SMOOTH CONJUGACY OF ANOSOV DIFFEOMORPHISMS ON HIGHER DIMENSIONAL TORI

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ABSTRACT. Let L be a hyperbolic automorphism of $\mathbb{T}^d$, $d \geq 3$. We study the smooth conjugacy problem in a small C^1-neighborhood $\mathcal{U}$ of L.

The main result establishes $C^{1+\nu}$ regularity of the conjugacy between two Anosov systems with the same periodic eigenvalue data. We assume that these systems are C^1-close to an irreducible linear hyperbolic automorphism L with simple real spectrum and that they satisfy a natural transitivity assumption on certain intermediate foliations.

We elaborate on the example of de la Llave of two Anosov systems on $\mathbb{T}^4$ with the same constant periodic eigenvalue data that are only Hölder conjugate. We show that these examples exhaust all possible ways to perturb $C^{1+\nu}$ conjugacy class without changing periodic eigenvalue data. Also we generalize these examples to majority of reducible toral automorphisms as well as to certain product diffeomorphisms of $\mathbb{T}^4$ C^1-close to the original example.

Contents

1. Introduction and statements

Consider an Anosov diffeomorphism f of a compact smooth manifold. Structural stability asserts that if a diffeomorphism g is C^1 close to f then f and g are topologically conjugate. The conjugacy h is unique in the neighborhood of identity.

$$h \circ f = g \circ h$$

It is known that h is Hölder continuous.

There are simple obstructions for h to be smooth. Namely, let x be a periodic point of f, $f^p(x) = x$ then $g^p(h(x)) = h(x)$ and if h were differentiable then

$$Df^p(x) = (Dh(x))^{-1} Dg^p(h(x))Dh(x)$$

i. e. $Df^p(x)$ and $Dg^p(h(x))$ are conjugate. We see that every periodic point carries a modulus of smooth conjugacy.

Suppose that for every periodic point x, $f^p(x) = x$, differentials of return maps $Df^p(x)$ and $Dg^p(h(x))$ are conjugate then we say that *periodic data* (p. d.) of f and g coincide.

Question 1. *Suppose that p. d. coincide, is h differentiable? If it is then how smooth is it?*

1.1. Positive answers.

We describe situations when p. d. form full set of moduli of C^1 conjugacy.

The only surface that supports Anosov diffeomorphisms is two dimensional torus. For Anosov diffeomorphisms of $\mathbb{T}^2$ the complete answer was given by de la Llave, Marco and Moriyón.

Theorem ([LMM88], [L92])**.** *Let f and g be C^r, $r > 1$, Anosov diffeomorphisms of $\mathbb{T}^2$ that are topologically conjugate,*

$$h \circ f = g \circ h.$$

Suppose that p. d. coincide. Then h is $C^{r-\varepsilon}$ where $\varepsilon > 0$ is arbitrarily small.

De la Llave [L92] also observed that the answer is negative for Anosov diffeomorphisms of $\mathbb{T}^d$, $d \geq 4$. He constructed two diffeomorphisms with the same p. d. which are only Hölder conjugate. We describe this example in Section 2.

In dimension three the only manifold that supports Anosov diffeomorphisms is three dimensional torus. Moreover, all Anosov diffeomorphisms of $\mathbb{T}^3$ are topologically conjugate to the linear automorphisms of $\mathbb{T}^3$. Nevertheless the answer to the Question 1 is not known.

Conjecture 1. *Let f and g be C^r, $r > 1$, Anosov diffeomorphisms of $\mathbb{T}^3$ that are topologically conjugate,*

$$h \circ f = g \circ h.$$

Suppose that p. d. coincide. Then h is at least C^1.

There are partial results that support this conjecture.

Theorem ([GG08])**.** *Let L be a hyperbolic automorphism of $\mathbb{T}^3$ with real eigenvalues. Then there exists a C^1-neighborhood $\mathcal{U}$ of L such that any f and g in $\mathcal{U}$ having the same p. d. are $C^{1+\nu}$ conjugate.*

Theorem ([KS07])**.** *Let L be a hyperbolic automorphism of $\mathbb{T}^3$ that has one real and two complex eigenvalues. Then any f sufficiently C^1 close to L that has the same p. d. as L is C^∞ conjugate to L.*

In higher dimensions not much is known. In recent years big progress has been made (see [L02], [KS03], [L04], [F04], [S05], [KS07]) in the case when stable and unstable foliations carry invariant conformal structures. To ensure existence of these conformal structures one has at least to assume that every periodic orbit has only one positive and one negative Lyapunov exponent. This is a very restrictive assumption on p. d.

