

PARTIALLY HYPERBOLIC DIFFEOMORPHISMS WITH COMPACT CENTER FOLIATIONS

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ABSTRACT. Let $f: M \rightarrow M$ be a partially hyperbolic diffeomorphism such that all of its center leaves are compact. We prove that Sullivan's example of a circle foliation that has arbitrary long leaves cannot be the center foliation of f . This is proved by thorough study of the accessible boundaries of the center-stable and the center-unstable leaves.

Also we show that a finite cover of f fibers over an Anosov toral automorphism if one of the following conditions is met:

1. the center foliation of f has codimension 2, or
2. the center leaves of f are simply connected leaves and the unstable foliation of f is one-dimensional.

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1. INTRODUCTION

1.1. Let M be a smooth compact manifold without boundary. A diffeomorphism $f: M \rightarrow M$ is called *partially hyperbolic* if there exists a Riemannian metric on M along with a Df -invariant continuous splitting $TM = E^s \oplus E^c \oplus E^u$ of the tangent bundle such that for all $x \in M$ all unit vectors $v^\sigma \in E^\sigma(x)$, $\sigma = s, c, u$, satisfy the following properties

$$\begin{aligned} \|D_x f(v^s)\| &< 1, \\ \|D_x f(v^u)\| &> 1, \\ \|D_x f(v^s)\| &< \|D_x f(v^c)\| < \|D_x f(v^u)\|. \end{aligned}$$

It is known that there are f -invariant foliations \mathcal{W}^s and \mathcal{W}^u , the *stable* and *unstable* foliations, that are tangent to the distributions E^s and E^u , respectively. In general, the center distribution E^c does not necessarily integrate to a foliation. In this paper we will study partially hyperbolic diffeomorphisms whose *center distribution E^c uniquely integrates to a foliation \mathcal{W}^c with compact leaves*. Foliation with all leaves compact are called *compact*.

The following question was posed by Charles Pugh (Problem 48 in [RHRHU07]).

Main Question. *Consider a partially hyperbolic diffeomorphism f that has a compact center foliation. Is it true that the volume of the center leaves is uniformly bounded? Is it true that f can be finitely covered by a partially hyperbolic diffeomorphism $\tilde{f}: \tilde{M} \rightarrow \tilde{M}$ so that there is a fibration $p: \tilde{M} \rightarrow N$ whose fibers are the center leaves and an Anosov diffeomorphism $\tilde{f}: N \rightarrow N$ such that p is a semiconjugacy between \tilde{f} and \tilde{f} ?*

We have modified the question from [RHRHU07] to accommodate some simple examples with finite covers.

An affirmative answer to the Main Question would reduce the classification problem of partially hyperbolic diffeomorphisms with compact center foliations to the classification problem of Anosov diffeomorphisms. This paper provides positive results for partially hyperbolic diffeomorphisms under each of the following additional assumptions:

- 1) the center foliation \mathcal{W}^c has codimension 2;
- 2) the leaves of \mathcal{W}^c are simply connected;
- 3) $\dim E^c = 1$, $\dim E^s \leq 2$, $\dim E^u \leq 2$.

1.2. We can place the Main Question into a wider framework of the classification problem. In general, the classification of partially hyperbolic diffeomorphisms beyond dimension 3 seems to be a hopeless problem. However one can try to classify (dynamically coherent) partially hyperbolic diffeomorphisms based on the properties of the center foliation.

The leaves of the stable and unstable foliations are all homeomorphic to Euclidean balls. The center foliation however displays a variety of behaviors and one

can pose classification questions under various assumptions on the center foliation. From this perspective the Main Question is probably the first question that comes to mind.

Question 1.1. *Assume that the center foliation \mathcal{W}^c is a foliation with all leaves diffeomorphic to \mathbb{R} . Is it true that a finite cover of f is center conjugate to an affine map of G/Γ , $x\Gamma \mapsto g \cdot A(x)\Gamma$?*

Question 1.2. *Assume that $\dim \mathcal{W}^c = 1$ and that \mathcal{W}^c has countable many leaves that are diffeomorphic to S^1 and the rest diffeomorphic to \mathbb{R} . Is it true that a finite cover of f is center conjugate to a time-1 map of an Anosov flow?*

Question 1.3. *Assume that $\dim \mathcal{W}^c = 1$ and that the map induced on the space of center leaves is identity. Is it true that a finite cover of f is center conjugate to a time-1 map of an Anosov flow?*

For positive results one may start by introducing additional assumptions such as $\dim E^u = 1$.

1.3. Foliation theory perspective. Compact foliations were studied for their own sake. Below is background to help better understand the Main Question.

1.3.1. Given a compact foliation \mathcal{W} of a compact manifold M we can form the *leaf space* X by collapsing each leaf to a point. We equip X with the quotient topology, which makes it a compact space.

A Riemannian metric induces a Riemannian volume on the leaves of \mathcal{W} and hence a volume function $vol: X \rightarrow (0, +\infty)$. Exact values of vol depend on the choice of the Riemannian metric. However, the property of vol being bounded (or unbounded) is independent of such a choice.

Proposition 1.4. *Function $vol: X \rightarrow (0, +\infty)$ is lower semi-continuous and it is continuous on an open dense subset of X . Function vol is bounded if and only if X is Hausdorff.*

For a proof see, for example, [E72, E76]. The proof of semi-continuity given in [E72] is for the case $\dim M = 3$, $\dim \mathcal{W} = 1$. However, the argument can be adjusted to the higher dimensional setup with only minor modifications.

Remark 1.5. We will write *length* instead of vol when $\dim \mathcal{W} = 1$.

1.3.2. It is a corollary of the Reeb Stability Theorem that a foliation with simply connected leaves is a fibration and, hence, its vol is bounded.

1.3.3. It is not known whether a compact foliation with leaves of arbitrarily large volume can have simply connected leaves (see Question A.1.2 in [L92]).

1.3.4. The Main Question is already very interesting in the case when the center foliation is a foliation by circles. Sullivan [S76] gave a beautiful example of a smooth foliation by circles of a compact 5-manifold with arbitrarily long circles (hence the function *length* is unbounded). Later similar example was constructed [EV78] on a compact 4-manifold. This is optimal since Epstein [E72] had shown that lengths of the circles that foliate a 3-manifold are uniformly bounded. Also Vogt [V76] and Edwards, Millet, Sullivan [EMS77] had independently generalized Epstein's result to codimension 2 foliations.

1.4. Seifert fibrations. Let \mathcal{W} be a compact foliation on M . Associated to every leaf $\mathcal{W}(x)$ is the group $G_x(\mathcal{W})$ of germs of holonomy homeomorphisms of a small transversal centered at x . We say that \mathcal{W} is a *Seifert fibration* if there are finitely many *exceptional fibers* $\mathcal{W}(x)$ (leaves of \mathcal{W}) for which $G_x(\mathcal{W})$ is a non-trivial finite group isomorphic to a subgroup of the orthogonal group $O(\dim M - \dim \mathcal{W})$ and the other leaves have trivial holonomy group.

We also note that in the case when $G_x(\mathcal{W})$ is finite it can actually be identified with a group of homeomorphisms of a transversal about x .

1.5. Partially hyperbolic skew products. Let X be a compact *topological manifold*, that is, a Hausdorff second countable metric space with locally Euclidean structure given by continuous charts. Let M be a compact smooth manifold and $p: M \rightarrow X$ be a locally trivial fibration such that $\mathcal{W}^c \stackrel{\text{def}}{=} \{p^{-1}(x), x \in X\}$ is a continuous foliation with C^1 leaves (see Section 2 for the definition). A C^1 partially hyperbolic diffeomorphism of M with $E^c \stackrel{\text{def}}{=} T\mathcal{W}^c$ being the center distribution is called *partially hyperbolic skew product*. Hence a partially hyperbolic skew product f fits into the commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ p \downarrow & & p \downarrow \\ X & \xrightarrow{\tilde{f}} & X \end{array}$$

We say that f *fibers over a hyperbolic automorphism* if the induced homeomorphism $\tilde{f}: X \rightarrow X$ is topologically conjugate to a hyperbolic automorphism.

The basic examples of partially hyperbolic skew products are skew products over Anosov diffeomorphisms with fiber expansion/contraction dominated by the base expansion/contraction. Furthermore, by Hirsch-Pugh-Shub structural stability for partially hyperbolic diffeomorphisms, we can perturb these basic examples in C^1 topology to get more examples of partially hyperbolic skew products. Examples where M is a non-trivial fiber bundle are also possible.

1.6. Statements of results. We say that a partially hyperbolic diffeomorphism is *dynamically coherent* if there exists a foliation \mathcal{W}^c tangent to E^c and any curve tangent to E^c is contained in a leaf of \mathcal{W}^c .

Now we are ready to give the precise statements of our results.

Theorem 1.6. *Let $f: M \rightarrow M$ be a C^1 dynamically coherent partially hyperbolic diffeomorphism with compact center foliation \mathcal{W}^c . Assume that $\dim E^s = \dim E^u = 1$. Then volumes of the center leaves are uniformly bounded and f admits a finite covering diffeomorphism $\tilde{f}: \tilde{M} \rightarrow \tilde{M}$ which is a partially hyperbolic skew product. Moreover, \tilde{f} fibers over a hyperbolic automorphism of the 2-torus.*

Theorem 1.7. *Let f be a C^1 dynamically coherent partially hyperbolic diffeomorphism with compact center foliation \mathcal{W}^c . Assume that the leaves of \mathcal{W}^c are simply connected. Then volumes of center leaves are uniformly bounded and f is a partially hyperbolic skew product.*

Assume additionally that $\dim E^u = 1$, then f fibers over a hyperbolic automorphism of the torus.

The following is our main result.

Main Theorem. *Let f be a C^1 dynamically coherent partially hyperbolic diffeomorphism with compact center foliation \mathcal{W}^c . Assume that $\dim E^u \leq 2$, $\dim E^s \leq 2$ and $\dim E^c = 1$ so that \mathcal{W}^c is a foliation by circles. Then the lengths of the center leaves are uniformly bounded. Moreover, every center leaf has a finite holonomy group.*

Question 1.8. *It is not clear to us whether the assumption $\dim E^c = 1$ is crucial for our approach to work. Can one generalize our techniques to higher dimensional center foliation?*

Based on the Main Theorem and a theorem of Bohnet on codimension 4 finite holonomy center foliations [B11, Theorem 2.64, pp. 116-117], it is easy to establish the following.

