound give

$$\|D_{(x_0,y_n)}(\widehat{x}^0, \widehat{y}^0)\| \le 2.$$

Consequently,

$$\begin{aligned}&\big\|D_{(x_0,y_n)}\big((\widehat{x},\widehat{y})-(\widehat{x}^0,\widehat{y}^0)\big)\big\|_\infty\\&\le\big\|\big(I-D\mathcal{S}_n\big)^{-1}\big\|_\infty\big\|D\mathcal{S}_n-D\mathcal{S}_n^0\big\|_\infty\big\|D_{(x_0,y_n)}(\widehat{x}^0,\widehat{y}^0)\big\|_\infty\le 4C\mu^{2n}.\end{aligned}$$

Hence,

$$\max_{0\le i\le n}\big(\|\widehat{x}_i-\widehat{x}_i^0\|_{C^1}+\|\widehat{y}_i-\widehat{y}_i^0\|_{C^1}\big)\le 4C\mu^{2n}.$$

Here, again, we view the local passages $(\widehat{x}_i,\widehat{y}_i)$ and $(\widehat{x}_i^0,\widehat{y}_i^0)$ as functions of (x_0,y_n). Since $x_n=\mu^n\widehat{x}_n$ and $y_0=\mu^n\widehat{y}_0$, this implies

$$\|x_n-x_n^0\|_{C^1}+\|y_0-y_0^0\|_{C^1}\le C\mu^{3n}. \tag{3.11}$$

Step 3: *Asymptotics of the truncated fixed point.*

In this step we derive asymptotics for the F_0-passage. Set $u=\mu^n\widehat{x}_i^0\widehat{y}_i^0$, which is independent of i. Since $\widehat{x}_0^0=x_0$ and $\widehat{y}_n^0=y_n$, we have

$$\widehat{x}_n^0=x_0B(u)^n,\qquad \widehat{y}_0^0=y_nB(u)^n,\qquad u=sB(u)^n,$$

where $s=\mu^nx_0y_n$. Note that $u=O(s)$. Since

$$B(u)^n=1-nau+O(n^2u^2),$$

the equation $u=sB(u)^n$ gives

$$u=s-nasu+O(n^2su^2)=s+O(ns^2).$$

Substituting this estimate back gives

$$\begin{aligned}u&=s-nas\big(s+O(ns^2)\big)+O(n^2s^3)\\&=s-nas^2+O(n^2s^3).\end{aligned}$$

It follows that

$$\begin{aligned}\widehat{x}_n^0&=x_0B(u)^n\\&=x_0\left(1-nas+\left(nb+\frac{n(3n-1)}{2}a^2\right)s^2+O(n^3s^3)\right)\\&=x_0\Pi_n(s)+O(n^3s^3).\end{aligned}$$

Analogously,

$$\widehat{y}_0^0=y_n\Pi_n(s)+O(n^3s^3).$$

The remainders and their first derivatives with respect to s are bounded by Cn^3s^3 and Cn^3s^2, respectively. Hence, after substituting $s=\mu^nx_0y_n$, the identities $\partial_{x_0}s=\mu^ny_n$ and $\partial_{y_n}s=\mu^nx_0$ show that the derivatives of remainders with respect to (x_0,y_n) are bounded by $Cn^3\mu^{3n}$. Therefore,

$$\max\big\{\|\widehat{x}_n^0-\widehat{x}_0\Pi_n(s)\|_{C^1},\|\widehat{y}_0^0-\widehat{y}_n\Pi_n(s)\|_{C^1}\big\}\le Cn^3\mu^{3n}.$$

Equivalently,

$$\max\big\{\|x_n^0-\mu^nx_0\Pi_n(s)\|_{C^1},\|y_0^0-\mu^ny_n\Pi_n(s)\|_{C^1}\big\}\le Cn^3\mu^{4n}. \tag{3.12}$$

Step 4: *Assembling everything together.*

The C^1 bounds (3.11) and (3.12) give

$$\begin{aligned}&\max\{\|x_n-\mu^nx_0\Pi_n(s)\|_{C^1},\|y_0-\mu^ny_n\Pi_n(s)\|_{C^1}\}\\&\le C\big(\mu^{3n}+n^3\mu^{4n}\big)\le C\mu^{3n}.\end{aligned}$$

The two differences on the first line define $\Delta_{1,n}$ and $\Delta_{2,n}$. This finishes the proof of the lemma. □

Lemma 3.3. *Recall that $I_n = x_0 y_0$. There exists a constant $c > 0$ such that*

$$I_n = c\mu^n - nac^2\mu^{2n} + O(\mu^{2n}). \tag{3.13}$$

Moreover, $I_n - x_n y_n = O(\mu^{3n})$.

Proof. Let $s_n = \mu^n x_0 y_n = O(\mu^n)$. By Lemma 3.2,

$$I_n = x_0 y_0 = s_n \Pi_n(s_n) + O(\mu^{3n}), \qquad x_n y_n = s_n \Pi_n(s_n) + O(\mu^{3n}), \tag{3.14}$$

where recall

$$\Pi_n(s) = 1 - nas + O(n^2 s^2).$$

In particular, $I_n = O(\mu^n)$ and $x_n y_n - I_n = O(\mu^{3n})$.

