Exotic Topology in Geometry and Dynamics

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Abstract

The existence of exotic differentiable structures on high dimensional spheres was one of most important discoveries in topology. Another surprising and deep discovery was the non triviality of the stable pseudoisotopy space $\mathcal{P}(\mathbb{S}^1)$ of the circle $\mathbb{S}^1$; specifically here the exotic objects are non null-homotopic maps $\mathbb{S}^k \to \mathcal{P}(\mathbb{S}^1)$, for certain values of k. In the last few years the authors have used these exotic objects to obtain results in geometry and dynamics. The results in geometry concern the topology of the space of negatively curved metrics, and in dynamics the topology of the space of Anosov diffeomorphisms. In this paper we survey these results.

1 The space of negatively curved metrics.

Let M be a closed smooth manifold. We will say that a Riemannian metric on M is *hyperbolic* if all sectional curvatures are equal to -1. We denote by $\mathcal{MET}(M)$ the space of all smooth Riemannian metrics on M and we consider $\mathcal{MET}(M)$ with the smooth topology. Note that the space $\mathcal{MET}(M)$ is contractible. A subspace of metrics whose sectional curvatures lie in some interval will be denoted by placing a superscript on $\mathcal{MET}(M)$. For example, $\mathcal{MET}^{sec<\epsilon}(M)$ denotes the space all Riemannian metrics on M that have all sectional curvatures less that ϵ and $\mathcal{MET}^{sec=-1}(M)$ is the space of all hyperbolic metrics on M.

We also denote by $\mathrm{DIFF}(M)$ the group of all smooth self-diffeomorphisms of M and by $\mathcal{D}(M)$ the group $\mathbb{R}^+ \times \mathrm{DIFF}(M)$. The group $\mathcal{D}(M)$ acts on $\mathcal{MET}(M)$ by scaling and pulling-back metrics.

A natural question about a closed negatively curved manifold M is the following: is the space $\mathcal{MET}^{\,sec\,<\,0}(M)$ of negatively curved metrics on M path

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connected? This problem had been around for some time and had been posed several times in the literature. See for instance K. Burns and A. Katok ([BK85], Question 7.1). For another motivation that comes from an approach to the "topological" Lawson-Yau question see [F08, p. 167].

For 2-dimensional surfaces uniformization results of Earle and Eells [EE69] (as well as Hamilton Ricci flow [H82]) imply that the space of hyperbolic metrics $\mathcal{MET}^{sec=-1}(M^2)$ is a deformation retract of $\mathcal{MET}^{\,sec\,<\,0}(M^2)$. Also recall that $\mathcal{MET}^{sec=-1}(M^2)$ fibers over the Teichmüller space $\mathcal{T}(M^2)\cong\mathbb{R}^{6g-6}$ (here g is the genus of M^2), with contractible fiber $\mathcal{D}=\mathbb{R}^+\times \mathrm{DIFF}(M^2)$ [EE69]. Therefore $\mathcal{MET}^{sec=-1}(M^2)$ and $\mathcal{MET}^{\,sec\,<\,0}(M^2)$ are contractible.

In [FO10a] Farrell and Ontaneda proved that the space $\mathcal{MET}^{\,sec\,<\,0}(M^n)$ is never path-connected for $n\geq 10$; in fact, it has infinitely many path-components. Moreover they showed that the homotopy groups $\pi_{2p-4}(\mathcal{MET}^{\,sec\,<\,0}(M^n))$ are non-trivial for every prime number $p\in[3,(n+4)/6]$. Furthermore, they also showed that $\pi_1(\mathcal{MET}^{\,sec\,<\,0}(M^n))$ contains the infinite sum $(\mathbb{Z}_2)^\infty$ when $n\geq 14$. These results about homotopy groups are true for each path component of $\mathcal{MET}^{sec<0}(M^n)$; i.e., relative to any base point. We state explicitly the main result in [FO10a] as Theorem 1.1 below.

Note that $\mathrm{DIFF}(M)$ leaves invariant all spaces $\mathcal{MET}^{sec\in I}(M)$, for any $I\subset\mathbb{R}$. For any metric g on M we denote by $\mathrm{DIFF}(M)\,g$ the orbit of g by the action of $\mathrm{DIFF}(M)$. Define the map $\Lambda_g:\mathrm{DIFF}(M)\to\mathcal{MET}(M)$ by $\Lambda_g(\phi)=\phi_*g$. Then the image of Λ_g is the orbit $\mathrm{DIFF}(M)\,g$ of g.

Theorem 1.1 ([FO10a]). *Let M be a closed smooth n-manifold and let g be a negatively curved Riemannian metric on M. Then we have that:*

i. the map

$$\pi_0(\Lambda_g):\pi_0(\,\mathrm{DIFF}(M)\,)\to\pi_0(\,\mathcal{MET}^{\,sec\,<\,0}(M)\,)$$

is not constant, provided $n\geq 10$.

ii. the homomorphism

$$\pi_1(\Lambda_g):\pi_1(\,\mathrm{DIFF}(M)\,)\to\pi_1(\,\mathcal{MET}^{\,sec\,<\,0}(M)\,)$$

is non-zero, provided $n\geq 14$.

iii. For $k=2p-4$, p prime integer and $1<k\leq\frac{n-8}{3}$, the homomorphism

$$\pi_k(\Lambda_g):\pi_k(\,\mathrm{DIFF}(M)\,)\to\pi_k(\,\mathcal{MET}^{\,sec\,<\,0}(M)\,)$$

is non-zero. (See Remark 1.4 below.)

Addendum to Theorem 1.1. *We have that the image of $\pi_0(\Lambda_g)$ is infinite and in cases (ii.), (iii.) mentioned in Theorem 1.1, the image of $\pi_k(\Lambda_g)$ is not finitely generated. In fact we have:*

i. *For* $n \geq 10$, $\pi_0(\,\mathrm{DIFF}(M)\,)$ *contains* $(\mathbb{Z}_2)^\infty$, *and* $\pi_0(\Lambda_g)|_{(\mathbb{Z}_2)^\infty}$ *is one-to-one.*

ii. *For* $n \geq 14$, *the image of* $\pi_1(\Lambda_g)$ *contains* $(\mathbb{Z}_2)^\infty$.

iii. *For* $k = 2p - 4$, p *prime integer and* $1 < k \leq \frac{n-8}{3}$, *the image of* $\pi_k(\Lambda_g)$ *contains* $(\mathbb{Z}_p)^\infty$. *See Remark 1.4 below.*

A key ingredient in the proof of Theorem 1.1 is the non-triviality (and structure) of certain homotopy groups $\pi_k\mathcal{P}(\mathbb{S}^1)$ of the stable pseudoisotopy space of the circle $\mathbb{S}^1$.

For any a, b, $a < b < 0$, the map Λ_g factors through the inclusion map $\mathcal{MET}^{\,a \,\leq\, sec \,\leq\, b}(M) \hookrightarrow \mathcal{MET}^{sec<0}(M)$ provided that $g \in \mathcal{MET}^{\,a \,\leq\, sec \,\leq\, b}(M)$. Therefore the above results hold also if the decoration "$sec < 0$" is replaced by "$a \leq sec \leq b$ ", that is, they hold for the space $\mathcal{MET}^{\,a \,\leq\, sec \,\leq\, b}(M^n)$. This is stated in the next corollary.

