

CENTER LYAPUNOV EXPONENTS IN PARTIALLY HYPERBOLIC DYNAMICS

ANDREY GOGOLEV AND ALI TAHZIBI

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1. INTRODUCTION

Let M be a compact smooth Riemannian manifold. Denote by $\text{Diff}^r(M)$, $r \geq 1$, the space of C^r diffeomorphisms of M endowed with C^r topology and by $\text{Diff}_m^r(M)$ the subspace that consists of diffeomorphisms which preserve volume m .

Conjecture 1.1 (Pesin [P77, P07]). *Consider a diffeomorphism $f \in \text{Diff}_m^r(M)$, $r > 1$. Then arbitrarily close to f in $\text{Diff}_m^r(M)$ there exists a diffeomorphism g , which has nonzero Lyapunov exponents on a set of positive volume. Moreover, there is an open set U containing g and a residual subset of U such that any h in this subset has non-zero Lyapunov exponents on a positive measure set.*

This conjecture was motivated by the success of Pesin theory which studies diffeomorphisms with non-zero exponents and, in particular, establishes existence of an ergodic component of positive volume.

Pesin theory associates a rather subtle structure to a diffeomorphism with non-zero exponents: Pesin sets, local stable and unstable manifolds which vary only measurably and whose size deteriorates along the orbits. Even though it is a widespread belief that non-uniformly hyperbolic diffeomorphisms that carry this structure are abundant in $\text{Diff}_m^r(M)$, concrete examples of such diffeomorphisms are not very easy to come by. The subject of this survey is partially hyperbolic diffeomorphisms with non-zero center exponents. These systems provide a rather hands-on setting where some of the features of genuinely non-uniformly hyperbolic behavior are present.

The C^r , $r > 1$, topology in the second part of the above conjecture is crucial. By Mañé-Bochi's result we have that C^1 -generic volume preserving diffeomorphisms of any surface is either Anosov or the Lyapunov exponents of almost all points vanish. As the 2-torus is the only surface admitting Anosov diffeomorphism we conclude that for a C^1 -generic volume preserving diffeomorphism of any surface different from the 2-torus, the Lyapunov exponents of almost all points vanish. From ergodic point of view C^1 dynamics is very different since stable and unstable foliations could be non-absolutely continuous, which results in the failure of Pesin theory. We remark that it is not even known whether or not C^1 Anosov diffeomorphisms are ergodic and it is likely that they are not.

For large r ($r > 2 \dim M + 1$) one cannot expect to find a dense set of diffeomorphisms whose exponents are non-zero on a set of full volume since there are diffeomorphisms with persistent regions that are occupied by codimension 1 invariant tori (see [CS89, Her90, X92, Y92]). Still, for small r , e.g., $r = 2$, one can hope for a dense set of non-uniformly hyperbolic diffeomorphisms.

This survey is by no means a comprehensive one. Our goal is to give an overview of the field and explain several major ideas in simplified setups. Many important topics were omitted or only touched upon briefly. We assume that the

reader is familiar with the basic definitions and results from partially hyperbolic dynamics. This can be found in many sources, e.g., [HP06, P04, RHRHU11].

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2. ABUNDANCE OF NON-ZERO LYAPUNOV EXPONENTS

Consider a partially hyperbolic diffeomorphism f of a 3-manifold M with a fixed Finsler metric. By the Oseledets Theorem, the Lyapunov exponents

$$\lambda^\sigma(f)(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \log J^\sigma(f^n)(x), \quad \sigma = s, c, u,$$

are well defined a.e. with respect to an invariant measure μ . If μ is ergodic then by the Birkhoff theorem

$$\lambda^\sigma(f)(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log J^\sigma(f)(f^i x) = \int_M \log J^\sigma(f) d\mu, \quad \sigma = s, c, u, \quad (2.1)$$

for a.e. x . Using the last expression one can easily check that the Lyapunov exponents are independent of the choice of the Finsler metric. Also, this formula implies that λ^σ , $\sigma = s, c, u$, depend continuously on the diffeomorphism from $\text{Diff}_\mu^r(M)$ in C^1 topology.

Now consider a partially hyperbolic diffeomorphism $f: M \rightarrow M$ of an n -dimensional manifold that preserves the splitting $TM = E^s \oplus E^c \oplus E^u$ with $\dim(E^\sigma) = d_\sigma$, $\sigma = s, c, u$. Again by the Oseledets Theorem for a.e. point $x \in M$ the Lyapunov exponents λ_i^σ , $1 \leq i \leq d_\sigma$, $\sigma = s, c, u$ exist and

$$\sum_{i=1}^{d_\sigma} \lambda_i^\sigma(f)(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \log J^\sigma(f^n)(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log J^\sigma(f)(f^i x),$$

A weaker form of hyperbolicity is the existence of a dominated splitting. A Df -invariant splitting $TM = E_1 \oplus E_2 \cdots \oplus E_k$ is called *dominated* if there exists $l \in \mathbb{N}$ such that

$$\frac{\|Df^l(x)v\|}{\|Df^l(x)w\|} \leq 1/2$$

for any $x \in M$ and any $v \in E_i, w \in E_{i+1}, 1 \leq i < k$.

It is easy to conclude from the definition that the largest Lyapunov exponent corresponding to E_i is strictly less than the smallest exponent in the E_{i+1} direction. So any diffeomorphism with dominated splitting of k -subbundles admits at least k distinct Lyapunov exponents for a.e. point. However, some or all Lyapunov exponents along a fixed subbundle of the dominated splitting may coincide.

Next we will be concerned with the important case when μ is a smooth measure. We will discuss techniques for removing zero center exponent by a small perturbation in $\text{Diff}_\mu^r(M)$.

2.1. Removing zero exponent for smooth measures by a global perturbation. We describe the construction of Shub and Wilkinson [SW00] and incorporate some of the simplifications introduced in [BDV05, p. 139]. Start with an algebraic partially hyperbolic automorphism of \mathbb{T}^3 . It turns out that for certain very explicit perturbations the computations are workable and the center exponent can be estimated using (2.1).

Let $A: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be an Anosov automorphism with positive eigenvalues $\lambda < \lambda^{-1}$ and let L_0 be a partially hyperbolic automorphism of \mathbb{T}^3 given by

$$L_0(x_1, x_2, y) = (A(x_1, x_2), y). \quad (2.2)$$

Our goal is to construct a small volume preserving perturbation f of L_0 that has non-zero center exponent.

Denote by e_s , e_c and e_u the unit constant vector fields on \mathbb{T}^3 that correspond to the eigenvalues λ , 1 and λ^{-1} respectively.

Consider a stably ergodic partially hyperbolic skew product $L_\varphi: \mathbb{T}^3 \rightarrow \mathbb{T}^3$ of the form

$$L_\varphi(x_1, x_2, y) = (A(x_1, x_2), y + \varphi(x_1, x_2)), \quad (2.3)$$

where $\varphi: \mathbb{T}^2 \rightarrow \mathbb{T}^1$ is a smooth function C^r close to 0.

Remark 2.1. *Burns and Wilkinson [BW99] showed that stably ergodic skew products are open and dense in the space of skew products.*

Also consider a fibration $\pi: \mathbb{T}^3 \rightarrow \mathbb{T}^1$ given by

$$\pi(x_1, x_2, y) = y - x_1.$$

(If we think of the tori $y = \text{const}$ as “horizontal tori” then the fibers of π are “diagonal tori” inside \mathbb{T}^3 .)

There exist unique numbers a and b such that vector fields $e_s + ae_c$ and $e_u + be_c$ are tangent to the fibers of π . Let $\hat{\psi}: \mathbb{T}^1 \rightarrow \mathbb{R}$ be a non-constant smooth function C^r close to 0. Lift $\hat{\psi}$ to $\psi: \mathbb{T}^3 \rightarrow \mathbb{R}$ along the fibers: $\psi = \hat{\psi} \circ \pi$. Finally define a diffeomorphism $h: \mathbb{T}^3 \rightarrow \mathbb{T}^3$ by the formula

$$h(x_1, x_2, y) = (x_1, x_2, y) + \psi(x_1, x_2, y)(e_s + ae_c).$$

Diffeomorphism h is a small translation in the direction $e_s + ae_c$ which is constant on the fibers of π .

Let $f = h \circ L_\varphi$. Diffeomorphism f is partially hyperbolic because it is a small perturbation of L_φ . The derivative of f can be conveniently computed in the

coordinates $(e_s + ae_c, e_c, e_u + be_c)$:

$$DL_\varphi(x_1, x_2, y) = \begin{pmatrix} \lambda & 0 & 0 \\ \alpha + \varphi_s(x_1, x_2) & 1 & \beta + \varphi_u(x_1, x_2) \\ 0 & 0 & \lambda^{-1} \end{pmatrix}$$

where $\alpha = a(1 - \lambda)$, $\beta = b(1 - \lambda^{-1})$, φ_s and φ_u are derivatives of φ in the direction e_s and e_u respectively;

$$Dh = \begin{pmatrix} 1 & \psi_c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where ψ_c is the derivative of ψ in the direction e_c ; then

$$Df = Dh \circ DL_\varphi = \begin{pmatrix} \lambda + (\alpha + \varphi_s)\psi_c \circ L_\varphi & \psi_c \circ L_\varphi & (\beta + \varphi_u)\psi_c \circ L_\varphi \\ \alpha + \varphi_s & 1 & \beta + \varphi_u \\ 0 & 0 & \lambda^{-1} \end{pmatrix}$$

We can see that f is volume preserving. Notice also that f preserves center-stable distribution E^{cs} of L_0 spanned by e_s and e_c . Since center-stable distribution of f is the unique continuous 2-dimensional f -invariant distribution C^0 close to E^{cs} we have that E^{cs} is indeed the center-stable distribution of f . Therefore, the center distribution of f is spanned by a vector field of the form $(\varepsilon, 1, 0)^t$ (written in the coordinates $(e_s + ae_c, e_c, e_u + be_c)$). Function $\varepsilon: \mathbb{T}^3 \rightarrow \mathbb{R}$ is continuous and takes values in a small neighborhood of 0.

