A COUNTEREXAMPLE TO MARKED LENGTH SPECTRUM SEMI-RIGIDITY

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Abstract. Given a closed orientable negatively curved Riemannian surface (M, g), we show how to construct a perturbation (M, g') such that each closed geodesic becomes longer, and yet there is no diffeomorphism $f\colon (M, g') \to (M, g)$ which contracts every tangent vector.

1. Introduction

Let M be a smooth closed manifold and let g be a Riemannian metric on M with everywhere negative sectional curvature. It is well known that inside of every non-trivial free homotopy class of loops σ there exists a unique closed geodesic. Denoting the length of a curve γ with respect to the metric g by $\ell_g(\gamma)$, we define the *marked length spectrum* to be the function which takes a free homotopy class σ and returns the length of the unique closed geodesic γ_σ:

$$\mathrm{MLS}(g, \sigma) \coloneqq \ell_g(\gamma_\sigma).$$

This function has attracted a lot of attention due to the marked length spectrum rigidity conjecture, which says that $\mathrm{MLS}(g, \sigma) = \mathrm{MLS}(g', \sigma)$ for all σ implies that there is a diffeomorphism $f : M \to M$ such that $f^*(g') = g$ [3, Conjecture 3.1]. It is known that the conjecture holds in dimension two [6, 12], and in dimension three and higher when the manifold is locally symmetric [2, 11] or when the metrics are sufficiently close in a fine topology [13, 14]. In general, the conjecture is still open.

Variations of the marked length spectrum rigidity conjecture have been considered in recent years. For example, Butt considered an "approximate version," where one seeks to approximately recover the metric knowing the lengths of finitely many closed geodesics [5]. In this paper, we wish to explore a "semi-rigidity" question: what can be said if $\mathrm{MLS}(g) \le \mathrm{MLS}(g')$? Similar to the marked length spectrum rigidity conjecture, it is conjectured that $\mathrm{MLS}(g) \le \mathrm{MLS}(g')$ implies $\mathrm{Vol}(g) \le \mathrm{Vol}(g')$, with equality holding if and only if g is isometric to g' [8]. It is known that this holds if M is a surface or if g is conformal to g' [7, Theorems 1.1 and 1.2].

Using [7, Theorem 1.1] and Moser's homotopy trick [15], one can show that if M is a surface and g and g' are two negatively curved metrics on M with $\mathrm{MLS}(g) \le \mathrm{MLS}(g')$, then there exists a *volume shrinking diffeomorphism*, in the sense that the Jacobian is bounded above by one. Along the same lines, it is natural to ask whether or not an inequality on the marked length spectrum implies the existence of a (length) *shrinking diffeomorphism* between g and g', i.e., a diffeomorphism $f : (M, g') \to (M, g)$ such that

$$\|D_x f(v)\| \le \|v\|' \text{ for all } (x, v) \in TM.$$

Somewhat surprisingly, we will show that the answer is "no" in a rather strong sense.

Theorem 1. *Let M be a closed, connected, orientable surface and let g be a negatively curved metric on M. Then g admits arbitrarily C^∞-small perturbations g' for which there exists $\varepsilon > 0$ so that $MLS(g') > (1 + \varepsilon)MLS(g)$ and for which there does not exist a shrinking diffeomorphism $f : (M, g') \to (M, g)$.*

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Remark 2.

(1) Although the result is stated for Riemannian metrics, we note that our arguments work in the setting of Finsler metrics as well. We also note that the result gives higher dimensional examples by embedding the surface as a totally geodesic submanifold.

(2) We say that a homeomorphism $f : (M, g') \to (M, g)$ is shrinking if

$$d_{g'}(f(a), f(b)) \leq d_g(a, b) \text{ for all } a, b \in M.$$

Although the results and arguments are stated for the case where f is a diffeomorphism, they can also be adapted to the case where f is a homeomorphism.

We now give a brief and informal description of an example; the full proof will follow in Section 3. Fixing a negatively curved metric g, let γ be a g-geodesic with a single self-intersection, so that it forms a "figure eight." Suppose that we have constructed a metric g' in such a way that for every shrinking diffeomorphism $f : (M, g') \to (M, g)$ we have that $\gamma_f := f \circ \gamma$ is homotopic to γ. Indeed, if γ is the shortest figure eight g-geodesic and has multiplicity one in the length spectrum of (M, g), then γ_f is homotopic to γ, provided g' is sufficiently close to g. Furthermore, we suppose that g' is constructed in such a way that one loop of γ gets shorter by some amount ξ_1 while the other gets longer by some amount ξ_2, the marked length spectrum of g' is strictly larger than the marked length spectrum of g, and γ is a g'-geodesic after a reparameterization. We also assume that g' has been constructed in such a way that ξ_2 can be made arbitrarily close to ξ_1 without affecting the above properties.