Corollary 1.9. *Let f be as in the Main Theorem. Then there is a finite cover $\tilde{f}: \tilde{M} \rightarrow \tilde{M}$ with center foliation $\tilde{\mathcal{W}}^c$ such that $\tilde{\mathcal{W}}^c$ is a Seifert fibration on \tilde{M} . The holonomy groups of exceptional fibers are products of two cyclic groups.*

Question 1.10. *In Corollary 1.9 passing to a finite cover is needed in order to orient \mathcal{W}^u and \mathcal{W}^s . Is it possible to eliminate exceptional fibers by passing to further finite covers? More generally, can one have a center foliation which is a Seifert fibration with exceptional fibers such that there is no finite cover in which the exceptional fibers disappear?*

Examples in [B11] show that it is necessary to pass to a finite cover that orients \mathcal{W}^u and \mathcal{W}^s , otherwise one might have infinitely many leaves with non-trivial finite holonomy.

Remarks 1.11.

1. Passing to a finite cover in Theorem 1.6 is only needed to make sure that foliations \mathcal{W}^s , \mathcal{W}^c and \mathcal{W}^u are orientable. For an example of a partially hyperbolic skew product with non-orientable foliations see [BW05]. In this example the center foliation is a Seifert fibration over the ‘‘pillow-case’’ orbifold.
2. The first part of Theorem 1.6 for 3-dimensional manifolds is a corollary of results from [BW05].
3. The center foliation \mathcal{W}^c in Theorem 1.6 is a codimension 2 foliation. One may wonder if the volumes of the center leaves are uniformly bounded without making any dynamical assumptions. As discussed in 1.3.4 this is indeed the case. However, the results in [EMS77, V76] use C^1 -smoothness of the foliation in a crucial way. Therefore we cannot apply them in our setting since the center foliation is only continuous transversally.
4. Note that in Theorem 1.6 we do not assume that the leaves of the center foliation are all homeomorphic. In fact, in the setup of Theorem 1.6 the center leaves for f are not necessarily homeomorphic. (The center leaves for \tilde{f} are, of course, homeomorphic.)
5. The first part of Theorem 1.7 is actually immediate from Reeb Stability Theorem.
6. Most of our arguments in the proof of the Main Theorem work for stable and unstable foliations of arbitrary dimension. The assumption on dimension is only used at the end of Section 7 in Proposition 7.14.

1.7. Work of Bohnet and Carrasco. In recent theses by Doris Bohnet [B11] and Pablo Carrasco [C10] the same topic was pursued independently of each other and of our work. There is a certain overlap in results established. In particular, Theorem 1.6 is also established in both theses. Moreover, it is shown in [B11] that \widetilde{M} from Theorem 1.6 is actually a two or one fold cover and the center foliation of f is a Seifert fibration with either 4 or 0 exceptional fibers whose holonomy groups has two elements $\{id, -id\}$. A similar description is provided in [B11] for codimension 1 partially hyperbolic diffeomorphisms whose compact center foliation has finite holonomy. This generalizes our Theorem 1.7 to other foliations with finite holonomy (whose leaves are not necessarily simply connected).

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2. PRELIMINARIES FROM FOLIATION THEORY

Here we collect well known results on foliations that we will need. We will consider continuous foliations with C^1 leaves. A foliation \mathcal{W} is a *continuous foliation with C^1 leaves* if it is given by continuous charts, the leaves $\mathcal{W}(x)$, $x \in M$, are C^1 immersed submanifolds and the tangent space $T_x\mathcal{W}(x)$ depends continuously on $x \in M$.

2.1. Reeb stability and foliations with simply connected leaves. Let \mathcal{W} be a continuous foliation with C^1 leaves on a compact manifold M . Then given a point x one can define the germinal holonomy group $G_x(\mathcal{W})$ that consists of germs of holonomy homeomorphisms of a small transversal centered at x . One also has a surjective homomorphism

$$h^c: \pi_1(\mathcal{W}(x), x) \rightarrow G_x(\mathcal{W}).$$

Theorem 2.1 (Reeb Stability). *If a compact leaf $\mathcal{W}(x)$ has trivial holonomy $G_x(\mathcal{W})$ then there is a neighborhood of $\mathcal{W}(x)$ in M that is a union of leaves that are homeomorphic to $\mathcal{W}(x)$.*

Corollary 2.2. *If $\mathcal{W}(x)$ is a compact simply connected leaf then there is a \mathcal{W} -saturated neighborhood of $\mathcal{W}(x)$ in M that is homeomorphic to $T \times \mathcal{W}(x)$ via a homeomorphism that takes the leaves of \mathcal{W} to the fibers $\{\cdot\} \times \mathcal{W}(x)$.*

Theorem 2.3 (Generalized Reeb Stability). *If a compact leaf $\mathcal{W}(x)$ has a finite holonomy group $G_x(\mathcal{W})$ then there is an arbitrarily small foliated normal neighborhood V of $\mathcal{W}(x)$ and a projection $p: V \rightarrow \mathcal{W}(x)$ such that $(V, \mathcal{W}|_V, p)$ is a foliated bundle with all leaves compact. Furthermore, each leaf $\mathcal{W}(y) \subset V$ has finite holonomy group of order at most $|G_x(\mathcal{W})|$ and the covering $p|_{\mathcal{W}(y)}: \mathcal{W}(y) \rightarrow \mathcal{W}(x)$ has k sheets, where $k \leq |G_x(\mathcal{W})|$.*

We refer to the book [CC00] for a detailed discussion and the proofs of the above results.

2.2. The bad set of a compact foliation. Consider a compact foliation \mathcal{W} on a manifold M , the leaf space X and the volume function $vol: X \rightarrow (0, +\infty)$ as in 1.3.1. Define the lifting $\overline{vol}: M \rightarrow \mathbb{R}$ by $\overline{vol}(x) = vol(\mathcal{W}(x))$.

Define the bad set

$$\mathcal{B} \stackrel{\text{def}}{=} \{x \in M : \overline{vol} \text{ is not locally bounded at } x\}.$$

Clearly \mathcal{B} is compact. Also consider the set $\mathcal{B}' \supset \mathcal{B}$ — the set of points at which \overline{vol} is not continuous. Recall that by Proposition 1.4 vol is lower semi-continuous. Clearly \overline{vol} is also lower semi-continuous. Any lower semi-continuous function is the limit of an increasing sequence of continuous functions and the Baire Category Theorem implies that \mathcal{B}' has empty interior. Thus \mathcal{B} is a compact set with empty interior.

3. PRELIMINARIES FROM PARTIALLY HYPERBOLIC DYNAMICS

In this section we collect various preparatory results, which apply to a wider class of partially hyperbolic diffeomorphisms than just partially hyperbolic diffeomorphisms with compact center foliations.

3.1. Notation. Define $E^{cs} \stackrel{\text{def}}{=} E^c \oplus E^s$ and $E^{cu} \stackrel{\text{def}}{=} E^c \oplus E^u$.

We write \mathcal{W}^s and \mathcal{W}^u for the stable and unstable foliations, that is, foliations tangent to E^s and E^u , respectively. These are continuous foliations with C^1 leaves. It is known that the leaves $\mathcal{W}^s(x), \mathcal{W}^u(x)$, $x \in M$, are diffeomorphic to $\mathbb{R}^{\dim E^s}$ and $\mathbb{R}^{\dim E^u}$ respectively. Define the local leaves

$$\mathcal{W}^\sigma(x, \varepsilon) \stackrel{\text{def}}{=} \{y \in \mathcal{W}^\sigma(x) : d^\sigma(y, x) < \varepsilon\}, \quad \sigma = s, u,$$

where d^σ is the metric induced on the leaves of \mathcal{W}^σ by the Riemannian metric. If $\varepsilon > 0$ is small we will sometimes refer to the local leaves as *plaques*.

If f is dynamically coherent (see Section 1.6 for the definition) then we can also define the center local leaves $\mathcal{W}^c(x, \varepsilon)$ in a similar way.

3.2. The center-stable and center-unstable leaves.

Proposition 3.1. *Let f be a partially hyperbolic diffeomorphism and let B^c be a C^1 embedded disk tangent to E^c ($\dim B^c = \dim E^c$) at every point. Then for any $\varepsilon > 0$ the set*

$$\mathcal{W}^\sigma(B^c, \varepsilon) \stackrel{\text{def}}{=} \bigcup_{y \in B^c} \mathcal{W}^\sigma(y, \varepsilon), \quad \sigma = s, u,$$

is a C^1 immersed submanifold tangent to $E^{c\sigma}$, that is, for every $x \in \mathcal{W}^\sigma(B^c, \varepsilon)$ $T_x \mathcal{W}^\sigma(B^c, \varepsilon) = E^{c\sigma}$. If ε is sufficiently small then $\mathcal{W}^\sigma(B^c, \varepsilon)$ is injectively immersed.

For a proof see, e.g., Proposition 3.4 in [BBI04]. They considered the low dimensional situation, but the proof extends to higher dimensions straightforwardly.

Corollary 3.2. *Given a complete center leaf \mathcal{C} of a partially hyperbolic diffeomorphism and $\varepsilon > 0$ define*

$$\mathcal{W}^\sigma(\mathcal{C}, \varepsilon) \stackrel{\text{def}}{=} \bigcup_{y \in \mathcal{C}} \mathcal{W}^\sigma(y, \varepsilon), \quad \sigma = s, u. \tag{3.1}$$

Then $\mathcal{W}^\sigma(\mathcal{C}, \varepsilon)$ is a C^1 immersed submanifold tangent to $E^{c\sigma}$.

Next we show that $\mathcal{W}^\sigma(\mathcal{C}, \varepsilon)$ is also “foliated” by the local center leaves in the sense which is made precise below in Proposition 3.3.

We say that E^c is *weakly integrable* if for every $x \in M$ there is an immersed complete C^1 manifold $\mathcal{W}^c(x)$ which contains x and is tangent to E^c everywhere, that is, $T_y\mathcal{W}^c(x) = E(y)$ for each $y \in \mathcal{W}^c(x)$.

Proposition 3.3. *Assume that E^c is weakly integrable. Given a complete center leaf \mathcal{C} of a partially hyperbolic diffeomorphism define $\mathcal{W}^\sigma(\mathcal{C}, \varepsilon)$ by (3.1). Then for every $x \in \mathcal{W}^\sigma(\mathcal{C}, \varepsilon)$ there exists a ball $B^c \subset \mathcal{W}^\sigma(\mathcal{C}, \varepsilon)$, that contains x and is tangent to E^c at every point.*

Proof. Let $\bar{\sigma} = u$ if $\sigma = s$ and $\bar{\sigma} = s$ if $\sigma = u$.