We equip $T\mathbb{T}^3$ with Finsler metric given by the sup-norm in the coordinates $(e_s + ae_c, e_c, e_u + be_c)$ so that $Df(\varepsilon, 1, 0)^t = J^c(f)(\varepsilon \circ f, 1, 0)^t$. By a direct computation

$$Df \begin{pmatrix} \varepsilon \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \varepsilon\lambda + \varepsilon(\alpha + \varphi_s)\psi_c \circ L_\varphi + \psi_c \circ L_\varphi \\ \varepsilon(\alpha + \varphi_s) + 1 \\ 0 \end{pmatrix}$$

Thus we obtain that $J^c(f) = 1 + \varepsilon(\alpha + \varphi_s)$ and

$$\varepsilon \circ f = \psi_c \circ L_\varphi + \frac{\varepsilon\lambda}{J^c(f)}.$$

From the last equation we see that ε is a non-zero function.

Because $\alpha + \varphi_s > 0$, we have that $J^c(f) > 1$ when $\varepsilon > 0$, and $J^c(f) < 1$ when $\varepsilon < 0$. It follows that $\varepsilon\lambda/J^c(f) \leq \varepsilon\lambda$ and the equality holds if and only if $\varepsilon = 0$. Therefore $\varepsilon \circ f \leq \psi_c \circ L_\varphi + \varepsilon\lambda$ and after integrating we have a strict inequality:

$$\int_{\mathbb{T}^3} \varepsilon \circ f dm < \int_{\mathbb{T}^3} \psi_c \circ L_\varphi dm + \lambda \int_{\mathbb{T}^3} \varepsilon dm = \lambda \int_{\mathbb{T}^3} \varepsilon dm.$$

Because $\lambda \in (0, 1)$ we conclude that $\int_{\mathbb{T}^3} \varepsilon dm < 0$.

Now we can estimate the center Lyapunov exponent

$$\begin{aligned} \lambda^c(f) &= \int_{\mathbb{T}^3} \log J^c(f) dm = \int_{\mathbb{T}^3} \log(1 + \varepsilon(\alpha + \varphi_s)) dm \\ &< \int_{\mathbb{T}^3} \varepsilon(\alpha + \varphi_s) dm = \alpha \int_{\mathbb{T}^3} \varepsilon dm < 0. \end{aligned} \quad (2.4)$$

2.2. Removing zero exponent for smooth measures by localized C^1 -perturbations for general partially hyperbolic diffeomorphisms. Here we will work with general partially hyperbolic diffeomorphisms and expose a result of Baraviera and Bonatti [BB03] that allows to perturb the center exponent by a C^1 -local perturbation. Their method allows to move any (center) exponent, not necessarily a zero exponent. Here the context is wider than that of the previous subsection but the perturbations are in C^1 -topology. Similar construction in a more restricted setting was carried out by Dolgopyat [D04, Appendix E].

Theorem 2.2 ([BB03]). *Let (M, m) be a compact manifold endowed with a C^r volume m , $r \geq 2$. Let f be a C^1 m -preserving diffeomorphism that admits a dominated splitting $TM = E_1 \oplus \cdots \oplus E_k, k > 1$. Then there are m -preserving C^r -diffeomorphisms g , arbitrary C^1 -close to f for which*

$$\int_M \log J^i(g)(x) dm \neq 0$$

where $J^i(g)(x) := |\det Dg(x)|_{E_i}|$.

Recall that $\int_M \log J^i(g)(x) dm$ is equal to the integral of the sum of all Lyapunov exponents corresponding to the subbundle E_i . For partially hyperbolic diffeomorphisms with one dimensional center bundle the above theorem implies:

Corollary 2.3. *There exists an open and dense subset \mathcal{N} in the space of volume preserving partially hyperbolic diffeomorphisms with one dimensional center such that for any $f \in \mathcal{N}$ we have*

$$\int_M \lambda^c(f)(x) dm \neq 0.$$

Let us mention that C. Liang, W. Sun and J. Yang [LSY12] showed that if f is partially hyperbolic with $TM = E^s \oplus E^c \oplus E^u$ with $\dim(E^c) = d_c$ then there is a volume preserving diffeomorphism g C^1 -close to f such that

$$\int_M \lambda_i^c(g)(x) dm \neq 0 \quad \text{for all } 1 \leq i \leq d_c,$$

where $\lambda_i^c(g)(x), 1 \leq i \leq d_c$, are the Lyapunov exponents corresponding to the center bundle. The idea is to perturb f to a diffeomorphism f_1 so that the center bundle has finest dominated splitting $E^{c_1} \oplus \cdots \oplus E^{c_k}$ in robust fashion. This

means that every diffeomorphism g close to f_1 has the finest dominated splitting of E^c into the same number of subbundles with dimensions $d_i, 1 \leq i \leq k$. By Baraviera-Bonatti result we can also suppose that $\int_M \log J^i(g)(x)dm \neq 0$ for any such g . Since E^{c_i} are elements of the finest dominated splitting, using a Bochi-Viana's argument, it is possible to perform a C^1 -perturbation of f_1 to obtain the desired property for Lyapunov exponents.

Now we will present a detailed proof of the Baraviera-Bonatti's result in the setting where f is a linear partially hyperbolic automorphism of \mathbb{T}^3 . Then we will explain what adjustments are needed to obtain the result in the full generality.

Let $f: \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be a linear partially hyperbolic diffeomorphism with eigenvalues $0 < \lambda_s < \lambda_c < \lambda_u$. Let $E^s \oplus E^c \oplus E^u$ be the corresponding splitting. We also consider a coordinate system corresponding to the this splitting with notation (x, y, z) .

Our goal is to perturb f in a small ball B_r of radius r around a non-periodic point p_0 and obtain a new volume preserving diffeomorphism f_r such that

$$\int_{\mathbb{T}^3} \lambda^c(f_r)(x)dm > \int_{\mathbb{T}^3} \lambda^c(f)(x)dm = \log(\lambda_c).$$

In fact we construct a one parameter family of volume preserving local perturbations $f_r, r \in [0, r_0]$, of f such that the average of unstable Lyapunov exponent of f_r (which is equal to $\int \log J_{f_r}^u dm$) is strictly less than the unstable exponent of f and the stable Lyapunov exponent of f remains unchanged (the claim about stable exponent can be guaranteed only when f is linear) after perturbation. Because f_r is volume preserving the sum of the Lyapunov exponents must be zero and consequently

$$\int_{\mathbb{T}^3} \lambda^c(f_r)(x)dm > \int_{\mathbb{T}^3} \lambda^c(f)(x)dm.$$

As mentioned above after a local perturbation of linear partially hyperbolic automorphism we can decrease the unstable exponent. It is interesting to point out that it is not possible to increase it by a perturbation. In fact in the next section we show that the unstable Lyapunov exponent of any partially hyperbolic diffeomorphism on \mathbb{T}^3 is less than or equal to the unstable exponent of its linearization.

2.2.1. Construction of the perturbation. Let $h: B(0, 1) \rightarrow B(0, 1)$ be a volume preserving diffeomorphism coinciding with the identity map on a neighborhood of the boundary of the unit ball and $B_r, 0 < r < 1$ an small ball around a non-periodic point p_0 . We denote by $h_r = \phi_r^{-1} \circ h \circ \phi_r$ where $\phi_r: B(0, r) \rightarrow B(0, 1)$ is a homothety of ratio $1/r$. By definition h_r is supported on B_r . We view B_r as ball of radius r about p_0 and extend h_r by identity to the rest of \mathbb{T}^3 . Let $f_r = f \circ h_r$. Then f_r is a small C^1 volume preserving perturbation of f .

Observe that $\|h_r - Id\|_{C^1} = \|h - Id\|_{C^1}$ but $\|h_r - Id\|_{C^k}, k > 1$ depends on r and consequently this method does not provide C^k -perturbations to remove zero Lyapunov exponent. The following question remains open:

Question 2.4. *Is it possible to make a local C^r -perturbation ($r > 1$) to remove zero Lyapunov exponent of the volume?*

We require that h preserves E^s coordinate, that is, $\pi_s h(x, y, z) = x$. Let us prove a local estimate which sheds some light on the behavior of center Lyapunov exponent of f_r . Let $Dh(p)(e_u) = h^u(p)e_u + h^c(p)e_c$ for any $p \in B(0, 1)$ where e_u and e_c stand for unit vectors in E^u and E^c . Also define $h_r^u(q) := h^u(\phi_r(q))$.