Corollary 1.2. *Let* M *be a closed smooth* n*-manifold,* $n \geq 10$. *Let* $a < b < 0$ *and assume that* $\mathcal{MET}^{\,a \,\leq\, sec \,\leq\, b}(M)$ *is not empty. Then the inclusion map* $\mathcal{MET}^{\,a \,\leq\, sec \,\leq\, b}(M) \hookrightarrow \mathcal{MET}^{sec<0}(M)$ *is not null-homotopic. Indeed, the induced maps, at the* k*-homotopy level, are not constant for* $k = 0$, *and non-zero for the cases (ii.), (iii.) of Theorem 1.1. Furthermore, the image of these maps satisfy a statement analogous to the one in the Addendum to Theorem 1.1.*

When $a = b = -1$ we obtain

Corollary 1.3. *Let* M *be a closed hyperbolic* n*-manifold,* $n \geq 10$. *Then the inclusion map* $\mathcal{MET}^{sec=-1}(M) \hookrightarrow \mathcal{MET}^{sec<0}(M)$ *is not null-homotopic. Indeed, the induced maps, at the* k*-homotopy level, are not constant for* $k = 0$, *and non-zero for the cases (ii.), (iii.) of Theorem 1.1. Furthermore, the image of these maps satisfy a statement analogous to the one in the Addendum to Theorem 1.1.*

Hence, taking $k = 0$ in Corollary 1.3, we get that for any closed hyperbolic manifold (M^n, g), $n \geq 10$, there is a hyperbolic metric g' on M such that g and g' cannot be joined by a path of negatively curved metrics.

Also, by taking $b = -1$ and $a = -1 - \epsilon$ ($\epsilon > 0$) in Corollary 1.2 we obtain that the space $\mathcal{MET}^{\,-1-\epsilon \,\leq\, sec \,\leq -1}(M^n)$ of ϵ-pinched negatively curved Riemannian metrics on M has infinitely many path components, provided it is not empty and $n \geq 10$. The higher homotopy groups $\pi_k(\mathcal{MET}^{\,-1-\epsilon \,\leq\, sec \,\leq -1}(M))$, are non-zero for the cases (ii.), (iii.) of Theorem 1.1.

Remark 1.4. *The restriction on* $n = dim\, M$ *given in Theorem 1.1, its addendum and its corollaries are certainly not optimal. In particular, in (iii.) it can be improved to* $1 < k < \frac{n-10}{2}$ *by using Igusa's "Surjective Stability Theorem" ([I88], p. 7).*

Let M^n be a closed smooth manifold of dimension $dim\, M = n$. It follows Theorem 1.1 that $\pi_k \mathcal{MET}^{\,sec\,<\,0}(M^n)$ is non-trivial for certain pairs (n, k), provided $\mathcal{MET}^{\,sec\,<\,0}(M)$ is not the empty set, that is, M admits a negatively curved metric. Let $\mathcal{MET}^{\,sec\,\leq\,0}(M)$ be the subspace of $\mathcal{MET}(M)$ of all non-positively curved Riemannian metrics on M. In [FO09b] Farrell and Ontaneda generalized to $\mathcal{MET}^{\,sec\,\leq\,0}(M)$ the main result in [FO10a] (stated as Theorem 1.1 here), provided $\pi_1 M$ is (word) hyperbolic:

Theorem 1.5 ([FO09b]). *Let M^n be a closed smooth manifold with hyperbolic fundamental group $\pi_1 M$. If the space $\mathcal{MET}^{\,sec\,\leq\,0}(M)$ is non-empty, then (i.), (ii.), (iii.) in the statement of Theorem 1.1 (and its Addendum) hold when we replace $\mathcal{MET}^{\,sec\,<\,0}(M)$ by $\mathcal{MET}^{\,sec\,\leq\,0}(M)$.*

Remark 1.6. *If M^n is smooth and closed and $\mathcal{MET}^{\,sec\,<\,0}(M)$ is non-empty, it follows from the Theorem above and its proof, that the inclusion map*

$$\mathcal{MET}^{\,sec\,<\,0}(M) \hookrightarrow \mathcal{MET}^{\,sec\,\leq\,0}(M)$$

is "very non-trivial" at the π_k-level, for certain n and k.

This result is quite surprising because non-positive curvature is a "non-stable" property, while negative curvature is stable (it is an open set in $\mathcal{MET}(M)$). It is not even known whether $\mathcal{MET}^{\,sec\,\leq\,0}(M)$ is locally contractible or even locally connected. Moreover, in our specific case, there are two additional obstacles to pass from negative curvature to non-positive curvature. First, since in the non-positively curved case there can exist parallel geodesic rays emanating perpendicularly from a closed geodesic, the obstructions in the negatively curved case defined in [FO10a] (that lie in the pseudoisotopy space of $\mathbb{S}^1 \times \mathbb{S}^{n-2}$) may not be homeomorphisms at infinity. Farrell and Ontaneda show in [FO09b] that the obstructions for the non-positively curved case now lie in $\mathrm{CELL}(\mathbb{S}^1 \times \mathbb{S}^{n-2} \times [0, 1])$, where $\mathrm{CELL}(L)$ is the space of cellular self maps of L.

The second problem is that there may now exist a whole family of closed geodesics freely homotopic to a given one. Farrell and Ontaneda's earlier papers ([FO09a], [FO10a], [FO10b], [FO10c]) strongly relied on the fact there is a *unique* such closed geodesic. Moreover, they strongly used the fact that such unique closed geodesics depend smoothly on the metric. This does not happen in non-positive curvature. Even worse: there are examples of smooth families g_t, $t \in [0, 1]$, of non-positively curved metrics such that there is no continuous path of closed g_t-geodesics joining a closed g_1-geodesic to a closed g_0-geodesic (all closed geodesics in the same free homotopy class). See for instance the "swinging neck" in Appendix A of [FO09b]. Farrell and Ontaneda dealt with this by incorporating the closed geodesics into the system, but there is a price for this: instead of dealing with discs (to prove that an element is zero in a homotopy group) they had to deal with more complicated spaces which they called "cellular discs". Because of this the use of shape theory ([DS78], [MS82]) became necessary.

2 The Teichmüller and the moduli spaces of negatively curved metrics.

The quotient space $\mathcal{M}(M) = \mathcal{MET}(M)/\mathcal{D}(M)$ is called the *moduli space of metrics on M*. Denote by κ the quotient map

$$\mathcal{MET}(M) \xrightarrow{\kappa} \mathcal{M}(M).$$

We say that a property of Riemannian metrics is *geometric* if it is invariant under isometries, that is, by the action of $\mathrm{DIFF}(M)$ on $\mathcal{MET}(M)$ (recall that if $\phi \in \mathrm{DIFF}(M)$ and $g \in \mathcal{MET}(M)$ then $\phi : (M, \phi^* g) \to (M, g)$ is an isometry). Hence if two Riemannian metrics represent the same element in $\mathcal{M}(M)$ then they possess the same geometric properties. Clearly, the study of the moduli space of metrics is of fundamental importance not just in geometry but in other areas of mathematics as well. (See, for instance, [Bess87] Ch. 4)

As in the case of $\mathcal{MET}(M)$, it is also interesting to consider subspaces of $\mathcal{M}(M)$ that represent some geometric property. One obvious choice is to consider metrics with constant curvature. For instance, let M_g be an orientable two-dimensional manifold of genus $g > 1$. Consider the moduli space of all hyperbolic metrics on M_g, that is, the subspace of $\mathcal{M}(M_g)$ formed by elements that are represented by Riemannian metrics of constant sectional curvature equal to -1. The moduli space of all hyperbolic metrics is the quotient of another well known space: the Teichmüller space of M_g. This space is a subspace of the quotient of $\mathcal{MET}(M_g)$ by the subgroup of $\mathrm{DIFF}(M_g)$ formed by all smooth self-diffeomorphisms of M_g which are homotopic to the identity; namely, it is the subspace represented by hyperbolic metrics. Then the moduli space is the quotient of the Teichmüller space by the action of $\mathrm{Out}\,(\pi_1(M_g))$, the group of outer automorphisms of the fundamental group of M_g.

We want to generalize the definition of the Teichmüller space to higher dimensions. The obvious choice for a definition would be the quotient of the space of all hyperbolic metrics by the action of the group of all smooth self-diffeomorphisms which are homotopic to the identity. But Mostow's Rigidity Theorem implies that, in dimensions ≥ 3, this space contains (at most) one point.

