Rigidity from the strong unstable foliation for higher-dimensional Anosov and partially hyperbolic flows

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Abstract

Recently Burniol Clotet [BC25] proved that two transitive Anosov flows on 3-manifolds with equivalent strong unstable foliations must be conjugate up to time rescaling. We further explore the mechanism of Burniol Clotet's argument and treat the case of higher-dimensional Anosov flows and their isometric extensions by compact Lie groups.

1 Results

Let M be a closed manifold and let $X^t\colon M \to M$ be a C^r, $r \geq 1$, Anosov flow. We will consider X^t to be the "model" flow and another Anosov flow Y^t related to the model X^t in the following way. We say that their strong unstable foliations are *equivalent* if there exists a homeomorphism $\varphi\colon N \to M$ such that

$$\varphi\big(W^u_Y(y)\big) = W^u_X\big(\varphi(y)\big), \qquad y \in N.$$

We call such a homeomorphism an *unstable equivalence*. The question is then to recover a direct link between dynamics of X^t and Y^t.

The genesis of this type of questions goes back at least to Ratner who proved that measurable isomorphisms between the horocycle flows of finite-volume hyperbolic surfaces are algebraic [R82] and, hence, the surfaces must be isometric. This result was later generalized to higher dimensional hyperbolic manifolds by Flaminio and Spatzier [F87, FS90]. The more direct predecessor to the topological question we consider here is the work of Marcus [M83], who showed that if X^t and Y^t are geodesic flows on negatively curved surfaces which admit an unstable equivalence then Y^t is conjugate to a rescaled flow $X^{\lambda t}$, where λ is a fixed positive constant. In the same paper he claimed the same theorem for 3-dimensional suspension flows and posed the general question. Being unaware of Marcus' work Abe also proved the same result [A95]. The proofs have a lot in common with Marcus' being more geometric, he argues that the equivalence preserves weak unstable foliation by looking at unstable horocycles on the universal cover of the surface, and Abe's being more dynamical. Recently Burniol Clotet generalized the arguments and resolved the question of Marcus in the setting of transitive 3-dimensional Anosov flows [BC25]. Here we built upon the arguments of Burniol Clotet and some ideas of [GRH23, GRH25] and pursue the same question in higher dimensions.

Theorem 1. *Let $X^t\colon M \to M$ and $Y^t\colon N \to N$ be transitive Anosov flows. Additionally assume that the strong unstable distribution E^u_X is C^1. If their strong unstable foliations are equivalent, then we have the following dichotomy:*

(i) *either there exist $\lambda > 0$ and a homeomorphism $\psi\colon N \to M$ such that*

$$\psi \circ Y^t = X^{\lambda t} \circ \psi, \qquad t \in \mathbb{R};$$

(ii) *or there is a non-zero continuous DX^t-invariant subbundle $E \subset E^s_X$ which integrates to an X^t-invariant foliation $\mathcal{F} \subset W^s_X$; moreover, $E \oplus E^u_X$ integrates to a X^t-invariant foliation subfoliated by both $\mathcal{F}$ and W^u_X.*

Without the C^1 assumption on E^u_X, one still obtains the following result.

Addendum 2. *Let $X^t\colon M \to M$ and $Y^t\colon N \to N$ be transitive Anosov flows. Suppose that X^t is locally su-non-degenerate. If their strong unstable foliations are equivalent, then there exist $\lambda > 0$ and a homeomorphism $\psi\colon N \to M$ such that*

$$\psi \circ Y^t = X^{\lambda t} \circ \psi, \qquad t \in \mathbb{R}.$$

Here local *su*-non-degeneracy roughly means that stable and unstable directions do not integrate together on any local configuration; see Definition 2.8 for the precise definition.

It is unknown to the author if the second alternative in Theorem 1 or the *su*-degenerate case of the addendum can be further promoted to joint integrability of strong distributions E^s_X and E^u_X. Still it is possible under some additional assumptions.

Theorem 3. *Let $X^t\colon M \to M$ and $Y^t\colon N \to N$ be transitive Anosov flows. Suppose that their strong stable foliations are one-dimensional. If their strong unstable foliations are equivalent, then there exist $\lambda > 0$ and a homeomorphism $\psi\colon N \to M$ such that*

$$\psi \circ Y^t = X^{\lambda t} \circ \psi, \qquad t \in \mathbb{R}.$$

Corollary 4. *Assume that M is equipped with a smooth volume and has dimension at least 4. Let $\mathcal{A}(M)$ be the space of smooth volume-preserving Anosov flows which have one-dimensional strong unstable subbundle. There exists a C^1-open and C^∞-dense subset $\mathcal{V} \subset \mathcal{A}(M)$ with the following property. If $X^t \in \mathcal{V}$ and $Y^t \in \mathcal{A}(M)$ have equivalent strong unstable foliations, then there exist $\lambda > 0$ and a C^1 diffeomorphism $\psi\colon M \to M$ such that*

$$\psi \circ Y^t = X^{\lambda t} \circ \psi, \qquad t \in \mathbb{R}.$$

This corollary is the unstable-foliation version of generic smooth rigidity for codimension-one volume-preserving Anosov flows [GRH23]. We note that both Corollary 4 and Theorem 3 apply to suspensions of codimension-one Anosov diffeomorphisms and to the new examples of Fenley–Mann–Potrie [FMP26].

Theorem 5. *Let $\bar{X}^t\colon B \to B$ be an Anosov flow and let $\pi\colon M \to B$ be a principal fiber bundle whose fibers are given by a right action of a compact Lie group K. Suppose that $X^t\colon M \to M$ is an isometric extension of $\bar{X}^t$. Assume that the strong unstable foliation W^u_X is minimal and that X^t is locally su-non-degenerate. Let $Y^t\colon N \to N$ be a dynamically coherent partially hyperbolic flow. If the strong unstable foliations of X^t and Y^t are equivalent, then there exist $\lambda > 0$, a continuous homomorphism $k\colon \mathbb{R} \to K$, and a homeomorphism $\psi\colon N \to M$ such that*

$$\psi \circ Y^t = R^t \circ X^{\lambda t} \circ \psi, \qquad t \in \mathbb{R},$$

where $R^t(y) = y \cdot k(t)$, $y \in M$.

In general, the drift along the compact group in Theorem 5 cannot be removed. Indeed, for every continuous homomorphism $k\colon \mathbb{R} \to K$, the flows X^t and $R^t \circ X^t$ have the same strong unstable foliation.

Combining our results with marked-length-spectrum rigidity [H99, GL19] also yields the following geometric corollaries.

Corollary 6. *Let (M, g_0) and (N, g) be negatively curved n-manifolds, $n \geq 3$, and assume that (M, g_0) is locally symmetric. Let $X^t \colon T^1M \to T^1M$ and $Y^t \colon T^1N \to T^1N$ be their geodesic flows. If their strong unstable foliations are equivalent, then there exist an isometry*

$$F \colon (N, \lambda^2 g) \longrightarrow (M, g_0),$$

where $\lambda > 0$ is a constant.

Corollary 7. *Let g_0 be a negatively curved metric on M, $\dim M = n \geq 2$, and fix an integer $k > 3n/2 + 8$. There exists $\varepsilon > 0$ with the following property. Let g be a negatively curved metric such that*

$$\|g - g_0\|_{C^k} < \varepsilon.$$

Suppose that the strong unstable foliations of the geodesic flows of g_0 and g are equivalent via a homeomorphism homotopic to the identity. Then there exist an isometry

$$F \colon (M, \lambda^2 g) \longrightarrow (M, g_0),$$

where $\lambda > 0$ is a constant.