Fix a point $x \in \mathcal{W}^\sigma(\mathcal{C}, \varepsilon)$. By weak integrability there exists a small ball \tilde{B}^c around x that is tangent to E^c . Then by Proposition 3.1, $\mathcal{W}^{\bar{\sigma}}(\tilde{B}^c, \delta)$, $\delta > 0$, is a C^1 submanifold tangent to $E^{\bar{\sigma}^c}$. Clearly $\mathcal{W}^\sigma(\mathcal{C}, \varepsilon) \cap \mathcal{W}^{\bar{\sigma}}(\tilde{B}^c, \delta)$ is a C^1 immersed submanifold tangent to E^c . To finish the proof we let B^c be the connected component of x in $\mathcal{W}^\sigma(\mathcal{C}, \varepsilon) \cap \mathcal{W}^{\bar{\sigma}}(\tilde{B}^c, \delta)$. \square

3.3. The center-stable and center-unstable leaves for a dynamically coherent partially hyperbolic diffeomorphisms. Given a center leaf \mathcal{C} define

$$\mathcal{W}^\sigma(\mathcal{C}) \stackrel{\text{def}}{=} \bigcup_{\varepsilon > 0} \mathcal{W}^\sigma(\mathcal{C}, \varepsilon), \quad \sigma = s, u. \quad (3.2)$$

Clearly $\mathcal{W}^\sigma(\mathcal{C})$ is a C^1 immersed submanifold as well.

Let us assume now that f is dynamically coherent (the center distribution E^c integrates uniquely to a foliation \mathcal{W}^c). Then Proposition 3.3 implies that for any center leaf $\mathcal{W}^c(x)$ and any $y \in \mathcal{W}^\sigma(\mathcal{W}^c(x))$ the local center manifold $\mathcal{W}^c(y, \delta)$ is contained in $\mathcal{W}^\sigma(\mathcal{W}^c(x))$ for sufficiently small δ . Note that it does not follow that the whole leaf $\mathcal{W}^c(y)$ is in $\mathcal{W}^\sigma(\mathcal{W}^c(x))$. The intersection $\mathcal{W}^c(y) \cap \mathcal{W}^\sigma(\mathcal{W}^c(x))$ is some open subset of $\mathcal{W}^c(y)$.

3.4. The accessible boundary. Consider the collection \mathcal{A}^σ of smooth curves $\alpha: [0, 1] \rightarrow M$ such that

$$\dot{\alpha} \in E^c, \quad \alpha([0, 1]) \subset \mathcal{W}^\sigma(\mathcal{C}), \quad \alpha(1) \notin \mathcal{W}^\sigma(\mathcal{C}), \quad \sigma = s, u.$$

Equip \mathcal{A}^σ with the following equivalence relation. Two curves α_0 and α_1 are equivalent if there is a continuous homotopy $\alpha_t, t \in [0, 1]$ connecting α_0 and α_1 such that for all $t \in [0, 1]$ $\alpha_t \in \mathcal{A}^\sigma$ and $\alpha_t(1) = \alpha_0(1)$.

Define the *accessible boundary* of $\mathcal{W}^\sigma(\mathcal{C})$ as the set of equivalence classes

$$\partial\mathcal{W}^\sigma(\mathcal{C}) \stackrel{\text{def}}{=} \{[\alpha] : \alpha \in \mathcal{A}^\sigma\}, \quad \sigma = s, u.$$

Also define the *closure* $\mathcal{W}^\sigma(\mathcal{C})^{\text{cl}} \stackrel{\text{def}}{=} \mathcal{W}^\sigma(\mathcal{C}) \sqcup \partial\mathcal{W}^\sigma(\mathcal{C})$. We equip the closure with the obvious topology. Note also that there is a natural projection $\pi: \partial\mathcal{W}^\sigma(\mathcal{C}) \rightarrow M$, $[\alpha] \rightarrow \alpha(1)$. However $\partial\mathcal{W}^\sigma(\mathcal{C})$ cannot be identified with a subset of M since in general π is not injective.

The following proposition is easy to prove. (See, e.g., Proposition 1.13 in [BW05] for a proof in dimension 3.)

Proposition 3.4. *The boundary $\partial\mathcal{W}^\sigma(\mathcal{C})$ is saturated by the leaves of \mathcal{W}^σ , $\sigma = s, u$. This means that for every $[\alpha] \in \partial\mathcal{W}^\sigma(\mathcal{C})$ there exists a continuous map $\mathcal{W}^\sigma(\alpha(1)) \rightarrow \partial\mathcal{W}^\sigma(\mathcal{C})$ that fits into the commutative diagram*

$$\begin{array}{ccc} \mathcal{W}^\sigma(\alpha(1)) & \longrightarrow & \partial\mathcal{W}^\sigma(\mathcal{C}) \\ & \searrow i & \downarrow \pi \\ & & M \end{array}$$

where i is the natural immersion of $\mathcal{W}^\sigma(\alpha(1))$, $\sigma = s, u$.

Corollary 3.5. *If $\dim E^c = 1$ then $\mathcal{W}^\sigma(\mathcal{C})^{\text{cl}}$ is a manifold with smooth boundary $\partial\mathcal{W}^\sigma(\mathcal{C})$ which is a union of leaves of \mathcal{W}^σ (some of which can be repeated), $\sigma = s, u$.*

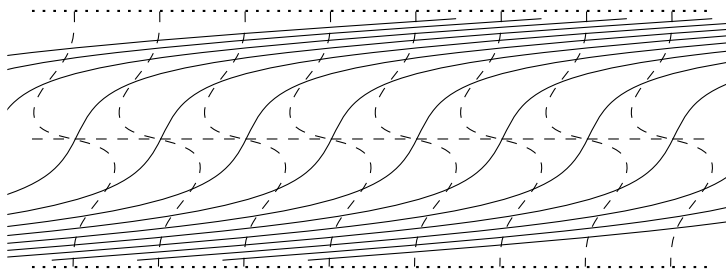


FIGURE 1. The solid curves represent the stable leaves and the dashed curves represent the center leaves. The horizontal dashed line \mathcal{C} is tangent to the center distribution at every point. However \mathcal{C} is not a leaf of the invariant center foliation. Still one can form $\mathcal{W}^s(\mathcal{C})$ which is an open strip with two boundary components drawn by dotted lines. Note that on the manifold these two boundary components are represented by the same stable leaf.

Remark 3.6. To the best of our knowledge currently there is no example of a dynamically coherent partially hyperbolic diffeomorphism of a compact manifold with a center leaf \mathcal{C} (not necessarily compact) such that $\partial\mathcal{W}^s(\mathcal{C}) \neq \emptyset$. However, an example of a partially hyperbolic diffeomorphism with a one dimensional center distribution that does not integrate uniquely was constructed [RHRHU11]. In this example there is a curve \mathcal{C} tangent to E^c and diffeomorphic to \mathbb{R} such that $\mathcal{W}^s(\mathcal{C})$ is an open strip with two boundary components as shown on Figure 1.

Remark 3.7. In Section 7.1 the construction of the accessible boundary will be carried out in a different setting. In fact, in this paper we will not use directly the accessible boundary defined above. Instead we will use the version that appears in Section 7.1. That construction may appear to be somewhat artificial. Thus we have included the description of the “usual” accessible boundary above as a preparation.

3.5. Volume recurrent center leaves. A compact center leaf \mathcal{C} is called *volume recurrent* if

$$\liminf_{n \rightarrow \infty} \text{vol}(f^n \mathcal{C}) < +\infty.$$

The following proposition will not be used in the proofs but we include it since it allows a better description of the accessible boundary of the lifted center-stable manifolds $\partial\widehat{\mathcal{W}}^s(\widehat{\mathcal{C}})$ (see Section 7.1) and thus may be useful to readers interested in extending the results.

Proposition 3.8. *Let f be a dynamically coherent partially hyperbolic diffeomorphism with a compact volume recurrent center leaf \mathcal{C} . Then for any $x \in \mathcal{C}$ the intersection $\mathcal{W}^\sigma(x) \cap \mathcal{C}$ is a finite set. Moreover, the number of points of intersection, $j_\sigma(x) \stackrel{\text{def}}{=} \#(\mathcal{W}^\sigma(x) \cap \mathcal{C})$, is an lower semi-continuous function on \mathcal{C} , $\sigma = s, u$.*

Proof. Choose a sufficiently small $\varepsilon > 0$ so that the following property holds: for any $y \in M$ and for any $z \in \mathcal{W}^s(y, \varepsilon)$, $z \neq y$, the local center leaves $\mathcal{W}^c(y, \varepsilon)$ and $\mathcal{W}^c(z, \varepsilon)$ are disjoint. Define

$$\mu \stackrel{\text{def}}{=} \inf_{y \in M} \text{vol}(\mathcal{W}^c(y, \varepsilon)).$$

Clearly $\mu > 0$.

Consider a finite set $\{x_1, x_2, \dots, x_k\} \subset \mathcal{W}^s(x) \cap \mathcal{C}$. Then there exists a sufficiently large N such that $\forall n \geq N$ the set $\{f^n x_1, f^n x_2, \dots, f^n x_k\}$ will be contained in a local stable manifold of size ε and hence the local center manifolds $\mathcal{W}^c(f^n x_1, \varepsilon), \mathcal{W}^c(f^n x_2, \varepsilon), \dots, \mathcal{W}^c(f^n x_k, \varepsilon)$ will be disjoint. Note that these local center manifolds are subsets of the leaf $f^n \mathcal{C}$. It follows that

$$\forall n \geq N \text{vol}(f^n \mathcal{C}) \geq k\mu.$$

This yields an upper bound on k

$$k \leq \frac{1}{\mu} \liminf_{n \rightarrow \infty} \text{vol}(f^n \mathcal{C}).$$

Hence $\mathcal{W}^s(x) \cap \mathcal{C}$ is a finite set. Finally notice that continuity of \mathcal{W}^s inside $\mathcal{W}^s(\mathcal{C})$ implies that for any $x \in \mathcal{C}$ and any $y \in \mathcal{C}$ which is sufficiently close to x one has $\#(\mathcal{W}^s(y) \cap \mathcal{C}) \geq \#(\mathcal{W}^s(x) \cap \mathcal{C})$. Therefore function j_s is lower semi-continuous.

By reversing the time we also see that $\#(\mathcal{W}^u(x) \cap \mathcal{C})$ is bounded and lower semi-continuous. \square

3.6. The splitting of the center holonomy. The following statement is an immediate corollary of the uniqueness of the center leaves.