Let us go back to dimension two for a moment. Hamilton's Ricci flow [H82] shows that every negatively curved metric on M_g, $g > 1$, can be canonically deformed (through negatively curved metrics) to a hyperbolic metric. This deformation commutes with the action of $\mathrm{DIFF}(M_g)$, therefore the Teichmüller space of M_g is canonically a deformation retract of the space which is the quotient of all negatively curved Riemannian metrics on M_g by the action of the group of all smooth self-diffeomorphisms which are homotopic to the identity. Also, instead of considering the space of all negatively curved metrics we can consider the space of all *pinched* negatively curved metrics, or for that matter, the space of all Riemannian metrics. These are the concepts that we will generalize. Next, we give detailed definitions and introduce some notation.

Denote by $\mathrm{DIFF}_0(M)$ the subgroup of $\mathrm{DIFF}(M)$ of all smooth diffeomorphisms of M which are homotopic to the identity id_M. Also, denote by $\mathcal{D}_0(M)$ the group $\mathbb{R}^+ \times \mathrm{DIFF}_0(M)$. We call the quotient space $\mathcal{T}(M) = \mathcal{MET}(M)/\mathcal{D}_0(M)$ the *Teichmüller space of metrics* on M. Therefore we have the following diagram of quotient spaces:

$$\mathcal{MET}(M) \xrightarrow{\xi} \mathcal{T}(M) \xrightarrow{\varsigma} \mathcal{M}(M)$$

where the first arrow is the quotient map induced by the action of $\mathcal{D}_0(M)$ on $\mathcal{MET}(M)$ and the second arrow is the quotient map induced by the action of $\mathrm{DIFF}(M)/\mathrm{DIFF}_0(M)$ on $\mathcal{T}(M)$. And we have $\kappa = \varsigma\xi$.

Introduce the following abbreviated notation

$$\mathcal{MET}^{\epsilon}(M) = \mathcal{MET}^{-1-\epsilon \leq sec \leq -1}(M)$$

The quotient space $\mathcal{M}^{\epsilon}(M) = \mathcal{MET}^{\epsilon}(M)/\mathcal{D}(M)$ is called the moduli space of ϵ-pinched negatively curved metrics on M.

Remark 2.1. *The space $\mathcal{MET}^{\epsilon}(M)$ is not $\mathcal{D}(M)$ invariant (the problem is the scaling). So, by $\mathcal{MET}^{\epsilon}(M)/\mathcal{D}(M)$ we mean that two elements in $\mathcal{MET}^{\epsilon}(M)$ are identified if they are related by an element in $\mathcal{D}(M)$. For an alternative "invariant" (but a bit longer) definition see [FO09a].*

Also $\mathcal{T}^{\epsilon}(M) = \mathcal{MET}^{\epsilon}(M)/\mathcal{D}_0(M)$ is called the Teichmüller space of ϵ-pinched negatively curved metrics on M. In particular, $\mathcal{T}^{sec<0}(M) = \mathcal{T}^{\infty}(M)$ is the Teichmüller space of all negatively curved metrics on M, and $\mathcal{T}^{sec=-1}(M) = \mathcal{T}^{0}(M)$ is the Teichmüller space of all hyperbolic metrics (in this case $\mathcal{T}^{sec=-1}(M) = \mathcal{MET}^{sec=-1}(M)/\mathrm{DIFF}_0(M)$, see remark 2.1). And similarly for the moduli space $\mathcal{M}$. Of particular interest are the spaces:

- $\mathcal{MET}^{\,sec\,<\,0}(M)$, the space of negatively curved metrics on M
- $\mathcal{T}^{\,sec\,<\,0}(M)$, the Teichmüller space of negatively curved metrics on M and
- $\mathcal{M}^{\,sec\,<\,0}(M)$, the moduli space of negatively curved metrics on M.

Note that the inclusions $\mathcal{MET}^{\epsilon}(M) \hookrightarrow \mathcal{MET}(M)$ induce inclusions $\mathcal{T}^{\epsilon}(M) \hookrightarrow \mathcal{T}(M)$. Also note that, for $\delta \geq \epsilon$, these inclusions factor as follows: $\mathcal{MET}^{\epsilon}(M) \hookrightarrow \mathcal{MET}^{\delta}(M) \hookrightarrow \mathcal{MET}(M)$ and $\mathcal{T}^{\epsilon}(M) \hookrightarrow \mathcal{T}^{\delta}(M) \hookrightarrow \mathcal{T}(M)$.

Remark 2.2. *If M_g is an orientable two-dimensional manifold of genus $g > 1$, then the original Teichmüller space of M is (in our notation) $\mathcal{T}^{sec=-1}(M_g)$, and $\mathcal{T}^{sec=-1}(M_g)$ is homeomorphic to $\mathbb{R}^{6g-6}$ (see [EL81]). Hence $\mathcal{T}^{sec=-1}(M_g)$ is contractible. By the uniformization techniques mentioned above ([EE69], [H82]), it follows that $\mathcal{T}^{\epsilon}(M_g)$, $\mathcal{T}^{sec<0}(M_g)$, $\mathcal{T}(M_g)$ are all contractible. (This is also true for non-orientable surfaces of Euler characterisitc < 0.)*

Remark 2.3. *Let M be a closed hyperbolic manifold. If $\dim M \geq 3$, Mostow's Rigidity Theorem [Mo67] implies that $\mathcal{T}^{sec=-1}(M) = *$; i.e., $\mathcal{T}^{sec=-1}(M)$ contains exactly one point. Therefore $\mathcal{MET}^{sec=-1}(M) = \mathrm{DIFF}_0(M)$. It also follows that $\mathcal{T}^{sec=-1}(M)$ is contractible when $\dim M \geq 2$.*

In dimensions two and three it is known that $\mathcal{D}_0(M)$ (and hence $\mathcal{MET}^{sec=-1}(M)$) is contractible. (This is due to Earle and Eells [EE69] in dimension two and to Gabai [G01] in dimension three.) This is certainly false in dimensions ≥ 11, because $\pi_0(\mathcal{D}_0(M))$ is not finitely generated (see [FJ89b], Corollary 10.16 and 10.28), and it is reasonable to conjecture that $\mathcal{D}_0(M)$ is also not contractible for dimension n, $5 \leq n \leq 10$.

Remark 2.4. *Let M be a hyperbolic manifold. Then the action of $\mathcal{D}_0(M)$ on $\mathcal{MET}(M)$ is free (see Lemma 1.1 of [FO09a]). Since $\mathcal{MET}(M)$ is contractible by Ebin's Slice Theorem [E68] we have that $\mathcal{D}_0(M) \to \mathcal{MET}(M) \to \mathcal{T}(M)$ is a principal $\mathcal{D}_0(M)$-bundle and $\mathcal{T}(M)$ is the classifying space $B\mathcal{D}_0(M)$ of $\mathcal{D}_0(M)$.*

Let M is a closed hyperbolic manifold. Then $\mathcal{MET}^{\epsilon}(M)$ interpolates between $\mathcal{MET}^0(M) = \mathcal{MET}^{sec=-1}(M)$ (which is equal to $\mathcal{D}_0(M)$) and $\mathcal{MET}(M)$ (which is contractible). Likewise $\mathcal{T}^{\epsilon}(M)$ interpolates between $\mathcal{T}(M)$ (which is equal to $B\mathcal{D}_0(M)$) and $\mathcal{T}^{sec=-1}(M)$ (which is contractible). We have the following commutative diagram:

$$\begin{array}{ccccccc}
\mathcal{MET}^{sec=-1}(M) & \hookrightarrow & \mathcal{MET}^{\epsilon}(M) & \hookrightarrow & \mathcal{MET}^{\infty}(M) & \hookrightarrow & \mathcal{MET}(M) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{T}^{sec=-1}(M) & \hookrightarrow & \mathcal{T}^{\epsilon}(M) & \hookrightarrow & \mathcal{T}^{\infty}(M) & \hookrightarrow & \mathcal{T}(M)
\end{array}$$

All vertical arrows represent quotient maps by the action of the group $\mathcal{D}_0(M)$.