In contrast to above we will study smooth conjugacy problem in proximity of a hyperbolic automorphism $L : \mathbb{T}^d \to \mathbb{T}^d$ with simple spectrum. Namely, with exception of Theorem B we will always assume that the eigenvalues of L are real and have different absolute values. For the sake of notation we assume that the eigenvalues of L are positive. This is not restrictive.

Let l be the dimension of the stable subspace of L and k be the dimension of the unstable subspace of L, $k + l = d$. Consider L-invariant splitting

$$T\mathbb{T}^d = F_l \oplus F_{l-1} \oplus \ldots \oplus F_1 \oplus E_1 \oplus E_2 \oplus \ldots \oplus E_k$$

along the eigendirections with corresponding eigenvalues

$$\mu_l < \mu_{l-1} < \ldots < \mu_1 < 1 < \lambda_1 < \lambda_2 < \ldots < \lambda_k.$$

Let $\mathcal{U}$ be a C^1-neighborhood of L. Precise choice of $\mathcal{U}$ is described in Section 6.1. Theory of partially hyperbolic dynamical systems guarantees that for any f in $\mathcal{U}$ the invariant splitting survives (e. g. see [Pes04])

$$T\mathbb{T}^d = F_l^f \oplus F_{l-1}^f \oplus \ldots \oplus F_1^f \oplus E_1^f \oplus E_2^f \oplus \ldots \oplus E_k^f.$$

We will see these one dimensional invariant distributions integrate uniquely to foliations U_l^f, U_{l-1}^f,... U_1^f, V_1^f, V_2^f,... V_k^f.

Given a foliation $\mathcal{F}$ on $\mathbb{T}^d$ and an open set B define

$$\mathcal{F}(B) = \bigcup_{y \in B} \mathcal{F}(y).$$

We will be assuming the following property of f

Property $\mathcal{A}$. For every $x \in \mathbb{T}^d$ and every open ball $B \ni x$

$$\begin{aligned} &\overline{U_{l-1}^f(B)} = \overline{U_{l-2}^f(B)} = \ldots = \overline{U_1^f(B)} \\ &\quad = \overline{V_1^f(B)} = \overline{V_2^f(B)} = \ldots = \overline{V_{k-1}^f(B)} = \mathbb{T}^d. \end{aligned} \tag{$\mathcal{A}$}$$

We discuss this property in Section 4.1.

Theorem A. *Let L be a hyperbolic automorphism of $\mathbb{T}^d, d \geq 3$, with simple real spectrum. Assume that characteristic polynomial of L is irreducible over $\mathbb{Z}$. There exists a C^1-neighborhood $\mathcal{U} \subset \mathrm{Diff}^r(\mathbb{T}^d)$, $r \geq 2$, of L such that any $f \in \mathcal{U}$ satisfying $\mathcal{A}$ and any $g \in \mathcal{U}$ with the same p. d. are $C^{1+\nu}$ conjugate.*

Remark. We will see in Section 4.1 that irreducibility of characteristic polynomial of L is necessary for f to satisfy $\mathcal{A}$. Formally, we could have omitted the irreducibility assumption above. Theorem B below shows that irreducibility of L is a necessary assumption for the conjugacy to be C^1. We believe that Theorem A holds when L is irreducible without assuming that f satisfies $\mathcal{A}$.

Remark. Number ν is a small positive number. It is possible to estimate ν from below in terms of eigenvalues of L and the size of $\mathcal{U}$.

Remark. Obviously analogous result holds on finite factors of tori. But we do not know how prove it on nilmanifolds. The problem is that for an algebraic Anosov automorphism of a nilmanifold various intermediate distributions may happen to be non-integrable.

Theorem A is a generalization of the theorem from [GG08] quoted above. Our method does not lead to higher regularity of the conjugacy (see the last section of [GG08] for an explanation). Nevertheless we conjecture that the situation is the same as in dimension two.

Conjecture 2. *In the setup of Theorem A one can actually conclude that f and g are $C^{r-\varepsilon}$ conjugate, where ε is an arbitrarily small positive number.*

Simple examples of diffeomorphisms that possess Property $\mathcal{A}$ include $f = L$ and any $f \in \mathcal{U}$ when $\max(k, l) \le 2$ (see Section 4.1). In addition we construct a C^1-open set of Anosov diffeomorphisms of $\mathbb{T}^5$ and $\mathbb{T}^6$ close to L that have Property $\mathcal{A}$. It seems that this construction can be extended to arbitrary dimension.

We describe this open set when $l = 2$ and $k = 3$. Given $f \in \mathcal{U}$ denote by D_f^{wu} the derivative of f along V_1^f. Choose $f \in \mathcal{U}$ in such a way that

$$\forall x \neq x_0 \quad D_f^{wu}(x) > D_f^{wu}(x_0),$$

where x_0 is a fixed point of f. Then any diffeomorphism sufficiently C^1 close to f possess Property $\mathcal{A}$.