Remark 8. *Let (M_i, g_i), $i = 1, 2$, be negatively curved n-manifolds, $n \geq 3$, and let X^t and Y^t be their oriented frame flows, respectively. Assume that W^u_X is minimal (for example, when n is odd, $n \neq 7$, [BG80], and when g_1 has constant negative curvature [B75]). If their strong unstable foliations are equivalent, then the geodesic flows on (M_2, g_2) and (M_1, g_1) are conjugate up to time rescaling.*

Comments on AI

Upon reading [BC25] the author realized that the arguments could be generalized to the codimension one setting and also to yield the dichotomy of Theorem 1. At the time, in July 2026, the author only began experimenting with AI. Upon feeding two pages of notes outlining the proof to Claude Opus 4.8, it produced 31 page AI-slop "paper" proving Theorems 1 and 3. To the author's surprise (at the time), while terribly written the "paper" was mostly correct. After some expositional improvements by Claude Opus and then ChatGPT 5.6 Sol, the author had to largely rewrite the paper which included fixing some mistakes and a lot of streamlining of the arguments. Needless to say, the author bears full responsibility for correctness of this paper.

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2 Preliminaries

We recall well-known definitions and results, and at the same time establish some notation.

Definition 2.1. A flow X^t on M is *Anosov* if there is a continuous DX^t-invariant splitting $TM = E^s \oplus \mathbb{R}X \oplus E^u$ and constants $C, \chi > 0$ such that, for $t \geq 0$,

$$\|DX^t v^s\| \leq Ce^{-\chi t}\|v^s\|, \qquad \|DX^{-t} v^u\| \leq Ce^{-\chi t}\|v^u\|$$

for $v^s \in E^s$ and $v^u \in E^u$.

The strong stable and strong unstable foliations are denoted by W^s and W^u. The weak stable and weak unstable foliations tangent to $E^s \oplus \mathbb{R}X$ and $\mathbb{R}X \oplus E^u$ are denoted by W^{0s} and W^{0u}, respectively. For $\sigma \in \{s, u, 0s, 0u\}$, the notation $W^\sigma_{\mathrm{loc}}(x)$ means a fixed local plaque through x, while $W^\sigma_\rho(x)$ denotes the intrinsic radius-ρ plaque. We write d^s and d^u for the intrinsic distances on strong stable and strong unstable leaves. When two flows are compared, a subscript such as X or Y is used for clarity.

Definition 2.2. The strong stable and strong unstable foliations of an Anosov flow are *jointly integrable* if $E^s \oplus E^u$ integrates to an X^t-invariant foliation subfoliated by both W^s and W^u.

Theorem 2.3 (Anosov alternative [A67, P72])**.** *Let X^t be a transitive Anosov flow. Then either all strong stable and strong unstable manifolds are dense and X^t is topologically mixing, or X^t is a suspension over an Anosov diffeomorphism by a constant roof function.*

Theorem 2.4 (Plante [P72], Theorem 3.7)**.** *Let X^t be a transitive Anosov flow for which $\dim E^s = 1$ or $\dim E^u = 1$. If W^s and W^u are jointly integrable, then every leaf of the integrating foliation is a compact C^1 global cross-section with constant return time.*

Definition 2.5. A flow X^t on M is *partially hyperbolic* if there is a continuous DX^t-invariant splitting

$$TM = E^s \oplus E^c \oplus E^u, \qquad \mathbb{R}X \subset E^c,$$

such that E^s and E^u are uniformly contracted and expanded, respectively, and the center has intermediate behavior. That is, there exist $C, \chi > 0$ such that, for $t \geq 0$,

$$\|DX^t|_{E^s}\| \leq Ce^{-\chi t}, \qquad \|DX^{-t}|_{E^u}\| \leq Ce^{-\chi t},$$
$$\frac{\|DX^t|_{E^s}\|}{m(DX^t|_{E^c})} \leq Ce^{-\chi t}, \qquad \frac{\|DX^t|_{E^c}\|}{m(DX^t|_{E^u})} \leq Ce^{-\chi t}.$$

where $m(A) = \|A^{-1}\|^{-1}$. The flow is *dynamically coherent* if $E^s \oplus E^c$ and $E^c \oplus E^u$ integrate to invariant foliations W^{cs} and W^{cu}. In this case we let $W^c = W^{cs} \cap W^{cu}$.

For a partially hyperbolic flow, we also write W^s and W^u for the strong stable and strong unstable foliations tangent to E^s and E^u, respectively. For $\sigma \in \{s, u, cs, cu, c\}$, we use the same local plaque notation as in Anosov case.

Definition 2.6. Let $\bar{X}^t \colon B \to B$ be an Anosov flow and let $\pi \colon M \to B$ be a principal fiber bundle whose fiber is a compact Lie group K. We fix a K-invariant metric on M. An *isometric extension* of $\bar{X}^t$ is a flow X^t on M such that

$$\pi \circ X^t = \bar{X}^t \circ \pi, \qquad X^t \circ R_k = R_k \circ X^t, \qquad \forall k \in K,\ t \in \mathbb{R},$$

where $R_k(y) = y \cdot k$ is the principal right action. Then X^t is partially hyperbolic with $E^c = \mathbb{R}X \oplus \ker D\pi$.

We use the following notation for the local product bracket. For an Anosov flow, whenever x and y are sufficiently close, we denote

$$[x, y] := W^s_{\mathrm{loc}}(x) \cap W^{0u}_{\mathrm{loc}}(y).$$

For a dynamically coherent partially hyperbolic flow, we use the same notation with $W^{cu}_{\mathrm{loc}}(y)$ in place of $W^{0u}_{\mathrm{loc}}(y)$.

Definition 2.7. For an Anosov flow X^t, let $w \in W^s_{\mathrm{loc}}(x)$ and $z \in W^u_{\mathrm{loc}}(x)$. The *temporal distance* $\Delta(w, z)$ is the unique small time such that

$$X^{\Delta(w,z)}([z, w]) \in W^u_{\mathrm{loc}}(w).$$

Definition 2.8. Let X^t be a dynamically coherent partially hyperbolic flow. We call X^t *locally su-non-degenerate* if, whenever $w \in W^s_{\mathrm{loc}}(x) \setminus \{x\}$, the local strong-stable holonomy from $W^{cu}_{\mathrm{loc}}(x)$ to $W^{cu}_{\mathrm{loc}}(w)$ does not carry an open strong unstable plaque through x into the strong unstable plaque through w. This definition also applies to Anosov flow, in this case it says that for every such w the function

$$z \longmapsto \Delta(w, z), \qquad z \in W^u_{\mathrm{loc}}(x),$$

is not identically zero on any neighborhood of x in $W^u_{\mathrm{loc}}(x)$.

Definition 2.9. Let M have dimension $2n+1$. A smooth flow X^t on M, with generator X, is a *contact flow* if it is the Reeb flow of a smooth contact form α; that is,

$$\alpha \wedge (d\alpha)^n \neq 0, \qquad \alpha(X) = 1, \qquad \iota_X d\alpha = 0.$$

In particular, α is X^t-invariant. If X^t is also Anosov, we call it a *contact Anosov flow.*

For a contact Anosov flow, $E^s \oplus E^u = \ker \alpha$. Moreover, $d\alpha$ vanishes on each of E^s and E^u and pairs these two subbundles non-degenerately.

Proposition 2.10. *Every smooth contact Anosov flow is locally su-non-degenerate. More precisely, if $w \in W^s_{\mathrm{loc}}(x) \setminus \{x\}$, then there are $c > 0$ and $z_n \in W^u_{\mathrm{loc}}(x)$ with $z_n \to x$ such that*

$$|\Delta(w, z_n)| \geq c d^u(x, z_n)^2.$$

Moreover, if X^t is an isometric extension of a smooth contact Anosov flow $\bar{X}^t$, then X^t is locally su-non-degenerate as well.