In [FO09a] Farrell and Ontaneda proved that the last two horizontal arrows of the lower row of the diagram above are not in general homotopic to a constant map. In particular $\mathcal{T}^{\epsilon}$, $0 \leq \epsilon \leq \infty$, in general, is not contractible. More specifically, they proved that under certain conditions on the dimension n of the hyperbolic manifold M, the manifold M has a finite cover N (which depends on ϵ) such that $\pi_k\Big(\mathcal{T}^{\epsilon}(N)\Big) \to \pi_k\Big(\mathcal{T}(N)\Big)$ is non-zero. In particular, $\mathcal{T}^{\epsilon}(N)$ is not contractible. The requirements on the dimension n are implied by one of the following conditions: n is larger than some constant or n is larger than 5 but in this last case we need that $\Theta_{n+1} \neq 0$, where Θ_ℓ denotes the group of homotopy spheres of dimension ℓ. Here is a more detailed statement of this result:

Theorem 2.5 ([FO09a])**.** *For every integer $k_0 \geq 1$ there is an integer $n_0 = n_0(k_0)$ such that the following holds. Given $\epsilon > 0$ and a closed real hyperbolic n-manifold M with $n \geq n_0$, there is a finite-sheeted cover N of M such that, for every $1 \leq$*

$k \leq k_0$ with $n+k \equiv 3$ mod 4, the map $\pi_k\Big(\mathcal{T}^\epsilon(N)\Big) \to \pi_k\Big(\mathcal{T}(N)\Big)$, induced by the inclusion $\mathcal{T}^\epsilon(N) \hookrightarrow \mathcal{T}(N)$, is non-zero. Consequently $\pi_k\Big(\mathcal{T}^\epsilon(N)\Big) \neq 0$. In particular, $\mathcal{T}^\delta(N)$ is not contractible, for every δ such that $\epsilon \leq \delta \leq \infty$ (provided $k_0 \geq 4$).

Here (and in the corollary below) we consider the given hyperbolic metric as the basepoint for $\mathcal{T}(N)$, $\mathcal{T}^\epsilon(N)$.

For $k_0 = 1$ we can take $n_0(1) = 6$ and drop the condition $n+k \equiv 3$ *mod* 4. Hence we obtain the following corollary to (the proof of) Theorem 2.5.

Corollary 2.6. *Let M be a closed hyperbolic manifold of dimension n, $n \geq 6$. Assume that $\Theta_{n+1} \neq 0$. Then for every $\epsilon > 0$ there is a finite-sheeted cover N of M such that $\pi_1\Big(\mathcal{T}^\epsilon(N)\Big) \neq 0$. Therefore $\mathcal{T}^\epsilon(N)$ is not contractible.*

Recall that an n-dimensional π manifold is a manifold that embeds in $\mathbb{R}^{2n+2}$ with trivial normal bundle. Every real hyperbolic manifold has a finite-sheeted cover that is a π manifold (see [Su79], p.553). We have the following addition to the statements of Theorem 2.5 and the Corollary 2.6.

Addendum to Theorem 2.5 and the Corollary 2.6 . *We can choose $N = M$ in the statements of Theorem 2.5 and the Corollary 2.6, provided M is a π-manifold and the radius of injectivity of M at some point is sufficiently large (how large depending only on the dimension of M).*

A key ingredient in the proof of Theorem 2.5 is the existence of (exotic) non-trivial elements in the group Θ_n of homotopy n-spheres. These elements are used to construct non-nullhomotopic maps $\phi : \mathbb{S}^k \to \text{DIFF}_0(M)$, such that the map $u \mapsto \phi(u)^* g$ is nullhomotopic (here g is a fixed hyperbolic metric).

We now make some comments on Theorem 2.5 and the diagram above.

1. Since $\mathcal{MET}(M)$ is contractible, Theorem 2.5 implies that, for a general hyperbolic manifold M, the map $\pi_k\Big(\mathcal{MET}^\epsilon(M)\Big) \to \pi_k\Big(\mathcal{T}^\epsilon(M)\Big)$, induced by the second vertical arrow of the diagram, is not onto for some k.

2. By Remark 2.2, the lower row of the diagram above is homotopically trivial in dimension 2. In dimension 3 one could ask the same: is the lower row of the diagram above homotopically trivial in dimension 3? In view of a result of Gabai (see [G01]), this is equivalent to asking: is $\mathcal{T}^{sec<0}(M^3)$ contractible?

3. Let M be a hyperbolic manifold. Consider the upper row of the diagram. It follows from a result of Ye on the Ricci flow (see [Ye93]) that, provided the dimension of M is even, there is an $\epsilon_0 = \epsilon_0(M) > 0$ such that for all $\epsilon \leq \epsilon_0$ the inclusion map $\mathcal{MET}^\epsilon(M) \to \mathcal{MET}^{sec<0}(M)$ is $\mathcal{D}_0(M)$-equivariantly homotopic to a retraction $\mathcal{MET}^\epsilon(M) \to \mathcal{MET}^{sec=-1}(M) \subset \mathcal{MET}^{sec<0}(M)$. This has the following consequences. First the retraction above descends to a retraction $\mathcal{T}^{\epsilon}(M) \to \mathcal{T}^{sec=-1}(M)$, hence the inclusion map $\mathcal{T}^{\epsilon}(M) \to \mathcal{T}^{sec<0}(M)$ is homotopic to a constant map (provided $\epsilon \leq \epsilon(M)$), and thus induces the zero homomorphism $\pi_k\Big(\mathcal{T}^{\epsilon}(M)\Big) \to \pi_k\Big(\mathcal{T}(M)\Big)$ for all k. Second, the inclusion map $\mathcal{D}_0(M) = \mathcal{MET}^{sec=-1}(M) \to \mathcal{MET}^{\epsilon}(M)$ induces monomorphisms $\pi_k\Big(\mathcal{D}_0(M)\Big) = \pi_k\Big(\mathcal{MET}^{sec=-1}(M)\Big) \to \pi_k\Big(\mathcal{MET}^{\epsilon}(M)\Big)$, provided $\epsilon \leq \epsilon(M)$. Theorem 2.5 then shows that in many cases $\epsilon_0(M) < \infty$.

4. Let M be a hyperbolic manifold. Since $\mathrm{DIFF}(M)/\mathrm{DIFF}_0(M) \cong \mathrm{Out}\Big(\pi_1(M)\Big)$ we have that $\mathcal{M}(M) \cong \mathcal{T}(M)/\mathrm{Out}\Big(\pi_1(M)\Big)$ or, in general,

$$\mathcal{M}^{\epsilon}(M) \cong \mathcal{T}^{\epsilon}(M)/\mathrm{Out}\Big(\pi_1(M)\Big).$$

Note that $\mathrm{Out}(\pi_1(M))$ is a finite group, provided $dim\, M \geq 3$.

5. Let M be a hyperbolic manifold. We can consider the quotients of $\mathcal{MET}(M)$ and $\mathcal{MET}^{\epsilon}(M)$ by $\mathrm{DIFF}^0(M)$, the connected component of the identity 1_M in $\mathrm{DIFF}(M)$, instead of by the larger group $\mathrm{DIFF}_0(M)$. Since the quotient group $\mathrm{DIFF}_0(M)/\mathrm{DIFF}^0(M)$ is discrete, it can be easily checked from the proofs of results of Farrell and Ontaneda that the statement of Theorem 2.5 also holds for the inclusion of the quotients: $\mathcal{MET}^{\epsilon}(M)/\mathrm{DIFF}^0(M) \to \mathcal{MET}(M)/\mathrm{DIFF}^0(M)$, with the strengthened restriction "$2 \leq k \leq k_0$" and proviso "(provided $k_0 \geq 5$)".