Write $G = (G_1, G_2)$. Since $p_0 = G(p_n)$,

$$G_2(x_n, y_n) = y_0 = O(\mu^n).$$

Homoclinic transversality gives $\frac{\partial G_2}{\partial y}(p_{\text{out}}) \neq 0$. Hence the implicit function theorem gives $y_n = y_{\text{out}} + O(\mu^n)$. Also $x_0 = G_1(x_n, y_n) = x_{\text{in}} + O(\mu^n)$ because $G(p_{\text{out}}) = p_{\text{in}} = (x_{\text{in}}, 0)$. Thus, if $c = x_{\text{in}} y_{\text{out}}$, then

$$x_0 y_n = c + O(\mu^n)$$

and therefore $s_n = c\mu^n + O(\mu^{2n})$. Substituting this into the first equation in (3.14) gives

$$I_n = c\mu^n - nac^2\mu^{2n} + O(\mu^{2n}).$$

□

Lemma 3.4. *There exists a constant d such that*

$$T_n = nT_0 + T_{\text{exc}}(p_{\text{out}}) + dI_n + nK(p, T_0)I_n + O(n\mu^{2n}). \tag{3.15}$$

Proof. Iterating forward in the first coordinate and backward in the second gives (cf. [GRH22, Lemma 4.1])

$$x_i = O(\mu^i), \qquad y_i = O(\mu^{n-i}), \qquad 0 \leq i \leq n.$$

Hence

$$\sum_{i=0}^{n-1} (x_i + y_i) = O(1).$$

Multiplying the two coordinates in (2.3) and using that ρ_1 and ρ_2 are C^1 and vanish at the origin gives

$$x_{i+1} y_{i+1} - x_i y_i = O\big((x_i y_i)^3 (x_i + y_i)\big). \tag{3.16}$$

Summing (3.16) gives

$$x_i y_i = I_n + O(\mu^{3n}), \qquad 0 \leq i \leq n. \tag{3.17}$$

Recall the splitting of T_n into the local passage and the excursion part (3.1). Using (2.5) we can rewrite this decomposition as

$$T_n = \sum_{i=1}^{n} (T_0 + \widehat{\tau}(x_i, y_i)) + \widehat{T}_{\text{exc}}(p_n),$$

where $\widehat{T}_{\mathrm{exc}}(p_n) = T_{\mathrm{exc}}(p_n) - v(p_n) + v(p_0)$. Again using using (2.5) and (3.17), we obtain

$$\sum_{i=0}^{n-1} \widehat{\tau}(x_i, y_i) = nK(p, T_0)I_n + O(n\mu^{2n}). \tag{3.18}$$

Hence

$$T_n = nT_0 + nK(p, T_0)I_n + O(n\mu^{2n}) + \widehat{T}_{\mathrm{exc}}(p_n).$$

It remains to understand the remaining error $\widehat{T}_{\mathrm{exc}}(p_n) - \widehat{T}_{\mathrm{exc}}(p_{\mathrm{out}}) = \widehat{T}_{\mathrm{exc}}(p_n) - T_{\mathrm{exc}}(p_{\mathrm{out}})$ (recall that v vanishes on the coordinate axes). To calculate this error we first calculate, the local curve on which the successive points p_n land (to the first order), and then apply the differential $D\widehat{T}_{\mathrm{exc}}(p_{\mathrm{out}})$ to appropriately scaled tangent vector to this curve. The same proof was used [GRH22, Lemma 4.3] to calculate asymptotics of excursion time, where linearizing coordinates were used, which simplifies the calculation. Here we cannot use linearized coordinates because we need a better control on the error.

To expand $\widehat{T}_{\mathrm{exc}}(p_n)$ at $p_{\mathrm{out}} = (0, y_{\mathrm{out}})$, for small $\varepsilon > 0$ define

$$\Phi_\varepsilon(\xi, \eta) = \big(\xi - G_1(\varepsilon\xi, \eta), G_2(\varepsilon\xi, \eta) - \varepsilon\eta\big)$$

for (ξ, η) near $(x_{\mathrm{in}}, y_{\mathrm{out}})$. We have $\Phi_0(x_{\mathrm{in}}, y_{\mathrm{out}}) = (0, 0)$ and

$$D\Phi_0(x_{\mathrm{in}}, y_{\mathrm{out}}) = \begin{pmatrix} 1 & -\dfrac{\partial G_1}{\partial y}(p_{\mathrm{out}}) \\ 0 & \dfrac{\partial G_2}{\partial y}(p_{\mathrm{out}}) \end{pmatrix}.$$

Its determinant is $\frac{\partial G_2}{\partial y}(p_{\mathrm{out}})$ doesn't vanish by homoclinic transversality. Therefore, the implicit function theorem therefore gives a local C^{r-3} curve

$$\varepsilon \longmapsto (\xi(\varepsilon), \eta(\varepsilon)), \qquad (\xi(0), \eta(0)) = (x_{\mathrm{in}}, y_{\mathrm{out}}),$$

such that $\Phi_\varepsilon(\xi(\varepsilon), \eta(\varepsilon)) = 0$.

Now let $\varepsilon = \mu^n$. It follows from (3.2) and (3.3) that

$$x_n = \varepsilon x_0 + O(n\varepsilon^2), \qquad y_0 = \varepsilon y_n + O(n\varepsilon^2).$$

Since $p_0 = G(p_n)$, we have

$$\begin{aligned}\Phi_\varepsilon(x_0, y_n) = \big(&G_1(x_n, y_n) - G_1(\varepsilon x_0, y_n), \\ &G_2(\varepsilon x_0, y_n) - G_2(x_n, y_n) + y_0 - \varepsilon y_n\big) = O(n\varepsilon^2).\end{aligned}$$

The local inverses in the implicit function theorem have uniformly bounded derivatives. Hence

$$(x_0, y_n) = (\xi(\varepsilon), \eta(\varepsilon)) + O(n\varepsilon^2).$$

Consequently,

$$p_n = \big(\varepsilon\xi(\varepsilon), \eta(\varepsilon)\big) + O(n\varepsilon^2).$$

Differentiating with respect to ε at zero gives

$$p_n = p_{\mathrm{out}} + \varepsilon\big(\xi(0), \eta'(0)\big) + O(n\varepsilon^2) = p_{\mathrm{out}} + \varepsilon\big(x_{\mathrm{in}}, \eta'(0)\big) + O(n\varepsilon^2).$$