With the above set up, we can prove the result by contradiction. Let $f : (M, g') \to (M, g)$ be a shrinking diffeomorphism and let p be the point of self-intersection for γ. Using the shrinking property of f, we have

$$\ell_g(\gamma) \leq \ell_g(\gamma_f) \leq \ell_{g'}(\gamma) = \ell_g(\gamma) + \xi_2 - \xi_1.$$

In particular, it is well-known that if ξ_2 is sufficiently close to ξ_1 then γ_f is C^0-close to γ (see Lemma 4). The contradiction now comes from the fact that we can use the very short g-geodesic connecting $f(p)$ to p along with γ_f to construct a curve in the same homotopy class as γ which has g-length shorter than γ. Namely, letting ν be the g-geodesic connecting $f(p)$ to p and letting γ_f^i be the ith loop, $i = 1, 2$, we have that the concatenated curve $\gamma_f^2 \nu \gamma_f^1 \nu^{-1}$ has g-length less than γ. Thus there cannot be a shrinking diffeomorphism in this case.

From this, we see that the challenge is showing that there is a perturbation of g such that all of the above properties hold. We also need to deal with the case where γ_f and γ are not homotopic.

2. Preliminaries

Throughout, let M be a closed, connected, orientable surface, let g be a negatively curved Riemannian metric on M, and let $p : TM \to M$ be the footprint map. The *connection map* is defined by

$$K : TTM \to TM, \quad K_{(x,v)}(\xi) := \frac{D_{p\circ V} V}{dt}(0),$$

where V is a curve on TM satisfying $V(0) = (x, v)$ and $\dot{V}(0) = \xi$. One can decompose the tangent space of the tangent bundle at $(x, v) \in TM$ using the connection and footprint maps:

$$T_{(x,v)}TM = \ker(K_{(x,v)}) \oplus \ker(D_{(x,v)}p).$$

The *Sasaki metric* g_{SM} is the metric on TTM induced by g which makes these spaces orthogonal:

$$(g_{SM})_{(x,v)}(\xi, \xi') := g_x(D_{(x,v)}p(\xi), D_{(x,v)}p(\xi')) + g_x(K_{(x,v)}(\xi), K_{(x,v)}(\xi')).$$

Note that we can lift a unit speed g-geodesic γ on M to a g_{SM}-geodesic on the unit tangent bundle S_gM by considering

$$\tilde{\gamma}(t) := g^t(\gamma(0), \dot{\gamma}(0)),$$

Let γ, η be two smooth curves on M. The *C^1-distance* between them is given by

$$d_{C^1}(\gamma, \eta) := d^{SM}_{C^0}(\tilde{\gamma}, \tilde{\eta}),$$

where d^{SM} is the metric induced by the Sasaki metric.

We recall the following two standard lemmas.

Lemma 3 (E.g. [1, Theorem 6]). *Let (M, g) be as above and let $\gamma : [0, 1] \to M$ be a g-geodesic. For every $\varepsilon > 0$ there is a $\delta > 0$ and a neighborhood $U \subseteq M$ so that if $\eta : [0, 1] \to M$ is a geodesic with $\max\{d(\gamma(0), \eta(0)), d(\gamma(1), \eta(1))\} < \delta$ and $Im(\eta) \subseteq U$, then $d_{C^1}(\gamma, \eta) < \varepsilon$.*

Lemma 4 (E.g. proof of [9, Theorem 2.2]). *Let (M, g) be as above, let g be a metric on M, and let γ be a closed g-geodesic. For all $\varepsilon > 0$, there is a $\delta > 0$ such that if η is a closed curve homotopic to γ satisfying $|\ell_g(\eta) - \ell_g(\gamma)| < \delta$, then $d_{C^0}(\eta, \gamma) < \varepsilon$.*

Next, given a g-geodesic γ, we are able to smoothly perturb the metric g to get a new metric g_s where a reparameterization of γ is still a g_s-geodesic.

Lemma 5. *Let M be a closed, connected, oriented surface, let g be a negatively curved metric on M, let γ be a g-geodesic on M, and let $p \in \gamma$. There exists an open neighborhood U of p such that for every s with $|s| < 1$ and for every $V \subseteq U$ we can find a closed neighborhood A of p satisfying $A \subseteq V$ and a smooth bump function $\kappa_s : S \to \mathbb{R}$ satisfying*

- $(\kappa_s)|_A \equiv (1 + s)^2$,
- $(\kappa_s)|_{V^c} \equiv 1$, *where V^c is the complement of V,*
- $g_s := \kappa_s g$ *defines a new metric with the property that γ defines a g_s-geodesic after a reparameterization.*

We omit the proof, as it is an easy calculation in Fermi coordinates.

Remark 6.

(1) As we vary s and V in the last lemma, we get smooth perturbations of g. Negative curvature is an open condition, so if we fix the neighborhood U coming from the claim then there is an s_0 so that if $|s| \le s_0$, then g_s is also a negatively curved metric. Thus there is an $s_0 > 0$ so that if $|s| \le s_0$ and $V \subseteq U$, then g_s is a negatively curved metric.

(2) It will be convenient to think of κ_s as the product of two real-valued functions so that $\text{supp}(\kappa_s - 1)$ is a box in local coordinates.