Proof. Let $x_n = X^n(x)$, $w_n = X^n(w)$, and $a_n = d^s(x_n, w_n) \to 0$, $n \to \infty$. In stable leafwise exponential coordinates write $w_n = \exp^s_{x_n}(\bar{w}_n)$. Since $d\alpha$ pairs E^s and E^u non-degenerately, we can choose $\bar{v}_n \in E^u(x_n)$ such that

$$\|\bar{v}_n\| = \|\bar{w}_n\| = a_n, \qquad |d\alpha(\bar{v}_n, \bar{w}_n)| \geq c_0 a_n^2.$$

Let $z'_n = \exp^u_{x_n}(\bar{v}_n)$. Then Liverani's temporal-distance estimate [L04, Lemma B.7] gives a smaller constant $c_1 > 0$ such that

$$|\Delta(w_n, z'_n)| \geq c_1 a_n^2$$

Let $z_n = X^{-n}(z'_n)$. Iterating backwards we have $d^u(x, z_n) \leq C a_n$, while flow invariance of temporal distance gives $\Delta(w, z_n) = \Delta(w_n, z'_n)$. The asserted estimate follows, and in particular the temporal-distance function cannot vanish on a neighborhood of x in $W^u_{\mathrm{loc}}(x)$.

For an isometric extension, the projection π maps local strong stable and strong unstable plaques of X^t onto the corresponding plaques of $\bar{X}^t$ and intertwines their holonomies. If a local stable holonomy upstairs carried an open strong unstable plaque into a strong unstable plaque, its projection would do the same downstairs, a contradiction. □

3 Anosov flows with C^1 unstable foliation

In this section we describe a general construction for an Anosov flow X^t with C^1 regular unstable distribution E^u. It is well-known that if E^u is C^1 then the unstable foliation W^u has C^1 holonomies. The author believes that this discussion has some independent interest. It could also be carried out for partially hyperbolic systems.

Recall that $E^{0s} = E^s \oplus \mathbb{R}X$. For $z \in W^u(y)$, let

$$\mathrm{Hol}^u_{y,z} \colon W^{0s}_{loc}(y) \longrightarrow W^{0s}_{loc}(z)$$

be the strong unstable holonomy between weak stable leaves. Unstable leaves are simply connected, so this holonomy is independent of the choice of path from y to z. It is a C^1 diffeomorphism between sufficiently small neighborhoods of its basepoints. The differential $D_y \operatorname{Hol}^u_{y,z}$ identifies $E^{0s}(y)$ with $E^{0s}(z)$.

For $\rho > 0$, let

$$E_\rho(y) = \bigcap_{z \in W^u_\rho(y)} \left(D_y \operatorname{Hol}^u_{y,z}\right)^{-1} E^s(z). \tag{3.1}$$

Define

$$E_{\min}(y) = \lim_{\rho \searrow 0} E_\rho(y), \qquad E_{\max}(y) = \lim_{\rho \to \infty} E_\rho(y) = \bigcap_{z \in W^u(y)} \left(D_y \operatorname{Hol}^u_{y,z}\right)^{-1} E^s(z). \tag{3.2}$$

We have

$$E_{\max}(y) \subset E_\rho(y) \subset E_{\min}(y) \subset E^s(y).$$

The following lemma is immediate from the definitions.

Lemma 3.1. *Both $E_{\min}$ and $E_{\max}$ are DX^t-invariant. Moreover, $E_{\max}$ is invariant under unstable holonomy:*

$$D_y \operatorname{Hol}^u_{y,z}\big(E_{\max}(y)\big) = E_{\max}(z), \qquad z \in W^u(y). \tag{3.3}$$

Lemma 3.2. *The function $d(y) = \dim E_{\max}(y)$ is upper semicontinuous. If W^u is minimal, then d is constant and there exists $L > 0$ such that*

$$E_{\max}(y) = E_L(y) \qquad \text{for every } y \in M. \tag{3.4}$$

Proof. Since $E_{\max}(y)$ is finite dimensional, we can choose finitely many $z_i \in W^u(y)$ whose pulled-back stable subspaces intersect in $E_{\max}(y)$. We choose such a collection of minimal cardinality, then the corresponding normal vectors are linearly independent. The corresponding holonomies are C^1 maps $h_i = \operatorname{Hol}^u_{y,z_i}$ which we can restrict to a common weak stable disk D^{0s} about y. For any $w \in D^{0s}$, the same holonomy map, can be regarded as unstable holonomy from w to $h_i(w)$. Consequently,

$$E_{\max}(w) \subset K(w) := \bigcap_i (D_w h_i)^{-1} E^s(h_i(w)). \tag{3.5}$$

The subspaces in this intersection vary continuously with w. After shrinking D^{0s} if needed, their normal vectors remain independent, so $\dim K(w) = \dim E_{\max}(y) = d(y)$. Hence $d(w) \le d(y)$ on D^{0s}. This proves upper semicontinuity on weak stable leaves. Unstable holonomy invariance then gives upper semicontinuity on M.

Suppose now that W^u is minimal. The minimum locus of d is nonempty, open, and saturated by entire unstable leaves. Its complement is closed and unstable-saturated, so minimality makes that complement empty. Thus d is constant, and the inclusion in (3.5) is now an equality.

The holonomies h_i persist on D^{0s}, and their unstable path lengths are bounded above by L_y. Hence we have

$$E_{\max}(w) = E_{L_y}(w)(y) \qquad \text{for every } w \in D^{0s}. \tag{3.6}$$

This is true for any y. Hence, by compactness we have a finite cover and the corresponding maximal value L which gives (3.6). □

Lemma 3.3. *If W^u is minimal, then*

$$E_{\min}(y) = E_{\max}(y) \qquad \text{for every } y \in M.$$

In fact, $E_\rho(y) = E_{\max}(y)$ for every $\rho > 0$.

Proof. By stabilization, $E_{\min}(y) = E_\rho(y)$ for some $\rho > 0$. Let L be given by Lemma 3.2. For sufficiently large $t > 0$, backward contraction along unstable leaves gives

$$X^{-t}\big(W^u_L(X^t y)\big) \subset W^u_\rho(y).$$

Hence,

$$DX^t E_\rho(y) \subset E_L(X^t y) = E_{\max}(X^t y).$$

and by flow invariance we conclude that $E_{\min}(y) = E_\rho(y) \subset E_{\max}(y)$, hence they are equal. □

Proposition 3.4. *Assume that W^u is minimal. The common subbundle $E = E_{\min} = E_{\max} \subset E^s$ is continuous and DX^t-invariant. It integrates to an X^t-invariant foliation $\mathcal{F} \subset W^s$ whose restriction to every stable leaf is a C^1 foliation. Moreover, $E \oplus E^u$ integrates to an X^t-invariant C^1 foliation subfoliated by $\mathcal{F}$ and W^u.*

Proof. The holonomies h_i from the proof of Lemma 3.2 provide local C^1 foliation charts tangent to E on weak stable disks. Thus E is continuous and integrates to a foliation $\mathcal{F} \subset W^s$. Because the maps h_i which locally define $\mathcal{F}$ (3.5) are unstable holonomies, $\mathcal{F}$ integrates jointly with W^u. □

4 Proofs

In this section we prove Theorem 1 and Addendum 2 and then deduce Theorem 3 and Corollary 4. We would like to emphasize that our proof is merely a crystallization of Burniol Clotet's proof [BC25], which, in turn, is based on Abe's [A95] and Marcus' [M83]: we replace the algebraic renormalization argument with a soft dynamical one and push the argument into higher dimensions. We first treat the case in which the unstable equivalence does not respect the weak unstable foliation. Renormalizing a non-trivial stable displacement produces a non-trivial local *su*-integrability relation. For the dichotomy, the assumption that E^u is C^1 makes the associated temporal-distance functions C^1 along stable leaves, allowing us to create an integrable invariant subbundle. We then treat, in Proposition 4.7, the case in which the equivalence respects the weak unstable foliation. There the transported flow preserves each weak unstable leaf of X^t, and correcting maps produce the conjugacy as in [BC25, A95].