Lemma 2.5. *Let $B(0, 1)$ be the unit ball of \mathbb{R}^3 . Consider any volume preserving diffeomorphism h of $B(0, 1)$ coinciding with the identity map on a neighborhood of the boundary of the ball and preserving E^s coordinate. Assume that $0 < \|h - Id\|_{C^1} < 1$ then*

$$\int_{B(0,1)} \log h^u(p) dm < 0.$$

To prove the above lemma we take any segment γ tangent to E^u direction and joining two boundary points of $B(0, 1)$. Let $\gamma : J \subset \mathbb{R} \rightarrow B(0, 1)$ be a parametrization by the arc length, that is $\|\gamma'(\xi)\| = 1$. Denote by π_u the orthogonal projection on $\gamma(J)$. Since h coincides with identity close to the boundary we have that the curve $\pi_u(h(\gamma))$ has the same image as γ , but, possibly, different parametrization. These curves have the same length, i.e.,

$$l(\gamma) = \int_J \|(\pi_u \circ h \circ \gamma)'(\xi)\| d\xi = \int_J h^u(\gamma(\xi)) d\xi.$$

By Jensen inequality for the probability measure $\frac{d\xi}{l(\gamma)}$,

$$\int_J \log h^u(\gamma(\xi)) \frac{d\xi}{l(\gamma)} \leq \log \int_J h^u(\gamma(\xi)) \frac{d\xi}{l(\gamma)} = 0.$$

We claim that the above inequality is strict. Indeed, in the case of equality h^u is constant equal to 1. Because h coincides with identity close to the boundary we conclude that h preserves E^u -coordinate. As h is volume preserving and preserves E^s and E^u coordinates, the derivative of h in E^c -direction is also equal to 1. Again, because h is identity close to the boundary, it preserves E^c -coordinate too. This implies that h is identity which is a contradiction.

2.2.2. An auxiliary cocycle. Now we calculate the average of the unstable Lyapunov exponent of f_r . Note that the unstable and center distributions are C^1 -close to, but different from those for f . This makes it difficult to estimate

$$\int_{\mathbb{T}^3} \log J^u(f_r)(p) dm$$

directly. The trick is to substitute $J_{f_r}^u$ by a function λ_r which is easier to analyze, but at the same time

$$\int_{\mathbb{T}^3} \log \lambda_r(p) dm = \int_{\mathbb{T}^3} \log J^u(f_r)(p) dm.$$

We will think of λ_r as the generator of a cocycle F_r over f_r acting on E^u ,

$$F_r: (p, e_u) \mapsto (f_r(p), \lambda_r(p)e_u).$$

This way $F_r(p, e_u) = \lambda_r(p)e_u$. By definition, the Lyapunov exponent of the cocycle at p is

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|F_r^n(e_u)\|.$$

Now let us define the cocycle:

- The action of F_r coincides with Df outside $B_r \cup f^{-1}(B_r)$ that is, $\lambda_r(p) := \lambda_u$.
- In B_r , the action of F_r is the action of Df on the projection of $Dh_r(e_u)$ on e_u parallel to the center bundle of f , that is, $\lambda_r(p) := \lambda_u h_r^u(p)$.
- For any $p \in f^{-1}(B_r)$ if the negative orbit of p is disjoint from B_r then set $\lambda_r(p) = \lambda_u$. Otherwise let $n(p)$ be the smallest integer for which there exist $q \in B_r$ such that $p = f^{n(p)}(q)$. Let $\tilde{q} = h^{-1}(q)$. Recall that

$$Dh_r(\tilde{q})(e_u) = h_r^u(\tilde{q})e_u + h_r^c(\tilde{q})e_c.$$

On one hand we have

$$F_r^{n(p)}(e_u) = h_r^u(\tilde{q})\lambda_u^{n(p)}e_u$$

and on the other hand, using chain rule for $f_r = f \circ h_r$, we have

$$Df_r^{n(p)}(e_u) = F_r^{n(p)}(e_u) + \lambda_c^{n(p)}h_r^c(\tilde{q})e_c.$$

Observe that the second summand above belongs to E^c which may be different from the center direction of the perturbed diffeomorphism. Let $w(p)$ be the projection of $\lambda_c^{n(p)}h_r^c(\tilde{q})e_c$ on e_u direction parallel to the new center direction at p for f_r . Observe that in the $n(p)$ iterates of the cocycle we ignored the vector in the e_c direction which should be considered for the new diffeomorphism. So consider the correction term

$$A(p) := \frac{\|F_r^{n(p)}(e_u) + w(p)\|}{\|F_r^{n(p)}(e_u)\|} = 1 + \frac{\|w(p)\|}{h_r^c(\tilde{q})\lambda_u^{n(p)}}$$

and define $\lambda_r(p) := A(p)\lambda_u$.

By the above definitions, for any point $p \in B_r$ and $N > 0$ such that $f^N(p) \in B_r$, we have that $F_r^N(e_u)$ is just the projection of $Df_r^N(e_u)$ on e_u direction along the new center stable bundle of f_r . As this bundle is transversal to e_u and $Df_r^N(e_u)$ belongs to an unstable cone field this projection affects the norm in uniformly

bounded way and the Lyapunov exponent of f_r along $E_{f_r}^u$ coincides with the Lyapunov exponent of the cocycle F_r .

Let us state the key proposition which shows the effect of the perturbation on the unstable Lyapunov exponent.

Proposition 2.6. *For sufficiently small $r > 0$, $\int_{\mathbb{T}^3} \log J^u(f_r) dm < \log(\lambda_u)$.*

By the above proposition, after perturbation the average of the unstable Lyapunov exponent decreases. By construction the perturbation does not change the stable Lyapunov exponent. At the same time the volume is preserved and the sum of Lyapunov exponents remains zero. Hence, after perturbation the center Lyapunov exponent increases.

To finish the proof of Theorem 2.2 it remains to prove Proposition 2.6.

Proof. As $\int_{\mathbb{T}^3} \log J^u(f_r)(x) dm = \int_{\mathbb{T}^3} \log \lambda_r(x) dm$ and $\lambda_r(p) = \lambda_u$ for $p \notin B_r \cup f^{-1}(B_r)$ we have

$$\begin{aligned} \log(\lambda_u) - \int_{\mathbb{T}^3} \log J^u(f_r) dm &= \int_{B_r} \log(\lambda_u) - \log(\lambda_r(p)) dm + \int_{f^{-1}B_r} \lambda_u - \log(\lambda_r(p)) dm \\ &= -\text{vol}(B_r) \int_{B(0,1)} \log(h^u(p)) dm - \int_{f^{-1}B_r} \log(A(p)) dm. \end{aligned}$$

where the last equality comes from the definition of λ_r . The first summand is the product of $-\text{vol}(B_r)$ and a negative constant that does not depend on r . We estimate the second term from above

$$\left| \int_{f^{-1}B_r} \log(A(p)) dm \right| \leq \text{vol}(B_r) \max_{p \in f^{-1}B_r} \log(A(p)).$$

Note that up till now we did not use “partial hyperbolicity” of f . Now we will use domination (here it is just $\lambda_c < \lambda_u$) to show that the maximum above is exponentially small. Let n_r be the least strictly positive integer n such that $f^{n_r} B_r \cap B_r \neq \emptyset$. Obviously $n_r \rightarrow \infty$ as $r \rightarrow 0$.

Lemma 2.7. *There is exists $0 < \alpha < 1$ and a constant $C > 0$ such that for any $r > 0$*

$$\max_{p \in f^{-1}B_r} |\log(A(p))| \leq C\alpha^{n_r}.$$

We show that $|A(p) - 1| \leq C_0\alpha^{n_r}$. By definition of $A(p)$ we have to show

$$\frac{\|w(p)\|}{h_r^u(\tilde{q})\lambda_u^{n(p)}} \leq C_0\alpha^{n_r}.$$

As the projection of e_c on e_u along the new center direction has a uniformly bounded norm for all points it is enough to show that

$$\frac{h_r^c(\tilde{q})\lambda_c^{n(p)}}{h_r^u(\tilde{q})\lambda_u^{n(p)}} \leq C_1\alpha^{n_r}.$$

Take $\alpha = \frac{\lambda_c}{\lambda_u}$. Then the inequality follows because $n(p) \leq n_r$ and the partial derivatives of h are bounded away from below and above on $B(0, 1)$. □

Remark 2.8. *Considering inverse of diffeomorphism f and similar perturbation as above it would be also possible to make the average of center exponent decrease after a small perturbation.*

The hypothesis of f being linear may seem to be very far from general partial hyperbolicity. However, if f is partially hyperbolic with non-periodic point p_0 then in a neighborhood of p_0 the splitting $E^s \oplus E^c \oplus E^u$ is close to constant splitting and the same conclusions about Lyapunov exponents of the perturbation can be made. One argues as follows:

The perturbation is again $f \circ h_r$ such that h_r is the conjugated by homothety to $h : B(0, 1) \rightarrow B(0, 1)$ which is defined exactly as before. Although h preserves the direction parallel to $E^s(p_0)$, the stable Lyapunov exponent of f_r is not necessarily equal to f . This is because the stable subbundle of f is not constant. However, in a very small scale the partially hyperbolic decomposition is almost constant. It can be proved that

$$\lim_{r \rightarrow 0} \frac{\int_{\mathbb{T}^3} \log J^u(f_r) dm - \int_{\mathbb{T}^3} \log J^u(f) dm}{\text{vol}(B_r)} = \int_{B(0,1)} \log h^u(p) dm < 0$$

and

$$\lim_{r \rightarrow 0} \frac{\int_{\mathbb{T}^3} \log J^s(f_r) dm - \int_{\mathbb{T}^3} \log J^s(f) dm}{\text{vol}(B_r)} = 0.$$

This yields that for any small enough $r > 0$

$$\int_{\mathbb{T}^3} \log J^u(f) dm - \int_{\mathbb{T}^3} \log J^u(f_r) dm > \left| \int_{\mathbb{T}^3} \log J^s(f_r) dm - \int_{\mathbb{T}^3} \log J^s(f) dm \right|.$$

As the volume is preserved, the center exponent should increase after perturbation.