Given $(x, v) \in TM$ we have a unique geodesic $\gamma_{x,v}$ defined in a local chart around x such that

$$\gamma_{x,v}(0) = x, \quad \dot{\gamma}_{x,v}(0) = v.$$

The manifold M is closed, so we are able to extend this to a unique g-geodesic $\gamma_{x,v} : \mathbb{R} \to M$. The *geodesic flow* is a flow on TM defined by

$$g^t(x, v) := (\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t)).$$

Let (X, d) be a metric space. Recall that a flow $\varphi^t : X \to X$ is *expansive* if for every $\varepsilon > 0$ there is a $\delta > 0$ with the property that if $d(\varphi^t(x), \varphi^{r(t)}(y)) < \delta$ for any pair of points $x, y \in X$ and for any continuous function $r : \mathbb{R} \to \mathbb{R}$ with $r(0) = 0$, then $y = \varphi^\tau(x)$ where $|\tau| < \varepsilon$. Recall that δ is called the *expansivity constant* of φ^t with respect to ε. It is well-known that negatively curved geodesic flows are hyperbolic, hence expansive.

Let g_s be a C^∞-family of negatively curved metrics. Fixing $\varepsilon > 0$, we see that each flow g_s^t will have a corresponding expansivity constant δ_s corresponding to ε. It is not hard to see that there is an $s_0 > 0$ so that if $|s| \le s_0$ then we have uniform constants δ_0 and ε_0 so that δ_0 is the expansivity constant of g_s^t with respect to ε_0.

We will need the following standard lemma which says that closed geodesics must be some distance away from each other. We include a brief proof based on expansivity for convenience.

Lemma 7. *Let M be as above and let g be a negatively curved metric. For every closed g-geodesic γ there is an open neighborhood U containing $Im(\gamma)$ such that if η is a closed g-geodesic satisfying $Im(\eta) \subseteq U$, then $Im(\eta) = Im(\gamma)$.*

Proof. As before, $p : S_gM \to M$ denotes the footprint map. Write $\gamma : [0, 1] \to M$ with $\gamma(0) = \gamma(1)$. Lifting γ, we get a g_{SM}-geodesic $\tilde{\gamma} : [0, 1] \to S_gM$. Fix some $\varepsilon > 0$ and let δ be the expansivity constant for g^t corresponding to ε. Using Lemma 3, we can find an open set U with $\mathrm{Im}(\gamma) \subseteq U$ such that if η is a closed geodesic with $\mathrm{Im}(\eta) \subseteq U$, then η is C^1-close to a reparameterization of γ. By making U small enough, this guarantees that the lift $\tilde{\eta}$ of η is within δ of a reparameterization of $\tilde{\gamma}$. By the definition of expansivity, this guarantees that $\mathrm{Im}(\gamma) = \mathrm{Im}(\eta)$. $\square$

3. Proof of Theorem 1

Throughout, let M be a closed, connected, oriented surface and let g be a negatively curved metric on M. As noted in [4, Theorem 4.2.4], there is at least one shortest g-geodesic with a self-intersection, and such a g-geodesic will have exactly one self-intersection. Let $\mathcal{F}$ be the collection of shortest g-geodesics with a single self-intersection; they all have the same length and there are finitely many of them. For each $\gamma \in \mathcal{F}$, denote the shorter loop by γ^1 and the other loop by γ^2 so that we have $\ell_g(\gamma^1) \le \ell_g(\gamma^2)$. Let $\gamma_{\text{short}} \in \mathcal{F}$ be such that $\ell_g(\gamma^1_{\text{short}}) \le \ell_g(\gamma^1)$ for all $\gamma \in \mathcal{F}$; in other words, γ_{short} has the shortest first loop.

3.1. **Outline of the Proof.** The goal is to perturb our metric g in such a way that we get a new metric g' which is C^∞-close to g and such that the following holds:

(1) for every shrinking diffeomorphism $f : (M, g') \to (M, g)$ and $\gamma \in \mathcal{F}$, we have that there is an $\eta \in \mathcal{F}$ so that $f \circ \gamma$ is homotopic to η.

(2) each $\gamma \in \mathcal{F}$ is a g'-geodesic after reparameterization,

(3) there are constants $0 < \xi_1 < \xi_2$ so that for every $\gamma \in \mathcal{F}$ we have
$$\ell_{g'}(\gamma^1) = \ell_g(\gamma^1) - \xi_1 \text{ and } \ell_{g'}(\gamma^2) = \ell_g(\gamma^2) + \xi_2,$$

(4) $\mathrm{MLS}(g') > (1 + \varepsilon)\mathrm{MLS}(g)$.

Our method of constructing these metrics will also ensure that we are able to further perturb g in such a way so that ξ_2 is arbitrarily close to ξ_1 and the above properties hold.