By (3.13), $I_n = c\varepsilon + O(n\varepsilon^2)$, where $c = x_{\mathrm{in}} y_{\mathrm{out}}$. Therefore,

$$p_n = p_{\mathrm{out}} + \frac{I_n}{c}\big(x_{\mathrm{in}}, \eta'(0)\big) + O(n\mu^{2n}).$$

Let

$$d = \frac{1}{c} D\widehat{T}_{\mathrm{exc}}(p_{\mathrm{out}})\big(x_{\mathrm{in}}, \eta'(0)\big).$$

Taylor expansion of $\widehat{T}_{\mathrm{exc}}$ at p_{out} now gives

$$\widehat{T}_{\mathrm{exc}}(p_n) = \widehat{T}_{\mathrm{exc}}(p_{\mathrm{out}}) + dI_n + O(n\mu^{2n}) = T_{\mathrm{exc}}(p_{\mathrm{out}}) + dI_n + O(n\mu^{2n}),$$

where, the last equality follows from $v(p_{\mathrm{in}}) = v(p_{\mathrm{out}}) = 0$. Combining this with (3.18) yields

$$T_n = nT_0 + T_{\mathrm{exc}}(p_{\mathrm{out}}) + dI_n + nK(p, T_0)I_n + O(n\mu^{2n}).$$

This proves the lemma. □

Proof of Proposition 3.1. Let $C = cd$. Substituting (3.13) into (3.15) and using $a = A(p, T_0)$ yields the posited formula

$$\begin{aligned} T_n = nT_0 + T_{\mathrm{exc}}(p_{\mathrm{out}}) + C\mu^n + cK(p, T_0)n\mu^n \\ - c^2 A(p, T_0)K(p, T_0)n^2\mu^{2n} + O(n\mu^{2n}). \end{aligned}$$

□

4. Asymptotic formula for the multipliers

We continue to work with the shadowing setup of Section 3, using the refined Birkhoff coordinates.

Proposition 4.1. *Let $\lambda_n > 1$ be the unstable multiplier of γ_n. There exist constants C_0, C_1 such that*

$$\begin{aligned} \log \lambda_n = & -n \log \mu + C_0 + C_1\mu^n + 2cA(p, T_0)n\mu^n \\ & - \frac{5}{2} c^2 A(p, T_0)^2 n^2 \mu^{2n} + O(n\mu^{2n}). \end{aligned} \tag{4.1}$$

Proof. As before, let $s_n = \mu^n x_0 y_n$. Let $\mathcal{A}_n$ be the derivative of the polynomial map from Lemma 3.2:

$$\begin{aligned} \mathcal{A}_n = (\alpha_{ij}) &= D_{(x_0, y_n)}\big(\mu^n x_0 \Pi_n(s_n), \mu^n y_n \Pi_n(s_n)\big) \\ &= \begin{pmatrix} \mu^n\big(\Pi_n(s_n) + s_n\Pi_n'(s_n)\big) & \mu^{2n} x_0^2 \Pi_n'(s_n) \\ \mu^{2n} y_n^2 \Pi_n'(s_n) & \mu^n\big(\Pi_n(s_n) + s_n\Pi_n'(s_n)\big) \end{pmatrix}. \end{aligned} \tag{4.2}$$

The estimates (3.11) and (3.12) show that the derivative of $(x_0, y_n) \mapsto (x_n, y_0)$ under F^n is $\mathcal{A}_n + O(\mu^{3n})$. Moreover,

$$\det \mathcal{A}_n = \mu^{2n} \Pi_n(s_n)\big(\Pi_n(s_n) + 2s_n\Pi_n'(s_n)\big) = O(\mu^{2n}).$$

Since $\alpha_{22} = \mu^n(1 + o(1))$, implicit differentiation applied to $F^n(x_0, y_0) = (x_n, y_n)$ gives

$$\alpha_{22} DF^n(x_0, y_0) = \left(\begin{smallmatrix} \det \mathcal{A}_n & \alpha_{12} \\ -\alpha_{21} & 1 \end{smallmatrix}\right) + O(\mu^{2n}) = \left(\begin{smallmatrix} 0 & \alpha_{12} \\ -\alpha_{21} & 1 \end{smallmatrix}\right) + O(\mu^{2n}). \tag{4.3}$$

Write $DG(p_n) = (\beta_{n,ij})$. Transversality of the homoclinic intersection gives

$$\frac{\partial G_2}{\partial y}(p_{\mathrm{out}}) \neq 0, \qquad \text{and} \qquad \beta_{n,22} \longrightarrow \frac{\partial G_2}{\partial y}(p_{\mathrm{out}}), n \to \infty.$$

Multiplying (4.3) by $DG(p_n)$ gives

$$\alpha_{22} DG(p_n) DF^n(p_0) = \begin{pmatrix} -\beta_{n,12}\alpha_{21} & \beta_{n,11}\alpha_{12} + \beta_{n,12} \\ -\beta_{n,22}\alpha_{21} & \beta_{n,21}\alpha_{12} + \beta_{n,22} \end{pmatrix} + O(\mu^{2n}).$$

The main matrix on the right converges to

$$\begin{pmatrix} 0 & \dfrac{\partial G_1}{\partial y}(p_{\mathrm{out}}) \\ 0 & \dfrac{\partial G_2}{\partial y}(p_{\mathrm{out}}) \end{pmatrix},$$

whose non-zero eigenvalue we denote by

$$P(0) = \frac{\partial G_2}{\partial y}(p_{\mathrm{out}}).$$

Our goal is to derive the asymptotics as $n \to \infty$ of the unstable eigenvalue λ_n. It is the larger eigenvalue of $DG(p_n)DF^n(p_0)$. The eigenvalue of the scaled matrices $\alpha_{22}DG(p_n)DF^n(p_0)$ converge: $\alpha_{22}\lambda_n \longrightarrow P(0)$, $n \to \infty$. The corresponding eigenvector can be normalized as $(w_n, 1)$, where w_n is uniformly bounded.