Question 2.9. *Take a symplectic partially hyperbolic diffeomorphism f with two dimensional center bundle and zero center Lyapunov exponents, i.e $\lambda_1^c(x, f) = \lambda_2^c(x, f) = 0$ a.e x . Is it possible to C^1 -perturb f to a symplectomorphism g such that $\int_M \lambda_1^c(x, g) dm(x) \neq 0$?*

2.3. Removing zero exponents for SRB measures. The original approach of Shub and Wilkinson in [SW00] was to consider a special one parameter family of volume preserving partially hyperbolic diffeomorphism $f_\varepsilon: \mathbb{T}^3 \rightarrow \mathbb{T}^3$ through the partially hyperbolic automorphism f_0 given by (2.2) and then to obtain a second order expansion for the center Lyapunov exponent for the volume

$$\lambda^c(f_\varepsilon) = K\varepsilon^2 + o(\varepsilon^2),$$

with an explicit non-zero K . This approach was further pursued by Ruelle [R03]. Ruelle considered general one parameter families $f_\varepsilon: \mathbb{T}^n \rightarrow \mathbb{T}^n$ through a partially hyperbolic automorphism with one dimensional center $f_0: \mathbb{T}^n \rightarrow \mathbb{T}^n$. He obtained explicit second order expansions for the center Lyapunov exponent for volume preserving families as well as dissipative families f_ε assuming that they have unique SRB measures μ_ε .

Dolgopyat [D04] studied the case where f_0 is the time one a geodesic flow on a closed surface of negative curvature. Based on earlier work on existence and uniqueness of SRB measures for partially hyperbolic diffeomorphisms [ABV00, D00] he proved the following result.

Theorem 2.10 ([D04]). *Let f_ε be a generic one parameter C^1 family of partially hyperbolic diffeomorphisms passing through f_0 . Then f_ε has a unique SRB measure μ_ε for small ε . Moreover, one has the following second order expansion for the center Lyapunov exponent of μ_ε*

$$\lambda^c(f_\varepsilon, \mu_\varepsilon) = K\varepsilon^2 + o(\varepsilon^2), K \neq 0.$$

3. LYAPUNOV EXPONENTS AND LINEARIZATIONS

In the previous section we described perturbations that remove zero center Lyapunov exponent. Recall that the idea in both (local and global perturbations) methods was to “borrow” exponents from the unstable bundle to the center bundle. However, as we remarked before, for all perturbations that we did, the Lyapunov exponent of unstable bundle decreased.

So let us shift our attention for a moment to the unstable Lyapunov exponent. Consider the functional

$$\Lambda : f \rightarrow \int_M \lambda^u(f)(x) dm(x). \quad (3.1)$$

In what follows we prove a result which, in particular, implies that linear partially hyperbolic diffeomorphisms are local maximum point for Λ .

Theorem 3.1 ([MT13]). *Let $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be a C^2 partially hyperbolic diffeomorphism and A be its linearization. Then $\lambda^u(f)(x) \leq \lambda^u(A)$ and $\lambda^s(f)(x) \geq \lambda^s(A)$ for Lebesgue almost every $x \in \mathbb{T}^3$.*

We recall that, using geometric growth of foliations, Saghin and Xia [SX09] proved that the unstable Lyapunov exponent does not increase after perturbation of a linear partially hyperbolic diffeomorphism. We prove a global version of their result. The claim for the stable Lyapunov exponent comes out just by taking f^{-1} instead of f .

Firstly, let us recall some useful facts on the partially hyperbolic diffeomorphisms of \mathbb{T}^3 . Every diffeomorphism of the torus $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ induces an automorphism of the fundamental group and there exists a unique linear diffeomorphism A which induces the same automorphism on $\pi_1(\mathbb{T}^3)$. The diffeomorphism A is called the linearization of f . One can choose a lift of f and A to the universal cover \mathbb{R}^3 which we denote again by f and A . The lifts are also partially hyperbolic and preserve invariant foliations \mathcal{F}^σ , $\sigma \in \{s, c, u\}$. An important geometric property of invariant foliations \mathcal{F}^σ (in the universal cover) is their *quasi-isometric property*: There exists a universal $Q > 0$ such that $\|x - y\| \geq Qd^\sigma(x, y)$ for any $x, y \in \mathbb{R}^3$, $y \in \mathcal{F}^\sigma(x)$, where d^σ stands for the inherited Riemannian metric along the leaves of \mathcal{F}^σ . See [H12] and [BBI].

Let us state two basic propositions which are used in the proof of the above theorem.

Proposition 3.2 ([H12]). *Let $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be a partially hyperbolic diffeomorphism and let $A : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be the linearization of f then for each $k \in \mathbb{Z}$ and $C > 1$ there is an $M > 0$ such that for $x, y \in \mathbb{R}^3$,*

$$\|x - y\| > M \Rightarrow \frac{1}{C} < \frac{\|f^k(x) - f^k(y)\|}{\|A^k(x) - A^k(y)\|} < C.$$

Proposition 3.3 ([MT13]). *Let $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be a partially hyperbolic diffeomorphism and $A : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ the linearization of f . For all $n \in \mathbb{Z}$ and $\epsilon > 0$ there exists M such that for $x, y \in \mathbb{R}^3$ with $y \in \mathcal{F}_x^\sigma$ and $\|x - y\| > M$ then*

$$(1 - \epsilon)e^{n\lambda^\sigma(A)}\|y - x\| \leq \|A^n(x) - A^n(y)\| \leq (1 + \epsilon)e^{n\lambda^\sigma(A)}\|y - x\|,$$

where $\lambda^\sigma(A)$ is the Lyapunov exponent of A corresponding to E^σ and $\sigma \in \{s, c, u\}$.

Proof. We prove the statement on $\lambda^u(f)$ of Theorem 3.1. Suppose by contradiction that there is a positive volume set $Z \subset \mathbb{R}^3$, such that, for every $x \in Z$ we have $\lambda^u(f)(x) > \lambda^u(A)$. We can take Z such that, there exists $\epsilon > 0$ such that $\lambda^u(f)(x) > \lambda^u(A) + 2\log(1 + 2\epsilon)$ for all $x \in Z$. Since f is C^2 , the unstable foliation \mathcal{F}^u for f is absolutely continuous. In particular there is a positive volume set $B \in \mathbb{R}^3$ such that for every point $p \in B$ we have

$$m_{\mathcal{F}_p^u}(\mathcal{F}_p^u \cap Z) > 0, \tag{3.2}$$

where $m_{\mathcal{F}_p^u}$ is the induced volume on the unstable leaf. Now consider a segment $[x, y]_u \subset \mathcal{F}_p^u$ satisfying $m_{\mathcal{F}_p^u}([x, y]_u \cap Z) > 0$ such that $d^u(x, y) \geq \frac{M}{Q}$, where M is

as required in propositions 3.2, 3.3 and Q is the quasi isometric constant. So, we have $\|x - y\| \geq M$ and by choosing M large enough we have:

$$\|Ax - Ay\| \leq (1 + \varepsilon)e^{\lambda^u(A)}\|y - x\|$$

and

$$\frac{\|fx - fy\|}{\|Ax - Ay\|} \leq 1 + \varepsilon.$$

The above equations imply that

$$\|fx - fy\| \leq (1 + \varepsilon)^2 e^{\lambda^u(A)}\|y - x\|.$$

Inductively, we assume that for $n \geq 1$ we have

$$\|f^n x - f^n y\| \leq (1 + \varepsilon)^{2n} e^{n\lambda^u(A)}\|y - x\|. \quad (3.3)$$

Since f expands uniformly on the u -direction we have $\|f^n x - f^n y\| > M$ and hence

$$\begin{aligned} \|f(f^n x) - f(f^n y)\| &\leq (1 + \varepsilon)\|A(f^n x) - A(f^n y)\| \\ &\leq (1 + \varepsilon)^2 e^{\lambda^u(A)}\|f^n x - f^n y\| \\ &\leq (1 + \varepsilon)^{2(n+1)} e^{(n+1)\lambda^u(A)}. \end{aligned}$$

For each $n > 0$, let $A_n \subset Z$ be the following set

$$A_n = \{x \in Z : \|D^u f^k\| \geq (1 + 2\varepsilon)^{2k} e^{k\lambda^u(A)} \text{ for any } k \geq n\}.$$

We have $m(Z) > 0$ and by the choice of ε , $A_n \uparrow Z$. Consider a large n and $\alpha_n > 0$ such that $m_{\mathcal{F}_p^u}([x, y]_u \cap A_n) = \alpha_n m_{\mathcal{F}_p^u}([x, y]_u)$. We have $\alpha_n \geq \alpha_0 > 0$ for every large $n > 1$. Then

$$\|f^n x - f^n y\| \geq Q \int_{[x, y]_u \cap A_n} \|D^u f^n(z)\| dm_{\mathcal{F}_p^u}(z) \geq \quad (3.4)$$

$$\geq Q(1 + 2\varepsilon)^{2n} e^{n\lambda^u(A)} m_{\mathcal{F}_p^u}([x, y]_u \cap A_n) \quad (3.5)$$

$$\geq \alpha_0 Q(1 + 2\varepsilon)^{2n} e^{n\lambda^u(A)} \|x - y\|. \quad (3.6)$$

The inequalities (3.3) and (3.6) contradict each other. Therefore we obtain $\lambda^u(f)(x) \leq \lambda^u(A)$, for almost every $x \in \mathbb{T}^3$. Considering the inverse f^{-1} we conclude that $\lambda^s(A) \leq \lambda^s(f)(x)$ for almost every $x \in \mathbb{T}^3$. \square

Conjecture 3.4. *Let f be a C^2 volume preserving Anosov diffeomorphism of \mathbb{T}^2 and suppose that f is a local maximum for the functional $\Lambda : \text{Diff}_m^2(\mathbb{T}^2) \rightarrow \mathbb{R}$ given by (3.1), then f is C^1 conjugate to a linear toral automorphism.*

We would like to point out that a “global version” of this conjecture holds true. To see this fix a hyperbolic toral automorphism $A: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ and let \mathcal{X}_A be the space of Anosov diffeomorphism that are homotopic to A . Then each $f \in \mathcal{X}_A$ is conjugate to A by a homeomorphism α_f . Notice that $\Lambda(f)$ is simply the metric entropy of f . Hence

$$\Lambda(f) = h_m(f) = h_{\alpha_f^* m}(L) \leq h_m(L) = h_{top}(L).$$

The last inequality is the variational principle. Hence Λ attains the global maximum on L . Moreover, if $h_{\alpha_f^* m}(L) = h_m(L)$ then the conjugacy α_f has to be volume preserving and, hence, by work of de la Llave, Marco and Moriyón [MM87, dlL92], α_f is $C^{1+\varepsilon}$, $\varepsilon > 0$. We conclude that Λ attains a global maximum on $f \in \mathcal{X}_A$ if and only if f is C^1 conjugate to A .