Given a shrinking diffeomorphism $f : (M, g') \to (M, g)$, we can use property (1) to find an $\eta \in \mathcal{F}$ such that $\gamma_f := f \circ \gamma_{\text{short}}$ is homotopic to η. Notice that we have
$$\ell_g(\eta) = \ell_g(\gamma_{\text{short}}) \le \ell_g(\gamma_f) \le \ell_{g'}(\gamma_{\text{short}}) = \ell_g(\eta) + \xi_2 - \xi_1.$$

Using Lemma 4, we may assume that ξ_2 is close enough to ξ_1 so that if q is the point of self-intersection for η and p is the point of self-intersection for γ_f, then for any shrinking diffeomorphism we have $d_g(q, p) < \xi_1/2$.

Assuming we can construct a metric g' close to g with the above properties, we have all of the ingredients to show that there cannot be a shrinking diffeomorphism $f : (M, g') \to (M, g)$.

Proof of Theorem 1. Assume for contradiction that $f : (M, g') \to (M, g)$ is shrinking. Let γ_f, η, p and q be as above. Since γ_f is homotopic to η, we must have that γ_f^1 is homotopic to either η^1 or η^2. Without loss of generality, assume it is homotopic to η^1. Let ν be the unique g-geodesic connecting p and q. By the above discussion, we have $\ell_g(\eta) < \xi_1/2$. Concatenating γ_f^1 with ν, ν^{-1}, and η^2, we get a new figure eight curve in the same free homotopy class as η. Using the length shrinking property, notice that the first loop has length

$$\ell_g(\nu^{-1}\gamma_f^1\nu) < \xi_1 + \ell_g(\gamma_f^1) \leq \xi_1 + \ell_{g'}(\gamma_{\text{short}}^1) = \ell_g(\gamma_{\text{short}}^1) \leq \ell_g(\eta^1).$$

Thus this new curve has g-length smaller than η, which contradicts the fact that η is the curve with the shortest length in its free homotopy class. □

The main difficulty is constructing the perturbation of g so that the above properties hold. We briefly describe the intuition behind the construction. Given a curve with one intersection $\gamma \in \mathcal{F}$, we can use Lemma 5 to construct a new metric which is C^∞-close to the original metric and which makes one loop shorter and the other loop longer. We refer to the neighborhood which shrinks a loop of γ as the "shrinking neighborhood" of γ and the neighborhood which expands a loop of γ as the "expanding neighborhood" of γ. We also construct it in such a way so that, outside of a small neighborhood of γ, we have that the new metric is uniformly larger than the original metric. In this set up, it is clear that any curve which stays outside of this neighborhood must get longer, so we have an inequality on the marked length spectrum as long as the geodesic does not cross the neighborhood which shrinks a loop.

The only problem now is if a geodesic crosses the shrinking neighborhood for γ, so that a portion of it gets shorter. By shrinking the neighborhood containing the curve further, we can ensure that either the crossing geodesic has to leave the neighborhood, and thus get longer by some amount, or it has to cross the expanding neighborhood for γ. In either case, carefully adjusting parameters ensures that every geodesic gets uniformly longer, so that we have an inequality on the marked length spectrum and so that we have the constants ξ_1 and ξ_2 described above.

By adjusting the shrinking and expanding parameters described above, we can guarantee that the new metric g' and the original metric g are C^∞-close. Furthermore, letting ξ_2 approach ξ_1 will only make g' closer to g, so we can freely assume they are arbitrarily close. We will show in Claim 8 that as long as g' and g are sufficiently close, then given any shrinking diffeomorphism $f : (M, g') \to (M, g)$ and any $\gamma \in \mathcal{F}$, we have that $f \circ \gamma$ is homotopic to some $\eta \in \mathcal{F}$. As long as there is only one curve in $\mathcal{F}$, we will see that we can adjust the parameters so that this metric proves Theorem 1.

Finally, we will observe that if $\mathcal{F}$ has more than one loop, then we only need to somewhat modify the above construction. This modification, along with the above argument, completes the proof of Theorem 1.

3.2. **Construction of the Metrics.** We start by establishing property (1).

Claim 8. *There exists a C^∞ neighborhood U of g so that if $g' \in U$ then for every shrinking diffeomorphism $f : (M, g') \to (M, g)$ and every $\gamma \in \mathcal{F}$ we have that $f \circ \gamma$ is homotopic to some $\eta \in \mathcal{F}$.*

Proof. For $T \in \mathbb{R}$, let

$$P_g(T) = \{\eta \mid \eta \text{ is a } g\text{-geodesic and } \ell_g(\eta) \leq T\}.$$

We break this up into a series of steps.

Step 1: Let η_0 be a g-geodesic of shortest length and let $T_0 = \ell_g(\eta_0)$. We claim that as long as g' is sufficiently C^∞-close to g then f preserves $P_g(T_0)$ up to permutation. In other words, if $\eta \in P_g(T_0)$ and $f \circ \eta$ is homotopic to η', then $\eta' \in P_g(T_0)$.