The second coordinate of the eigenvalue equation gives

$$\alpha_{22}\lambda_n = \beta_{n,22} + \beta_{n,21}\alpha_{12} - \beta_{n,22}\alpha_{21}w_n + O(\mu^{2n}).$$

Note that, by (4.2), we have $\alpha_{12} = O(n\mu^{2n})$ and $\alpha_{21} = O(n\mu^{2n})$. Since w_n is uniformly bounded, these terms can be absorbed into the error term, and hence

$$\alpha_{22}\lambda_n = \beta_{n,22} + O(n\mu^{2n}).$$

Multiplying the preceding identity by $\mu^n\Pi_n(s_n)/\alpha_{22}$ and using (4.2) gives

$$\mu^n\Pi_n(s_n)\lambda_n = \frac{\Pi_n(s_n)}{\Pi_n(s_n) + s_n\Pi_n'(s_n)}\beta_{n,22} + O(n\mu^{2n}). \tag{4.4}$$

The main task is to expand $\beta_{n,22}$ and the coefficient in front of it. Recall from the proof of Lemma 3.4 that p_n lies approximately on the curve $\varepsilon \mapsto (\varepsilon\xi(\varepsilon), \eta(\varepsilon))$ trough p_{out}:

$$p_n = \big(\mu^n\xi(\mu^n), \eta(\mu^n)\big) + O(n\mu^{2n}).$$

Also recall that G is C^{r-3}. Therefore,

$$\beta_{n,22} = \frac{\partial G_2}{\partial y}(p_n) = \frac{\partial G_2}{\partial y}\big(\mu^n\xi(\mu^n), \eta(\mu^n)\big) + O(n\mu^{2n}).$$

Recall that by Lemmas 3.2 and 3.3,

$$I_n = s_n\Pi_n(s_n) + O(\mu^{3n}). \tag{4.5}$$

Also, by (3.13),

$$\mu^n = \frac{I_n}{c} + O(n\mu^{2n}).$$

Define

$$P(I) = \frac{\partial G_2}{\partial y}\left(\frac{I}{c}\xi\left(\frac{I}{c}\right), \eta\left(\frac{I}{c}\right)\right),$$

which agrees with $P(0)$ value defined earlier. The function P is C^2, and the preceding expansion reads

$$\beta_{n,22} = P(I_n) + O(n\mu^{2n}).$$

Taylor expanding and using (3.3), (4.5), and (3.13), gives

$$\frac{\Pi_n(s_n)}{\Pi_n(s_n) + s_n\Pi_n'(s_n)} = 1 + acn\mu^n - a^2c^2n^2\mu^{2n} + O(n\mu^{2n}).$$

Plugging this expansion and the expansion for $\beta_{n,22}$ into (4.4) we have

$$\mu^n\Pi_n(s_n)\lambda_n = P(I_n)\left(1 + acn\mu^n - a^2c^2n^2\mu^{2n}\right) + O(n\mu^{2n}).$$

Taking logarithms and using (3.13) again gives

$$\log\big(\Pi_n(s_n)\lambda_n\big) = -n\log\mu + \log P(0) + c\frac{P'(0)}{P(0)}\mu^n + acn\mu^n - \frac{3}{2}a^2c^2n^2\mu^{2n} + O(n\mu^{2n}).$$

Recall that $s_n = c\mu^n + O(\mu^{2n})$. Therefore, by (3.3),

$$\log\Pi_n(s_n) = -acn\mu^n + a^2c^2n^2\mu^{2n} + O(n\mu^{2n}).$$

Subtracting this identity from the preceding expansion gives

$$\log\lambda_n = -n\log\mu + \log P(0) + c\frac{P'(0)}{P(0)}\mu^n + 2acn\mu^n - \frac{5}{2}a^2c^2n^2\mu^{2n} + O(n\mu^{2n}).$$

Recalling that $a = A(p, T_0)$, this proves the posited formula (4.1) with $C_0 = \log P(0)$, and $C_1 = cP'(0)/P(0)$. □

5. Proofs

Given a periodic point $p = X^{T_0}(p)$ we have derived the asymptotic formulae for the periods on for the Lyapunov exponents:

$$\begin{aligned} T_n &= nT_0 + T_{\mathrm{exc}}(p_{\mathrm{out}}) + C\mu^n \\ &\quad + cK(p,T_0)n\mu^n - c^2A(p,T_0)K(p,T_0)n^2\mu^{2n} + O(n\mu^{2n}), \\ \log\lambda_n &= -n\log\mu + C_0 + C_1\mu^n \\ &\quad + 2cA(p,T_0)n\mu^n - \frac{5}{2}c^2A(p,T_0)^2n^2\mu^{2n} + O(n\mu^{2n}). \end{aligned}$$

If volume is the measure of maximal entropy then the synchronization identity $\log\lambda_n = hT_n$ implies that, after multiplying the first equation by h, all coefficients in the above formulae must be equal. Coefficients by $n\mu^n$ give $hK(p,T_0) = 2A(p,T_0)$ while the coefficients by $n^2\mu^{2n}$ give $hK(p,T_0) = \frac{5}{2}A(p,T_0)$. Hence $A(p,T_0) = K(p,T_0) = 0$ for every periodic orbit. Hence, by the Livshits theorem, $[K] = 0$ and Theorem 2.4 implies that X^t is either a contact flow or a constant-roof suspension.

In the contact case, Foulon's proof [Fou01] applies and gives that X^t is C^{r_*} conjugate to a finite lift of the constant-speed geodesic flow of a closed hyperbolic surface.[4]

In the suspension case, let f be the return map to the torus section $\mathbb{T}^2$. Then f is area preserving and area is the measure of maximal entropy of f. Then de le Llave rigidity theorem implies that the continuous conjugacy to the linear model is, in fact, a C^{r_*} regular [dlL92]. Hence the proof of Theorem 1.1 is complete.