Proposition 3.9. *Assume that f is a dynamically coherent partially hyperbolic diffeomorphism. Then the holonomy group of the center foliation splits as a product $G_x(\mathcal{W}^c) = G_x^s(\mathcal{W}^c) \times G_x^u(\mathcal{W}^c)$, where $G_x^s(\mathcal{W}^c)$ and $G_x^u(\mathcal{W}^c)$ are the holonomy groups of \mathcal{W}^c inside $\mathcal{W}^s(\mathcal{W}^c(x))$ and $\mathcal{W}^u(\mathcal{W}^c(x))$ respectively.*

4. ANOSOV HOMEOMORPHISMS

A partially hyperbolic skew product $f: M \rightarrow M$ projects to a homeomorphism $\bar{f}: X \rightarrow X$. Clearly \bar{f} must be “uniformly hyperbolic” in some sense. However, there is no smooth structure on X which is compatible with dynamics. (In fact, it is not clear how to equip X with some smooth structure.) This motivates a more general definition of an Anosov homeomorphism.

4.1. Definition. Let (X, ρ) be a compact metric space. For a homeomorphism $h: X \rightarrow X$ define

$$\mathcal{W}^s(x, \varepsilon) \stackrel{\text{def}}{=} \{y \in X : \rho(h^n x, h^n y) \leq \varepsilon, \forall n \geq 0\},$$

$$\mathcal{W}^u(x, \varepsilon) \stackrel{\text{def}}{=} \{y \in X : \rho(h^{-n} x, h^{-n} y) \leq \varepsilon, \forall n \geq 0\}.$$

We say that $h: X \rightarrow X$ is an *Anosov homeomorphism* if there exist $\varepsilon > 0$, $\delta > 0$, and constants $C > 0$, $\lambda \in (0, 1)$ such that

$$\begin{aligned} \text{(AH1)} \quad y \in \mathcal{W}^s(x, \varepsilon) &\Rightarrow \rho(h^n x, h^n y) \leq C\lambda^n \rho(x, y), n \geq 0, \\ y \in \mathcal{W}^u(x, \varepsilon) &\Rightarrow \rho(h^{-n} x, h^{-n} y) \leq C\lambda^n \rho(x, y), n \geq 0. \end{aligned}$$

(AH2) If $\rho(x, y) \leq \delta$ then there is a unique point of intersection

$$[x, y] \stackrel{\text{def}}{=} \mathcal{W}^s(x, \varepsilon) \cap \mathcal{W}^u(y, \varepsilon)$$

and the map $[\cdot, \cdot]: \{(x, y) \in X^2 : \rho(x, y) \leq \delta\} \rightarrow X$ is continuous.

Remark 4.1. The original definition of Anosov homeomorphism was given by Bowen [B71] and by Alekseev and Yakobson [AY81] in order to axiomatize the conditions needed for the construction of Markov partitions. Alekseev and Yakobson use term “ $A^\#$ -homeomorphism.” Their definition is identical to the above one except that they require constant C to be equal to 1. Soon we will see that for our purposes the above definition is more convenient. The difference is actually very minor. The usual adapted metric construction is applicable in this setting: given an Anosov homeomorphism $h: (X, \rho) \rightarrow (X, \rho)$ one can construct an equivalent metric $\tilde{\rho}$ such that $h: (X, \tilde{\rho}) \rightarrow (X, \tilde{\rho})$ is $A^\#$ -homeomorphism in the sense of Alekseev and Yakobson.

Given an Anosov homeomorphism h define global stable and unstable sets in the standard way:

$$\mathcal{W}^s(x) \stackrel{\text{def}}{=} \{y : \rho(h^n x, h^n y) \rightarrow 0, n \rightarrow +\infty\},$$

$$\mathcal{W}^u(x) \stackrel{\text{def}}{=} \{y : \rho(h^{-n} x, h^{-n} y) \rightarrow 0, n \rightarrow +\infty\}.$$

Now let X be a topological manifold. We say that $h: X \rightarrow X$ is *Anosov homeomorphism with stable and unstable foliations* if h is an Anosov homeomorphism and the stable and unstable sets form two continuous topologically transverse foliations of X . In this case it makes sense to speak about dimensions of \mathcal{W}^s and \mathcal{W}^u .

4.2. Partially hyperbolic skew products and Anosov homeomorphisms.

Let us go back to the fibration $p: M \rightarrow X$, the partially hyperbolic skew product $f: M \rightarrow M$ and the factor $\bar{f}: X \rightarrow X$.

Endow X with the Hausdorff metric defined as

$$\rho(a, b) \stackrel{\text{def}}{=} \max \left(\max_{x \in \mathcal{W}^c(a)} \min_{y \in \mathcal{W}^c(b)} d(x, y), \max_{x \in \mathcal{W}^c(b)} \min_{y \in \mathcal{W}^c(a)} d(x, y) \right),$$

where d is the metric on M induced by the Riemannian metric. Here we slightly abuse the notation by writing $\mathcal{W}^c(\cdot)$ for $p^{-1}(\cdot)$.

Proposition 4.2. *Homeomorphism $\bar{f}: (X, \rho) \rightarrow (X, \rho)$ is an Anosov homeomorphism with stable and unstable foliations.*

Proof. Notice that the local stable and unstable manifolds of $a \in X$ are simply the projections of $\mathcal{W}^s(\mathcal{W}^c(a), \varepsilon)$ and $\mathcal{W}^u(\mathcal{W}^c(a), \varepsilon)$ on X . It immediately follows that the stable and unstable sets of \bar{f} are, in fact, transverse continuous foliations.

Property (AH2) can also be established rather easily. Consider two points $a, b \in X$ with $\rho(a, b) < \delta$. Then transversality implies that for an appropriate value of ε (more precisely, $\varepsilon = C\delta$, where C depends on the lower bounds of the angles between E^s , E^c and E^u), $\mathcal{W}^s(\mathcal{W}^c(a), \varepsilon)$ and $\mathcal{W}^u(\mathcal{W}^c(a), \varepsilon)$ intersect at a single center leaf. This leaf depends continuously on (a, b) and, hence, (AH2) follows.

Verification of (AH1) requires some work. Let \mathcal{A} be the collection of piecewise smooth paths $\alpha: [0, 1] \rightarrow M$ such that $\dot{\alpha}$ lies in one of the distributions E^s , E^c or E^u whenever α is differentiable. Define a new metric, d^{scu} , on M in the following way

$$d^{scu}(x, y) \stackrel{\text{def}}{=} \inf_{\alpha \in \mathcal{A}, \alpha(0)=x, \alpha(1)=y} (\text{length}(\alpha)).$$

Define the corresponding Hausdorff metric on X :

$$\rho^{scu}(a, b) \stackrel{\text{def}}{=} \max \left(\max_{x \in \mathcal{W}^s(a)} \min_{y \in \mathcal{W}^c(b)} d^{scu}(x, y), \max_{x \in \mathcal{W}^c(b)} \min_{y \in \mathcal{W}^s(a)} d^{scu}(x, y) \right).$$

Since d^{scu} is equivalent to d we also have that ρ is equivalent to ρ^{scu} . Hence it is clear that homeomorphism \bar{f} being Anosov with respect to ρ^{scu} implies \bar{f} being Anosov with respect to ρ — one only needs to change the values of C , ε and δ from the definition of Anosov homeomorphism. Therefore, it is sufficient to show that $\bar{f}: (X, \rho^{scu}) \rightarrow (X, \rho^{scu})$ satisfies (AH1).

The proof of the first inequality (the second one is analogous) from (AH1) is based on the following:

1. Given any $\kappa > 0$ one can find a sufficiently small $\varepsilon > 0$ such that the following claim holds. For any point $b \in M$ and any $a \in \mathcal{W}^s(\mathcal{W}^c(b), \varepsilon)$ that is also ε -close to b ($d(a, b) < \varepsilon$) one can choose a sequence of paths $\{\alpha_n\}$ from \mathcal{A} that connect a and b such that

$$\lim_{n \rightarrow \infty} \text{length}(\alpha_n) = d^{scu}(a, b)$$

and the lower limit of the ratio of the total length of components of α_n that are tangent to E^s or E^c to the length of α_n is greater than $1 - \kappa$.

Then one can upgrade the above observation to the following:

2. Given any $\kappa > 0$ one can find a sufficiently small $\varepsilon > 0$ such that, if $a \in \mathcal{W}^s(\mathcal{W}^c(b), \varepsilon)$ realizes the minimum $\min_{x \in \mathcal{W}^c(a)} d^{scu}(x, b)$ and $\{\alpha_n\}$ is a sequence of paths connecting a and b as above, then the lower limit of the ratio of the total length of components of α_n that lie in \mathcal{W}^s to the length of α_n is greater than $1 - \kappa$.

We omit the estimate itself as it is rather standard. \square

4.3. The classification of codimension 1 Anosov homeomorphisms with stable and unstable foliations. Recall that given a homeomorphism $h: X \rightarrow X$, a point $x \in X$ is called a *wandering point* if there exists an open neighborhood \mathcal{U} of x such that the sets $h^n(\mathcal{U})$, $n \geq 0$, are mutually disjoint. Otherwise x is said to be a *non-wandering point*. The set of non-wandering points of h will be denoted by $NW(h)$.

The following are generalizations of the theorems of Franks [F68] and Newhouse [N70] to the setting of Anosov homeomorphisms.

Theorem 4.3. *Let $h: X \rightarrow X$ be a codimension 1 Anosov homeomorphism with stable and unstable foliations. Assume that $NW(h) = X$ then X is homeomorphic to a torus and h conjugate to an Anosov automorphism of the torus.*

Theorem 4.4. *If homeomorphism $h: X \rightarrow X$ is a codimension 1 Anosov homeomorphism with stable and unstable foliations then $NW(h) = X$.*

Short proofs of the Franks-Newhouse Theorems were given by Hiraide [H01]. These proofs transfer rather easily to the topological setting.

The proof of the Newhouse Theorem starts with the spectral decomposition for the Anosov diffeomorphism and then proceeds with a soft topological argument that does not require any alternation in the topological setting. The spectral decomposition in our setting comes from the spectral decomposition for a subshift of finite type since Anosov homeomorphisms admit Markov partitions (see, e.g., [AY81]).

The proof of the Franks Theorem starts with establishing global product structure of the stable and unstable foliations on the universal cover. Assuming that the unstable foliation is one dimensional, a “nice” measure class μ^u is constructed on the leaves of the unstable foliation which is invariant under the holonomy along the stable foliation. This step works well in the topological setting. The remaining arguments in [H01] are devoted to the construction of the group of deck transformations. They apply well to Anosov homeomorphisms with stable and unstable foliations with only minor adjustments.