4. NON-REMOVABLE ZERO EXPONENTS

The evidence presented earlier leads to the belief that generically partially hyperbolic diffeomorphisms have non-zero center exponents with respect to natural measures such as volume or an SRB measure. However, if one considers all ergodic measures then it is natural to expect existence of a measure with some (or all) zero center exponents.

Consider a partially hyperbolic automorphism $L: \mathbb{T}^3 \rightarrow \mathbb{T}^3$

$$L(x_1, x_2, y) = (A(x_1, x_2), y) \tag{4.1}$$

where A is an Anosov automorphism. And let \mathcal{U}_L be a C^1 small neighborhood of L (precise definition of \mathcal{U}_L appears later).

Question 4.1. *Given an ergodic volume preserving diffeomorphism $f \in \mathcal{U}_L$, is it true that the space of f -invariant ergodic measures equipped with weak* topology is path connected?*

Remark 4.2. *The space of ergodic measures of a transitive Anosov diffeomorphism is path connected [Sig77].*

Remark 4.3. *Let $g: S^1 \rightarrow S^1$ be a diffeomorphism with two hyperbolic fixed points, an attractor and a repeller. Then the space of ergodic measures of diffeomorphism $A \times g: \mathbb{T}^3 \rightarrow \mathbb{T}^3$ has two connected components. Moreover, this property is C^1 -stable—the space of ergodic measures of any sufficiently small perturbation of $A \times g$ has at least two connected components. This shows that it is essential that f is conservative for the above question.*

The motivation for Question 4.1 comes from the following observation. Pick a diffeomorphism $f \in \mathcal{U}_L$ such that the restriction of f to an invariant center leaf has a hyperbolic attracting point a and a hyperbolic repelling point b . Assume that there is a path μ_t , $t \in [0, 1]$, of ergodic measures connecting the atom at a and the atom at b .

The center Lyapunov exponent

$$\lambda^c(\mu_t) = \int_{\mathbb{T}^3} \log J_f^c d\mu_t$$

depends continuously on t and, by the intermediate value theorem, there exists $t_0 \in [0, 1]$ such that the center exponent of μ_{t_0} is zero.

Note that the fixed points a and b have continuations in a small C^1 neighborhood \mathcal{V} of f and, hence, the same argument applies for diffeomorphisms in \mathcal{V} .

Thus, for example, a positive answer to Question 4.1 implies that there exists an open set of volume preserving diffeomorphisms near L each of which has an ergodic measure with zero center exponent.

Remark 4.4. *In fact, the above argument can be applied to any $f \in \mathcal{U}_L$. Indeed, if f has positive (negative) center Lyapunov exponents for all invariant measures then it must be uniformly hyperbolic [AAS03].*

Even though the answer to Question 4.1 is unknown one can still proceed with a similar idea, which is to consider a sequence (rather than a path) of measures whose center exponent tend to zero and derive results on existence of measures with zero center exponents.

4.1. The results. We consider an automorphism $L: \mathbb{T}^3 \rightarrow \mathbb{T}^3$ given by (4.1) and a small C^1 neighborhood \mathcal{U}_L of L in $\text{Diff}^1(\mathbb{T}^3)$ chosen so that for every $f \in \mathcal{U}_L$ Hirsch-Pugh-Shub structural stability applies and yields a homeomorphism $H: \mathbb{T}^3 \rightarrow \mathbb{T}^3$ that conjugates L and f on the space of center leaves. From now on neighborhood \mathcal{U}_L will be fixed.

Theorem 4.5 ([GI99, GI00, GIKN05, KN07]). *There exists a C^1 open set $\mathcal{V} \subset \mathcal{U}_L$ such that for every $f \in \mathcal{V}$ there exists an ergodic measure μ_f of full support with zero center Lyapunov exponent.*

The measure μ_f is constructed as weak* limit of atomic measures which are supported on certain periodic orbits of f . The construction rests on so called Gorodetski-Ilyashenko strategy that aims at showing that properties of random dynamical systems (higher rank free actions) could be observed in partially hyperbolic diffeomorphisms and, more generally, locally maximal partially hyperbolic invariant sets.

We will explain this strategy and outline the proof of Theorem 4.5 in the next subsection. So far the work on realizing Gorodetski-Ilyashenko strategy was devoted to higher rank free actions on the circle S^1 . This corresponds to partially hyperbolic attractors whose center foliation is a foliation by circles. It would be very interesting to go beyond one dimensional case.

Recently Gorodetski and Díaz have announced that one can make sure that set \mathcal{V} from Theorem 4.5 contains diffeomorphism f constructed in Section 2.1. This

gives robust coexistence of (volume) non-uniform hyperbolicity and measures with zero Lyapunov exponents.

Bonatti, Gorodetski and Díaz have established presence of zero exponents for C^1 generic diffeomorphisms [DG09, BDG10]. In particular, they have the following.

Theorem 4.6. *For a C^1 residual set of diffeomorphisms f from $\text{Diff}^1(M)$ every homoclinic class with a one dimensional center direction and saddles of different indices is a support of an ergodic measure μ_f with zero center Lyapunov exponent.*

This result can be viewed as a step towards establishing a generic dichotomy “uniform hyperbolicity versus presence of zero Lyapunov exponents.” Returning to our setup: there is an open and dense subset of \mathcal{U}_L for which the homoclinic class is \mathbb{T}^3 . Hence we have the following.

Corollary 4.7. *There exists a C^1 residual subset \mathcal{S} of \mathcal{U}_L such that every $f \in \mathcal{S}$ has an ergodic measure μ_f of full support with zero center Lyapunov exponent.*

Recently Bochi, Bonatti and Diaz [BBD13] constructed C^2 open sets of step skew products (see Definition 4.9) on any manifold M , admitting fully-supported ergodic measures whose Lyapunov exponents along M are all zero. These measures are also approximated by measures supported on periodic orbits. This result is analogous to the result of [GIKN05] and is a major step towards an analogue of Theorem 4.5 with higher dimensional center foliation.

Finally, in the opposite direction, let us mention the result from [ABS10]: for a C^1 generic diffeomorphism generic measures supported on isolated homoclinic classes are ergodic and hyperbolic.

4.2. The setup and the reduction to Hölder skew products. We will be working in a more general setup than that of Theorem 4.5. This is the setup of partially hyperbolic locally maximal invariant sets rather than partially hyperbolic diffeomorphisms.

Let $h: M \rightarrow M$ be a diffeomorphism that has locally maximal hyperbolic set Λ . We call a skew product $F: \Lambda \times S^1 \rightarrow \Lambda \times S^1$

$$F(w, x) = (h(w), g_w(x)) \tag{4.2}$$

a *Hölder skew product* if there exists $C > 0$ and $\alpha > 0$ such that the fiber diffeomorphisms satisfy the following inequality

$$d_{C^0}(g_u, g_v) \leq Cd(u, v)^\alpha \tag{4.3}$$

for all $u, v \in \Lambda$.

The crucial step in Gorodetski-Ilyashenko strategy is the observation that every C^1 -small perturbation f of $f_0 = h \times Id$ in $\text{Diff}(M \times S^1)$ is conjugate to a Hölder skew product on the locally maximal partially hyperbolic set which is homeomorphic to $\Lambda \times S^1$.

To see this recall that by Hirsch-Pugh-Shub structural stability theorem there exists a locally maximal W_f^c -saturated invariant set $\Delta \subset M \times S^1$ and a homeomorphism $H: \Lambda \times S^1 \rightarrow \Delta$ that “straightens” center leaves (that is, $H_*W_{f_0}^c = W_f^c$) and conjugates the induced maps on the space of center leaves (that is, $H \circ f_0(W_{f_0}^c(\cdot)) = f \circ H(W_f^c(\cdot))$). This uniquely defines H as a map on the space of center leaves. Along the center leaves H can be arbitrary. Therefore we can request that H preserves the second coordinate. In other words, we choose H so that it has the form

$$H(w, x) = (H_1(w, x), x).$$

Define

$$F_f = H^{-1} \circ f|_{\Delta} \circ H.$$

Then F_f is a skew product of the form (4.2) which we call the *rectification* of f .

Proposition 4.8 ([G06]). *Rectification F_f is a Hölder skew product. Moreover, constants C and α in (4.3) can be chosen independently of $f \in \mathcal{U}_L$. If f is $C^{1+\varepsilon}$ for some positive ε then (4.3) can be replaced with a stronger inequality*

$$d_{C^1}(g_u, g_v) \leq Cd(u, v)^\alpha$$

Also it is clear that our choice of H ensures that the circle diffeomorphisms g_w , $w \in \Lambda$, are C^1 close to identity.