5.1. **Proof of Corollary 1.2.** Denote by m the invariant area and let $\varphi = -\log J^u f$. By the assumption, there is a C^7 function ψ cohomologous to φ. A sufficiently long finite Birkhoff average of ψ is negative. This average is still C^7 and cohomologous to φ; call it $-\rho$. Then ρ is a positive C^7 function.

The invariant area m is the equilibrium state of the geometric potential φ, and $P(\varphi) = 0$. Since $-\rho$ is cohomologous to φ, m is also the equilibrium state of $-\rho$ and $P(-\rho) = 0$. The description of equilibrium states for suspensions [PP90, Proposition 6.1] implies that, for the suspension flow over f with roof ρ,

$$\frac{m \times dt}{\int \rho\, dm}$$

[4]Foulon states his theorem for C^∞ flows but the proof works for C^2 flows.

is its measure of maximal entropy. This is a smooth invariant measure, therefore, Theorem 1.1 applies.

A contact flow cannot admit a global invariant section. Thus it is a constant roof suspension and f is smoothly conjugate to a linear hyperbolic automorphism.

Appendix A. Normal forms

A.1. **Strong adapted charts.**

Proof of Proposition 2.2. Hurder and Katok [HK91, Proposition 4.2] constructed adapted charts with all needed properties except properties 2 and 4. Property 2 is easy to add by pushing the charts along the flow lines to align the axes with strong stable and unstable manofolds. Thus it remains to establish property 4. Denote these initial adapted charts by $\widehat{\Psi}_q$ and their coordinate Poincaré maps by $\widehat{\mathcal{F}}_{q,t}$.

Fix $T > 0$ sufficiently large that

$$\vartheta := \sup_{q\in M} \mu(q,T) < 1.$$

For $q \in M$, write $\mu = \mu(q,T)$. Hurder and Katok [HK91, Lemma 4.7] give the following form:

$$\widehat{\mathcal{F}}_{q,T}(x,y) = \big(\mu x + \mu P_{q,T}(x,y), \mu^{-1}y + \mu^{-1}Q_{q,T}(x,y)\big) + R_{q,T}(x,y),$$

where $P_{q,T}$ and $Q_{q,T}$ are homogeneous polynomials of degree 3 and $D^j R_{q,T}(0,0) = 0$, $0 \le j \le 3$.

Let us first determine the possible cubic terms. Axis invariance implies that the polynomial $P_{q,T}$ is divisible by x and $Q_{q,T}$ is divisible by y. The terms of degree two in the identity $\det D\widehat{\mathcal{F}}_{q,T} = 1$ give

$$\partial_x P_{q,T} + \partial_y Q_{q,T} = 0.$$

It follows that there are coefficients $\widehat{\alpha} = \widehat{\alpha}(q,T)$, $A = A(q,T)$ and $\widehat{\gamma} = \widehat{\gamma}(q,T)$ such that

$$\big(P_{q,T}(x,y), Q_{q,T}(x,y)\big) = \big(\widehat{\alpha}x^3 - Ax^2y + 3\widehat{\gamma}xy^2, -3\widehat{\alpha}x^2y + Axy^2 - \widehat{\gamma}y^3\big). \tag{A.1}$$

We now eliminate $\widehat{\alpha}$ and $\widehat{\gamma}$ by adjusting the charts.

Given functions κ, ν on M, let Θ_q be the time-one map of the Hamiltonian vector field generated by

$$H_q(x,y) = \kappa(q)x^3y + \nu(q)xy^3.$$

Explicitly, this vector field is

$$X_{H_q} = (\partial_y H_q)\partial_x - (\partial_x H_q)\partial_y = \big(\kappa(q)x^3 + 3\nu(q)xy^2\big)\partial_x - \big(3\kappa(q)x^2y + \nu(q)y^3\big)\partial_y.$$

The map Θ_q is an area-preserving local diffeomorphism tangent to the identity and preserves both axes. Moreover,

$$\Theta_q(x,y) = (x,y) + U_q(x,y) + O\big((|x|+|y|)^5\big), \tag{A.2}$$

where

$$U_q(x,y) = \big(\kappa(q)x^3 + 3\nu(q)xy^2, -3\kappa(q)x^2y - \nu(q)y^3\big).$$

Set $\Psi_q = \widehat{\Psi}_q \circ \Theta_q$. The corresponding coordinate map is

$$\mathcal{F}_{q,T} = \Theta^{-1}_{X^T(q)} \circ \widehat{\mathcal{F}}_{q,T} \circ \Theta_q.$$

Comparing cubic terms, the two non-resonant coefficients change according to

$$\alpha(q,T) = \widehat{\alpha}(q,T) + \kappa(q) - \mu^2\kappa(X^T(q)), \qquad \gamma(q,T) = \widehat{\gamma}(q,T) + \nu(q) - \mu^{-2}\nu(X^T(q)),$$

while A is unchanged.

To eliminate $\alpha(q,T)$, we solve

$$0 = \widehat{\alpha}(q,T) + \kappa(q) - \mu(q,T)^2\kappa(X^T(q)).$$

Summing into the future gives

$$\kappa(q) = -\widehat{\alpha}(q,T) - \sum_{n=1}^{\infty} \mu(q,nT)^2\widehat{\alpha}(X^{nT}(q),T). \tag{A.3}$$

The series converges uniformly because $\mu(q,nT) \le \vartheta^n$. After decreasing the Hölder exponent if necessary, the convergence also holds in the Hölder norm, and hence its sum defines a Hölder function κ on M. The κ-term removes the x^3 term in the first component and, at the same time, the corresponding x^2y term in the second component.

Similarly, to eliminate $\gamma(q,T)$, we solve

$$0 = \widehat{\gamma}(q,T) + \nu(q) - \mu(q,T)^{-2}\nu(X^T(q)).$$

Summing into the past gives

$$\nu(q) = \sum_{n=1}^{\infty} \mu(X^{-nT}(q),nT)^2\widehat{\gamma}(X^{-nT}(q),T). \tag{A.4}$$

This series also define a Hölder continuous function ν which removes the y^3 term in the second component together with the corresponding xy^2 term in the first component, without reintroducing the terms removed by the κ-term.