1.2. **When the coincidence of periodic data is not sufficient.** First let us briefly describe the counterexample of de la Llave.

Let $L : \mathbb{T}^4 \to \mathbb{T}^4$ be an automorphism of the product type

$$L(x, y) = (Ax, By), \quad (x, y) \in \mathbb{T}^2 \times \mathbb{T}^2, \tag{1}$$

where A and B are Anosov automorphisms. Let λ, λ^{-1} be the eigenvalues of A and μ, μ^{-1} be the eigenvalues of B. We assume that $\mu > \lambda > 1$. Consider perturbations of the form

$$\tilde{L} = (Ax + \vec{\varphi}(y), By), \tag{2}$$

where $\vec{\varphi} : \mathbb{T}^2 \to \mathbb{R}^2$ is a C^1-small C^r, $r > 1$, function. Obviously p. d. of L and $\tilde{L}$ coincide. We will see in Section 2 that majority of perturbations (2) are only Hölder conjugate to L. The following theorem is a simple generalization of this counterexample.

Theorem B. *Let $L : \mathbb{T}^d \to \mathbb{T}^d$ be a hyperbolic automorphism. Assume that characteristic polynomial of L factors over $\mathbb{Q}$. Then there exist C^∞ diffeomorphisms $\tilde{L} : \mathbb{T}^d \to \mathbb{T}^d$ and $\hat{L} : \mathbb{T}^d \to \mathbb{T}^d$ arbitrarily C^1-close to L with the same p. d. such that the conjugacy between $\tilde{L}$ and $\hat{L}$ is not Lipschitz.*

Remark. In majority of cases one can take $\hat{L} = L$. The need to take $\tilde{L}$ and $\hat{L}$ both to be different from L appears, for instance, when $L(x, y) = (Ax, Ay)$. It was shown in [L02] that p. d. form complete set of moduli for smooth conjugacy problem to L. This is a remarkable phenomenon due to invariance of conformal structures on stable and unstable foliations. Nevertheless we still have a counterexample if we go a little bit away from L.

Next we study smooth conjugacy problem in the neighborhood of (1) assuming that $\mu > \lambda > 1$. We show that perturbations (2) exhaust all possibilities. Before formulating the result precisely let us move to a slightly more general setting. Let A and B be as in (1) with $\mu > \lambda > 1$. Consider Anosov diffeomorphism

$$L(x, y) = (Ax, g(y)), \quad (x, y) \in \mathbb{T}^2 \times \mathbb{T}^2, \tag{3}$$

where g is an Anosov diffeomorphism sufficiently C^1-close to B so that L can be treated as a partially hyperbolic diffeomorphism with automorphism A acting in the central direction. Consider perturbations of the form

$$\tilde{L} = (Ax + \vec{\varphi}(y), g(y)). \tag{4}$$

As before, it is obvious that p. d. of L and $\tilde{L}$ coincide. In Section 8 we will see that L and $\tilde{L}$ with non-linear g also provide a counterexample to Question 1.

Theorem C. *Given L as in (3) with $\mu > \lambda > 1$ there exists a C^1-neighborhood $\mathcal{U} \subset \mathrm{Diff}^r(\mathbb{T}^4)$, $r \geq 2$, of L such that any $f \in \mathcal{U}$ that has the same p. d. as L is $C^{1+\nu}$, $\nu > 0$, conjugate to a diffeomorphism $\tilde{L}$ of type (4).*

1.3. **Additional moduli of C^1 conjugacy in the neighborhood of the counterexample of de la Llave.** Let L be given by (1) with $\mu > \lambda > 1$ and let $\mathcal{U}$ be a small C^1-neighborhood of L. It is fruitful to think of diffeomorphisms from $\mathcal{U}$ as of partially hyperbolic diffeomorphisms with two dimensional central foliations. Consider $f, g \in \mathcal{U}$, $h \circ f = g \circ h$. According to the celebrated theorem of Hirsch, Pugh and Shub [HPS77] the conjugacy h maps the central foliation of f into the central foliation of g.

Assume that p. d. of f and g are the same. Then we show that h is $C^{1+\nu}$ along the central foliation. As described above it can still happen that h is not a C^1-diffeomorphism. This means that the conjugacy is not differentiable in the direction transverse to the central foliation. The geometric reason for this is mismatch between strong stable (unstable) foliations of f and g — the conjugacy h does not map strong stable (unstable) foliation of f into strong stable (unstable) foliation of g.

Motivated by this observation we introduce additional moduli of C^1-differentiable conjugacy. Roughly speaking these moduli measure the tilt of strong stable (unstable) leaves when compared to the model (1).