Since f is a shrinking diffeomorphism, we have

$$\ell_g(\eta') \leq \ell_g(f \circ \eta) \leq \ell_{g'}(\eta).$$

Let $\delta > 0$ be such that $P_g(T_0) \subseteq P_g(T_0 + \delta)$. Notice there are finitely many geodesics in $P_g(T_0 + \delta)$, so if we let

$$\Lambda := \min\{\ell_g(\eta') \mid \eta' \in P_g(T_0 + \delta) \setminus P_g(T_0)\},$$

then by taking g' sufficiently C^∞-close to g we can guarantee that for every shrinking diffeomorphism we have

$$\ell_g(\eta') \leq \ell_g(f \circ \eta) \leq \ell_{g'}(\eta) < \Lambda.$$

Thus we have that $\ell_g(\eta') = T_0$ and so f preserves $P_g(T_0)$ up to permutation.

Step 2: Let η_1 be a g-geodesic of second shortest length and let $T_1 = \ell_g(\eta_1)$. Repeating the argument above, by ensuring that g' is sufficiently close to g we have that every shrinking diffeomorphism preserves $P_g(T_1)$ up to permutation. In particular, intersecting the neighborhood from this step and the last step, we see that every shrinking diffeomorphism must preserve $P_g(T_1) \setminus P_g(T_0)$ up to permutation.

Step 3: Repeating Step 2 until we reach $\ell_g(\gamma)$ for $\gamma \in \mathcal{F}$, we have that for all sufficiently close g' if $f : (M, g') \to (M, g)$ is a shrinking diffeomorphism and η is homotopic to $f \circ \gamma$, then $\ell_g(\eta) = \ell_g(\gamma)$. Next, notice that for every shrinking diffeomorphism $f : (M, g') \to (M, g)$ we have

$$\ell_g(\gamma) = \ell_g(\eta) \leq \ell_g(f \circ \gamma) \leq \ell_{g'}(\gamma).$$

By making g' closer to g if needed, we can ensure that $|\ell_g(f \circ \gamma) - \ell_g(\eta)|$ is small for every shrinking diffeomorphism. Using Lemma 4, this forces $f \circ \gamma$ to be C^0 close to η. Noting that $f \circ \gamma$ has one self-intersection, we are able to ensure that η has a self-intersection by making g' even closer to g if needed, thus $\eta \in \mathcal{F}$.

□

We now describe the procedure for modifying the metric in the case where $\mathcal{F}$ has only one curve, say γ. Taking two points $p_1 \in \gamma^1$ and $p_2 \in \gamma^2$, let U_1 and U_2 be open neighborhoods of p_1 and p_2 such that Lemma 5 applies. Let W be an open set and let A be a closed set satisfying $\mathrm{Im}(\gamma) \subseteq A \subseteq W$. Consider the following functions.

- For sufficiently small $\mu > 0$ and for $\varepsilon_1 > 0$, $\rho_1 > 0$ such that $B_{\rho_1}(p_1) \subseteq U_1$, let $\kappa_{-\varepsilon_1,\mu}$ be the bump function coming from Lemma 5 where the width of the support is μ. We call $B_{\rho_1}(p_1)$ the *shrinking neighborhood* for γ and we refer to ε_1 as the *shrinking parameter*.
- For $\varepsilon_2 > 0$ and $\rho_2 > 0$ such that $B_{\rho_2}(p_2) \subseteq U_2$, let κ_{ε_2} be the bump function coming from Lemma 5. The width and height of the support can be anything as long as it satisfies the criteria. We call $B_{\rho_2}(p_2)$ the *expanding neighborhood* and we refer to ε_2 as the *expanding parameter*.
- For $\varepsilon_3 > 0$, let κ_{ε_3} be a bump function on M, where

$$(\kappa_{\varepsilon_3})|_{(W)^c} \equiv (1 + \varepsilon_3)^2 \text{ and } (\kappa_{\varepsilon_3})|_A \equiv 1.$$

We can define a family of metrics by setting

$$g_{W,A,\rho_1,\rho_2,\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu} := \kappa_{\varepsilon_3}\kappa_{\varepsilon_2}\kappa_{-\varepsilon_1,\mu}g.$$

By construction this gives us a smooth family of metrics parameterized by W, A, ρ_i, ε_j, and μ.

We start by fixing some W_0 open and some A_0 closed satisfying the above criteria. As noted earlier, there exists a uniform upper bound $t_1 > 0$ so that for all ε_j satisfying $\varepsilon_j < t_1$ we have that the metrics $g_{W,A,\rho_1,\rho_2,\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu}$ are negatively curved. We can also choose t_1 sufficiently small so that we can apply Claim 8 with all metrics satisfying this condition. Note that this is done independently of W, A, ρ_i, and μ. We now consider ε_j satisfying $\varepsilon_j < t_1$.