Now, recall that we have a diagram of quotient spaces:

$$\mathcal{MET}(M) \xrightarrow{\xi} \mathcal{T}(M) \xrightarrow{\varsigma} \mathcal{M}(M)$$

We make remarks about Theorems 1.1 and 2.5, and this diagram:

Remark 2.7. *The non-trivial elements in $\pi_k\, \mathcal{MET}^{\,sec\,<\,0}(M)$ mentioned in Theorem 1.1 and constructed in [FO10a] have trivial image by the map induced the quotient map by*

$$\xi : \mathcal{MET}^{\,sec\,<\,0}(M) \to \mathcal{T}^{\,sec\,<\,0}(M).$$

Remark 2.8. *The nonzero classes in $\pi_k\, \mathcal{T}^{\,sec\,<\,0}(M)$ given in Theorem 2.4 and constructed in [FO09a] are not in the image of the map induced by*

$$\xi : \mathcal{MET}^{\,sec\,<\,0}(M) \to \mathcal{T}^{\,sec\,<\,0}(M).$$

In [FO10b] Farrell and Ontaneda proved that the quotient map

$$\xi : \mathcal{MET}^{\,sec\,<\,0}(M) \to \mathcal{T}^{\,sec\,<\,0}(M)$$

is not trivial at some homotopy levels, provided the hyperbolic manifold M satisfies certain conditions. Hence there are elements in certain homotopy groups

$\pi_k \mathcal{MET}^{\,sec<0}(M)$ that survive in $\pi_k \mathcal{T}^{\,sec<0}(M)$. Moreover the results in [FO10b] hold for $k = 0$, a case not covered by Theorem 2.5 and [FO09a].

To state the results in [FO10b] consider the following relation between non-negative integers k and n

$$(\,*\,) \qquad \begin{cases} \mathbf{1}.\ \ k = 0 \ \text{ and } \ n \geq 10 \\ \mathbf{2}.\ \ k = 1 \ \text{ and } \ n \geq 12 \\ \mathbf{3}.\ \ k = 2p-4,\ p > 2 \text{ prime}, \ \text{ and } \ n \geq 3k+8 \end{cases}$$

Theorem 2.9 ([FO10b])**.** *For every closed hyperbolic n-manifold M there is a finite-sheeted cover N of M such that the maps*

$$\pi_k(\mathcal{MET}^{\,sec<0}(N)) \to \pi_k(\mathcal{T}^{\,sec<0}(N))$$

are nonzero, provided (n, k) satisfy $(*)$.

A similar result was also proven for homology.

Theorem 2.10 ([FO10b])**.** *For every closed hyperbolic n-manifold M there is a finite-sheeted cover N of M such that the maps*

$$H_k(\mathcal{MET}^{\,sec<0}(N)) \to H_k(\mathcal{T}^{\,sec<0}(N))$$

are nonzero, provided (n, k) satisfy $(*)$.

Addendum to Theorems 2.9 and 2.10. *The statements of Theorems 2.9 and 2.10 remain true if we replace the decoration "sec $<$ 0" on both $\mathcal{MET}^{\,sec<0}(M)$ and $\mathcal{T}^{\,sec<0}(M)$ by "-1-$\epsilon <$ sec $\leq$-1".*

The non-trivial classes in $\pi_k \mathcal{T}^{\,sec<0}(M)$ given in Theorem 2.9 besides coming from $\mathcal{MET}^{\,sec<0}(M)$ have a different nature and genesis: the classes given by Theorem 2.5 and [FO09a] come from the existence of exotic spheres, while the classes given in Theorem 2.9 arise from the non-triviality and structure of certain homotopy groups of the space of pseudoisotopies of the circle $\mathbb{S}^1$. The strength of the techniques used to prove Theorem 2.9 allowed Farrell and Ontaneda to prove also a homology version of Theorem 2.9, which is given in Theorem 2.10.

By taking $k = 0$ in Theorem 2.9 we have:

Corollary 2.11. *Let M be a closed hyperbolic n-manifold, $n > 9$. Then M admits a finite-sheeted cover N such that $\mathcal{T}^{\,sec<0}(N)$ is disconnected.*

Remark 2.12. *As mentioned before, the case $k = 0$ stated in Corollary 2.11 above was not covered by Theorem 2.5.*

Remark 2.13. *The Addendum to Theorem 2.9 implies that Corollary 2.11 remains true if we replace the decoration "sec $<$ 0 " by "-1-$\epsilon \leq$ sec $\leq$-1". In this case N depends not just on n but also on $\epsilon > 0$.*

So far we have studied the space of negatively curved metrics $\mathcal{MET}^{\,sec<0}(M)$, the Teichmüller space of negatively curved metrics $\mathcal{T}^{\,sec<0}(M)$ and the quotient map $\xi : \mathcal{MET}^{\,sec<0}(M) \to \mathcal{T}^{\,sec<0}(M)$. In general, it is very hard to get information on the quotient map

$$\mathcal{MET}^{\,sec<0}(M) \xrightarrow{\kappa} \mathcal{M}^{\,sec<0}(M) = \mathcal{T}^{\,sec<0}(M)/\mathrm{Out}\,\pi_1(M),$$

from the knowledge obtained about the map $\mathcal{MET}^{\,sec<0}(M) \xrightarrow{\kappa} \mathcal{T}^{\,sec<0}(M)$, because the action of $\mathrm{Out}\,\pi_1(M)$ on $\mathcal{T}^{\,sec<0}(M)$ could be quite general. But in [FO10c] Farrell and Ontaneda proved that the map $\pi_k\,\mathcal{MET}^{\,sec<0}(M) \to \pi_k\,\mathcal{M}^{\,sec<0}(M)$, induced by the quotient map $\kappa : \mathcal{MET}^{\,sec<0}(M) \to \mathcal{M}^{\,sec<0}(M)$, is not trivial provided the hyperbolic manifold M satisfies certain conditions. It is not known whether every closed hyperbolic manifold admits a finite cover that satisfies these conditions but it is proved in [FO10c] that this is true if M is non-arithmetic.

Theorem 2.14 ([FO10c])**.** *Let M be a closed non-arithmetic hyperbolic manifold and k a non-negative integer, with $(k, dim\,M)$ satisfying (*). Then M has a finite-sheeted cover N such that the maps*

$$\pi_k\Big(\mathcal{MET}^{\,sec<0}(N)\Big) \xrightarrow{\pi_k(\kappa)} \pi_k\Big(\mathcal{M}^{\,sec<0}(N)\Big)$$

$$H_k\Big(\mathcal{MET}^{\,sec<0}(N)\Big) \xrightarrow{H_k(\kappa)} H_k\Big(\mathcal{M}^{\,sec<0}(N)\Big)$$

are non-zero. In particular $\pi_k(\,\mathcal{M}^{\,sec<0}(N)\,)$ and $H_k(\,\mathcal{M}^{\,sec<0}(N)\,)$ are non-trivial.

The statements of Theorem 2.14 hold also for ϵ-pinched negatively curved metrics:

Addendum to Theorem 2.14. *The statement of Theorem 2.14 remains true if we:*

i. *replace the decoration "sec < 0" on $\mathcal{MET}^{\,sec<0}(M)$ in Theorem 2.14 by "-1-ϵ < sec ≤-1".*

ii. *replace the decoration "sec < 0" on both $\mathcal{MET}^{\,sec<0}(M)$ and $\mathcal{M}^{\,sec<0}(M)$ in Theorem 2.14 by "-1-ϵ < sec ≤-1".*

3 Bundles with Negatively Curved Fibers

Let M be a closed smooth manifold. By a *smooth bundle over X, with fiber M*, we mean a locally trivial bundle for which the change of coordinates between two local sections over, say, $U_\alpha, U_\beta \subset X$ is given by a continuous map

$U_\alpha \cap U_\beta \to \mathrm{DIFF}(M)$. A *smooth bundle map* between two such bundles over X is a bundle map such that, when expressed in a local chart as $U \times M \to U \times M$, the induced map $U \to \mathrm{DIFF}(M)$ is continuous. In this case we say that the bundles are *smoothly equivalent*. Smooth bundles over a space X, with fiber M, modulo smooth equivalence, are classified by $\Big[X, B\Big(\mathrm{DIFF}(M)\Big)\Big]$, the set of homotopy classes of continuous maps from X to the classifying space $B\Big(\mathrm{DIFF}(M)\Big)$.