Note that we can dispose of the jointly integrable case. Indeed, the foliation tangent to $E^s \oplus E^u$ is subfoliated by W^s and W^u, so the second alternative of Theorem 1 holds. Joint integrability is also impossible under the hypothesis of Addendum 2.

Thus we consider the case in which W^s and W^u are not jointly integrable. The suspension case in Theorem 2.3 is then ruled out, so X^t is topologically mixing and W^u is minimal; we will use this repeatedly.

4.1 The local displacement

Fix an unstable equivalence $\varphi\colon N \to M$. It is convenient to transplant the flow Y^t to M. Define

$$\widehat{Y}^t = \varphi \circ Y^t \circ \varphi^{-1}.$$

Thus $\widehat{Y}^t$ is a continuous flow on M and it preserves the strong unstable foliation of X^t:

$$\widehat{Y}^t\big(W^u(y)\big) = W^u\big(\widehat{Y}^t(y)\big).$$

Lemma 4.1. *For a sufficiently small $\delta > 0$ there exist continuous maps*

$$p, q\colon M \times (-\delta, \delta) \longrightarrow M \quad \text{and} \quad \beta\colon M \times (-\delta, \delta) \longrightarrow \mathbb{R}$$

such that

$$p(y,t) = [y, \widehat{Y}^t(y)] = W^s_{\mathrm{loc}}(y) \cap W^{0u}_{\mathrm{loc}}\big(\widehat{Y}^t(y)\big), \tag{4.1}$$

$$q(y,t) = X^{\beta(y,t)}(p(y,t)) \in W^u(\widehat{Y}^t(y)). \tag{4.2}$$

Moreover,

$$\sup_{y\in M}\big(d^s(y,p(y,t)) + |\beta(y,t)|\big) \longrightarrow 0 \qquad \text{as } t \longrightarrow 0, \tag{4.3}$$

and

$$d^u\big(q(y,t), \widehat{Y}^t(y)\big) \le C_0 \tag{4.4}$$

for a uniform constant C_0. For fixed t the restriction

$$q(\,\cdot\,,t)|_{W^u(y)} \colon W^u(y) \longrightarrow W^u(\widehat{Y}^t(y))$$

is a homeomorphism.

Proof. For sufficiently small t, p, q and β are all uniquely defined by local product structure. Uniform local product structure gives continuity and all listed properties. □

Fix $\delta_0 > 0$ so that the local decomposition in Lemma 4.1 is defined for $|t| < \delta_0$. Call such t *centered* if

$$p(y,t) = W^s_{\mathrm{loc}}(y) \cap W^{0u}_{\mathrm{loc}}\big(\widehat{Y}^t(y)\big) = y \qquad \text{for every } y \in M. \tag{4.5}$$

Lemma 4.2. *If $|t| < \delta_0$ and $p(y,t) = y$ for some $y \in M$, then t is centered.*

Proof. The set $S_t = \{y : p(y,t) = y\}$ is closed. Local uniqueness of the intersection defining p, together with the fact that $\widehat{Y}^t$ preserves W^u, shows that S_t is uniformly locally W^u-saturated. Hence it is W^u-saturated. By minimality, if S_t is nonempty, then $S_t = M$. □

Lemma 4.3. *If there are non-zero centered times arbitrarily close to zero, then every t with $|t| < \delta_0$ is centered. Moreover, φ maps the weak unstable foliation of Y^t onto that of X^t.*

Proof. Centeredness is equivalent to $\widehat{Y}^t(y) \in W^{0u}(y)$ for every y, and the set of centered times is closed by continuity. If t_1 and t_2 are centered and $|t_1 + t_2| < \delta_0$, then $\widehat{Y}^{t_1+t_2}(y) \in W^{0u}(y)$ for every y, so $t_1 + t_2$ is centered. Let $t_n \to 0$ be non-zero centered times. For any $|t| < \delta_0$, choose integers k_n such that $k_n t_n \to t$. Since $k_n t_n$ is centered, so is t. Subdividing an arbitrary time now gives $\widehat{Y}^t(y) \in W^{0u}_X(y)$ for every $t \in \mathbb{R}$. Since φ maps W^u_Y-leaves onto W^u_X-leaves, this gives $\varphi(W^{0u}_Y(x)) = W^{0u}_X(\varphi(x))$. □

4.2 Renormalization of a non-centered time

Given a small unstable plaque $D^u(y) \subset W^u(y)$, denote its local stable saturation by

$$\mathrm{Sat}^s_{\mathrm{loc}}(D^u(y)) = \bigcup_{z\in D^u(y)} W^s_{\mathrm{loc}}(z).$$

For $y' \in W^s_{\mathrm{loc}}(y)$ write $y' \in \mathcal{I}(y)$ if for a sufficiently small unstable plaque

$$D^u(y') \subset \mathrm{Sat}^s_{\mathrm{loc}}\big(D^u(y)\big). \tag{4.6}$$

This is a local joint integrability condition. After shrinking plaques appropriately, the same condition holds with y and y' interchanged.

Lemma 4.4. *If there are non-centered times arbitrarily close to 0, then there exist $y \in M$ and $y' \in \mathcal{I}(y) \setminus \{y\}$.*

Proof. Recall that by Lemma 4.3, every sufficiently small non-zero time is non-centered. Choose $t_n \searrow 0$. Fix $y_0 \in M$ and let

$$p_n = p(y_0, t_n) \in W^s_{\text{loc}}(y_0).$$

Then $p_n \neq y_0$. Fix a small $L > 0$. Flowing backwards expands intrinsic stable distance d^s, so there is a first time $T_n > 0$ such that

$$d^s\big(X^{-T_n}(y_0), X^{-T_n}(p_n)\big) = L. \tag{4.7}$$

Equation (4.3) implies $T_n \to \infty$. Passing to a subsequence, we have

$$y_n := X^{-T_n}(y_0) \longrightarrow y, \qquad y'_n := X^{-T_n}(p_n) \longrightarrow y'.$$

By (4.7),

$$y' \in W^s_{\text{loc}}(y), \qquad d^s(y, y') = L.$$

For $w \in W^u(y_n)$ set $z = X^{T_n}(w) \in W^u(y_0)$ and define

$$\begin{aligned} F_n(w) &= X^{-T_n}\big(\widehat{Y}^{t_n}(z)\big), \\ P_n(w) &= X^{-T_n}(p(z, t_n)), \\ Q_n(w) &= X^{-T_n}(q(z, t_n)). \end{aligned}$$

The map F_n is a homeomorphism from $W^u(y_n)$ onto an unstable leaf, and Q_n is the weak stable holonomy homeomorphism between the same two leaves. Backward contraction along W^u and (4.4) yield

$$\sup_{w \in W^u(y_n)} d^u\big(F_n(w), Q_n(w)\big) \longrightarrow 0,\ n \to \infty. \tag{4.8}$$

Also, (4.2) gives the exact flow displacement

$$Q_n(w) = X^{\beta(z,t_n)}(P_n(w)), \qquad \sup_{w \in W^u(y_n)} d\big(Q_n(w), P_n(w)\big) \longrightarrow 0, \quad n \to \infty, \tag{4.9}$$

where the convergence follows from (4.3).