5. THE PROOF OF THEOREM 1.6

Let \widetilde{M} be a finite cover of M such that the stable, center and unstable foliations lift to orientable foliations $\widetilde{\mathcal{W}}^s$, $\widetilde{\mathcal{W}}^c$ and $\widetilde{\mathcal{W}}^u$. Then f lifts to a partially hyperbolic diffeomorphism $\tilde{f}: \widetilde{M} \rightarrow \widetilde{M}$. We will show that \tilde{f} is the partially hyperbolic skew products posited in Theorem 1.6. Since f and \tilde{f} are dynamically coherent the discussion in Sections 3.6 and 3.3 applies and we can speak about the stable and unstable holonomy.

Lemma 5.1. *For any $x \in \widetilde{M}$ the holonomy groups $G_x^\sigma(\widetilde{\mathcal{W}}^c)$, $\sigma = s, u$, are trivial.*

The lemma and Proposition 3.9 imply that the full holonomy groups $G_x(\widetilde{\mathcal{W}}^c)$ are trivial. Then the Reeb Stability Theorem implies that \tilde{f} is a partially hyperbolic skew product and the leaf space of $\widetilde{\mathcal{W}}^c$ is a 2-dimensional topological manifold X . By Proposition 4.2 diffeomorphism \tilde{f} projects to an Anosov homeomorphism $h: X \rightarrow X$. And by Theorems 4.3 and 4.4 homeomorphism h is conjugate to a hyperbolic automorphism of \mathbb{T}^2 .

Hence to finish the proof of Theorem 1.6 we only need to prove the above lemma.

Proof. Assume that $G_x^\sigma(\widetilde{\mathcal{W}}^c)$ is not trivial. Choose $\alpha \in \pi_1(\widetilde{\mathcal{W}}^c(x), x)$ such that $h^c(\alpha)$ is non-trivial in $G_x^\sigma(\widetilde{\mathcal{W}}^c)$. For a sufficiently small $r > 0$ the corresponding center holonomy

$$H^c(\alpha): \widetilde{\mathcal{W}}^\sigma(x, r) \rightarrow \widetilde{\mathcal{W}}^\sigma(x)$$

that represents $h^c(\alpha)$ is well defined. Map $H^c(\alpha)$ is an orientation preserving homeomorphism onto its image. Also $H^c(\alpha)$ is not identity. This implies that either forward or backward $H^c(\alpha)$ -orbit of a point consists of distinct points and is contained in $\widetilde{\mathcal{W}}^\sigma(x, r)$. Denote this orbit by $\{x_n; n \geq 1\} \subset \widetilde{\mathcal{W}}^\sigma(x, r)$. Obviously $x_n \in \widetilde{\mathcal{W}}^c(x_1)$. Also it is clear that for sufficiently small $\varepsilon > 0$ the center plaques

$\widetilde{\mathcal{W}}^c(x_n, \varepsilon)$, $n \geq 1$, are disjoint. This gives a contradiction since $\text{vol}(\widetilde{\mathcal{W}}^c(x_1))$ is finite. \square

6. THE PROOF OF THEOREM 1.7

By Corollary 2.2 foliation \mathcal{W}^c is a locally trivial fibration over the leaf space X . Hence f is a partially hyperbolic skew product over $h: X \rightarrow X$. Homeomorphism h is Anosov by Proposition 4.2. Finally since $\dim E^u = 1$ Theorems 4.3 and 4.4 apply: h is conjugate to a hyperbolic automorphism of a torus.

7. ACCESSIBLE BOUNDARY IS EMPTY

Consider a complete center leaf \mathcal{C} of a partially hyperbolic diffeomorphism and the union $\mathcal{W}^s(\mathcal{C})$ of the stable leaves that pass through \mathcal{C} (3.2). If $\mathcal{W}^s(\mathcal{C})$ is complete then Proposition 3.3 implies that every point $y \in \mathcal{W}^s(\mathcal{C})$ is a center of a center-stable plaque of a fixed size which is independent of y . If $\mathcal{W}^s(\mathcal{C})$ is not complete then it has a non-empty accessible boundary $\partial\mathcal{W}^s(\mathcal{C})$ defined by (3.4). Proposition 3.3 still applies, but the size of the center-stable plaque decreases as the center of the plaque y approaches $\partial\mathcal{W}^s(\mathcal{C})$.

Standing assumption: *Partially hyperbolic diffeomorphism $f: M \rightarrow M$ is dynamically coherent with oriented one-dimensional compact center foliation \mathcal{W}^c . The center leaf \mathcal{C} is length recurrent.*

7.1. A reduction. Recall that by Corollary 3.5 the closure $\mathcal{W}^s(\mathcal{C})^{\text{cl}}$ is a manifold whose boundary is a union of stable leaves.

For any $x \in \mathcal{W}^s(\mathcal{C})$ the set $\mathcal{C} \cap \mathcal{W}^s(x)$ is either finite or countable. Each intersection point in $\mathcal{C} \cap \mathcal{W}^s(x)$ depends continuously on $x \in \mathcal{W}^s(\mathcal{C})$. Therefore the set

$$\widehat{\mathcal{W}}^s(\widehat{\mathcal{C}}) \stackrel{\text{def}}{=} \{(x, q), x \in \mathcal{W}^s(\mathcal{C}), q \in \mathcal{C} \cap \mathcal{W}^s(x)\}$$

inherits the topology, the smooth structure and the Riemannian metric from $\mathcal{W}^s(\mathcal{C})$.

The map $p: \widehat{\mathcal{W}}^s(\widehat{\mathcal{C}}) \rightarrow \mathcal{W}^s(\mathcal{C})$

$$p((x, q)) \stackrel{\text{def}}{=} x$$

is a local isometry. Therefore the stable and the center foliations lift to foliations $\widehat{\mathcal{W}}^s$ and $\widehat{\mathcal{W}}^c$ in $\widehat{\mathcal{W}}^s(\widehat{\mathcal{C}})$. Each stable leaf $\mathcal{W}^s(x), x \in \mathcal{C}$, lifts to $\#(\mathcal{W}^s(x) \cap \mathcal{C})$ stable leaves $\widehat{\mathcal{W}}^s((x, q)) = (\mathcal{W}^s(x), q), q \in \mathcal{W}^s(x) \cap \mathcal{C}$. It follows that $\widehat{\mathcal{W}}^s(\widehat{\mathcal{C}})$ is saturated by complete stable leaves and can be viewed as the disjoint union

$$\widehat{\mathcal{W}}^s(\widehat{\mathcal{C}}) = \bigsqcup_{x \in \widehat{\mathcal{C}}} \widehat{\mathcal{W}}^s(x),$$

where

$$\widehat{\mathcal{C}} \stackrel{\text{def}}{=} \{(x, x), x \in \mathcal{C}\}.$$

Remark 7.1. Note that even though the center leaves lift locally to $\widehat{\mathcal{W}}^s(\widehat{\mathcal{C}})$ they do not necessarily lift globally as p might fail to be a covering map.

Now we are ready to define the accessible boundary of $\widehat{\mathcal{W}}^s(\widehat{\mathcal{C}})$. For each $x \in \widehat{\mathcal{W}}^s(\widehat{\mathcal{C}})$ the center leaf $\widehat{\mathcal{W}}^c(x)$ is either oriented circle or an open segment $\alpha_x: (0, 1) \rightarrow$

$\widehat{\mathcal{W}}^s(\widehat{\mathcal{C}})$. In the latter case we complete α_x by adding two points $\alpha_x(0)$ and $\alpha_x(1)$ and define

$$\partial\widehat{\mathcal{W}}^s(\widehat{\mathcal{C}}) \stackrel{\text{def}}{=} \bigcup_{\widehat{\mathcal{W}}^c(x) \sim (0,1)} \{\alpha_x(0), \alpha_x(1)\}.$$

It is clear that if p is one-to-one then the above definition coincides with the one in Section 3.4. We also define $\widehat{\mathcal{W}}^s(\widehat{\mathcal{C}})^{\text{cl}} \stackrel{\text{def}}{=} \widehat{\mathcal{W}}^s(\widehat{\mathcal{C}}) \sqcup \partial\widehat{\mathcal{W}}^s(\widehat{\mathcal{C}})$ and metrize it in the following way. Let \hat{d} be the metric induced by the Riemannian metric on $\widehat{\mathcal{W}}^s(\widehat{\mathcal{C}})$, that is $\hat{d}(x, y) \stackrel{\text{def}}{=} \inf(\text{length}(\gamma))$, where the infimum is taken over all smooth curves that connect x and y . If $x, y \in \widehat{\mathcal{W}}^s(\widehat{\mathcal{C}})^{\text{cl}}$ then let α_x and α_y be corresponding parametrizations of the center leaves with $\alpha_x(t_x) = x$ and $\alpha_y(t_y) = y$. Extend \hat{d} in the following way

$$\hat{d}(x, y) \stackrel{\text{def}}{=} \lim_{t_1 \rightarrow t_x, t_2 \rightarrow t_y} \hat{d}(\alpha_x(t_1), \alpha_y(t_2)).$$

Now it is a routine exercise to check that $(\widehat{\mathcal{W}}^s(\widehat{\mathcal{C}})^{\text{cl}}, \hat{d})$ is the usual completion of $(\widehat{\mathcal{W}}^s(\widehat{\mathcal{C}}), \hat{d})$. Another routine exercise is to check that Corollary 3.5 holds in this more general setting; that is, $\widehat{\mathcal{W}}^s(\widehat{\mathcal{C}})^{\text{cl}}$ is a manifold with boundary where boundary component of $\alpha_x(\sigma)$ is identified with $\mathcal{W}^s(\lim_{t \rightarrow \sigma} p \circ \alpha(t))$, $\sigma = 0, 1$.

In the same way, for $n \geq 1$, define $\widehat{\mathcal{W}}^s(f^n \widehat{\mathcal{C}})$, $\partial\widehat{\mathcal{W}}^s(f^n \widehat{\mathcal{C}})$, $\widehat{\mathcal{W}}^s(f^n \widehat{\mathcal{C}})^{\text{cl}}$ and $f^n \widehat{\mathcal{C}}$. Finally define

$$\hat{f}: \widehat{\mathcal{W}}^s(f^{n-1} \widehat{\mathcal{C}}) \rightarrow \widehat{\mathcal{W}}^s(f^n \widehat{\mathcal{C}}), \quad \hat{f}: (x, q) \mapsto (fx, fq), q \in f^{n-1} \mathcal{C} \cap \mathcal{W}^s(x).$$

Note that both f and \hat{f} extend to the closures by continuity. We have the following commutative diagram:

$$\begin{array}{ccccc} \widehat{\mathcal{W}}^s(\widehat{\mathcal{C}})^{\text{cl}} & \xrightarrow{\hat{f}} & \widehat{\mathcal{W}}^s(f\widehat{\mathcal{C}})^{\text{cl}} & \xrightarrow{\hat{f}} & \dots \\ \downarrow & & \downarrow & & \\ \mathcal{W}^s(\mathcal{C})^{\text{cl}} & \xrightarrow{f} & \mathcal{W}^s(f\mathcal{C})^{\text{cl}} & \xrightarrow{f} & \dots \end{array}$$

Notice that, a priori, “ \hat{f} dynamics” does not come from an ambient diffeomorphism. However, if one studies “ f dynamics” in the forward orbit of $\widehat{\mathcal{W}}^s(\widehat{\mathcal{C}})^{\text{cl}}$ then the tools that come from partial hyperbolicity (the center and stable foliations) are available.