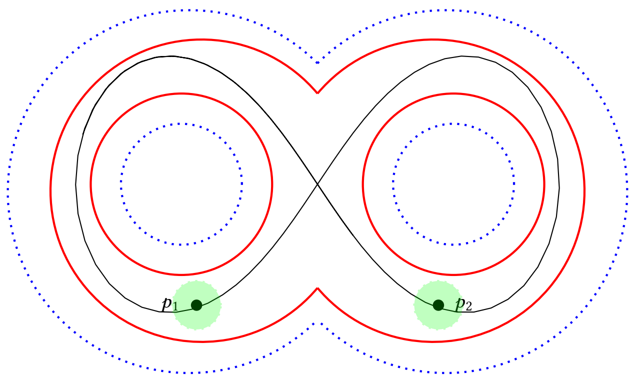

FIGURE 1. The blue dotted lines represent the neighborhood W, the red lines represent the boundary of the closed neighborhood A, and the green disks represent the sets $B_{\rho_i}(p_i)$.

Next, we fix $\rho_2 > 0$. Let W_1 be an open set such that $\mathrm{Im}(\gamma) \subseteq W_1 \subseteq W_0$. Also suppose that if W is open and $\mathrm{Im}(\gamma) \subseteq W \subseteq W_1$, then the following holds:

- if U is the open set coming from Lemma 7, then $W \subsetneq U$,
- $\mathrm{supp}(\kappa_{\varepsilon_2} - 1) \nsubseteq W$, i.e., the width of the support of $\kappa_{\varepsilon_2} - 1$ extends beyond W.

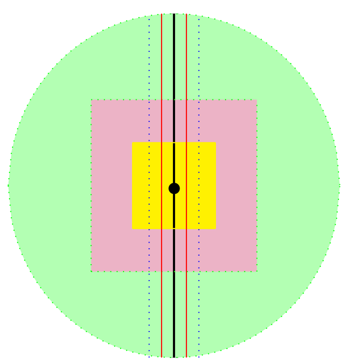

FIGURE 2. An example of $B_{\rho_2}(p_2)$ in local coordinates.

Our next step is to further adjust the size of the neighborhood W so that a $g_{W,A,\rho_1,\rho_2,\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu}$-geodesic segment contained in W is not able to "turn around." We make this precise with the following claim.

Claim 9. *There exists open set W_2 satisfying $Im(\gamma) \subseteq W_2 \subseteq W_1$ and there is a $\rho_1 > 0$ such that for all open W with $Im(\gamma) \subseteq W \subseteq W_2$ and for any closed set A with $Im(\gamma) \subseteq A \subseteq W$ there exists a $t_2 < t_1$ (depending on W and A) such that if $\varepsilon_j < t_2$, then we have the property that if η' is a closed $g_{W,A,\rho_1',\rho_2,\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu}$-geodesic such that*

- $\eta'|_{[0,T]} \subseteq W$,
- $\eta'|_{[0,T]} \cap B_{\rho_1'}(p_1)^c \neq \emptyset$,
- $\eta'(0), \eta'(T) \in B_{\rho_1'}(p_1)$,

then there is a $t \in (0,T)$ so that $\eta'(t) \in B_{\rho_2}(p_2)$.

Proof. Using Lemma 3, we see that we can find an open set W_2 and a $\rho_1 > 0$ so that if $\mathrm{Im}(\gamma) \subseteq W_2 \subseteq W_1$ and if η is a g-geodesic with segment $\eta|_{[0,T]}$ satisfying the above properties, then $\eta|_{[0,T]}$ is C^1-close to $\gamma|_I$ for some interval I. By making ρ_1 and W_2 smaller if needed, this guarantees that η is C^1-close to γ up until the self-intersection. Since it is C^1-close, we have that it cannot turn around. Adjusting ρ_1 and W_2 more if necessary, we guarantee that η must continue following the curve γ until it intersects $B_{\rho_2}(p_2)$.

Choose an open set W satisfying $\mathrm{Im}(\gamma) \subseteq W \subseteq W_2$ and choose a closed set A with $\mathrm{Im}(\gamma) \subseteq A \subseteq W$. Once these are fixed, we can find $0 < t_2 < t_1$ such that if $\varepsilon_j < t_2$ then $g_{W,A,\rho_1,\rho_2,\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu}$-geodesics are C^∞-close to their g-geodesic counterparts. In particular, we can choose t_2 so that if η' is a $g_{W,A,\rho_1,\rho_2,\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu}$-geodesic satisfying $\mathrm{Im}(\eta') \subseteq W$ and η is its g-geodesic counterpart, then $\mathrm{Im}(\eta) \subseteq W_2$. By making t_2 smaller, we can guarantee that η' must also intersect $B_{\rho_2}(p_2)$. □

Let W_2 and ρ_1 be as in Claim 9. Fix an open set W satisfying $\mathrm{Im}(\gamma) \subseteq W \subseteq W_2$. Fix a closed set A satisfying $\mathrm{Im}(\gamma) \subseteq A \subseteq W$ and satisfying $\zeta := d(\partial W, A) > 0$. We can adjust ρ_1 further so that $B_{\rho_1}(p_1) \subseteq W$. Let $T_1 > 0$ be such that everything holds for $\mu \leq T_1$. Notice that this gives us an upper bound on μ, however, we are free to make μ smaller without interfering with any of the above properties. Also notice this does not cause any issues with Claim 9, since making μ smaller can only make the metric closer to g. With the quantities W, A, and ρ_i now fixed, we now consider the family of metrics

$$g_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu} := g_{W,A,\rho_1,\rho_2,\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu}$$

for $\varepsilon_j < t_2$ and $\mu \leq T_1$.