Proof (Sketch). Consider two nearby points $(u, x), (v, x) \in \Lambda \times S^1$. Then we have

$$|g_u(x) - g_v(x)| = |f_2(H(u, x)) - f_2(H(v, x))| \leq A_1 d(H(u, x), H(v, x)), \quad (4.4)$$

where $f_2(\cdot)$ stands for the S^1 -coordinate of $f(\cdot)$ and A_1 is a constant close to 1 that depends only on $d_{C^1}(f, f_0)$.

Let $D^c f = \|Df|_{E_f^c}\|$. Under appropriate choice of Riemannian metric $|g'_w(x)| = D^c f(H(w, x))$ and

$$\begin{aligned} |g'_u(x) - g'_v(x)| &= |D^c f(H(u, x)) - D^c f(H(v, x))| \\ &\leq A_2 \text{dist}(E_f^c(H(u, x)), E_f^c(H(v, x))) \leq A_2 A_3 d(H(u, x), H(v, x))^\beta, \end{aligned} \quad (4.5)$$

where A_2 is again a constant that depends on $d_{C^1}(f, f_0)$ and the last inequality is due to Hölder continuity of E_f^c .

Combining (4.4) and (4.5) we have

$$d_{C^1}(g_u, g_v) \leq A_4 \sup_{x \in S^1} d(H(u, x), H(v, x))^\beta$$

and to obtain the desired inequality (4.3) we need to have

$$d(H(u, x), H(v, x)) \leq A_5 d(u, v)^\gamma$$

for all $x \in S^1$. The proof of this inequality is not hard and can be carried out in the same way as the proof of Hölder continuity of the conjugacy from structural stability (see, e.g., [KH95, Theorem 19.1.2]).

Finally let us remark that all the constants above depend only on $d_{C^1}(f, f_0)$ and hence can be chosen uniformly in $f \in \mathcal{U}_L$. \square

4.3. Step skew products. We have passed from a diffeomorphism f to its rectification $F_f: \Lambda \times S^1 \rightarrow \Lambda \times S^1$. Skew product F_f is easier to handle since dynamics on Λ admits symbolic description. Symbolic dynamics takes the simplest form in the case when Λ is the hyperbolic invariant set of the horseshoe.

Therefore we now assume that Λ is the hyperbolic set of the horseshoe. (Ultimately we are interested in the case $\Lambda = \mathbb{T}^2$ and we discuss this case later.) Then Λ is homeomorphic to Σ^N , the space of all bi-infinite words $\omega = \dots\omega_{-1}\omega_0\omega_1\dots$, $\omega_i \in \{1, 2, \dots, N\}$, and $h: \Lambda \rightarrow \Lambda$ becomes the left shift $\sigma: \Sigma^N \rightarrow \Sigma^N$. The rectification F_f takes the form

$$F_f(\omega, x) = (\sigma\omega, g_\omega x).$$

Recall that our goal is to find certain periodic points of F_f that would give us sought measure with zero center exponent in the limit. Thus we need to come up with periodic words ω , $\sigma^k\omega = \omega$, such that corresponding fiber diffeomorphisms $g_{\sigma^{p-1}\omega} \circ g_{\sigma^{p-2}\omega} \circ \dots \circ g_\omega$ can be shown to possess desired properties such as existence of a fixed point. It would be nice if we can choose diffeomorphism $g_\omega, g_{\sigma\omega}, \dots, g_{\sigma^{p-1}\omega}$ independently to produce these properties. However the problem is that, a priori, these diffeomorphisms do depend on each other. This heuristics motivates the introduction of step skew products.

Definition 4.9. *A skew product $F: \Sigma^N \times S^1 \rightarrow \Sigma^N \times S^1$ is called step skew product if it has the form*

$$F(\omega, x) = (\sigma\omega, g_{\omega_0}x),$$

where ω_0 is the zeroth letter of ω .

Remark 4.10. *The term “step skew product” comes from the analogy with a step function as $\omega \mapsto g_{\omega_0}$ takes only finitely many values. Another common term is “iterated function system.”*

For a step skew product the fiber diffeomorphism $g_{\sigma^{m-1}\omega} \circ g_{\sigma^{m-2}\omega} \circ \dots \circ g_\omega$ becomes $g_{\omega_{m-1}} \circ g_{\omega_{m-2}} \circ \dots \circ g_{\omega_0}$ and thus we can paste together diffeomorphisms g_1, g_2, \dots, g_N any way we please to obtain orbits of F with desired properties.

Example 4.11. *Take $N = 2$ and let g_1 be a rotation by a very small angle and g_2 be a diffeomorphism with two fixed points, an attractor and a repeller. This choice makes it easy to create periodic words of g_1 and g_2 that give periodic points with small (positive or negative) center exponents. Indeed, take a small interval $I \subset S^1$ and start rotating it using g_1 . Once the interval is near the attracting fixed point of g_2 we can slightly shrink it with g_2 and then continue rotating until it comes inside the original interval I . Then there must be a fixed point in I for*

corresponding composition of g_1 -s and g_2 -s. This point is a periodic point with small center exponent for the step skew product F .

This type of arguments are very fruitful and can go a long way. In fact, for the above example, one can show that periodic points with arbitrarily small center exponent are dense in $\Sigma^2 \times S^1$. Also one can construct dense orbits of F with prescribed center exponent λ in some small interval $(-\varepsilon, \varepsilon)$. These properties are C^1 stable in the space of step skew products and, more importantly, in the space of Hölder skew products [GI99, GI00].

4.4. Sketch of the proof of Theorem 4.5. The first step is to choose certain step skew product F by specifying the circle diffeomorphisms g_1, g_2, \dots, g_N . Then there exists a diffeomorphism $f: M \times S^1 \rightarrow M \times S^1$ such that $F_f = F$. Diffeomorphisms g_1, g_2, \dots, g_N are chosen so that we can find a sequence of periodic points $\{p_n; n \geq 1\}$ which gives an ergodic measure μ , $\text{supp } \mu = \Sigma^N \times S^1$, with zero center exponent in the limit (cf. Example 4.11).

One has to prove that the construction of $\{p_n; n \geq 1\}$ is robust under sufficiently C^1 -small perturbations of F in the space of Hölder skew products. So that, by Proposition 4.8, if \mathcal{V} is sufficiently small C^1 neighborhood of f then any $g \in \mathcal{V}$ would have an ergodic measure with zero center exponent supported on locally maximal partially hyperbolic set.

Next give an outline of this argument without going into technical details.

For any periodic point p of F let $\lambda^c(p)$ be the center exponent at p and let $\nu(p)$ be the atomic measure supported on the orbit of p , that is,

$$\nu(p) = \frac{1}{|\mathcal{O}(p)|} \sum_{q \in \mathcal{O}(p)} \delta_q.$$

We have to construct a sequence of periodic points $\{p_n; n \geq 1\}$ such that $\lambda^c(p_n) \rightarrow 0$, $n \rightarrow \infty$, and any weak* accumulation point μ of the sequence $\{\nu(p_n); n \geq 1\}$ is an ergodic measure of full support. Note that, since the space of measures on a compact space is compact in weak* topology, the sequence $\{\nu(p_n); n \geq 1\}$ has at least one accumulation point μ . Measure μ is the measure we seek. Indeed, by the Birkhoff ergodic theorem

$$\lambda^c(\mu) = \int_{\Sigma^N \times S^1} \log \frac{\partial F}{\partial x} d\mu = \lim_{n \rightarrow \infty} \int_{\Sigma^N \times S^1} \log \frac{\partial F}{\partial x} d\nu(p_n) = \lim_{n \rightarrow \infty} \lambda^c(p_n) = 0.$$

Periodic points p_n are constructed inductively. Ergodicity of the limit is guaranteed by certain similarity condition on periodic orbits $\mathcal{O}(p_n)$, $n \geq 1$. Very roughly, this condition says that for large n and all $m > n$ the (non-invariant) measures

$$\frac{1}{n} \sum_{i=1}^n \delta_{F^i(q)}, \quad q \in \mathcal{O}(p_m),$$

are weak* close to $\nu(p_n)$ for majority of points $q \in \mathcal{O}(p_m)$.

The Σ^N -coordinate of a periodic point p_n is a periodic word with some period α . Then Σ^N -coordinate of the next periodic point p_{n+1} is a periodic word with period $\alpha^k\beta$, where β is much shorter than α^k . Word β is a “correction term” that, in particular, ensures inequality $|\lambda^c(p_{n+1})| < c|\lambda^c(p_n)|$ for some fixed $c \in (0, 1)$.

Another thing to take care of in the inductive construction is to make sure that orbits $\mathcal{O}(p_n)$ become well-distributed in $\Lambda \times S^1$ to guarantee $\text{supp } \mu = \Lambda \times S^1$. For words this means that every finite word γ eventually appears as a subword of $\alpha = \alpha(p_n)$. Note that once we pass to $\alpha^k\beta$ word γ would appear k times. Thus γ will maintain the same positive proportion in all subsequent words and corresponding open subset of Λ (the γ -cylinder) will have some positive μ -measure. (Of course, one has to take care of S^1 -coordinate as well.)

We see that there are many things that has to be carefully tracked during the induction step. Moreover, the procedure must be C^1 robust. For step skew products such inductive procedure was carried out in [GIKN05] and it is sufficient to use only two diffeomorphisms and work over Σ^2 . It is harder to carry out this scheme C^1 -robustly in the space of Hölder skew products, this was done in [KN07] who need at least 5 symbols to play with.