Remark 3.1. *In what follows we will be considering everything pointed: X comes with a base point x_0, the bundles come with smooth identifications between the fibers over x_0 and M, and the bundle maps preserve these identifications. Also, classifying maps are base point preserving maps.*

From now on we assume that X is simply connected. Then we obtain a reduction in the structural group of these bundles: smooth bundles over a simply connected space X, with fiber M, modulo smooth equivalence, are classified by $\Big[X, B\Big(\mathrm{DIFF}_0(M)\Big)\Big]$, where $\mathrm{DIFF}_0(M)$ is the space of all self diffeomorphisms of M that are homotopic to the identity id_M. In what follows we assume X to be simply connected. If we assume in addition that M is aspherical with $\pi_1(M)$ centerless (e.g. admits a negatively curved metric) then old results of Borel [Bor83], Conner-Raymond [CR77] say that $\mathrm{DIFF}_0(M)$ acts freely on $\mathcal{MET}(M)$. Moreover, Ebin's Slice Theorem [E68] assures us that $\mathrm{DIFF}_0(M) \to \mathcal{MET}(M) \to \Big(\mathcal{MET}(M)/\mathrm{DIFF}_0(M)\Big)$ is a locally trivial bundle. Hence, since $\mathcal{MET}(M)$ is contractible, we can write $B\Big(\mathrm{DIFF}_0(M)\Big) = \mathcal{MET}(M)/\mathrm{DIFF}_0$. Recall that $\mathcal{T}(M) = \mathcal{MET}(M)\Big/\Big(\mathbb{R}^+ \times \mathrm{DIFF}_0(M)\Big)$ is the Teichmüller Space of Riemannian Metrics on M. Since $\mathcal{T}(M)$ is homotopy equivalent to $\mathcal{MET}(M)/\mathrm{DIFF}_0(M)$ we can also write $B\Big(\mathrm{DIFF}_0(M)\Big) = \mathcal{T}(M)$. Therefore smooth bundles over a simply connected space X, with aspherical fiber M and $\pi_1(M)$ centerless, modulo smooth equivalence, are classified by $\Big[X, \mathcal{T}(M)\Big]$.

Let $\mathcal{S}$ be a complete collection of local sections of the bundle $\mathcal{MET}(M) \to \mathcal{T}(M)$. Using $\mathcal{S}$ and a given map $f : X \to \mathcal{T}(M)$ we can explicitly construct a smooth bundle E over X, with fiber M. Yet, with these data we seem to get a little more: we get a Riemannian metric on each fiber E_x of the bundle E. This collection of Riemannian metrics does depend on $\mathcal{S}$, but it is uniquely defined (i.e. independent of the choice of $\mathcal{S}$) up to smooth equivalence.

Of course, any bundle with fiber M admits such a fiberwise collection of Riemannian metrics because $\mathcal{MET}(M)$ is contractible, so we seem to have gained nothing. On the other hand, in the presence of a geometric condition we do get a meaningful notion. We explain this next.

If we are given a map $X \to \mathcal{T}^{sec<0}(M)$, we get a smooth bundle E with fiber M, and in addition, as mentioned before, we get a collection of Riemannian metrics, one on each fiber E_x, $x \in X$. And, since now the target space is $\mathcal{T}^{sec<0}(M)$, these Riemannian metrics are all negatively curved. We call such a bundle a *bundle with negatively curved fibers*. Still, to get a bona fide bundle theory we have to introduce the following concept. We say that two bundles E_0, E_1 over X, with negatively curved fibers, are *negatively curved equivalent* if there is a bundle E over $X \times [0,1]$, with negatively curved fibers, such that $E|_{X\times\{i\}}$ is smoothly equivalent to E_i, $i = 0, 1$, via bundle maps that are isometries between fibers. Then, bundles with negatively curved fibers over a (simply connected) space X, modulo negatively curved equivalence, are classified by $\Big[X, \mathcal{T}^{sec<0}(M)\Big]$. And the inclusion map $F : \mathcal{T}^{sec<0}(M) \hookrightarrow \mathcal{T}(M)$ gives a relationship between the two bundle theories:

$$\Big[X, \mathcal{T}^{sec<0}(M)\Big] \xrightarrow{F_X} \Big[X, \mathcal{T}(M)\Big]$$

and the map F_X is the "forget the negatively curved structure" map. The "kernel" $\mathcal{K}_X$ of this map between the two bundle theories is given by bundles over X, with negatively curved fibers, that are smoothly trivial. Every bundle in $\mathcal{K}_X$ can be represented by the choice of a negatively curved metric on each fiber of the trivial bundle $X \times M$, that is, by a map $X \to \mathcal{MET}^{sec<0}(M)$. Note that this representation is not unique, because smoothly equivalent representations give rise to the same bundle with negatively curved fibers. In any case, $\mathcal{K}_X$ is the image of $\Big[X, \mathcal{MET}^{sec<0}(M)\Big]$ by the map $\Big[X, \mathcal{MET}^{sec<0}(M)\Big] \longrightarrow \Big[X, \mathcal{T}^{sec<0}(M)\Big]$, induced by the quotient map $\mathcal{MET}^{sec<0}(M) \to \mathcal{T}^{sec<0}(M)$. Note that we can think of $\Big[X, \mathcal{MET}^{sec<0}(M)\Big]$ as a bundle theory: the "bundles" here are choices of negatively curved metrics, one for each fiber of the trivial bundle $X \times M$, modulo the following weak version of negatively curved equivalence. Two "bundles" E_0, E_1, here are equivalent if there is a "bundle" E over $X \times I$ such that $E|_{X\times\{i\}} = E_i$, $i = 0, 1$. Summarizing, we get the following exact sequence of bundle theories:

$$(**) \qquad \Big[X, \mathcal{MET}^{sec<0}(M)\Big] \xrightarrow{R_X} \Big[X, \mathcal{T}^{sec<0}(M)\Big] \xrightarrow{F_X} \Big[X, \mathcal{T}(M)\Big]$$

where the map R_X is the "representation map", which assigns the set $R_X^{-1}(E)$ of representations of E of the form $X \to \mathcal{MET}^{sec<0}(M)$ to each smoothly trivial bundle with negatively curved fibers $E \in \mathcal{K}_X$.

It is natural to inquire about the characteristics of these maps. For instance: are they non-constant? are they one-to-one? are they onto? If, in (**), we specify $X = \mathbb{S}^k$, $k > 1$ (recall we are using basepoint preserving maps), we obtain

$$(***) \qquad \pi_k(\mathcal{MET}^{sec<0}(M)) \to \pi_k(\mathcal{T}^{sec<0}(M)) \to \pi_k(\mathcal{T}(M))$$

Some information about these maps between homotopy groups was given in Theorems 1.1, 2.5, 2.9 and 2.10 above. We explain next what is the relationship between Theorems 1.1, 2.5, 2.9 and 2.10 and the sequences $(**)$ and $(***)$.

1. Recall that Theorem 1.1 says that $\pi_2(\mathcal{MET}^{sec<0}(M))$ is never trivial, provided $\mathcal{MET}^{sec<0}(M) \neq \emptyset$ and $dim\, M > 13$. But the non-zero elements in $\pi_2(\mathcal{MET}^{sec<0}(M))$, constructed in [FO10a], are mapped to zero by the map $\pi_2(\mathcal{MET}^{sec<0}(M)) \to \pi_2(\mathcal{T}^{sec<0}(M))$. Hence the first arrow in $(***)$ is not one-to-one when $k = 2$. Therefore the representation map $R_{\mathbb{S}^2}$ in $(**)$ is never one-to-one, provided $\mathcal{MET}^{sec<0}(M) \neq \emptyset$ and $dim\, M > 13$.