The advantage of considering \hat{f} instead of f is that for every $z \in \widehat{\mathcal{W}}^s(f^n \widehat{\mathcal{C}})$ the intersection $\widehat{\mathcal{W}}^s(z) \cap f^n \widehat{\mathcal{C}}$ is a single point.

A notational convention: *To avoid heavy notation we take off the hats. In other words, we will write $\mathcal{W}^s(\mathcal{C})$ for $\widehat{\mathcal{W}}^s(\widehat{\mathcal{C}})$, f for \hat{f} , and so on. For the center foliation the abuse is more severe. Clearly, the leaves of $\widehat{\mathcal{W}}^c$ in $\widehat{\mathcal{W}}^s(f^d \widehat{\mathcal{C}})$, $d \geq 0$, fall into two categories—the leaves homeomorphic to a circle and the leaves homeomorphic to an open interval. We will need to introduce a different notation for the center leaves that are homeomorphic to an open interval. A convenient way of doing so is to pretend that the manifolds $\widehat{\mathcal{W}}^s(f^d \widehat{\mathcal{C}})$, $d \geq 0$, are immersed into an ambient manifold and $\widehat{\mathcal{W}}^c$ is the restriction of the center foliation by circles to the manifolds $\widehat{\mathcal{W}}^s(f^d \widehat{\mathcal{C}})$, $d \geq 0$. This way the leaves of $\widehat{\mathcal{W}}^c$ that are homeomorphic to an open interval become proper subsets of “the center leaves” and will be denoted by $\mathcal{D}^c(\cdot)$ in the sequel. Thus in what follows we will assume that there is a “center foliation” by circles \mathcal{W}^c that restricts to a center foliation ($= \widehat{\mathcal{W}}^c$) on $\mathcal{W}^c(f^d \mathcal{C}) (=$*

$\widehat{\mathcal{W}}^s(f^d\widehat{\mathcal{C}})$, $d \geq 0$. It will be clear that no harm is done as all the action takes place in $\mathcal{W}^s(\mathcal{C})^{\text{cl}} (= \widehat{\mathcal{W}}^s(\widehat{\mathcal{C}})^{\text{cl}})$.

7.2. The holonomy projection. Pick a point $x \in f^d\mathcal{C}$, $d \geq 0$, and a point $y \in \mathcal{W}^s(x)$. Recall that $\mathcal{W}^c(y) \cap \mathcal{W}^s(f^d\mathcal{C})$ is an open subset of $\mathcal{W}^c(y)$. Let $\mathcal{D}^c(y)$ be the connected component of y in $\mathcal{W}^c(y) \cap \mathcal{W}^s(f^d\mathcal{C})$. In other words $\mathcal{D}^c(y)$ is the leaf of $\mathcal{W}^c|_{\mathcal{W}^s(f^d\mathcal{C})}$. If the set $\mathcal{D}^c(y)$ is distinct from $\mathcal{W}^c(y)$ then $\mathcal{D}^c(y)$ is homeomorphic to an open interval. Also note that $\mathcal{D}^c(fy) = f(\mathcal{D}^c(y))$.

Define the holonomy projection along the stable leaves $\mathcal{H}_y^s: \mathcal{D}^c(y) \rightarrow f^d\mathcal{C}$ by taking the intersection

$$\mathcal{H}_y^s(z) \stackrel{\text{def}}{=} \mathcal{W}^s(z) \cap f^d\mathcal{C}.$$

This is well defined according to the discussion in Section 7.1.

Lemma 7.2. *If $\mathcal{D}^c(y) = \mathcal{W}^c(y)$ then \mathcal{H}_y^s is a covering map with a finite number of sheets.*

Proof. Clearly the the holonomy projection is a local homeomorphism. Any local homeomorphism between circles is a covering map. \square

We use notation $(z_1, z_2)^c$ to denote a positively oriented open interval inside of a center leaf with endpoints z_1 and z_2 . And we write $\overline{(z_1, z_2)^c}$ for a negatively oriented center interval.

Lemma 7.3. ¹ *If $\mathcal{D}^c(y_0) \neq \mathcal{W}^c(y_0)$, $y_0 \in \mathcal{W}^s(\mathcal{C})$, then $\mathcal{H}_{y_0}^s: \mathcal{D}^c(y_0) \rightarrow f^d\mathcal{C}$ is a covering map with countable number of sheets.*

Proof. Let $\tau: S^1 \rightarrow \mathcal{C}$ be the positively oriented generator of $\pi_1(\mathcal{C}) = \mathbb{Z}$. As in the proof of the previous lemma choose $\varepsilon > 0$ sufficiently small so that $\mathcal{W}^s(f^n\mathcal{C}, \varepsilon)$, $n \geq d$, is a fiber bundle with small stable plaques as fibers.

Since \mathcal{C} is length recurrent we can choose an increasing sequence of times $\{k_i; i \geq 1\}$ such that

$$\forall i \geq 1 \text{ length}(f^{k_i}\mathcal{C}) < C \tag{7.1}$$

for some $C > 0$.

Claim. *For any $m \geq 1$ there exists $\delta > 0$ such that for any $i \geq 1$ and any $z_0 \in \mathcal{W}^s(f^{k_i}\mathcal{C}, \delta)$ the loops $f^{k_i} \circ \tau^m$ and $f^{k_i} \circ \tau^{-m}$ lift along the stable plaques to open intervals in $\mathcal{W}^c(z_0)$ so that z_0 is an end-point for both intervals. We denote these intervals by $(z_0, z_m)^c$ and $\overline{(z_0, z_{-m})^c}$, respectively.*

This claim is a direct corollary of (7.1) and the uniformity of the local product structure.

Choose a sufficiently large i so that $z_0 \stackrel{\text{def}}{=} f^{k_i}y_0 \in \mathcal{W}^s(f^{k_i}\mathcal{C}, \delta)$. Then, by the above claim, the loops $f^{k_i} \circ \tau^m$ and $f^{k_i} \circ \tau^{-m}$ lift to $(z_0, z_m)^c$ and $\overline{(z_0, z_{-m})^c}$. By pulling back with f^{k_i} we get that τ^m and τ^{-m} lift to $(y_0, y_m)^c$ and $\overline{(y_0, y_{-m})^c}$. This way the sequence $\{y_m; m \in \mathbb{Z}\}$ is defined. It is obvious that $\{y_m; m \in \mathbb{Z}\}$ is an increasing sequence in $\mathcal{D}^c(y_0)$.

Let $a = \lim_{m \rightarrow \infty} y_{-m}$ and $\varkappa(a) = \lim_{m \rightarrow \infty} y_m$. It is clear from our construction that $\mathcal{H}_{y_0}^s: (a, \varkappa(a))^c \rightarrow \mathcal{C}$ is a covering map and that $(\mathcal{H}_{y_0}^s)^{-1}(x) \supset \{y_m; m \in \mathbb{Z}\}$.

¹The proof of Lemma 7.3 is the only place where the length recurrence assumption is used.

Assume for a moment that $a \in \mathcal{D}^c(y_0)$. The points y_{-m} accumulate at a as $m \rightarrow \infty$ and $\mathcal{H}_{y_0}^s(y_m) = x$ for all m . This contradicts to the fact that $\mathcal{H}_{y_0}^s$ must be injective when restricted to a sufficiently small neighborhood of a in $\mathcal{D}^c(y_0)$.

Thus we see that a is a boundary point of $\mathcal{D}^c(y_0)$. Analogously $\varkappa(a)$ is the other boundary point of $\mathcal{D}^c(y_0)$. We conclude that $\mathcal{D}^c(y_0) = (a, \varkappa(a))^c$ and $\mathcal{H}_{y_0}^s : \mathcal{D}^c(y_0) \rightarrow \mathcal{C}$ is a covering map. \square

Remark 7.4. The sequence $\{y_m, m \in \mathbb{Z}\}$ and the endpoints $a, \varkappa(a)$ constructed in the proof above will be used throughout the current section.

7.3. The center holonomy. Consider the flow generated by the positively oriented vector field tangent to the center foliation \mathcal{W}^c . Here we will argue that the first return map to $\mathcal{W}^s(y_0)$ of this flow is a well defined homeomorphism $\mathcal{H}_{y_0}^c : \mathcal{W}^s(y_0) \rightarrow \mathcal{W}^s(y_0)$.

It is clear from Lemma 7.3 that if we start at y_0 and follow the leaf $\mathcal{W}^c(y_0)$ in positive direction then we will return to $\mathcal{W}^s(y_0)$ at y_1 and this is the first return, i.e., $(y_0, y_1)^c \cap \mathcal{W}^s(y_0) = \emptyset$. Similarly for any $z_0 \in \mathcal{W}^s(y_0)$ with $\mathcal{D}^c(z_0) \neq \mathcal{W}^c(z_0)$ the simple argument in the proof of Lemma 7.3 can be repeated and yields a well defined point of first return $z_1 \in \mathcal{W}^s(y_0)$. For any such z_0 define $\mathcal{H}_{y_0}^c(z_0) = z_1$.

If $\mathcal{D}^c(z_0) = \mathcal{W}^c(z_0)$ for a point $z_0 \in \mathcal{W}^s(y_0)$ then we have that the holonomy projection $\mathcal{H}_{z_0}^s : \mathcal{W}^c(z_0) \rightarrow \mathcal{C}$ is a finite covering by Lemma 7.2. Thus the first return time is well defined as well.

Finally, we remark that continuity of \mathcal{W}^c implies that $\mathcal{H}_{y_0}^c$ is a homeomorphism.

7.4. Structure of the center holonomy. Here we reveal some topological structure on $\mathcal{W}^s(y_0)$ which is respected by the center holonomy $\mathcal{H}_{y_0}^c$.