Choose a small $\rho > 0$ and consider the unstable plaques

$$D^u_n := W^u_\rho\big(F_n(y_n)\big).$$

Since $P_n(y_n) = y'_n$, equations (4.8)–(4.9) show that the centers $F_n(y_n)$ and y'_n converge together to y'. Thus D^u_n and the plaques $W^u_\rho(y'_n)$ both converge in the C^0 plaque topology to $W^u_\rho(y')$.

The holonomies Q_n are uniformly proper on $W^u_{loc}(y_n)$. Hence, after shrinking ρ, there exists $\rho' > \rho$ such that every sequence $\{z_n\} \in D^u_n$ can be written as $z_n = F_n(w_n)$ with $w_n \in W^u_{\rho'}(y_n)$. By passing to a subsequence, we have $w_n \to w \in W^u_{\text{loc}}(y)$. By (4.8)–(4.9), the points z_n and $P_n(w_n)$ have the same limit. But

$$P_n\big(W^u_{\rho'}(y_n)\big) \subset \text{Sat}^s_{\text{loc}}\big(W^u_{\rho'}(y_n)\big),$$

so continuity of the stable foliation places this limit in $W^s_{\text{loc}}(w)$. Since $\{z_n\}$ was arbitrary, it follows that

$$W^u_\rho(y') \subset \text{Sat}^s_{\text{loc}}\big(W^u_{\text{loc}}(y)\big).$$

Thus $y' \in \mathcal{I}(y)$. □

Lemma 4.5. *Assume that* $\dim E^s = 1$. *If there are non-centered times arbitrarily close to zero, then* W^s *and* W^u *are jointly integrable.*

Proof. Use the notation from the proof of Lemma 4.4, and let $J_n \subset W^s_{\mathrm{loc}}(y_n)$ be the stable interval with endpoints y_n and y'_n. These intervals converge to the stable interval $J \subset W^s_{\mathrm{loc}}(y)$ with endpoints y and y'.

Fix $y'' \in J$ and choose $y''_n \in J_n$ with $y''_n \to y''$. Set $I_n = [\min\{0, t_n\}, \max\{0, t_n\}]$. The map

$$s \longmapsto X^{-T_n}\big(p(y_0, s)\big), \qquad s \in I_n,$$

is a continuous path in the one-dimensional stable leaf joining y_n to y'_n. Hence there is s_n between 0 and t_n such that

$$X^{-T_n}\big(p(y_0, s_n)\big) = y''_n.$$

In particular, $s_n \to 0$. Repeating the final plaque-convergence argument of Lemma 4.4, with s_n in place of t_n and the same T_n, gives

$$W^u_\rho(y'') \subset \mathrm{Sat}^s_{\mathrm{loc}}\big(W^u_{\mathrm{loc}}(y)\big)$$

after shrinking ρ if necessary. Thus $J \subset \mathcal{I}(y)$.

The stable displacements corresponding to sufficiently small positive and negative times lie on opposite sides of y_0. Indeed, otherwise continuity would give arbitrarily small $a, b > 0$ such that $p(y_0, -a) = p(y_0, b)$. Then $\widehat{Y}^{-a}(y_0)$ and $\widehat{Y}^{b}(y_0)$ would lie in the same weak unstable leaf, so

$$p\big(\widehat{Y}^{-a}(y_0), a + b\big) = \widehat{Y}^{-a}(y_0).$$

By Lemma 4.2, $a + b$ would be centered, contradicting Lemma 4.3 and the hypothesis.

Using the same T_n as above, backward expansion along W^s and the intermediate value theorem give $\bar{t}_n > 0$ such that $\bar{t}_n \to 0$ and

$$d^s\big(X^{-T_n}(y_0), X^{-T_n}(p(y_0, -\bar{t}_n))\big) = L,$$

where $X^{-T_n}(p(y_0, -\bar{t}_n))$ lies on the side opposite to y'_n. After passing to a subsequence,

$$X^{-T_n}(p(y_0, -\bar{t}_n)) \longrightarrow \bar{y}'$$

for some $\bar{y}'$ on the side of y opposite to y'. Repeating the argument above with $-\bar{t}_n$ in place of t_n, and with the same T_n, shows that the stable interval $\bar{J}$ with endpoints y and $\bar{y}'$ is contained in $\mathcal{I}(y)$. Hence y is an interior point of $J \cup \bar{J}$, and the unstable plaques through a smaller interval about y form a local product disk saturated by both W^s and W^u.

Let U be the set of points admitting such a disk. It is open and X^t-invariant. Since y_0 was arbitrary and $y \in \overline{X^{\mathbb{R}}(y_0)} \cap U$, the closure of every orbit meets U. The set $M \setminus U$ is closed and invariant, hence, it must be empty. Therefore $U = M$, and W^s and W^u are jointly integrable. □

4.3 The subbundle step

Recall that, by definition, $\mathcal{I}(y) \subset W^s_{\mathrm{loc}}(y)$, and if $w \in \mathcal{I}(y)$, then a local strong unstable plaque through w is contained in the local stable saturation of a strong unstable plaque through y.

Lemma 4.6. *Let* X^t *be an Anosov flow whose strong unstable foliation is minimal, and assume that* E^u *is* C^1. *If* $\mathcal{I}(y) \setminus \{y\} \neq \varnothing$ *for some* y, *then there is a non-zero continuous* DX^t*-invariant subbundle* $E \subset E^s$. *The subbundle* E *integrates to an* X^t*-invariant foliation* $\mathcal{F} \subset W^s$. *Moreover,* $E \oplus E^u$ *integrates to an* X^t*-invariant foliation subfoliated by* $\mathcal{F}$ *and* W^u.

Proof. The subbundle E was already constructed in Section 3, Proposition 3.4; it remains only to prove that $E = E_{\min}$ is non-zero. Suppose, to the contrary, that E is trivial, and take $y' \in \mathcal{I}(y) \setminus \{y\}$. Thne there is an $r > 0$ such that

$$\mathrm{Hol}^u_{y,z}(y') \in W^s_{\mathrm{loc}}(z) \qquad \text{for every } z \in W^u_r(y).$$

Choose $t_n \to +\infty$ such that $x_n := X^{t_n}(y) \longrightarrow x$, $n \to \infty$. Since $y' \in W^s_{\mathrm{loc}}(y)$, we also have $x'_n := X^{t_n}(y') \longrightarrow x$.

By Lemma 3.3, we can pick a small $\rho > 0$ there are finitely many points $z_1, \dots, z_k \in W^u_\rho(x)$, including $z_1 = x$, such that

$$\bigcap_{i=1}^{k} \big(D_x \,\mathrm{Hol}^u_{x,z_i}\big)^{-1} E^s(z_i) = E(x) = \{0\}.$$

Let $\mathcal{G}_i$ be the pullback of the strong stable foliation by the corresponding unstable holonomy. The product of local submersions defining the $\mathcal{G}_i$ has injective differential at x. Consequently, after restricting to a local product box about x, the common plaques of the continued foliations $\mathcal{G}_i$ contain no two distinct points sufficiently close to x. By the C^1 dependence of unstable holonomy, the same conclusion holds, uniformly, on every weak stable plaque obtained from $W^{0s}_{\mathrm{loc}}(x)$ by a sufficiently short unstable holonomy, using the same points $z_1, \dots, z_k$.