It remains to pass from the fixed time T to arbitrary time. Denote the two non-resonant coefficients in the adjusted charts by $\alpha(q,t)$ and $\gamma(q,t)$. Calculating the cubic terms in

$$\mathcal{F}_{q,t+T} = \mathcal{F}_{X^t(q),T} \circ \mathcal{F}_{q,t} = \mathcal{F}_{X^T(q),t} \circ \mathcal{F}_{q,T}$$

gives

$$\alpha(q,t) = \mu(q,T)^2\alpha(X^T(q),t), \qquad \gamma(q,t) = \mu(q,T)^{-2}\gamma(X^T(q),t).$$

Iterating the first identity into the future and the second one into the past shows that

$$\alpha(q,t) = \gamma(q,t) = 0$$

for every $q \in M$ and every $t > 0$. Consequently,

$$\mathcal{F}_{q,t}(x,y) = \big(\mu(q,t)x(1 - A(q,t)xy), \mu(q,t)^{-1}y(1 + A(q,t)xy)\big) + R_{q,t}(x,y).$$

This is precisely (2.1). □

A.2. **The finite-order Birkhoff normal form.** The analytic normal form was proved by Moser [Mos56]. The first-order finite-regularity normalization is carried out in [GS90]; see also [GS00]. Higher-order finite-regularity versions appeared in [Gon13, DGG15]. Neither reference gives precisely the statement we need, because we require a C^{r-3} regularity and remainders $x^3y^2\rho_1$, $x^2y^3\rho_2$ with $\rho_1, \rho_2 \in C^{r-6}$ vanishing at the origin. We follow essentially the same argument as in the above references, keeping track of the regularity and the remainders. After each change of coordinates, we abuse notation and continue to denote the conjugated return map by F.

Proof of Proposition 2.5.
Step 1: Area preservation, invariant axes and their linearization. Choose a C^r transversal $\mathcal{T}_p$ through p which contains the local strong stable and strong unstable manifolds of p. In local coordinates (u, v) on $\mathcal{T}_p$, write

$$(\iota_X \Omega)|_{\mathcal{T}_p} = \rho(u, v)\, du \wedge dv.$$

The function ρ is C^r. After changing coordinates by

$$\xi = u, \qquad \eta = \int_0^v \rho(u, s)\, ds,$$

the transverse area is $d\xi \wedge d\eta$. We now write the local stable manifold of $(0, 0)$ as $\eta = s(\xi)$ and straighten it by the area-preserving shear

$$(\xi, \eta) \longmapsto (\xi, \eta - s(\xi)).$$

We similarly straighten the local unstable manifold of F. These changes do not lose any regularity. Hence the return map $F = (F_1, F_2)$ is C^r, area-preserving and preserves both axes, and we will keep using (x, y)-variables.

We first record the elementary decomposition used at every step. If h is C^2, then

$$h(x, y) = h(x, 0) + h(0, y) - h(0, 0) + xy \int_0^1 \int_0^1 \frac{\partial^2 h}{\partial x \partial y}(tx, sy)\, dt\, ds. \tag{A.5}$$

The one-dimensional linearization gives C^r coordinates χ_s and χ_u, such that

$$\chi_s(F_1(x, 0)) = \mu \chi_s(x), \qquad \chi_u(F_2(0, y)) = \mu^{-1} \chi_u(y).$$

The first coordinate is extended by

$$(x, y) \longmapsto \left(\chi_s(x), \frac{y}{\chi_s'(x)} \right),$$

and the second one by

$$(x, y) \longmapsto \left(\frac{x}{\chi_u'(y)}, \chi_u(y) \right).$$

Both changes preserve area and both axes. Here we lose one derivative. Thus, in the resulting C^{r-1} coordinates,

$$F_1(x, 0) = \mu x, \qquad F_2(0, y) = \mu^{-1} y.$$

Area preservation on the two axes also gives

$$\partial_x F_1(0, y) = \mu, \qquad \partial_y F_2(x, 0) = \mu^{-1}.$$

Therefore,

$$F_1(x, y) = \mu x + x^2 y h_1(x, y), \qquad F_2(x, y) = \mu^{-1} y + x y^2 h_2(x, y),$$

where $h_1, h_2 \in C^{r-4}$. Applying (A.5) to h_1 and h_2 gives

$$\begin{aligned} F_1(x, y) &= \mu x \big(1 + (c_1 + \varphi_1(x) + \psi_1(y)) xy\big) + x^3 y^2 R_1(x, y), \\ F_2(x, y) &= \mu^{-1} y \big(1 + (c_2 + \varphi_2(x) + \psi_2(y)) xy\big) + x^2 y^3 R_2(x, y). \end{aligned} \tag{A.6}$$

where explicitely

$$\begin{aligned} c_1 &= \mu^{-1} h_1(0, 0), & c_2 &= \mu h_2(0, 0), \\ \varphi_1(x) &= \mu^{-1} \big(h_1(x, 0) - h_1(0, 0)\big), & \varphi_2(x) &= \mu \big(h_2(x, 0) - h_2(0, 0)\big), \\ \psi_1(y) &= \mu^{-1} \big(h_1(0, y) - h_1(0, 0)\big), & \psi_2(y) &= \mu \big(h_2(0, y) - h_2(0, 0)\big). \end{aligned}$$

Hence $\varphi_j, \psi_j \in C^{r-4}$ and vanish at 0. Functions R_1 and R_2 are C^{r-6}.