We define these moduli precisely. Let W_L^{ss}, W_L^{ws}, W_L^{wu} and W_L^{su} be the foliations by straight lines along the eigendirections with eigenvalues μ^{-1}, λ^{-1}, λ and μ respectively. For any $f \in \mathcal{U}$ these invariant foliations survive. We denote them by W_f^{ss}, W_f^{ws}, W_f^{wu} and W_f^{su}. Also we write W_f^s and W_f^u for two dimensional stable and unstable foliations.

Let h_f be the conjugacy to the linear model, $h_f \circ f = L \circ h_f$. Then

$$h_f(W_f^\sigma) = W_L^\sigma, \quad \sigma = s, u, ws, wu. \tag{5}$$

Fix orientation of W_L^σ, $\sigma = ss, ws, wu, su$. Then for every $x \in \mathbb{T}^4$ there exists a unique orientation preserving isometry $\mathcal{I}^\sigma(x) : W_L^\sigma(x) \to \mathbb{R}$, $\mathcal{I}^\sigma(x) = 0$, $\sigma = ss, ws, wu, su$.

Define $\Phi_f^u : \mathbb{T}^4 \times \mathbb{R} \to \mathbb{R}$ by the formula

$$\Phi_f^u(x,t) = \mathcal{I}^{wu}\big(\mathcal{I}^{su}(x)^{-1}(t)\big)\big(h_f(W_f^{su}(h_f^{-1}(x))) \cap W_L^{wu}(\mathcal{I}^{su}(x)^{-1}(t))\big).$$

The geometric meaning is transparent and illustrated on Figure 1. Image of strong unstable manifold $h_f(W_f^{su}(x))$ can be viewed as a graph of function $\Phi_f^u(x,\cdot)$ over $W_L^{su}(x)$. Analogously we define $\Phi_f^s : \mathbb{T}^4 \times \mathbb{R} \to \mathbb{R}$.

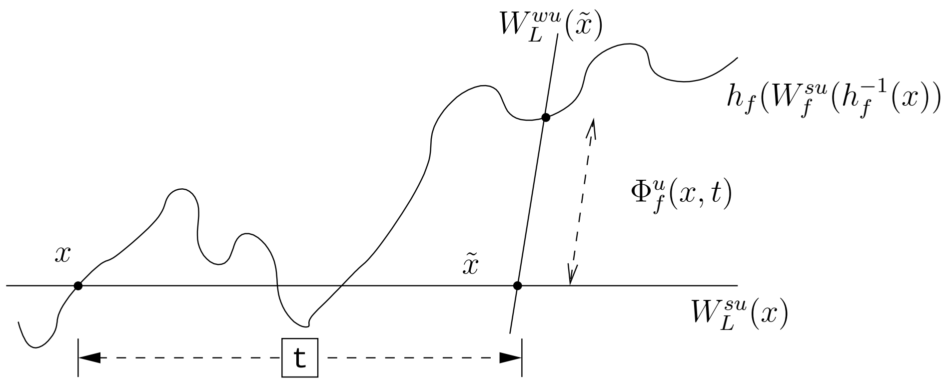

FIGURE 1. Geometric meaning of Φ_f^u. Here $\tilde{x} = \mathcal{I}^{su}(x)^{-1}(t)$.

Clearly $\Phi_f^{s/u}$ are moduli of C^1 conjugacy. Indeed, assume that f and g are C^1 conjugate by h. Then $h(W_f^{su}) = h(W_g^{su})$ and $h(W_f^{ss}) = h(W_g^{ss})$ since strong stable and unstable foliations are characterized by the speed of convergence which is preserved by C^1 conjugacy. Hence $\Phi_f^{s/u} = \Phi_g^{s/u}$.

It is possible to choose a subfamily of these moduli in an efficient way. We say that f and g from $\mathcal{U}$ have the same *strong unstable foliation moduli* if

$$\exists t \neq 0 \quad \text{such that} \quad \forall x \in \mathbb{T}^4, \quad \Phi_f^u(x,t) = \Phi_g^u(x,t) \tag{6}$$

or

$$\exists x \in \mathbb{T}^4 \quad \text{and} \quad \exists I = (a,b) \subset \mathbb{R} \quad \text{such that} \quad \forall t \in I \quad \Phi_f^u(x,t) = \Phi_g^u(x,t). \tag{7}$$

Definition of *strong stable foliation moduli* is analogous.

Theorem D. *Given L as in (1) with $\mu > \lambda > 1$ there exists a C^1-neighborhood $\mathcal{U} \subset \mathrm{Diff}^r(\mathbb{T}^4)$, $r \geq 2$ of L such that if $f, g \in \mathcal{U}$ have the same p. d. and the same strong unstable and strong stable foliation moduli. Then f and g are $C^{1+\nu}$ conjugate.*

Remark. In this case $C^{1+\nu}$-differentiability is in fact the optimal regularity.