2. The case $k = 1$ in Theorem 1.1 proves that the forget structure map $F_{\mathbb{S}^2}$ is not onto. To see this just glue two copies of $\mathbb{D}^2 \times M$ along $\mathbb{S}^1$ using a non-trivial element in $\pi_1(\mathcal{MET}^{sec<0}(M))$. Thus, there are (nontrivial) smooth bundles E over $\mathbb{S}^2$ which do not admit a collection of negatively curved Riemannian metrics on the fibers of E. Applying the same argument for other values of k in Theorem 1.1 we can conclude that the same is true for $\mathbb{S}^k$, $k = 2p - 3$, $p > 2$.

3. Theorem 2.5 asserts that there are examples of closed hyperbolic manifolds for which $\pi_k(\mathcal{T}^{sec<0}(M))$ is non-zero. Here M depends on k and always $k > 0$. However no conclusion was reached on the case $k = 0$ (i.e. about the connectedness of $\mathcal{T}^{sec<0}(M)$). Also, the images of these elements by the inclusion map $\mathcal{T}^{sec<0}(M) \to \mathcal{T}(M)$ are non-zero. Hence the forget structure map $F_{\mathbb{S}^k}$ is, in general, non-trivial. This means that there are bundles with negatively curved fibers that are not smoothly trivial, i.e. the representation map $R_{\mathbb{S}^k}$ is not onto in these cases.

4. Theorem 2.9 implies that the forget structure map $F_{\mathbb{S}^k}$ is, in general, not one-to-one, for $k = 2p - 4$, p prime.

Remark 3.2. *In all above discussion we can replace "negatively curved metrics" by "ϵ-pinched negatively curved metrics".*

We expect that bundles with negatively curved fibers are, in fact, topologically rigid. That is, that the forget structure map F_X is constant (we assume X is simply connected).

Conjecture 3.3. *Let $M \to E \to X$ be a fiber bundle whose base X is simply connected and whose fibers can be equipped with continuously varying metrics of negative curvature. Then the bundle is topologically equivalent to a product $M \times X$, i.e., there exists a fiber preserving homeomorphism $E \to X \times M$.*

In [FG13b], using classical dynamical systems techniques, Farrell and Gogolev established a related result.

Theorem 3.4. *Let X be a closed simply connected manifold and $p\colon E \to X$ be a bundle with negatively curved fibers. Then its (fiberwise) associated sphere bundle $S(p)\colon SE \to X$ is topologically trivial.*

Here the *associated sphere bundle* is the bundle of all unit tangent vectors to the fibers of the original bundle.

4 Exotic topology and hyperbolic dynamical systems

Recall that given a compact smooth Riemannian manifold M an *Anosov diffeomorphism* f is a diffeomorphism that preserves a continuous splitting $TM = E^s \oplus E^u$, uniformly contracts the stable subbundle E^s and uniformly expands the unstable subbundle E^u. The number $\min\{\dim E^s, \dim E^u\}$ is called *the codimension* of the Anosov diffeomorphism f. An Anosov diffeomorphism is called *conformal* if the stable quasi-conformal distortion

$$K^s(x,n) = \frac{\max\{\|Df^n(v)\| : v \in E^s, \|v\| = 1\}}{\min\{\|Df^n(v)\| : v \in E^s, \|v\| = 1\}}$$

and analogously defined unstable distortion $K^u(x,n)$ are uniformly bounded in $x \in M$ and $n \in \mathbb{Z}$.

Also recall that a self-covering $f\colon M \to M$ is called an *expanding map* if the tangent map Df expands all non-zero tangent vectors.

Smale provided a general construction of Anosov automorphisms of many (but not all) compact nilmanifolds [Sm67]. (Smale himself credits A. Borel.) This construction generalizes to give affine Anosov diffeomorphisms of infranilmanifolds. All currently known examples of manifolds that support Anosov diffeomorphisms are homeomorphic to infranilmanifolds.

Question 4.1. *Given an Anosov diffeomorphism $f\colon M \to M$, is it true that f is conjugate to an affine hyperbolic map L of an infranilmanifold N?*

Note that a positive answer would imply that M is homeomorphic to an infranilmanifold. This question goes back to Anosov and Smale [An69, Sm67] and is an outstanding open problem in smooth dynamics.

For expanding maps the analogous question was resolved positively. Shub [Sh69] proved that an expanding endomorphism of a closed manifold M is topologically conjugate to an affine expanding endomorphism of an infranilmanifold if and only if the fundamental group $\pi_1(M)$ contains a nilpotent subgroup of finite index. Franks [Fr70] showed that if M admits an expanding endomorphism then $\pi_1(M)$ has polynomial growth. Finally, in 1981, Gromov [Gr81] completed the classification by showing that any finitely generated group of polynomial growth contains a nilpotent subgroup of finite index. Hence any expanding map is topologically conjugate to an affine expanding endomorphism of an infranilmanifold. In particular, any manifold that supports an expanding endomorphism is homeomorphic to an infranilmanifold.

4.1 The Dictionary

We suggest the following illuminating albeit somewhat vague dictionary.

Geometry	Dynamics
Hyperbolic metric on M	Conformal Anosov diffeomorphism of N
Negatively curved metric g on M	Anosov diffeomorphism f of N
The space $\mathcal{MET}^{sec<0}(M)$	The space $\mathfrak{X}_f$ of Anosov diffeomorphism homotopic to f
Pullback of a negatively curved metric g by a diffeomorphism $h\colon M \to M$	Conjugation of Anosov diffeomorphism f by a diffeomorphism $h\colon N \to N$

The similarity is confirmed by various results and conjectures. For example, the analogue of Mostow rigidity is the following result of Kalinin and Sadovskaya, which is based on work of Benoist and Labourie.

Theorem 4.2 ([KS03])**.** *Let f be a transitive Anosov diffeomorphism of a compact manifold N which is conformal on the stable and unstable distributions. Suppose that both distributions have dimension at least three. Then f is smoothly conjugate to an affine Anosov automorphism of a flat Riemannian manifold.*

Corollary 4.3. *Let $f, g\colon N \to N$ be homotopic conformal Anosov diffeomorphisms whose stable and unstable distributions are at least 3-dimensional. Then f and g are smoothly conjugate.*

4.2 Anosov diffeomorphism and expanding maps on exotic infranilmanifolds

Farrell and Jones showed that certain hyperbolic manifolds admit exotic smooth structures that are compatible with negative curvature. Namely, they proved the following result.

Theorem 4.4 ([FJ89a])**.** *If M is a hyperbolic manifold and Σ is an exotic sphere, then M has a finite covering $\tilde{M}$ such that the connected sum $\tilde{M}\#\Sigma$ is not diffeomorphic to $\tilde{M}$ and admits a Riemannian metric of negative curvature.*

This result refutes the original Lawson-Yau conjecture (see [FJO07, FO04] for a discussion).

The parallel development in the Anosov world actually precedes this result by 10 years. Indeed, in 1978 Farrell and Jones [FJ78b] constructed Anosov diffeomorphisms on exotic tori $\mathbb{T}^n\#\Sigma$. A different construction was carried out by

Farrell and Gogolev [FG12a] to give Anosov diffeomorphisms on many other exotic infranilmanifolds $N\#\Sigma$. This newer construction was done for a larger class of exotic infranilmanifolds. Another advantage of the construction in [FG12a] is that it yields higher-codimension Anosov diffeomorphisms meanwhile the original Farrell-Jones construction only gives codimension-one diffeomorphisms.