As discussed in Section 2, there is a t_2 so that we have uniform constants $\lambda, \delta > 0$, where δ is the expansivity constant of $g^t_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu}$ with respect to λ. Using these constants and making W smaller if necessary, we can use Lemma 7 to guarantee that if η is a closed $g_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu}$-geodesic with $\mathrm{Im}(\eta) \subseteq W$, then $\mathrm{Im}(\eta) = \mathrm{Im}(\gamma)$.

In the next claim, we will be studying how lengths of curves change under the new metric $g_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu}$ in order to get a bound on the marked length spectrum. In particular, we will be interested in lower bounds on the change in length. If a segment of a geodesic is contained in W and crosses $B_{\rho_2}(p_2)$, then by construction it must cross the entire height of the box defining $\mathrm{supp}(\kappa_{\varepsilon_2} - 1)$. We now fix ρ_1 and ρ_2, and we also assume that the height of the support of κ_{ε_2} is the same as the height of the support of $\kappa_{-\varepsilon_1,\mu}$. The main focus of the next part is to adjust the width of the support of $\kappa_{-\varepsilon_1,\mu} - 1$ inside of $B_{\rho_1}(p_1)$.

Let $\tilde{O}_{\varepsilon_2} = \mathrm{supp}(\kappa_{\varepsilon_2} - 1)$ and let

$$\xi_2 := \int_{\gamma^{-1}(\tilde{O}_{\varepsilon_2})} \|\dot{\gamma}(t)\|_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu} dt - \int_{\gamma^{-1}(\tilde{O}_{\varepsilon_2})} \|\dot{\gamma}(t)\| dt.$$

Notice that this gives us a lower bound on the change in length of any geodesic segment in W which crosses $\tilde{O}_{\varepsilon_2}$ – the $g_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu}$-length of a curve η which crosses $\tilde{O}_{\varepsilon_2}$ must increase by at least ξ_2. Furthermore, ξ_2 only depends on ε_2.

To get a lower bound on geodesics which cross $\mathrm{supp}(\kappa_{-\varepsilon_1,\mu} - 1)$ is trickier. Let $\tilde{O}_{\varepsilon_1,\mu} = \mathrm{supp}(\kappa_{-\varepsilon_1,\mu} - 1)$ and let

$$\xi_1 := \int_{\gamma^{-1}(\tilde{O}_{\varepsilon_1,\mu})} \|\dot{\gamma}(t)\|_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu} dt - \int_{\gamma^{-1}(\tilde{O}_{\varepsilon_1,\mu})} \|\dot{\gamma}(t)\| dt.$$

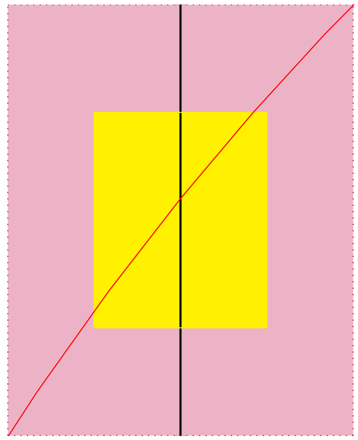

FIGURE 3. The red curve spends more time in $\tilde{O}_{\varepsilon_1,\mu}$ than γ does, so the difference between its $g_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu}$-length and g-length is larger.

Just like with ξ_2, we have that ξ_1 only depends on ε_1. In particular, it does not depend on μ.

Since γ does not cross the diameter of the box $\tilde{O}_{\varepsilon_1,\mu}$, we cannot use $-\xi_1$ to give us a lower bound on the change in length of a geodesic crossing $\tilde{O}_{\varepsilon_1,\mu}$ – a geodesic might spend more time in $\tilde{O}_{\varepsilon_1,\mu}$ compared to γ, and thus its $g_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu}$-length decreases more than γ's does. Using a compactness argument, it is not hard to see that there is a g-geodesic segment η which has the greatest change in length among all g-geodesic segments in $\tilde{O}_{\varepsilon_1,\mu}$. We define

$$A(\varepsilon_1,\mu) := \left[\int_{\eta^{-1}(\tilde{O}_{\varepsilon_1,\mu})} \|\dot{\eta}(t)\|_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu} dt - \int_{\eta^{-1}(\tilde{O}_{\varepsilon_1,\mu})} \|\dot{\eta}(t)\| dt\right] - \xi_1.$$

Notice that $A(\varepsilon_1,\mu)+\xi_1$ gives an lower bound on how much the $g_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu}$-length of a g-geodesic segment in $\tilde{O}_{\varepsilon_1,\mu}$ can shrink. Furthermore, letting μ approach 0, we see that η must get closer to γ, hence for fixed ε_1 we have $A(\varepsilon_1,\mu)\to 0$ as $\mu\to 0$. We also note that for fixed μ we have $A(\varepsilon_1,\mu)\to 0$ as $\varepsilon_1\to 0$.