1.4. **Organization of the paper and a remark on terminology.** In Section 2 we describe the counterexample of de la Llave in a way that allows us to generalize it to Theorem B in Section 3. Sections 2 and 3 are independent of the rest of the paper.

In Sections 4 and 5 we discuss Property $\mathcal{A}$ and construct examples of diffeomorphisms that satisfy Property $\mathcal{A}$. These sections are self-contained.

Section 6 is devoted to the proof of our main result, Theorem A. It is self-contained but in number of places we refer to [GG08] where three dimensional version of Theorem A was established.

Theorem C is proved in Section 7. It is independent of the rest of the paper with an exception of a reference to Proposition 10.

Proof of Theorem D appears in Section 8 and relies on some technical results from [GG08].

Throughout the paper we will be proving that various maps are $C^{1+\nu}$-differentiable. This should be understood in the usual way: the map is C^1 differentiable and the derivative is Hölder continuous with some positive exponent ν. Number ν is not the same in different statements.

When we say that a map is $C^{1+\nu}$-differentiable along foliation $\mathcal{F}$ we mean that restrictions of the map to the leaves of $\mathcal{F}$ are $C^{1+\nu}$-differentiable and the derivative is a Hölder continuous function on the manifold, not only on the leaf.

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2. The counterexample on $\mathbb{T}^4$

Here we describe the example of de la Llave of two Anosov diffeomorphisms of $\mathbb{T}^4$ with the same p. d. that are only Hölder conjugate. Understanding of the example is important for the proof of Theorem B.

Recall that we start with an automorphism $L:\mathbb{T}^4\to\mathbb{T}^4$

$$L(x,y)=(Ax,By),\ \ (x,y)\in\mathbb{T}^2\times\mathbb{T}^2,$$

where A and B are Anosov automorphisms, $Av=\lambda v$, $A\tilde v=\lambda^{-1}\tilde v$, $Bu=\mu u$, $B\tilde u=\mu^{-1}\tilde u$. We assume that $\mu\ge\lambda>1$.

To simplify computations we consider a special perturbation of the form

$$\tilde L=(Ax+\varphi(y)v,By).$$

We look for the conjugacy h of the form

$$h(x,y)=(x+\psi(y)v,y). \tag{8}$$

The conjugacy equation $h\circ\tilde L=L\circ h$ transforms into a cohomological equation on ψ

$$\varphi(y)+\psi(By)=\lambda\psi(y). \tag{9}$$

Let us solve for ψ using the recurrent formula

$$\psi(y)=\lambda^{-1}\varphi(y)+\lambda^{-1}\psi(By).$$

We get a continuous solution to (9)

$$\psi(y)=\lambda^{-1}\sum_{k\ge0}\lambda^{-k}\varphi(B^ky). \tag{10}$$

Hence the conjugacy is indeed given by the formula (8).

In the following proposition we denote by subscript u the partial derivative in the direction of u.

Proposition 1. *Assume that $\mu > \lambda > 1$. Then function ψ is Lipschitz in the direction of u if and only if*

$$\sum_{k\in\mathbb{Z}} \left(\frac{\mu}{\lambda}\right)^k \varphi_u(B^k y) = 0, \tag{11}$$

i. e. the series on the left converge in the sense of distribution convergence and the limit is equal to zero.

Proof. First assume (11). Let us consider series (10) as series of distributions that converge to ψ. Then as a distribution ψ_u is obtained by differentiating (10) termwise.

$$\psi_u = \lambda^{-1} \sum_{k\geq 0} \lambda^{-k}\mu^k \varphi_u(B^k). \tag{12}$$

Applying (11) we get

$$\psi_u = \lambda^{-1} \sum_{k<0} \lambda^{-k}\mu^k \varphi_u(B^k).$$

Since $\mu > \lambda$ the above series converge and the distribution is regular. Hence ψ is differentiable in the direction of u.

Now assume that ψ is u-Lipschitz. By differentiating (9) we get cohomological equation on ψ_u

$$\varphi_u(x) + \mu\psi_u(By) = \lambda\psi_u(y)$$

that is satisfied on a B-invariant set of full measure. We solve it using the recurrent formula

$$\psi_u(y) = -\frac{1}{\mu}\varphi_u(B^{-1}y) + \frac{\lambda}{\mu}\psi_u(B^{-1}y).$$

Hence

$$\psi_u = \lambda^{-1} \sum_{k<0} \lambda^{-k}\mu^k \varphi_u(B^k). \tag{13}$$

On the other hand we know that as a distribution ψ_u is given by (12). Combining (12) and (13) we get the desired equality (11). □

If $\mu = \lambda$ then the argument above works only in one direction. We will see that in this case L and $\tilde{L}$ do not provide a counterexample since p. d. are different.