For all sufficiently large n, let $q_n \in W^u_{\mathrm{loc}}(x)$ be the point such that $x_n \in W^{0s}_{\mathrm{loc}}(q_n)$. Then $q_n \to x$, and both x_n and x'_n lie in $W^{0s}_{\mathrm{loc}}(q_n)$ and are sufficiently close to x. For each i, the point

$$\mathrm{Hol}^u_{q_n,z_i}(x_n) \in W^u(x_n)$$

is at uniformly bounded unstable distance from x_n. Hence backward contraction gives

$$X^{-t_n}\big(\mathrm{Hol}^u_{q_n,z_i}(x_n)\big) \in W^u_r(y) \qquad \text{for all sufficiently large } n.$$

Equivariance of unstable holonomy under the flow therefore implies

$$\mathrm{Hol}^u_{q_n,z_i}(x'_n) \in W^s_{\mathrm{loc}}\big(\mathrm{Hol}^u_{q_n,z_i}(x_n)\big), \qquad i = 1, \dots, k.$$

Thus x_n and x'_n lie in the same plaque of every continued $\mathcal{G}_i$. Hence $x_n = x'_n$ for all sufficiently large n, a contradiction. Hence E is non-trivial. □

4.4 The weak unstable preserving alternative

The following proposition is the remaining ingredient in the proof of Theorem 1.

Proposition 4.7. *If every sufficiently small time is centered, then there exist $\lambda > 0$ and a homeomorphism $\psi\colon N \to M$ such that*

$$\psi \circ Y^t = X^{\lambda t} \circ \psi, \qquad t \in \mathbb{R}.$$

Proof. For every sufficiently small t, centeredness gives a unique time $\tau(y,t)$ such that

$$X^{-\tau(y,t)}\big(\widehat{Y}^t(y)\big) \in W^u_X(y).$$

The function $\tau(\cdot,t)$ is constant on strong unstable leaves. By minimality it is constant on M, and we can write it as $\tau(t)$. Composition of the local leaf relations gives

$$\tau(t+s) = \tau(t) + \tau(s)$$

whenever s, t, and $s+t$ are sufficiently small. Therefore $\tau(t) = \lambda t$ for small t and some $\lambda \in \mathbb{R}$. Given arbitrary t, choose n so large that t/n is in this range and iterate the local leaf relation n times. This gives

$$\varphi\big(W^u_Y(Y^t(x))\big) = X^{\lambda t}\big(W^u_X(\varphi(x))\big). \tag{4.10}$$

Since Y^t expands W^u_Y it is fairly clear from the above relation that $X^{\lambda t}$ should also expand W^u_X and, hence, λ must be positive. Nevertheless, let us give a detailed argument for this; at the same time we will prepare for building the conjugacy. Define

$$\Theta_T = Y^{-T} \circ \varphi^{-1} \circ X^{\lambda T}, \qquad T \geq 0.$$

For $z \in M$ and $0 \leq s \leq 1$, (4.10) gives

$$Y^{-s}\big(\varphi^{-1}(X^{\lambda s}(z))\big) \in W^u_Y\big(\varphi^{-1}(z)\big).$$

The intrinsic unstable distance between these two points is uniformly bounded by compactness. Taking $z = X^{\lambda T}(y)$ and applying backward contraction along W^u_Y therefore yields

$$d^u_Y\big(\Theta_{T+s}(y), \Theta_T(y)\big) \leq Ce^{-\chi T}, \qquad y \in M,\ 0 \leq s \leq 1.$$

Hence Θ_T converges uniformly to a continuous onto map $\Theta\colon M \to N$ satisfying

$$\Theta \circ X^{\lambda t} = Y^t \circ \Theta. \tag{4.11}$$

If $\lambda = 0$, then (4.11) and surjectivity imply that Y^t is the identity flow, which is impossible.

If $\lambda < 0$ and $y' \in W^u_X(y)$, then

$$d^u_X(X^{\lambda T}(y'), X^{\lambda T}(y)) \longrightarrow 0,\ T \to \infty.$$

The images of corresponding points under φ^{-1} lie on the same Y-unstable leaf and their intrinsic distance also tends to zero. Applying Y^{-T} only further contracts this distance. Hence $\Theta(y') = \Theta(y)$. Minimality of W^u_X would make Θ constant, contradicting surjectivity. Thus $\lambda > 0$.

Now let

$$\Phi_T = X^{-\lambda T} \circ \varphi \circ Y^T = \Theta_T^{-1}.$$

Since $\lambda > 0$, the same convergence argument, now using backward contraction along W^u_X, shows that Φ_T converges uniformly to a continuous map $\psi\colon N \to M$. Since Φ_T and Θ_T are mutual inverses and both converge uniformly, their limits are inverse:

$$\psi = \Theta^{-1}.$$

Hence ψ is a homeomorphism. Equation (4.11) gives: $\psi \circ Y^t = X^{\lambda t} \circ \psi$. □

4.5 Proofs of Theorems 1 and 3, Addendum 2, and Corollary 4

We can now summarize the argument. If every sufficiently small time is centered, then Proposition 4.7 gives the first alternative. Otherwise there are non-centered times arbitrarily close to zero, so Lemma 4.4 gives $y' \in \mathcal{I}(y) \setminus \{y\}$, $y' \neq y$, and Lemma 4.6 gives the second alternative. This proves Theorem 1.

Suppose now that E^u_X is not necessarily C^1, but that X^t is locally *su*-non-degenerate. Then Lemma 4.4 rules out non-centered times arbitrarily close to zero. Hence every sufficiently small time is centered, and Proposition 4.7 gives the conjugacy. This proves Addendum 2.

Now we prove Theorem 3. Suppose first that W^s_X and W^u_X are not jointly integrable. Then the preceding argument applies again. If there are non-centered times arbitrarily close to zero, Lemma 4.5 gives joint integrability, a contradiction. Hence every sufficiently small time is centered, and Proposition 4.7 gives the desired conjugacy.

It remains to consider the jointly integrable case. By Theorem 2.4, X^t is the suspension of a codimension-one Anosov diffeomorphism $f\colon \mathbb{T}_X \to \mathbb{T}_X$ with constant roof C_X. By Newhouse's theorem [N70], $\mathbb{T}_X$ is a torus. The constant-height tori of the suspension are precisely the closures of the strong unstable leaves of X^t. Then $\mathbb{T}_Y := \varphi^{-1}(\mathbb{T}_X)$ is the closure of a strong unstable leaf of Y^t. In particular, W^u_Y is not minimal, and by Theorem 2.3 we have that Y^t is also a suspension of a codimension-one Anosov diffeomorphism $g\colon \mathbb{T}_Y \to \mathbb{T}_Y$ with constant roof C_Y.

Just as in [BC25, Section 3.2], one can see that φ intertwines monodromies on $\mathbb{T}_Y$ and $\mathbb{T}_X$:

$$\varphi_* \circ g_* = f_* \circ \varphi_*.$$

Franks' classification [F70] therefore gives a homeomorphism $h\colon \mathbb{T}_Y \to \mathbb{T}_X$ such that $h \circ g = f \circ h$. Suspending h and taking $\lambda = C_X/C_Y$ gives $\psi \circ Y^t = X^{\lambda t} \circ \psi$ with $\lambda > 0$. This proves Theorem 3.

We finish by proving Corollary 4. Recall that if $X^t \in \mathcal{A}(M)$, then E^u_X is C^1; this follows by applying [GRH23, Section 2.1] to the reversed flow. Hence Theorem 1 applies to such flows. Further, [GRH23, Proposition 4.1], again applied after reversing time, gives a C^1-open and C^∞-dense subset $\mathcal{V} \subset \mathcal{A}(M)$ such that, for $X^t \in \mathcal{V}$, there is no non-trivial continuous DX^t-invariant subbundle $E \subset E^s_X$ which integrates to a foliation $\mathcal{F} \subset W^s_X$ and for which $E \oplus E^u_X$ integrates to a foliation subfoliated by $\mathcal{F}$ and W^u_X.