So we consider a C^1 -small perturbation \tilde{F} , $\tilde{F}(\omega, x) = (\sigma\omega, \tilde{g}_\omega x)$, of the step skew product F . The difficulty is that circle diffeomorphisms \tilde{g}_ω depend on the whole word ω . This difficulty can be overcome using so called “predictability property” of Hölder skew products: for any $m \geq 1$ the composition $\tilde{g}_{\sigma^{m-1}\omega} \circ \tilde{g}_{\sigma^{m-2}\omega} \circ \dots \circ \tilde{g}_\omega$ can be determined approximately from first m letters of ω . More precisely, for any $m \geq 1$, given two words ω and ω' with $\omega_i = \omega'_i$ for $i = 0, 1, \dots, m-1$ we have

$$d_{C^0}(\tilde{g}_{\sigma^{m-1}\omega} \circ \tilde{g}_{\sigma^{m-2}\omega} \circ \dots \circ \tilde{g}_\omega, \tilde{g}_{\sigma^{m-1}\omega'} \circ \tilde{g}_{\sigma^{m-2}\omega'} \circ \dots \circ \tilde{g}_{\omega'}) \leq K\delta^\beta,$$

where $\delta = d_{C^1}(F, \tilde{F})$; $K > 0$ and $\beta > 0$ are some fixed constants and σ is the left shift.

4.5. The case $\Lambda = \mathbb{T}^2$. In the case when $\Lambda = \mathbb{T}^2$ and $h: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is an Anosov automorphism $A: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ Gorodetski-Ilyashenko strategy cannot be applied in a straightforward way. One still has symbolic dynamics which is now a subshift of finite type. The major problem is that step skew products cannot be realized as partially hyperbolic diffeomorphisms contrary to the horseshoe case.

One way around this difficulty is to pass to a power A^m of A so that there exists an embedded A^m -invariant horseshoe $\bar{\Lambda} \subset \mathbb{T}^2$. Then define a skew product F over A^m as a step skew product over $\bar{\Lambda}$ and extend smoothly to the rest of \mathbb{T}^2 . Then the strategy outlined above goes through and the only alteration to be made is to make sure that periodic orbits $\mathcal{O}(p_n)$ spend some time in the complement of $\bar{\Lambda} \times S^1$ to guarantee that $\text{supp } \mu = \mathbb{T}^3$. Such arguments appear in [N08] where the author discusses the case when Λ is the Smale-Williams solenoid.

5. NON-ZERO LYAPUNOV EXPONENTS AND PATHOLOGICAL FOLIATIONS

Presence of non-zero Lyapunov exponents may lead to certain measure-theoretic “pathology” of the center foliation of a partially hyperbolic diffeomorphism. This observation is due to Mañé and first appeared in [SW00].

Let $f: \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be the partially hyperbolic diffeomorphism constructed in Section 2.1. By Oseledets Theorem there exists a full volume set $\Lambda \subset \mathbb{T}^3$ of Lyapunov regular points whose center exponent $\lambda^c = \lambda^c(f)$ is positive. Provided that f is sufficiently close to the linear automorphism L the center distribution E^c integrates to a foliation W^c which is also a circle fibration.

Recall that we can decompose Λ as $\Lambda = \cup_{k \geq 1} \Lambda_k$ so that on sets Λ_k (called *Pesin sets*) we have uniform hyperbolicity. That is, $\forall x \in \Lambda_k$ and $\forall n > 0$

$$\|Df^n v\| \geq \frac{1}{k} e^{n(\lambda^c - \varepsilon)} \|v\|, \quad v \in E^c,$$

where $\varepsilon \in (0, \lambda^c)$.

Each leaf $\mathcal{C} \in W^c$ intersects a set Λ_k , $k \geq 1$, at a set of leaf Lebesgue measure zero since otherwise the lengths of $f^n(\mathcal{C})$ would grow to infinity. *Thus (a full volume set) Λ intersects every leaf of W^c at a set of leaf measure zero.*

This argument can be generalized to the higher dimensional setup and gives the following result.

Theorem 5.1 ([HP07]). *Let f be a C^2 volume preserving partially hyperbolic diffeomorphism. Assume that the center distribution E^c integrates to an invariant foliation W^c with compact leaves. Also assume that f is W^c -dissipative, that is, the sum of center exponents is different from zero on a set of full volume. Then there exists a set of positive volume that meets every leaf of W^c at a set of zero leaf volume.*

Remark 5.2. *The first example of pathological center foliation was constructed by A. Katok in early nineties. This examples also lives on \mathbb{T}^3 , but has zero center exponents. See, e.g., [HP07, G08] for a description of Katok’s example and [M97] for a different (non-dynamical) construction of a pathological foliation that was inspired by Katok’s construction.*

5.1. Conditional measures and absolute continuity. There are several way to define absolute continuity of foliations. We say that a foliation is absolutely continuous if for each foliation box the factor measure of the volume is absolutely continuous with respect to the induced volume on a transversal and the conditional measures of the volume on the plaques of the foliation are absolutely continuous with respect to the Lebesgue measure on the plaques. This property is weaker than absolute continuity of holonomy maps. However, it can be proved that the above absolute continuity property implies that the holonomy maps between almost all pairs of transversals are absolutely continuous. Note

that the definition of absolute continuity is independent of the particular choice of the volume form. Also note that the definition does not require presence of any dynamics. See [P04] for a detailed discussion of absolute continuity.

5.2. Pathological foliations with compact center leaves. If M is the total space of a fiber bundle with compact fibers then one can speak about conditional measures on the fibers without considering foliation boxes. This is because in this case the partition into fibers is measurable. Repetition of the argument given in the beginning of the current section immediately yields the following result.

Theorem 5.3. *Consider a partially hyperbolic diffeomorphism f whose center foliation is a circle fibration. Assume that f preserves an ergodic measure μ (e.g., volume) with negative (or positive) center Lyapunov exponent. Then the conditional measures of μ on the leaves of the center foliation are singular with respect to the Lebesgue measure on the leaves.*

This result was generalized in [HP07] to the setting of partially hyperbolic diffeomorphisms with higher dimensional compact center leaves. If the sum of center Lyapunov exponents is non-zero then the center foliation is non-absolutely continuous. The sum of center Lyapunov exponents can be perturbed away from zero by Theorem 2.2.

Next theorem generalizes Theorem 5.3 and describes the conditional measures on the leaves of the center foliation.

Theorem 5.4 ([RW01, HP07]). *Consider a dynamically coherent partially hyperbolic diffeomorphism f whose center leaves are fibers of a (continuous) fiber bundle. Assume that the all center Lyapunov exponents are negative (or positive) then the conditional measures of μ on the leaves of the center foliation are atomic with p , $p \geq 1$, atoms of equal weight on each leaf.*

It is interesting to obtain examples beyond circle fibrations to which Theorem 5.4 applies.

Question 5.5. *Start with a partially hyperbolic skew product with fibers of dimension ≥ 2 that has zero center Lyapunov exponents. Is there an ergodic perturbation with only negative center Lyapunov exponents?*

Theorem 5.4 is a “fiber bundle”-generalization of the following proposition.

Proposition 5.6. *Let f be a diffeomorphism of a compact manifold that preserves an ergodic measure μ . Assume that all Lyapunov exponents of μ are negative (or positive). Then μ is an atomic measure on a periodic sink (or source).*

To illustrate the basic idea we prove the proposition first and then Theorem 5.4.

Proof of Proposition 5.6. Consider a Pesin set Λ_k of positive measure. Since all Lyapunov exponents are negative the stable manifolds of points in Λ_k are open

balls in M . Let $\delta_k > 0$ be the lower bound on the size of the stable manifolds of $x \in \Lambda_k$.

Choose a ball B of radius $\delta_k/10$ such that $\mu(B \cap \Lambda_k) > 0$. Then, by Poincaré recurrence, there exists an $x \in B \cap \Lambda_k$ that returns to $B \cap \Lambda_k$ infinitely often. The ball $B(x, \delta_k)$ is in the stable manifold of x . Thus, for sufficiently large return time n_m ,

$$f^{n_m}(B(x, \delta_k)) \subset B(f^{n_m}(x), \delta_k/2) \subset B(x, \delta_k).$$

Since f^{n_m} preserves μ we conclude that the restriction of μ to $B(x, \delta_k)$ is an atom at $y = \bigcap_{i \geq 0} f^{in_m}(B(x, \delta_k))$. Then y must be periodic and ergodicity implies that μ is an atomic measure that sits on the orbit of y . \square

Proof of Theorem 5.4. As earlier we consider Pesin sets Λ_k , $k \geq 1$, and we denote by $\delta_k > 0$ the lower bound on the size of stable manifolds of $x \in \Lambda_k$. To avoid possible confusion we point out that in this proof we use term “stable manifold” for Pesin stable manifold as opposed to the manifold tangent to E^s .

By $B^c(x, r)$ we denote a ball inside $W^c(x)$ centered at x of radius r . Since the dimension of stable manifolds is equal to $\dim E^c + \dim E^s$, a simple growth argument (which utilizes dynamical coherence) shows that the stable manifolds are contained in center-stable leaves. Thus, for $x \in \Lambda_k$ and sufficiently small r , e.g., $r = \delta_k/2$ the ball $B^c(x, r)$ is contained in the stable manifold of x .

By $\mu_x, x \in M$, we denote the conditional measure on the center leaf $W^c(x)$. By invariance, we have

$$\mu_{f(x)} = f_*\mu_x$$

for μ a.e. $x \in M$.

Recall that the union of Pesin sets $\bigcup_{k \geq 1} \Lambda_k$ has full measure. Choose a sufficiently large k so that $\mu(\Lambda_k) > 1/2$. Then the set

$$A = \bigcup_{x \in M: \mu_x(W^c(x) \cap \Lambda_k) \geq 1/2} W^c(x)$$

has positive measure. Consider the set $B \subset A$ that consists of points that return to A infinitely many times both in positive and negative time. Clearly B is saturated by the center leaves and, by the Poincaré recurrence, $\mu(B) = \mu(A)$. Let $F: B \rightarrow B$ be the first return map of f . Then, obviously, F is a bijection, $F(W^c(x)) = W^c(F(x))$ and $F_*\mu_x = \mu_{F(x)}$.