Remark 4.5. *Of course, one can multiply the Anosov diffeomorphism of Farrell and Jones on* $\mathbb{T}^n\#\Sigma$ *by an Anosov automorphism of an infranilmanifold* N *to obtain higher-codimension Anosov diffeomorphism on* $\mathbb{T}^n\#\Sigma \times N$*, which is still exotic. The construction in [FG12a] gives Anosov diffeomorphisms on manifolds with irreducible smooth structure; i.e., on manifolds that are not diffeomorphic to a smooth Cartesian product of two lower dimensional closed smooth manifolds.*

Remark 4.6. *In the above discussion we (probably unjustly) ignored the expanding maps on exotic infranilmanifolds. In fact, expanding maps on exotic tori* $\mathbb{T}^n\#\Sigma$ *were constructed first [FJ78a]. Recently Farrell and Gogolev also constructed expanding maps on PL-exotic tori [FG13a].*

Remark 4.7. *Recall that in order to equip a hyperbolic manifold with an exotic smooth structure and a negatively curved metric one has to pass to a sufficiently large finite cover first. (This is needed to employ Farrell-Jones warping trick [FJ89a].) Similarly, in order to construct an Anosov diffeomorphism on an exotic infranilmanifold* $N\#\Sigma$ *one has to pass to a sufficiently large self-cover of* N *first.*

We conclude this subsection by posing some open problems.

Problem 4.8. *Prove that all exotic tori admit expanding maps.*

Problem 4.9. *Construct Anosov diffeomorphisms on PL-exotic tori.*

Problem 4.10. *Show that the choice of smooth structure can obstruct existence of Anosov diffeomorphism. More specifically, construct an Anosov automorphism* $L\colon M \to M$ *and endow the nilmanifold* M *with a smooth structure* ω *such that* (M, ω) *does not admit Anosov diffeomorphisms homotopic to (homeomorphism)* L*. Then show that certain exotic nilmanifolds do not admit Anosov diffeomorphisms despite the fact that the underlying standard nilmanifolds do admit hyperbolic automorphisms.*

4.3 The space of Anosov diffeomorphisms

In [FG12b] we undertook the study of the space of Anosov diffeomorphisms which turned out to be analogous to the study of the space of negatively curved metrics from Section 1. We proceed to describe the results.

Fix a hyperbolic automorphism $L\colon \mathbb{T}^d \to \mathbb{T}^d$, $d \geq 2$. Denote by $\mathcal{X}_L$ *the space of* C^∞ *Anosov diffeomorphisms* of $\mathbb{T}^d$ which are homotopic to L. In other words, an Anosov diffeomorphism f belongs to $\mathcal{X}_L$ if and only if there exists a continuous path of maps $f_t\colon \mathbb{T}^d \to \mathbb{T}^d$ such that $f_0 = L$ and $f_1 = f$.

Theorem 4.11. *Let $L\colon \mathbb{T}^2 \to \mathbb{T}^2$ be a hyperbolic automorphism of the 2-torus. Then $\mathcal{X}_L$ is homotopy equivalent to $\mathbb{T}^2$.*

To prove this result we realize $\mathbb{T}^2$ as the collection of affine Anosov diffeomorphisms of the form $x \mapsto L(x) + v$, $v \in \mathbb{T}^2$. We use standard Gibbs states theory (see, e.g., [KH95, Chapter 20]) to show that any k-loop $\mathbb{S}^k \to \mathcal{X}_L$ can be homotoped to a k-loop with values in $\mathbb{T}^2 \subset \mathcal{X}_L$. Then, by J. H. C. Whitehead's Theorem, we conclude that $\mathcal{X}_L$ is homotopy equivalent to $\mathbb{T}^2$.

In higher dimensions the situation is completely different.

Theorem 4.12. *If $d \geq 10$ and $L\colon \mathbb{T}^d \to \mathbb{T}^d$ is a hyperbolic automorphism then $\mathcal{X}_L$ has infinitely many connected components.*

Remark 4.13. *In [FG12b] we consider a more general case when L is infranilmanifold automorphism. We also show that $\mathcal{X}_L$ is rich in higher homotopy groups.*

In the proof of the above theorem we rely on the following result. Below $\mathrm{TOP}_0(\mathbb{T}^d)$ ($\mathrm{DIFF}_0(\mathbb{T}^d)$) stands for the group of homeomorphism (diffeomorphisms) of $\mathbb{T}^d$ that are homotopic to identity.

Proposition 4.14 ([Hat78])**.** *If $d \geq 10$ then*

$$\pi_0(\mathrm{DIFF}_0(\mathbb{T}^d)) \simeq (\mathbb{Z}/2\mathbb{Z})^\infty \oplus G,$$

where G is a finite abelian group. Moreover, $(\mathbb{Z}/2\mathbb{Z})^\infty$ maps monomorphically into $\pi_0(\mathrm{TOP}_0(\mathbb{T}^d))$ via the map induced by inclusion of $\mathrm{DIFF}_0(\mathbb{T}^d)$ into the space of homeomorphism $\mathrm{TOP}_0(\mathbb{T}^d)$.

Now we explain how to use this proposition to obtain Theorem 4.12 in the special case when L has only one fixed point. Equip $\mathrm{DIFF}_0(\mathbb{T}^d)$ and $\mathcal{X}_L$ with the C^∞ topology and $\mathrm{TOP}_0(\mathbb{T}^d)$ with the C^0 topology. By global structural stability of Franks and Manning [Fr70, M74], for each $f \in \mathcal{X}_L$ there exists $h_f \in \mathrm{TOP}_0(\mathbb{T}^d)$ such that $f = h_f \circ L \circ h_f^{-1}$. Moreover, since the automorphism L has only one fixed point, it should have trivial centralizer in $\mathrm{TOP}_0(\mathbb{T}^d)$. Hence h_f is uniquely determined by f. Moreover, by (local) structural stability, h_f depends continuously on f.

Consider $h \in \mathrm{DIFF}_0(\mathbb{T}^d)$ such that $[h] \in (\mathbb{Z}/2\mathbb{Z})^\infty \subset \pi_0(\mathrm{DIFF}_0(\mathbb{T}^d))$ is nontrivial. Let $f = h \circ L \circ h^{-1}$. If f is isotopic to L then, by global structural stability, we obtain a C^0 path connecting $h_f = h$ and $h_L = id_{\mathbb{T}^d}$. Therefore f and L belong to different connected components of $\mathcal{X}_L$. The same argument shows that the map $h \mapsto h \circ L \circ h^{-1}$ induces a monomorphism on $(\mathbb{Z}/2\mathbb{Z})^\infty \subset \pi_0(\mathrm{DIFF}_0(\mathbb{T}^d))$ and, hence, proves the theorem.

Remark 4.15. *By Moser's homotopy trick (see, e.g., [KH95, Chapter 5]), the space $\mathrm{DIFF}_0^{vol}(\mathbb{T}^d)$ of all volume-preserving diffeomorphisms homotopic to $id_{\mathbb{T}^d}$ is a deformation retraction of $\mathrm{DIFF}_0(\mathbb{T}^d)$. Hence we have the same result for the space of volume-preserving Anosov diffeomorphisms.*

We conclude this subsection by posing some open questions.

Question 4.16. *For which $h \in \mathrm{DIFF}_0(\mathbb{T}^d)$ does the connected component of $h \circ L$ in $\mathrm{DIFF}(\mathbb{T}^d)$contain an Anosov diffeomorphism?*

It would be natural to proceed a study of Teichmüller and Moduli spaces of Anosov diffeomorphism which us analogous to the study surveyed in Section 2. Indeed, by using the Dictionary we obtain the following definition of the Teichmuller space. The *Teichmüller space of Anosov diffeomorphisms* $\mathcal{T}_L$ is the quotient of $\mathfrak{X}_L$ by the action of DIFF_0, i.e., it is the space of smooth conjugacy classes of Anosov diffeomorphisms. It is natural to equip $\mathcal{T}_L$ with the quotient C^r, $r \geq 1$, topology.

Recall that

$$\mathcal{D}_0(M) \to \mathcal{MET}^{\,sec\,<\,0}(M) \to \mathcal{T}^{\,sec\,<\,0}(M)$$

is a locally trivial principal fiber bundle by the work of Ebin [E68]. In the setting of Anosov diffeomorphism an analogous result is not available. Even the answer to the following question is not known.

Question 4.17. *Is $\mathcal{T}_L$ a Hausdorff space?*

If L is an Anosov automorphism of the 2-torus $\mathbb{T}^2$ then the answer is "yes", as implied by work of de la Llave, Marco and Moriyón [LMM87].

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