Theorem 1 applies to X^t and Y^t, and the second alternative is ruled out by the property of $\mathcal{V}$. Hence we obtain $\lambda > 0$ and a conjugacy $\psi\colon M \to M$ such that

$$\psi \circ Y^t = X^{\lambda t} \circ \psi.$$

Finally, [GRH23, Theorem 1.2] yields C^1 regularity of the conjugacy ψ.

5 Proof of Theorem 5

We now prove Theorem 5. Throughout this section, $\pi\colon M \to B$ is a principal K-bundle and X^t is an isometric extension of the Anosov flow $\bar{X}^t$.

Fix an unstable equivalence $\varphi\colon N \to M$. As in the previous proof, transplant Y^t to M:

$$\widehat{Y}^t = \varphi \circ Y^t \circ \varphi^{-1}.$$

It is a continuous flow preserving the strong unstable foliation of X^t.

5.1 Centering the transplanted flow

Fix $\delta_1 > 0$ so that the following local product intersection is defined for $|t| < \delta_1$, and let

$$p(y,t) = [y, \widehat{Y}^t(y)] = W^s_{\mathrm{loc}}(y) \cap W^{cu}_{\mathrm{loc}}\big(\widehat{Y}^t(y)\big). \tag{5.1}$$

Call t *centered* if $p(y,t) = y$ for every $y \in M$. As in Lemma 4.2, the set of y satisfying $p(y,t) = y$ is closed and W^u-saturated. By minimality assumption if $p(y,t) = y$ for some $y \in M$ then t is centered.

For $(s,k) \in \mathbb{R} \times K$ let

$$\Phi_{s,k} = R_k \circ X^s. \tag{5.2}$$

For small s the maps $\Phi_{s,k}$ give local charts on the center leaves.

Lemma 5.1. *Every sufficiently small time is centered.*

Proof. This is almost the same as Lemma 4.4; we include a brief proof for completeness. Suppose otherwise and choose non-centered times with $0 < |t_n| \to 0$. Local product structure inside W^{cu} gives unique small $\sigma(y,t) \in \mathbb{R}$ and $\kappa(y,t) \in K$ such that

$$q(y,t) := \Phi_{\sigma(y,t),\kappa(y,t)}(p(y,t)) \in W^u(\widehat{Y}^t(y)).$$

These coordinates tend uniformly to $(0, \mathrm{id}_K)$ as $t_n \to 0$, $d^u(q(y,t_n), \widehat{Y}^{t_n}(y))$ is uniformly bounded, and, for fixed t_n, $q(\,\cdot\,, t_n)$ restricts to a center-stable holonomy homeomorphism on each unstable leaf.

Fix y_0 and let $p_n = p(y_0, t_n) \neq y_0$. As in Lemma 4.4, choose $T_n \to \infty$ so that, after passing to a subsequence,

$$X^{-T_n}(y_0) \longrightarrow y, \qquad X^{-T_n}(p_n) \longrightarrow y' \in W^s_{\mathrm{loc}}(y) \setminus \{y\}.$$

Commutation of X^t with the fiber action gives

$$\sup_z d\big(X^{-T_n}(q(z,t_n)), X^{-T_n}(p(z,t_n))\big) \longrightarrow 0,$$

while the uniform unstable bound and backward contraction give

$$\sup_z d^u\big(X^{-T_n}(q(z,t_n)), X^{-T_n}(\widehat{Y}^{t_n}(z))\big) \longrightarrow 0;$$

here the suprema are taken over $z \in W^u(y_0)$. The plaque-limit argument from Lemma 4.4 therefore produces a non-trivial local *su*-integrability relation. This contradicts local *su*-non-degeneracy, so every sufficiently small time is centered. □

5.2 The center homomorphism

By Lemma 5.1, the path $t \mapsto \widehat{Y}^t(y)$ lies in $W^{cu}_{\mathrm{loc}}(y)$. Hence there is a unique pair

$$(\tau(y,t), k(y,t)) \in \mathbb{R} \times K$$

near $(0, \mathrm{id}_K)$ such that

$$\widehat{Y}^t(y) \in W^u\big(\Phi_{\tau(y,t),k(y,t)}(y)\big). \tag{5.3}$$

Because $\widehat{Y}^t$, X^t, and the isometric action all preserve the strong unstable foliation, local uniqueness implies that $\tau(y,t)$ and $k(y,t)$ are constant along strong unstable leaves. Minimality of W^u implies that they are independent of y; we write them as $\tau(t)$ and $k(t)$.

Local uniqueness of the center coordinate gives

$$\tau(t+s) = \tau(t) + \tau(s), \qquad k(s)k(t) = k(t+s) = k(t)k(s)$$

whenever s, t, and $s+t$ are sufficiently small. Consequently

$$\tau(t) = \lambda t$$

for small t and some $\lambda \in \mathbb{R}$, and the local map k extends uniquely to a continuous homomorphism $k\colon \mathbb{R} \to K$. Define the associated rotation flow by $R^t(y) = y \cdot k(t)$. Iterating the local leaf relation shows that, for every $t \in \mathbb{R}$,

$$\varphi\big(W^u_Y(Y^t(x))\big) = W^u(Z^t(\varphi(x))), \tag{5.4}$$

where

$$Z^t := R^t \circ X^{\lambda t}. \tag{5.5}$$

The proof that $\lambda > 0$ is the same as in Proposition 4.7, with very minor adjustments.

5.3 The correcting maps

With Z^t as in (5.5), $\lambda > 0$, define the correcting maps

$$\varphi_T = Z^{-T} \circ \varphi \circ Y^T, \qquad T \geq 0, \tag{5.6}$$

for which (5.4) gives $\varphi_T(x) \in W^u(\varphi(x))$. As in the proof of Proposition 4.7—now using Z^{-T}, which also contracts W^u, —the maps φ_T converge uniformly to a continuous $\psi\colon N \to M$ such that

$$\psi(x) \in W^u(\varphi(x)), \qquad \sup_x d^u(\psi(x), \varphi(x)) < \infty, \tag{5.7}$$

and, passing to the limit in $\varphi_T \circ Y^t = Z^t \circ \varphi_{T+t}$, we have

$$\psi \circ Y^t = Z^t \circ \psi = R^t \circ X^{\lambda t} \circ \psi. \tag{5.8}$$

Finally, set $\Theta_T = Y^{-T} \circ \varphi^{-1} \circ Z^T$. The same argument shows that Θ_T converges uniformly to a continuous onto map Θ. Since $\varphi_T = \Theta_T^{-1}$ and both families converge uniformly, the limits are mutually inverse; hence ψ is a homeomorphism. This finishes the proof of Theorem 5.

6 Geometric applications

The corollaries follow by combining the dynamical results above with the standard marked-length-spectrum rigidity theorems. We give the short reductions to these well-known results.