There exists a number m such that every center leaf can be covered by at most m balls of radius $\delta_k/2$. Thus for every $x \in B$ we can pick a ball $B_x^c = B^c(y(x), \delta_k/2)$ such that $y(x) \in W^c(x)$, $B_x^c \cap \Lambda_k \neq \emptyset$ and $\mu_x(B_x^c) \geq 1/2m$.

Now let $B_{n,x}^c = F^n(B_{F^{-n}(x)}^c)$ and note that $\mu_x(B_{n,x}^c) = \mu_{F^{-n}(x)}(B_{F^{-n}(x)}^c) \geq 1/2m$. At the same time $\text{diam}(B_{n,x}^c) \rightarrow 0$ as $n \rightarrow +\infty$ uniformly in x since every ball B_x^c is contained in a stable manifold of some point from Λ_k .

We conclude that for every $x \in B$ we have a sequence of shrinking sets inside $W^c(x)$ of positive μ_x measure. Clearly, a converging subsequence gives an atom of μ_x . Thus for μ a.e $x \in B$ measure μ_x has an atom of weight $\geq 1/2m$.

Measure μ can be decomposed as a sum of measure μ^a which has atomic conditional measures on the center leaves and measure μ^{na} which has atomless conditional measures on the center leaves. Both measures are invariant and we have shown that $\mu^a(M) > 0$. Thus ergodicity of μ implies that μ^{na} is a trivial measure, i.e., $\mu^{na}(M) = 0$. Furthermore, in similar fashion it follows from ergodicity that for μ a.e. x measure μ_x has p atoms of equal weight, where p does not depend on x . \square

Remark 5.7. *It is easy to construct examples with higher dimensional center leaves and non-zero average center exponent for which the conditional measures are singular but non-atomic.*

Remark 5.8. *Note that the proof of Theorem 5.4 does not provide information on the number of atoms p . It was shown in [Hom10] that p can be any positive integer. More precisely, it was proved that if an ergodic volume preserving perturbation L of partially hyperbolic automorphism $L_0: \mathbb{T}^3 \rightarrow \mathbb{T}^3$ (L_0 is given by (2.2)) has*

- (1) *negative center exponents*
- (2) *a fixed center fiber with a unique hyperbolic attracting fixed point and a unique hyperbolic repelling fixed point*
- (3) *minimal strong unstable and strong stable foliations*

then $p = 1$. The assumptions on L are known to hold for certain perturbations of the form (2.3).

Now, by passing to finite self covers of \mathbb{T}^3 one can get partially hyperbolic diffeomorphisms with prescribed number of atoms p .

5.3. Pathological foliations with non-compact leaves: near geodesic flow. For a general partially hyperbolic diffeomorphism the geometric structure of the support of disintegration measures is not clear.

Avila, Viana and Wilkinson [AVW11] showed that, for volume preserving perturbations of a time-one map $f_0: N \rightarrow N$ of the geodesic flow a closed negatively curved surfaces, the center foliation W^c has either atomic or absolutely continuous conditional measures. Recall that, as discussed in Section 2.3, center Lyapunov exponent of f_0 can be perturbed away from zero. The center Lyapunov exponent plays a key role in the proof of the “atomic-Lebesgue dichotomy” of [AVW11].

The fact that non-zero center exponent leads to atomic conditional measure on the center leaves follows easily from (the more general form of) Theorem 5.4.

Indeed, for a perturbation $f: N \rightarrow N$ of f_0 one can consider the extension

$$\begin{array}{ccc} S^1 & & S^1 \\ \downarrow & & \downarrow \\ E & \xrightarrow{\hat{f}} & E \\ \downarrow & & \downarrow \\ N & \xrightarrow{f} & N \end{array}$$

where fibers S_x^1 , $x \in N$, are the segments of center leaves with end-points identified, i.e., $S_x^1 = [x, f(x)]^c/x \sim f(x)$, and \hat{f} is induced by f the obvious way. This way non-compact center foliation can be compactified and [RW01] applies to give atomic conditionals on the fibers.

In the zero exponent case, the authors use in a subtle way the invariance principle proved in [ASV] (see also [AV]). The invariance principle is a non-trivial extension of seminal work of Ledrappier [Led86] to the context of smooth cocycles. Assuming that the center exponent is zero, the invariance principle yields a dichotomy into two cases; these two cases are characterized in terms of geometry of the supports of conditional measures along center leaves. In the first case, the support of conditional measures is a countable subset and there exist a full volume subset such that intersects almost all center leaves in exactly k orbits. In the second case, the supports of conditional measures coincide with the center leaves. In this case, using invariance of disintegration with respect to holonomies and regularity of holonomies (stable and unstable holonomies are C^1 between center leaves) the authors are able to prove that the conditional measures are absolutely continuous. Moreover, they prove that in this case, the diffeomorphism is time one map of a smooth flow and consequently, center foliation is smooth.

Question 5.9. *Is there an example f close to time one map of geodesic flow of a negatively curved surface, such that center exponents of f vanish and the center foliation is singular?*

5.4. Pathological foliations with non-compact leaves: Anosov and derived from Anosov case. It turns out that the argument given in the beginning of current section can be generalized to certain non-compact center foliations, which we proceed to describe.

Let L be an automorphism of \mathbb{T}^3 with three distinct Lyapunov exponents $\lambda^s(L) < \lambda^c(L) < \lambda^u(L)$. Assume that $\lambda^c(L) > 0$. Then the center foliation W_L^c is an expanding foliation. We have shown in Section 2.2 that there exists a perturbation g such that $\lambda^c(g) > \lambda^c(L)$. Define *geometric growth* of W_g^c as

$$\chi^c(g) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \text{length}(g^n(B_g^c(x, r))),$$

where $B_g^c(x, r)$ is a ball inside $W_g^c(x)$. Automorphism L is Anosov and thus, by the structural stability, L and g are conjugate. One can easily deduce that $\chi^c(g) = \chi^c(L) = \lambda^c(L)$. This property of g plays the same role as compactness of center leaves earlier. We can essentially repeat the argument given in the beginning of current section and conclude that the full volume set of Lyapunov regular points of g intersects every leaf of W_g^c by a set of zero length.

These arguments can be generalized to higher dimensional automorphisms. Namely, assume that automorphism L preserves a partially hyperbolic splitting $T\mathbb{T}^d = E_L^s \oplus E_L^c \oplus E_L^u$ and that the center foliation W_L^c is uniformly expanding.

Theorem 5.10 ([SX09]). *If $g \in \text{Diff}_m(\mathbb{T}^d)$ is a diffeomorphism C^1 -close to L and*

$$\lambda^c(g) = \int_{\mathbb{T}^d} \log J^c(g) dm > \int_{\mathbb{T}^d} \log J^c(L) dm = \lambda^c(L),$$

then foliation W_g^c is non-absolutely continuous.

Note that one can also consider diffeomorphisms g with $\lambda^c(g) < \lambda^c(L)$ and it is not clear if this inequality forces non-absolute continuity of W_g^c . However it is the case when $d = 3$. In the three dimensional case there is an if-and-only-if description for non-absolute continuity of W_g^c given in terms of eigenvalues at periodic points of g [G12]. The following theorem is a corollary of this description.

Theorem 5.11 ([G12]). *Let L be an automorphism of \mathbb{T}^3 with three distinct Lyapunov exponents $\lambda^s(L) < \lambda^c(L) < \lambda^u(L)$. Assume that $\lambda^c(L) > 0$. Let \mathcal{U}_L be a C^1 -small neighborhood of L in $\text{Diff}_m(\mathbb{T}^3)$. Then there is a C^1 -open and C^r -dense set $\mathcal{V} \subset \mathcal{U}_L$ such that $g \in \mathcal{V}$ if and only if the center foliation W_g^c is non-absolutely continuous with respect to the volume m .*

Recently, R. Varão [Va13] showed that there exist Anosov diffeomorphisms in \mathcal{V} with non-absolutely continuous center foliation which does not have atomic disintegration.

So far we have concentrated on diffeomorphism which are C^1 close to L . Recall that a diffeomorphism f is *derived from Anosov* (or simply DA) diffeomorphism if it is homotopic to an Anosov diffeomorphism. The following recent result of Ponce, Tahzibi and Varão [PTV13] establishes atomic disintegration for a class of (non-Anosov) partially hyperbolic DA diffeomorphisms.

Theorem 5.12. *Let L be a hyperbolic automorphism of \mathbb{T}^3 with three distinct Lyapunov exponents $\lambda^s(L) < \lambda^c(L) < \lambda^u(L)$. Let $f: \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be volume preserving DA diffeomorphism (homotopic to L). Assume that f is partially hyperbolic, volume preserving and ergodic. Also assume that $\lambda^c(f)\lambda^c(L) < 0$ then the disintegration of volume along center leaves of f is atomic and in fact there is just one atom per leaf.*

The above theorem can be verified for an open class of diffeomorphisms found by Ponce-Tahzibi in [PT13].

It is interesting to emphasize that conservative ergodic DA diffeomorphism on \mathbb{T}^3 show a feature that is not, so far, shared with any other known partially hyperbolic diffeomorphisms on dimension three, it admits all three disintegration of volume on the center leaf, namely: Lebesgue, atomic, and singular non-atomic (by a recent result of R. Varão [Va13]). We also remark that Katok's example (cf. Remark 5.2) can be modified to give singular non-atomic conditionals. However this example is not ergodic.

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