Recall that by Proposition 3.4 (and the discussion in Section 7.1) $\mathcal{W}^s(a) \sqcup \mathcal{W}^s(\varkappa(a)) \subset \partial\mathcal{W}^s(\mathcal{C})$. For any point $p \in \mathcal{W}^s(a)$ a sufficiently small center interval $(p, p')^c$ is contained in $\mathcal{W}^s(\mathcal{C})$. On the other hand, $\mathcal{W}^c(p) \neq \mathcal{D}^c(p')$ and hence there exists some $\varkappa(p) \in \mathcal{W}^c(p)$ such that $\mathcal{D}^c(p') = (p, \varkappa(p))^c$.

Lemma 7.5. *For any point $p \in \mathcal{W}^s(a)$ we have $\mathcal{W}^s(\varkappa(p)) = \mathcal{W}^s(\varkappa(a))$ and the map $\mathcal{W}^s(a) \ni p \mapsto \varkappa(p) \in \mathcal{W}^s(\varkappa(a))$ is a homeomorphism.*

Proof. Continuity of the center foliation implies that every point $p \in \mathcal{W}^s(a)$ has an open neighborhood $\mathcal{V}_p \subset \mathcal{W}^s(p)$ that maps homeomorphically to an open neighborhood $\mathcal{V}_{\varkappa(p)} \subset \mathcal{W}^s(\varkappa(p))$ of $\varkappa(p)$ via $x \mapsto \varkappa(x)$. It follows that the collection of open sets

$$\mathcal{U}_p \stackrel{\text{def}}{=} \{x \in \mathcal{W}^s(a) : \varkappa(x) = \varkappa(p)\}, \quad p \in \mathcal{W}^s(a)$$

is a cover of $\mathcal{W}^s(a)$ by open disjoint sets (we ignore repetitions). Since $\mathcal{W}^s(a)$ is connected we have that $\mathcal{U}_p = \mathcal{W}^s(a)$ for every $p \in \mathcal{W}^s(a)$ and $\varkappa(p) \in \mathcal{W}^s(\varkappa(a))$.

A similar argument shows that $\mathcal{W}^s(a) \ni p \mapsto \varkappa(p) \in \mathcal{W}^s(\varkappa(a))$ is onto and continuity of \mathcal{W}^c implies that it is a homeomorphism. \square

Recall that the point $a \in \partial\mathcal{W}^s(\mathcal{C})$ is uniquely defined by y_0 . Consider the set

$$\mathcal{S} = \mathcal{S}(y_0) \stackrel{\text{def}}{=} \bigcup_{p \in \mathcal{W}^s(a)} (p, \varkappa(p))^c.$$

It is clear that \mathcal{S} is an open subset of $\mathcal{W}^s(\mathcal{C})$. Also it is clear that \mathcal{S} is a trivial continuous fiber bundle over $\mathcal{W}^s(a)$.

Lemma 7.6. *Given a path $\alpha: [0, 1] \rightarrow \mathcal{W}^s(a)$ and a point $z_0 \in (\alpha(0), \varkappa(\alpha(0)))^c \cap \mathcal{W}^s(y_0)$ there exists a unique lift $\tilde{\alpha}: [0, 1] \rightarrow \mathcal{W}^s(y_0)$ such that $\tilde{\alpha}(0) = z_0$ and $\tilde{\alpha}(t) \in (\alpha(t), \varkappa(\alpha(t)))^c$. Moreover, if α is a loop then $\tilde{\alpha}$ is a loop as well.*

Proof. The intersection $\mathcal{W}^s(y_0) \cap \mathcal{S}$ is an injectively immersed submanifold of \mathcal{S} without boundary which is transverse to the fibers of \mathcal{S} . This implies the existence and uniqueness of the lift $\tilde{\alpha}$.

To prove the second part of the lemma recall that the set $\mathcal{W}^s(z_0) \cap \mathcal{D}^c(z_0)$ can be ordered into a sequence $\{z_m, m \in \mathbb{Z}\}$ as in the proof of Lemma 7.3 (for y_0). Then $\tilde{\alpha}(1) \in \{z_m, m \in \mathbb{Z}\}$.

Assume that $\tilde{\alpha}(1) = z_k, k \neq 0$. For concreteness also assume that $k < 0$.

Form the loop $\beta \stackrel{\text{def}}{=} (z_k, z_0)^c * \tilde{\alpha}$. Since the image of β is compact there exists a sufficiently large topological ball $B \subset \mathcal{W}^s(a)$ such that $\beta(S^1) \subset \mathcal{S}(B)$, where

$$\mathcal{S}(B) \stackrel{\text{def}}{=} \bigcup_{p \in B} (p, \varkappa(p))^c.$$

The set $\mathcal{S}(B)$ is homeomorphic to a ball. Therefore, β is contractible inside $\mathcal{W}^s(\mathcal{C})$. Since the image of this contraction is compact we can choose sufficiently large R so that the above contraction happens inside $\mathcal{W}^s(\mathcal{C}, R)$.

On the other hand, the map $P: \mathcal{W}^s(\mathcal{C}, R) \rightarrow \mathcal{C}, P(x) \stackrel{\text{def}}{=} \mathcal{W}^s(x) \cap \mathcal{C}$, is a continuous map that sends β to $\tau^{|k|}$ (recall that τ is the positive generator of \mathcal{C}). Therefore $\tau^{|k|} = P \circ \beta$ is contractible in \mathcal{C} . We have arrived at a contradiction and thus have established the second part of the lemma. \square

Denote by $\mathcal{U}(y_k)$ the path connected component of $\mathcal{S} \cap \mathcal{W}^s(y_0)$ that contains y_k . Clearly, sets $\mathcal{U}(y_k), k \in \mathbb{Z}$, are open.

Lemma 7.7. *Every point $z \in \mathcal{S} \cap \mathcal{W}^s(y_0)$ belongs to $\mathcal{U}(y_k)$ for some k .*

Proof. Let $(p, \varkappa(p))^c$ be the center interval that contains z . Let $\alpha: [0, 1] \rightarrow \mathcal{W}^s(a)$ be a path that connects p and a . Then by Lemma 7.6 α lifts to $\tilde{\alpha}$ that connects z and some point y_k . \square

Lemma 7.8. *Sets $\mathcal{U}(y_k), k \in \mathbb{Z}$, are mutually disjoint.*

Proof. Assume that $\mathcal{U}(y_i) = \mathcal{U}(y_j)$ with $i \neq j$. Then there is a path $\tilde{\alpha}$ that connects y_i and y_j inside $\mathcal{S} \cap \mathcal{W}^s(y_0)$. Path $\tilde{\alpha}$ projects along the center fibers to a loop $\alpha: S^1 \rightarrow \mathcal{W}^s(a), \alpha(0) = a$. But by Lemma 7.6 loop α lifts uniquely to a loop in $\mathcal{S} \cap \mathcal{W}^s(y_0)$. This gives us a contradiction since $\tilde{\alpha}$ is a lift of α which is not a loop. \square

Lemma 7.9. *The center holonomy $\mathcal{H}_{y_0}^c: \mathcal{W}^s(y_0) \rightarrow \mathcal{W}^s(y_0)$ cyclically permutes sets $\mathcal{U}(y_k), k \in \mathbb{Z}$, that is, $\mathcal{H}_{y_0}^c(\mathcal{U}(y_k)) = \mathcal{U}(y_{k+1}), k \in \mathbb{Z}$.*

Proof. Recall that $\mathcal{H}_{y_0}^c(y_k) = y_{k+1}$. Consider any point $z \in \mathcal{U}(y_k)$ and a path $\tilde{\alpha}: [0, 1] \rightarrow \mathcal{U}(y_k)$ that connects y_k and z . Then $\mathcal{H}_{y_0}^c \circ \tilde{\alpha}: [0, 1] \rightarrow \mathcal{W}^s(y_0)$ is a path that connects y_{k+1} and $\mathcal{H}_{y_0}^c(z)$. Thus $\mathcal{H}_{y_0}^c(\mathcal{U}(y_k)) \subset \mathcal{U}(y_{k+1})$. A similar argument shows that, in fact, $\mathcal{H}_{y_0}^c$ maps $\mathcal{U}(y_k)$ onto $\mathcal{U}(y_{k+1})$. \square

7.5. Wada Lakes structure on $\mathcal{W}^s(y_0)$. Wada Lakes are mutually disjoint connected open subsets of \mathbb{R}^d that share a common boundary. Clearly such open sets should have rather weird shapes. It is possible to construct Wada Lakes through an inductive procedure (see, e.g., p. 143 of [HY88]). Also Wada Lakes appear naturally in various dynamical contexts (see, e.g., [C06]).

Here we show that open sets $\mathcal{U}(y_k)$, $k \in \mathbb{Z}$, enjoy some properties very similar to those of Wada Lakes.

The set $\{z \in \mathcal{W}^s(y_0) : \mathcal{D}^c(z) \neq \mathcal{W}^c(z)\}$ is open because of continuity of the center foliation. Thus the set

$$\mathcal{K} \stackrel{\text{def}}{=} \{z \in \mathcal{W}^s(y_0) : \mathcal{W}^c(z) \subset \mathcal{W}^s(\mathcal{C})\}$$

is closed. Also note that \mathcal{K} is not empty since $x \in \mathcal{K}$.

Given a point $z \in \mathcal{K}$ denote the multiplicity of the finite covering projection $\mathcal{H}_z^s : \mathcal{W}^s(z) \rightarrow \mathcal{C}$ by $\text{per}(z)$. Then, obviously, $(\mathcal{H}_{y_0}^c)^{\text{per}(z)}(z) = z$. Therefore $\mathcal{H}_{y_0}^c|_{\mathcal{K}}$ is a point-wise periodic homeomorphism.

Lemma 7.10. *For any $k \in \mathbb{Z}$ the boundary $\partial\mathcal{U}(y_k)$ of the open set $\mathcal{U}(y_k)$ in $\mathcal{W}^s(y_0)$ is a subset of \mathcal{K} .*

Remark 7.11. Since $\mathcal{U}(y_k) \neq \mathcal{W}^s(y_0)$ it should have a non-empty boundary $\partial\mathcal{U}(y_k)$ in $\mathcal{W}^s(y_0)$.

Proof. Pick a point $z \in \partial\mathcal{U}(y_k)$. If $z \notin \mathcal{K}$ then it has a small open connected neighborhood $\mathcal{V}(z) \subset \mathcal{W}^s(y_0)$ such that for any point $z' \in \mathcal{V}(z)$ we have $\mathcal{D}^c(z') \neq \mathcal{W}^c(z')$.