6.1 Geodesic flows

Proof of Corollary 6. The geodesic flows are contact Anosov flows, so Proposition 2.10 and Addendum 2 give $\lambda > 0$ and a homeomorphism $\psi\colon T^1N \to T^1M$ satisfying $\psi \circ Y^t = X^{\lambda t} \circ \psi$. Since $n \geq 3$, this conjugacy induces an isomorphism $\rho\colon \pi_1(N) \to \pi_1(M)$ and hence a homotopy equivalence. Comparison of periodic orbits gives

$$\ell_{g_0}(\rho[\gamma]) = \lambda\, \ell_g([\gamma]) = \ell_{\lambda^2 g}([\gamma]), \qquad [\gamma] \in [\pi_1(N)].$$

Thus $(N, \lambda^2 g)$ and (M, g_0) have the same marked length spectrum. Marked length spectrum rigidity for negatively curved locally symmetric metrics [H99, BCG95] gives the asserted isometry. □

Proof of Corollary 7. Again Proposition 2.10 and Addendum 2 give $\lambda > 0$ and a conjugacy $\psi \circ Y^t = X^{\lambda t} \circ \psi$. The correcting maps in the proof of Addendum 2 show that ψ is homotopic to the given unstable equivalence, hence to the identity. It therefore preserves the marking, and comparison of periods gives again

$$\ell_{g_0}([\gamma]) = \lambda\, \ell_g([\gamma]) = \ell_{\lambda^2 g}([\gamma]), \qquad [\gamma] \in [\pi_1(M)].$$

For any fixed non-trivial $[\gamma_0]$, the same identity shows that $\lambda = \ell_{g_0}([\gamma_0])/\ell_g([\gamma_0])$, so $\lambda \to 1$ as $g \to g_0$. After decreasing ε, local marked-length-spectrum rigidity [GL19, Corollary 1.1] applies to $\lambda^2 g$ and g_0 and gives the asserted isometry. □

6.2 Frame flows

Proof of Remark 8. Each oriented frame flow is an isometric extension on the principal bundle $\mathrm{Fr}(M_i, g_i) \to T^1 M_i$ with fiber $SO(n-1)$; local *su*-non-degeneracy follows from Proposition 2.10. Write R_k^X and R_k^Y for the right $SO(n-1)$-actions on $\mathrm{Fr}(M_1, g_1)$ and $\mathrm{Fr}(M_2, g_2)$, respectively. The actions R^X and R^Y commute with X^t and Y^t, respectively. Thus Theorem 5 gives $\lambda > 0$, a homomorphism $\kappa\colon \mathbb{R} \to SO(n-1)$, and a homeomorphism $\psi\colon \mathrm{Fr}(M_2, g_2) \to \mathrm{Fr}(M_1, g_1)$ such that

$$\psi \circ Y^t = Z^t \circ \psi, \qquad Z^t = R^X_{\kappa(t)} \circ X^{\lambda t}.$$

We only need to check that ψ preserves the compact fibers. For $k \in SO(n-1)$, define

$$Q_k := \psi \circ R^Y_{k^{-1}} \circ \psi^{-1}.$$

Since $R^Y_{k^{-1}}$ commutes with Y^t, the map Q_k commutes with Z^t. When k is close to the identity element, the map Q_k is uniformly close to the identity map, and hyperbolicity implies that $Q_k(x) \in W^c_Z(x) = W^c_X(x)$. Hence there are continuous functions $a_k(x) \in SO(n-1)$ and $s_k(x) \in \mathbb{R}$ such that

$$Q_k(x) = R^X_{a_k(x)}\big(X^{s_k(x)}(x)\big).$$

Uniqueness of these center coordinates and preservation of strong unstable leaves imply that a_k and s_k are constant along W^u_X-leaves. Minimality makes them independent of x; write the resulting constants as $a(k)$ and $s(k)$.

We have $Q_{kh} = Q_k \circ Q_h$ for all $k, h \in SO(n-1)$. Finite-order elements are dense in $SO(n-1)$. Pick such an element k close to the identity such that $k^N = \mathrm{id}$ for some $N \geq 1$. Then

$$\mathrm{id} = Q_k^N = R^X_{a(k)^N} \circ X^{Ns(k)}.$$

Projecting to the base gives $\bar{X}^{Ns(k)} = \mathrm{id}$, and hence $s(k) = 0$. Such k are dense in the neighborhood of id; hence, s vanishes near the identity. Thus Q_k is vertical for k near the identity. Since $SO(n-1)$ is connected, every element is a finite product of elements in this neighborhood, and the group law shows that Q_k is vertical for every $k \in SO(n-1)$.

For $y \in \mathrm{Fr}(M_2, g_2)$, its fiber is its $SO(n-1)$-orbit, and

$$\psi\big(R^Y_k(y)\big) = Q_{k^{-1}}\big(\psi(y)\big), \qquad k \in SO(n-1).$$

Since $Q_{k^{-1}}$ is vertical, ψ maps the fiber through y into the fiber through $\psi(y)$. Its restriction to this fiber is a continuous injection between compact connected manifolds of the same dimension. By invariance of domain and compactness, ψ maps fibers onto fibers. Consequently ψ descends to a conjugacy of the base geodesic flows up to the constant time change. □

References

[A95] R. Abe, *Geometric approach to rigidity of horocycles*, Tokyo J. Math. 18 (1995), no. 2, 271–283.

[A67] D. V. Anosov, *Geodesic flows on closed Riemannian manifolds of negative curvature*, Trudy Mat. Inst. Steklov. 90 (1967), 3–210; English translation, Proc. Steklov Inst. Math. 90 (1967), 1–235.

[BCG95] G. Besson, G. Courtois and S. Gallot, *Entropies et rigidités des espaces localement symétriques de courbure strictement négative*, Geom. Funct. Anal. 5 (1995), no. 5, 731–799.

[B75] M. I. Brin, *Topological transitivity of one class of dynamical systems and flows of frames on manifolds of negative curvature*, Funct. Anal. Appl. 9 (1975), no. 1, 8–16.

[BC25] S. Burniol Clotet, *Rigidity of the unstable foliation*, arXiv:2511.22738, 2025.

[BG80] M. Brin and M. Gromov, *On the ergodicity of frame flows*, Invent. Math. 60 (1980), no. 1, 1–7.

[FMP26] S. Fenley, K. Mann and R. Potrie, *Exotic codimension one Anosov flows*, preprint, arXiv:2605.25082, 2026.

[F87] L. Flaminio, *An extension of Ratner's rigidity theorem to n-dimensional hyperbolic space*, Ergodic Theory Dynam. Systems 7 (1987), no. 1, 73–92.

[FS90] L. Flaminio and R. J. Spatzier, *Geometrically finite groups, Patterson–Sullivan measures and Ratner's rigidity theorem*, Invent. Math. 99 (1990), no. 3, 601–626.

[F70] J. Franks, *Anosov diffeomorphisms*, Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1970, 61–93.

[GRH23] A. Gogolev and F. Rodriguez Hertz, *Smooth rigidity for codimension one Anosov flows*, Proc. Amer. Math. Soc. 151 (2023), no. 7, 2975–2988.

[GRH25] A. Gogolev and F. Rodriguez Hertz, *Smooth rigidity for very non-algebraic Anosov diffeomorphisms of codimension one*, Israel J. Math. 269 (2025), no. 2, 801–852.

[GL19] C. Guillarmou and T. Lefeuvre, *The marked length spectrum of Anosov manifolds*, Ann. of Math. (2) 190 (2019), no. 1, 321–344.

[H99] U. Hamenstädt, *Cocycles, symplectic structures and intersection*, Geom. Funct. Anal. 9 (1999), no. 1, 90–140.

[L04] C. Liverani, *On contact Anosov flows*, Ann. of Math. (2) 159 (2004), no. 3, 1275–1312.

[M83] B. Marcus, *Topological conjugacy of horocycle flows*, Amer. J. Math. 105 (1983), no. 3, 623–632.

[N70] S. E. Newhouse, *On codimension one Anosov diffeomorphisms*, Amer. J. Math. 92 (1970), 761–770.

[P72] J. F. Plante, *Anosov flows*, Amer. J. Math. 94 (1972), 729–754.

[R82] M. Ratner, *Rigidity of horocycle flows*, Ann. of Math. (2) 115 (1982), no. 3, 597–614.

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