Proposition 2. *Assume that $\mu = \lambda$. Then (11) is a necessary assumption for ψ to be Lipschitz in the direction of u.*

Proof. As in the proof of Proposition 1, viewed as distribution, ψ_u is given by

$$\psi_u = \lambda^{-1} \sum_{k\geq 0} \varphi_u(B^k). \tag{14}$$

Assume that ψ is u-Lipschitz then analogously to (13) we get

$$\psi_u = \lambda^{-1} \sum_{-N\leq k<0} \varphi_u(B^k) + \psi(B^N). \tag{15}$$

Note that in the sense of distributions $\psi(B^N) \to 0$ as $N \to \infty$ since B is mixing. Hence, as a distibution, ψ_u is given by

$$\psi_u = \lambda^{-1} \sum_{k<0} \varphi_u(B^k). \tag{16}$$

Combining (14) and (16) we get (11). □

By rewriting condition (11) in terms of Fourier coefficients of φ one can see that it is an infinite codimension condition. Moreover, one can easily construct functions that do not satisfy (11). One only need to make sure that some Fourier coefficients of the sum (11) are non-zero. For instance, for any $\varepsilon > 0$ and positive integer p function

$$\varphi(y) = \varphi(y_1, y_2) = \varepsilon \sin(p\pi y_1) \tag{17}$$

will serve the purpose. Thus corresponding $\tilde{L}$ is not C^1 conjugate to L. Note that $\tilde{L}$ maybe chosen arbitrarily close to L.

Remark. Perturbations of the general type (2) can be treated analogously by decomposing $\vec{\phi} = \phi_1 v + \phi_2 \tilde{v}$.

Remark. Notice that the assumption $\mu \geq \lambda > 1$ is crucial in this construction.

Remark. By choosing appropriate λ and μ one can get any desired regularity of the conjugacy (see [L92] for details). For example, if $\mu^2 > \lambda > \mu > 1$ then the conjugacy is C^1 but not C^2.

From now on let us assume that $\mu = \lambda$. As we have remarked in the introduction L and $\tilde{L}$ do not provide a counterexample. Indeed, the derivative of $\tilde{L}$ in the basis $\{v, u, \tilde{v}, \tilde{u}\}$ is

$$\begin{pmatrix} \lambda & \varphi_u & 0 & \varphi_{\tilde{u}} \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & 0 & \lambda^{-1} \end{pmatrix}.$$

Let x be a periodic point, $\tilde{L}^p(x) = x$. Then the derivative of the returm map at x is

$$\begin{pmatrix} \lambda^p & \lambda^{p-1} \sum_{y \in \mathcal{O}(x)} \varphi_u(y) & 0 & * \\ 0 & \lambda^p & 0 & 0 \\ 0 & 0 & \lambda^{-p} & 0 \\ 0 & 0 & 0 & \lambda^{-p} \end{pmatrix}. \tag{18}$$

We see that it is likely to have a Jordan block while L is diagolizable. Hence L and $\tilde{L}$ have different p. d.

It is still easy to cook up a counterexample in the neighborhood of L. Let

$$\hat{L} = (Ax + \xi(y)v, By)$$

an let

$$h(x, y) = (x + \psi(y)v, y)$$

be the conjugacy between $\tilde{L}$ and $\hat{L}$

Proposition 3. *Condition*

$$\sum_{k \in \mathbb{Z}} (\xi - \varphi)_u(B^k y) = 0,$$

is necessary for ψ to be Lipschitz in the direction of u.

The proof is exactly the same as the one of Proposition 2.

Take φ that does not satisfy (11) as before and take $\xi = 2\varphi$. Then obviously the condition of Proposition 3 is not satisfied. Hence h is not Lipschitz. By looking at (18) it is obvious that our choice of ξ guarantees that Jordan normal forms of the derivatives of the return maps at periodic points of $\tilde{L}$ and $\hat{L}$ are the same.

Remark. Due to the special choice of ξ it was easy to ensure that p. d. of $\tilde{L}$ and $\hat{L}$ are the same. We could have taken a different and somewhat more general approach. It is possible to show that for many choices of φ the sum that appears over the diagonal in (18) is non-zero for every periodic point x. All correspoding diffeomorphism will have the same p. d. with a Jordan block at every periodic point.

3. Proof of Theorem B

Here we consider $L : \mathbb{T}^d \to \mathbb{T}^d$ with reducible characteristic polynomial. We show how to construct $\tilde{L}$ and $\hat{L}$ with the same p. d. which are not Lipschitz conjugate.

Assume that all real eigenvalues of L are positive. Otherwise we would consider L^2. Let $M : \mathbb{R}^d \to \mathbb{R}^d$ be the lift of L. And let $\{e_1, e_2, \dots e_d\}$ be the canonical basis so that $\mathbb{T}^d = \mathbb{R}^d / \mathrm{span}_{\mathbb{Z}}\{e_1, e_2, \dots e_d\}$.