Step 2: Removal of the non-resonant terms. Following [DGG15, Section 8], we first remove φ_1. We seek a canonical change of coordinates $(x, y) \to (\xi, \eta)$ generated by

$$V(x, \eta) = x\eta + \frac{1}{2}(x\eta)^2\omega(x),$$

where $\omega(0) = 0$ is to be determined. Thus

$$\xi = \partial_\eta V = x + x^2\eta\omega(x), \qquad y = \partial_x V = \eta + \frac{1}{2}\eta^2 \frac{d}{dx}\big(x^2\omega(x)\big).$$

Substituting this change in (A.6) and comparing the x^2y terms replaces $\varphi_1(x)$ by

$$\varphi_1(x) + \omega(\mu x) - \omega(x).$$

Therefore the change removes φ_1 provided that

$$\omega(\mu x) = \omega(x) - \varphi_1(x).$$

This equation is solved directly by summing into the future

$$\omega(x) = \sum_{k=0}^{\infty} \varphi_1(\mu^k x).$$

Indeed, the series converges in C^{r-4} because $\varphi_1(0) = 0$. We need to have better regularity of $x^2\omega(x)$. By the definition of φ_1,

$$x^2\omega(x) = \sum_{k=0}^{\infty} \mu^{-2k} \left(\mu^{-1}\partial_y F_1(\mu^k x, 0) - c_1(\mu^k x)^2\right).$$

It is easy to check that the series converges in C^{r-2}, and $x^2\omega(x)$ is C^{r-2}. Solving $y = \partial_x V$ for η gives the explicit coordinate change

$$\eta(x, y) = \frac{2y}{1 + \sqrt{1 + 2y\frac{d}{dx}(x^2\omega(x))}}, \qquad \xi(x, y) = x + \eta(x, y)x^2\omega(x). \tag{A.7}$$

Since $x^2\omega(x)$ is C^{r-2}, this defines an area-preserving local C^{r-3} diffeomorphism. By construction, this change removes φ_1. The symmetric construction gives a canonical coordinate change N_ψ which removes ψ_2. Note that we do not construct the second canonical change after the first one. Instead, the second change is constructed from the original C^{r-1} axis-linearized map and hence is also C^{r-3}. In this way there is no additional loss of regularity. We explain why the two changes can be applied successively without altering the coefficient removed by the other. Let N_φ denote the first change. Since $x^2\omega(x)$ has vanishing two-jet at the origin, we have

$$N_\varphi(0, y) = (0, y), \qquad DN_\varphi(0, y) = \mathrm{Id}\,.$$

Let $F_\varphi = N_\varphi \circ F \circ N_\varphi^{-1}$. Since $F(0, y) = (0, \mu^{-1}y)$,

$$DF_\varphi(0, y) = DN_\varphi(0, \mu^{-1}y)DF(0, y)DN_\varphi^{-1}(0, y) = DF(0, y).$$

Thus $\partial_x(F_\varphi)_2(0, y) = \partial_x F_2(0, y)$, so $c_2 + \psi_2(y)$ is unchanged. Similarly, N_ψ and its derivative are the identity along the x-axis, so it does not reintroduce φ_1. Both changes are C^{r-3}, and their composition is still C^{r-3}.

Now we examine the error terms after removing φ_1 and ψ_2. Since both $\omega(x)/x$ and $\omega'(x)$ are C^{r-5}, formula (A.7) gives

$$N_\varphi^{\pm 1}(x,y) = \big(x + x^3 y p_1^\pm(x,y), y + x^2 y^2 p_2^\pm(x,y)\big),$$
$$N_\psi^{\pm 1}(x,y) = \big(x + x^2 y^2 q_1^\pm(x,y), y + x y^3 q_2^\pm(x,y)\big),$$

where $p_j^\pm, q_j^\pm \in C^{r-5}$ and N_ψ denotes the symmetric change. Direct substitution, together with the two homological equations, gives

$$F_1(x,y) = \mu x\big(1 + (c_1 + \psi_1(y))xy\big) + x^3y^2 Q_1(x,y) + x^3 y E_1(x,y),$$
$$F_2(x,y) = \mu^{-1} y\big(1 + (c_2 + \varphi_2(x))xy\big) + x^2y^3 Q_2(x,y) + x y^3 E_2(x,y),$$

where $Q_1, Q_2 \in C^{r-6}$, $E_1, E_2 \in C^{r-5}$, and $E_1(x,0) = 0, E_2(0,y) = 0$. Consequently,

$$R_1(x,y) = Q_1(x,y) + \int_0^1 \partial_y E_1(x,ty)\,dt, \qquad R_2(x,y) = Q_2(x,y) + \int_0^1 \partial_x E_2(tx,y)\,dt$$

are C^{r-6} and

$$F_1(x,y) = \mu x\big(1 + (c_1 + \psi_1(y))xy\big) + x^3y^2 R_1(x,y),$$
$$F_2(x,y) = \mu^{-1} y\big(1 + (c_2 + \varphi_2(x))xy\big) + x^2y^3 R_2(x,y).$$

Area preservation, $\det DF = 1$, immediately gives

$$c_2 = -c_1, \qquad \varphi_2 = \psi_1 = 0.$$

Step 3: The second Birkhoff coefficient. Let $b = \mu^{-1} R_1(0,0)$. The coefficient of x^2y^2 in $\det DF = 1$ gives

$$3\big(b + \mu R_2(0,0) + c_1 c_2\big) = 0.$$

Thus $\mu R_2(0,0) = c_1^2 - b$. It remains to set $a = -c_1$ and

$$\rho_1(x,y) := R_1(x,y) - R_1(0,0), \qquad \rho_2(x,y) := R_2(x,y) - R_2(0,0).$$

to obtain the posited Birkhoff normal form (2.3).