We now study how the marked length spectrum relates to our choices of the ε_i and μ.

Claim 10. *Assume that $\varepsilon_j < t_3$ and $\mu < T_2$. For each ε_3 and μ, we can find a $t_4 > 0$ and an $s(\varepsilon_1,\mu) = s > 0$ so that if $\varepsilon_1 < t_4 + s < \varepsilon_2$ then*

$$MLS(g_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu}) > (1+\varepsilon) MLS(g) \text{ for some } \varepsilon = \varepsilon(\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu) > 0.$$

Proof. Let $\gamma_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu}$ be the reparameterization of γ so that it is a primitive $g_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu}$-geodesic. We break this up into a series of steps.

Step 1: Since $\mathrm{Im}(\gamma) = \mathrm{Im}(\gamma_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu})$ and the heights of $\tilde{O}_{\varepsilon_2}$ and $\tilde{O}_{\varepsilon_1,\mu}$ are the same, we have

$$\ell_{g_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu}}(\gamma_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu}) = \ell_g(\gamma) + \xi_2 - \xi_1 > \ell_g(\gamma)$$

as long as $\varepsilon_2 > \varepsilon_1$.

Step 2: Let η be a $g_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu}$-geodesic. Since geodesics are the unique length minimizers in their free homotopy class, we are done if we can show

$$\ell_{g_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu}}(\eta) > \ell_g(\eta).$$

The only problem occurs if η intersects $\tilde{O}_{\varepsilon_1,\mu}$, since everywhere else the metric $g_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu}$ makes curves longer. If $\mathrm{Im}(\eta) \subseteq W$ then the result follows by the prior step, so along with intersecting $\tilde{O}_{\varepsilon_1,\mu}$ we also assume that $\mathrm{Im}(\eta) \cap W^c \neq \emptyset$.

After intersecting $\tilde{O}_{\varepsilon_1,\mu}$, the curve has two options: either stay in W and intersect $\tilde{O}_{\varepsilon_1,\mu}$ again or leave W. If it leaves W, we use the fact that $\zeta = d(\partial W, A) > 0$. After leaving W and returning, the curve gets longer by a factor of at least $2\zeta\varepsilon_3$. In particular, once the curve returns to W, we have that the length of the curve is at least $2\zeta\varepsilon_3 - \xi_1 - A(\varepsilon_1, \mu)$. By setting t_4 to be sufficiently small, we can guarantee this is positive, so the geodesic gets longer.

By construction, we have that any geodesic staying in W and intersecting $\tilde{O}_{\varepsilon_2}$ must cross the entire length of the box. This gives us a lower bound of $\xi_2 - \xi_1 - A(\varepsilon_1, \mu)$. Notice that this is positive as long as $\varepsilon_2 > \varepsilon_1 + s$, where s is determined by $A(\varepsilon_1, \mu)$. In particular, we can find a uniform s so that if $\varepsilon_2 > t_4 + s > \varepsilon_1$ then the curve gets longer.

In either case, we see that the length gets longer. Combining the three cases lets us choose a uniform ε to bound the marked length spectrum.

□

Remark 11. Fixing ε_3 and ε_1 in the above, we can let ε_2 be arbitrarily close to ε_1 by adjusting μ.

As described in Section 3.1, this gives us the ingredients to prove the theorem provided there is only one curve in $\mathcal{F}$. If there is more than one curve in $\mathcal{F}$, then we apply the same construction in a neighborhood of each $\gamma \in \mathcal{F}$. If the curves do not intersect, the results and arguments are almost the same provided we choose the parameters for the curves in such a way so that ξ_1 and ξ_2 are uniform among all curves. Note that ε_3 will be chosen uniformly for all curves and the neighborhood W will be the union of all of the W neighborhoods for each curve.

The situation becomes trickier if two or more curves in $\mathcal{F}$ intersect. By choosing the centers of the shrinking and expanding neighborhoods away from the points of intersections and making all of the neighborhoods smaller, we are able to ensure that Claim 9 holds for each curve independently. In other words, if for $\gamma \in \mathcal{F}$ we have a geodesic intersects its shrinking neighborhood, then that geodesic must also intersect the expanding neighborhood for γ. Again, ε_3 will be chosen uniformly for all curves and W will be the union of all of the W neighborhoods for each curve, and we are able to repeat the argument in Claim 10 to ensure that the marked length spectrum gets longer. We also note that the shrinking and expanding parameters are again chosen in such a way so that ξ_1 and ξ_2 are uniform for each curve. Furthermore, by adjusting the expanding and shrinking parameters along with the widths of the shrinking neighborhoods, we may assume that ξ_2 is arbitrarily close to ξ_1. Notice that this procedure does not require any adjustments to ξ_1. This, along with the argument in Section 3.1, proves Theorem 1.

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