It is well known that characteristic polynomial of M factors over $\mathbb{Z}$ into the product of polynomials irreducible over $\mathbb{Q}$.

$$P(x) = P_1(x)P_2(x)\dots P_r(x), \quad r \geq 2.$$

Let λ be the eigenvalue of M with the smallest absolute value which is greater than one. Without loss of generality we assume that $P_1(\lambda) = 0$.

Let V_i be the invariant subspace that corresponds to the roots of P_i. Then $\dim V_i = \deg P_i$ and it is easy to show that

$$V_i = \mathrm{Ker}(P_i(M)).$$

Matrices of $P_i(M)$ have integer entries. Hence there is a basis $\{\tilde{e}_1, \tilde{e}_2, \dots \tilde{e}_d\}$, $\tilde{e}_i \in \mathrm{span}_{\mathbb{Z}}\{e_1, e_2, \dots e_d\}, i = 1, \dots d$, such that matrix of M in this basis has integer entries and is of a block diagonal form with blocks corresponding to invariant subspaces $V_i, i = 1, \dots r$.

We consider projection of M to $\tilde{\mathbb{T}}^d = \mathbb{R}^d / \mathrm{span}_{\mathbb{Z}}\{\tilde{e}_1, \tilde{e}_2, \dots \tilde{e}_d\}$. Denote by N the induced map on $\tilde{\mathbb{T}}^d$. We have the following commutative diagram where π is a finite-to-one projection.

$$\begin{array}{ccc}
\mathbb{R}^d & \xrightarrow{M} & \mathbb{R}^d \\
\big\downarrow & & \big\downarrow \\
\tilde{\mathbb{T}}^d & \xrightarrow{N} & \tilde{\mathbb{T}}^d \\
\pi \big\downarrow & & \pi \big\downarrow \\
\mathbb{T}^d & \xrightarrow{L} & \mathbb{T}^d
\end{array}$$

Notice that N has the form $N(x, y) = (Ax, By)$, $(x, y) \in \mathbb{T}^{\deg P_1} \times \mathbb{T}^{d - \deg P_1}$. Let μ be an eigenvalue of B. By construction λ, $|\lambda| \leq |\mu|$, is an eigenvalue of A.

With certain care the construction of Section 2 can be applied to N. We have to distinguish the following cases.

(1) λ and μ are real.
(2) λ is real and μ is complex.
(3) λ is complex and μ is real.
(4) λ and μ are complex.

Assume that $|\lambda| < |\mu|$. Then we take $\hat{L} = L$.

In the first case construction of Section 2 applies straightforwardly. We use function of the type (17) to produce $\tilde{N}$. Now we only need to make sure that $\tilde{N}$ can be projected to a map $\tilde{L} : \mathbb{T}^d \to \mathbb{T}^d$. Since π is a finite-to-one covering map this can be achieved by choosing suitable p in (17).

Other cases require heavier calculations but follow the same scheme of Proposition 1. We outline the construction in the case 4 that can appear, for instance, if A and B are hyperbolic automorphisms of four dimensional tori without real eigenvalues.

Let $V_A = \text{span}\{v_1, v_2\}$ be the two dimensional A-invariant subspace corresponding to λ and $V_B = \text{span}\{u_1, u_2\}$ be the two dimensional B-invariant subspace corresponding to μ. Then A acts on V_A by multiplication by $|\lambda| R_A$ and B acts on V_B by multiplication by $|\mu| R_B$, where R_A and R_B are rotation matrices expressed in bases $\{v_1, v_2\}$ and $\{u_1, u_2\}$ respectively.

We are following the construction from the previous section. Let

$$\tilde{N}(x, y) = (Ax + \vec{\varphi}(y)\vec{v}, By) \stackrel{\text{def}}{=} (Ax + \varphi_1(y)v_1 + \varphi_2(y)v_2, By).$$

Then we look for the conjugacy in the form

$$h(x, y) = (x + \vec{\psi}(y)\vec{v}, y) \stackrel{\text{def}}{=} (x + \psi_1(y)v_1 + \psi_2(y)v_2, y).$$

The conjugacy equation $h \circ \tilde{N} = N \circ h$ transforms into

$$\vec{\varphi}(y)\vec{v} + \vec{\psi}(By)\vec{v} = |\lambda| R_A \vec{\psi}(y). \tag{19}$$