Moreover, all the boundary points of $\mathcal{D}^c(z')$, $z' \in \mathcal{V}(z)$, lie on the same two boundary components from $\partial\mathcal{W}^s(\mathcal{C})$. (This can be seen from the argument in the proof of Lemma 7.5.) Since the intersection $\mathcal{V}(z) \cap \mathcal{U}(y_k)$ is not empty the neighborhood $\mathcal{V}(z)$ must be a subset of $\mathcal{S} \cap \mathcal{W}^s(y_0)$; moreover, $\mathcal{V}(z)$ must be a subset of the connected component of y_k , i.e., $\mathcal{V}(z) \subset \mathcal{U}(y_k)$. But this contradicts to our assumption that $z \in \partial\mathcal{U}(y_k)$. \square

Lemma 7.12. *Take any point $z \in \partial\mathcal{U}(y_k)$. Then $z \in \partial\mathcal{U}(y_{k+\ell\text{per}(z)})$ for all $\ell \in \mathbb{Z}$.*

Proof. By the previous lemma $z \in \mathcal{K}$ and thus $(\mathcal{H}_{y_0}^c)^{\ell\text{per}(z)}(z) = z$ for all $\ell \in \mathbb{Z}$. Recall that by Lemma 7.9 $(\mathcal{H}_{y_0}^c)^{\ell\text{per}(z)}(\mathcal{U}(y_k)) = \mathcal{U}(y_{k+\ell\text{per}(z)})$. The lemma follows. \square

Remark 7.13. We remark that it is not necessarily the case that $\mathcal{W}^s(y_0)$ is decomposed as a disjoint union of \mathcal{K} and $\mathcal{U}(y_k)$, $k \in \mathbb{Z}$. Recall that the sequence $\{\mathcal{U}(y_k), k \in \mathbb{Z}\}$ is associated to a pair of boundary components of $\mathcal{W}^s(\mathcal{C})$ — $\mathcal{W}^s(a)$ and $\mathcal{W}^s(\varkappa(a))$. The boundary $\partial\mathcal{W}^s(\mathcal{C})$ may have more, even infinitely many, components that can be paired in the same way and give rise to other sequences of disjoint open sets inside $\mathcal{W}^s(y_0)$.

Only now we specialize to the codimension 2 case.

Proposition 7.14. *Let \mathcal{C} be a length recurrent center leaf of a partially hyperbolic diffeomorphism f that satisfies assumptions of the Main Theorem. Assume that $\dim E^s = \dim E^u = 2$. Also assume that \mathcal{W}^c is orientable. Then $\partial\mathcal{W}^\sigma(\mathcal{C}) = \emptyset$, $\sigma = s, u$.*

Proof. Assume that $\partial\mathcal{W}^s(\mathcal{C}) \neq \emptyset$. Then there exists a point $y_0 \in \mathcal{W}^s(\mathcal{C})$ such that $\mathcal{D}^c(y_0) \neq \mathcal{W}^c(y_0)$ and all constructions above apply. In particular, we have the center holonomy $\mathcal{H}_{y_0}^c: \mathcal{W}^s(y_0) \rightarrow \mathcal{W}^s(y_0)$ and Wada Lakes structure associated to it.

Consider a smooth loop $\alpha: S^1 \rightarrow \mathcal{W}^s(x)$ that intersects the sets $\mathcal{U}(y_0)$ and $\mathcal{U}(y_1)$, and also passes through x . (Recall that $x = \mathcal{W}^s(y_0) \cap \mathcal{C}$.) Then $\mathcal{U}(y_0) \cap \alpha(S^1)$ is a union of disjoint open intervals. Let $\alpha((t, s))$ be one of these intervals. Obviously $\alpha(t) \neq \alpha(s)$. It is also clear that $\{\alpha(t), \alpha(s)\} \subset \partial\mathcal{U}(y_0)$. Therefore, by Lemma 7.10, $\alpha(t)$ and $\alpha(s)$ are in \mathcal{K} and must be fixed by some power of $\mathcal{H}_{y_0}^c$. Namely,

$$(\mathcal{H}_{y_0}^c)^{\text{per}(\alpha(t))}(\alpha(t)) = \alpha(t) \quad \text{and} \quad (\mathcal{H}_{y_0}^c)^{\text{per}(\alpha(s))}(\alpha(s)) = \alpha(s).$$

Consider the path $\beta: (t, s) \rightarrow \mathcal{W}^s(y_0)$ given by

$$\beta(t') \stackrel{\text{def}}{=} (\mathcal{H}_{y_0}^c)^k(\alpha(t')), \quad t' \in (t, s),$$

where $k \stackrel{\text{def}}{=} \text{per}(\alpha(t))\text{per}(\alpha(s))$. Obviously, β has the same endpoints as α and, by Lemma 7.9, we have $\beta((t, s)) \in \mathcal{U}(y_k)$.

We concatenate α and β to form a loop $\ell \subset \mathcal{W}^s(y_0)$ half of which is in $\mathcal{U}(y_0)$ and the other half in $\mathcal{U}(y_k)$.

Since $\mathcal{W}^s(y_0)$ is diffeomorphic to \mathbb{R}^2 , the Jordan Curve Theorem tells us that ℓ divides $\mathcal{W}^s(y_0)$ into the interior and exterior parts.

We can find a point $z \in \partial\mathcal{U}(y_0)$ in the interior of ℓ and a point $\bar{z} \in \partial\mathcal{U}(y_0)$ in the exterior of ℓ . Indeed, just pick $z_0 \in \ell \cap \mathcal{U}(y_0)$ and $z_1 \in \ell \cap \mathcal{U}(y_k)$ and connect z_0 and z_1 by a curve that lies in the interior of ℓ . Then point $z \in \partial\mathcal{U}(y_0)$ can be found on this curve. Point $\bar{z} \in \partial\mathcal{U}(y_0)$ in the exterior of ℓ can be found in a similar way.

Both z and \bar{z} are fixed by $(\mathcal{H}_{y_0}^c)^{m\text{per}(z)\text{per}(\bar{z})}$, $m \in \mathbb{Z}$. Thus, by Lemma 7.9, both z and \bar{z} also belong to the boundaries of $\mathcal{U}(y_{m\text{per}(z)\text{per}(\bar{z})})$, $m \in \mathbb{Z}$. But this is impossible since z and \bar{z} lie on different sides of ℓ and $\mathcal{U}(y_{m\text{per}(z)\text{per}(\bar{z})})$ is path connected and is different from $\mathcal{U}(y_0)$ and $\mathcal{U}(y_k)$ for a sufficiently large m . We have arrived at a contradiction. \square

8. THE PROOF OF THE MAIN THEOREM

We can assume that \mathcal{W}^c is orientable. Otherwise we can pass to a double cover of M .

Let \mathcal{B} be the bad set of \mathcal{W}^c . Assume that $\mathcal{B} \neq \emptyset$. It is clear from the definition of \mathcal{B} that $f\mathcal{B} = \mathcal{B}$.

Consider any f -invariant measure μ supported inside \mathcal{B} . Choose C large enough so that the set

$$A_C \stackrel{\text{def}}{=} \{x \in \mathcal{B} : \text{length}(\mathcal{W}^c(x)) \leq C\}$$

has positive μ -measure. Then the Poincare Recurrence implies that there exists a length recurrent center leaf $\mathcal{C} \subset A_C$.

Assume that $\dim E^s = 2$. Then the discussion in Section 7 applies to $\widehat{\mathcal{W}}^s(\widehat{\mathcal{C}})$. In particular, given $x \in \widehat{\mathcal{C}}$ we have the center holonomy

$$\mathcal{H}_x^c: \widehat{\mathcal{W}}^s(x) \rightarrow \widehat{\mathcal{W}}^s(x).$$

By Proposition 7.14 $\widehat{\mathcal{W}}^s(\widehat{\mathcal{C}})$ does not have accessible boundary and hence the leaves $\widehat{\mathcal{W}}^c(y)$, $y \in \widehat{\mathcal{W}}^s(x)$, are complete circles embedded in $\widehat{\mathcal{W}}^s(\widehat{\mathcal{C}})$. Then Lemma 7.2 implies that every $y \in \widehat{\mathcal{W}}^s(x)$ is a periodic point of \mathcal{H}_x^c .

Thus \mathcal{H}_x^c is a point-wise periodic homeomorphism of $\widehat{\mathcal{W}}^s(x)$. By a classical theorem of Montgomery [M37] \mathcal{H}_x^c must be periodic. This means that $\widehat{\mathcal{C}}$ has a small foliated neighborhood V and that the holonomy group of $\widehat{\mathcal{C}}$ in $\widehat{\mathcal{W}}^s(\widehat{\mathcal{C}})$ (that is, $G_x^s(\widehat{\mathcal{W}}^c)$, $x \in \widehat{\mathcal{C}}$) is finite. Then $p(V)$ (map $p: \widehat{\mathcal{W}}^s(\widehat{\mathcal{C}}) \rightarrow \mathcal{W}^s(\mathcal{C})$ was defined in Section 7.1) is a foliated neighborhood of \mathcal{C} in $\mathcal{W}^s(\mathcal{C})$ and the holonomy group $G_{p(x)}^s(\mathcal{W}^c)$ is also finite.

If $\dim E^s = 1$ then the arguments from Section 7 are not needed. One can see that $G_x^s(\mathcal{W}^c)$, $x \in \mathcal{C}$, has order 1 or 2 (the latter in the case when \mathcal{W}^s is not orientable) by repeating the simple argument of Lemma 5.1.

Similarly we see that $G_x^u(\mathcal{W}^c)$, $x \in \mathcal{C}$, is also finite. Thus, by Proposition 3.9, the full holonomy group is a finite group of order k .

Then by the Generalized Reeb Stability Theorem \mathcal{C} has a small foliated neighborhood U such that every center leaf in U covers \mathcal{C} at most k times. This implies that

$$\forall z \in U \quad \text{length}(\mathcal{W}^c(z)) \leq k \cdot \text{length}(\mathcal{C}) + K$$

for some $K > 0$. But \mathcal{C} is in the bad set so length must be locally unbounded at \mathcal{C} . We have arrived at a contradiction. Hence the bad set \mathcal{B} is empty and the Main Theorem follows.

Remarks 8.1.

1. The assumption $\dim E^s \leq 2$, $\dim E^u \leq 2$ is only used in the proof of Proposition 7.14. Thus the Main Theorem would immediately generalize to the higher dimensional setup if one can show that Wada Lakes structure described in Section 7.5 cannot exist on leaves of dimension ≥ 3 as well.
2. Accessible boundary appears naturally whenever one looks at classification problems for partially hyperbolic diffeomorphism. See, e.g., [BI08, BW05]. It is also implicit in the proofs of Theorems 1.6 and 1.7.

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