Step 4: The return time. Let Ψ be the resulting C^{r-3} parametrization of $\mathcal{T}_p$, and let $T_0 + \tau$ be the return time. Since $\mathcal{T}_p$ contains the local strong stable and strong unstable manifolds we have

$$\tau(x,0) = \tau(0,y) = 0.$$

Moreover, $\tau \in C^{r-3}$. Dividing successively by x and y we get

$$\tau(x,y) = xy\, q(x,y),$$

where q is C^{r-5} and hence at least C^2. By chart independence of the periodic value of the longitudinal cocycle [GRH22, Lemma 2.2],

$$K(p,T_0) = \partial_{xy}\tau(0,0) = q(0,0).$$

Applying (A.5) to q gives

$$q(x,y) = q(x,0) + q(0,y) - K(p,T_0) + O(xy).$$

After multiplication by xy, the two nonconstant axis terms are precisely the terms beginning in degree three which are not already $O((xy)^2)$. We will remove them by adding a coboundary $v \circ F - v$, where v has the form $v = xyw$. Define

$$w(x,y) = \sum_{j=0}^{\infty} \big(q(\mu^j x, 0) - K(p,T_0)\big) + \sum_{j=1}^{\infty} \big(K(p,T_0) - q(0,\mu^j y)\big).$$

The series converge in C^{r-5}, and

$$w(\mu x, \mu^{-1} y) - w(x,y) = 2K(p,T_0) - q(x,0) - q(0,y).$$

Set $\widehat{\tau} = \tau + v \circ F - v$. Then

$$\begin{aligned}\big|\widehat{\tau}(x,y) - K(p,T_0)xy\big| \le |xy|\big|q(x,y) + w(\mu x, \mu^{-1}y) - w(x,y) - K(p,T_0)\big| \\ + \big|F_1(x,y)F_2(x,y)w(F(x,y)) - xyw(\mu x, \mu^{-1}y)\big|.\end{aligned}$$

The first term on the right is $O((xy)^2)$ by the preceding discussion. It remains to estimate the second term. By the triangle inequality,

$$\begin{aligned}&\big|F_1(x,y)F_2(x,y)w(F(x,y)) - xyw(\mu x, \mu^{-1}y)\big| \\ &\qquad \le \big|F_1(x,y)F_2(x,y) - xy\big|\big|w(F(x,y))\big| \\ &\qquad\qquad + |xy|\big|w(F(x,y)) - w(\mu x, \mu^{-1}y)\big|.\end{aligned}$$

The normal form (2.3) gives $F_1(x,y)F_2(x,y) = xy + O((xy)^3)$, which takes care of the first term. Since w is C^1, we have

$$\big|w(F(x,y)) - w(\mu x, \mu^{-1}y)\big| = O(x^2|y| + |x|y^2) = O(|xy|),$$

Hence the second term in the first inequality is $O((xy)^2)$, and

$$\widehat{\tau}(x,y) = K(p,T_0)xy + O((xy)^2).$$

□

References

[Bir27] G.D. Birkhoff, *Dynamical systems.* American Mathematical Society Colloquium Publications, Vol. 9, American Mathematical Society, New York, 1927.

[BR75] R. Bowen, D. Ruelle, *The ergodic theory of Axiom A flows.* Invent. Math. 29 (1975), no. 3, 181–202.

[DGG15] A. Delshams, M. Gonchenko, S. Gonchenko, *On dynamics and bifurcations of area-preserving maps with homoclinic tangencies.* Nonlinearity 28 (2015), no. 9, 3027–3071.

[DSLVY20] J. De Simoi, M. Leguil, K. Vinhage, Y. Yang, *Entropy rigidity for 3D conservative Anosov flows and dispersing billiards.* Geom. Funct. Anal. 30 (2020), no. 5, 1337–1369.

[EVY] A. Erchenko, K. Vinhage, Y. Yang, *Normal forms in a neighborhood of hyperbolic periodic orbits for flows in dimension 3.* arXiv:2512.08051 (2025).

[Fou01] P. Foulon, *Entropy rigidity of Anosov flows in dimension three.* Ergodic Theory Dynam. Systems 21 (2001), no. 4, 1101–1112.

[FH03] P. Foulon, B. Hasselblatt, *Zygmund strong foliations.* Israel J. Math. 138 (2003), 157–169.

[GRH22] A. Gogolev, F. Rodriguez Hertz, *Smooth rigidity for 3-dimensional volume preserving Anosov flows and weighted marked length spectrum rigidity.* arXiv:2210.02295 (2022).

[Gon13] M. Gonchenko, *Homoclinic phenomena in conservative systems.* Ph.D. thesis, Universitat Politècnica de Catalunya, 2013.

[GS90] S. V. Gonchenko, L. P. Shilnikov, *Invariants of Ω-conjugacy of diffeomorphisms with a structurally unstable homoclinic trajectory.* Ukrainian Math. J. 42 (1990), no. 2, 134–140.

[GS00] S. V. Gonchenko, L. P. Shilnikov, *On two-dimensional area-preserving diffeomorphisms with infinitely many elliptic islands.* J. Stat. Phys. 101 (2000), 321–356.

[HK91] S. Hurder, A. Katok, *Differentiability, rigidity and Godbillon-Vey classes for Anosov flows.* Inst. Hautes Études Sci. Publ. Math. No. 72 (1990), 5–61 (1991).

[Kat88] A. Katok, *Four applications of conformal equivalence to geometry and dynamics.* Ergodic Theory Dynam. Systems 8 (1988), 139–152.

[dlL92] R. de la Llave, *Smooth conjugacy and S-R-B measures for uniformly and non-uniformly hyperbolic systems.* Comm. Math. Phys. 150 (1992), no. 2, 289–320.

[Mos56] J. Moser, *The analytic invariants of an area-preserving mapping near a hyperbolic fixed point.* Comm. Pure Appl. Math. 9 (1956), 673–692.

[PP90] W. Parry, M. Pollicott, *Zeta functions and the periodic orbit structure of hyperbolic dynamics.* Astérisque No. 187–188 (1990), 268 pp.

[Sh26] R. Sharp, *Abelian Livšic theorems for Anosov flows.* Bull. Lond. Math. Soc. 58 (2026), no. 1, Paper No. e70258.

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