Solving for $\vec{\psi}$ gives

$$\vec{\psi}(y) = \sum_{k \geq 0} |\lambda|^{-k-1} R_A^{-k-1} \vec{\varphi}(B^k y),$$

which we would like to differentiate in the directions u_1 and u_2. We use the formula

$$\vec{\varphi}(By)_{\vec{u}} = \begin{pmatrix} \varphi_1(By)_{u_1} & \varphi_1(By)_{u_2} \\ \varphi_2(By)_{u_1} & \varphi_2(By)_{u_2} \end{pmatrix} = |\mu| \begin{pmatrix} (\varphi_1)_{u_1} & (\varphi_1)_{u_2} \\ (\varphi_2)_{u_1} & (\varphi_2)_{u_2} \end{pmatrix} (By) R_B = \vec{\varphi}_{\vec{u}}(By) R_B$$

to get that as a distribution

$$\vec{\psi}_{\vec{u}} = \sum_{k \geq 0} |\lambda|^{-k-1} |\mu|^k R_A^{-k-1} \vec{\varphi}_{\vec{u}}(B^k) R_B^k.$$

Now we assume that $\vec{\psi}$ is Lipschitz and we differentiate (19) in the directions u_1 and u_2

$$\vec{\varphi}_{\vec{u}}(y) + |\mu| \vec{\psi}_{\vec{u}}(By) R_B = |\lambda| R_A \vec{\psi}_{\vec{u}}(y).$$

Hence by the recurrent formula

$$\vec{\psi}_{\vec{u}} = \sum_{k < 0} |\lambda|^{-k-1} |\mu|^k R_A^{-k-1} \vec{\varphi}_{\vec{u}}(B^k) R_B^k.$$

Combining the expressions for $\vec{\psi}_{\vec{u}}$ we get

$$\sum_{k \in \mathbb{Z}} |\lambda|^{-k} |\mu|^k R_A^{-k} \vec{\varphi}_{\vec{u}}(B^k) R_B^k = 0.$$

Using Fourier decomposition one can find functions $\vec{\varphi}$ that do not satisfy the condition above. One also needs to make sure that the choice of $\vec{\varphi}$ allows to project $\tilde{N}$ down to $\tilde{L}$. We omit this analysis since it is routine.

This is a contradiction and therefore $\vec{\psi}$ (and hence h) is not Lipschitz.

If $|\lambda| = |\mu|$ but $\lambda \neq \mu$ then the scheme above still works. Obviously extra Jordan blocks do not appear in the normal forms at periodic points of $\tilde{L}$.

Finally, the case $\lambda = \mu$ must be treated separately. We use the same trick as in Section 2 to find $\tilde{L}$ and $\hat{L}$ with the same p. d. that are only Hölder conjugate. The trick works well in the case of complex eigenvalues as well. We omit the details.

4. On the Property $\mathcal{A}$

4.1. **Transitivity versus minimality.** Here we discuss Property $\mathcal{A}$. Let $\mathcal{F}$ be a foliation of a compact manifold M. As usually $\mathcal{F}(x)$ stands for the leaf of $\mathcal{F}$ that contains x and $\mathcal{F}(x, R)$ stands for the ball of radius R centered at x inside of $\mathcal{F}(x)$.

Definition 1. *Foliation $\mathcal{F}$ is called minimal if every leaf of $\mathcal{F}$ is dense in M.*

Definition 2. *Foliation $\mathcal{F}$ is called transitive if these exists a leaf of $\mathcal{F}$ that is dense in M.*

Definition 3. *Foliation $\mathcal{F}$ is called tubularly minimal if for every x and every open ball $B \ni x$*

$$\overline{\bigcup_{y \in B} \mathcal{F}(y)} = M.$$

Property $\mathcal{A}$ simply requires foliations U^f_{l-1}, U^f_{l-2},... U^f_1, V^f_1, V^f_2,... V^f_{k-1} to be tubularly minimal.

Property $\mathcal{A}'$. Foliations U^f_{l-1}, U^f_{l-2},... U^f_1, V^f_1, V^f_2,... V^f_{k-1} are minimal. $(\mathcal{A}')$

Proposition 4. *Foliation $\mathcal{F}$ is transitive if and only if it is tubularly minimal.*[1]

Proof. Transitivity obviously implies tubular minimality.

Assume that $\mathcal{F}$ is tubularly minimal. Let $\{B_n, n \geq 1\}$ be a countable basis for the topology of M. By the definition of tubular minimality sets $\mathcal{F}(B_n)$ are open and dense in M. Hence by Baire category theorem we have the set

$$B = \bigcap_{n \geq 1} \mathcal{F}(B_n)$$

is non-empty. For every $x \in B$ the leaf $\mathcal{F}(x)$ is dense in M. □

Remark. We define Property $\mathcal{A}$ in terms of tubular minimality as opposed to transitivity because we need denseness of the tubes to carry out the proof of Theorem A.

A priori, transitivity is weaker than minimality. Hence, a priori, Property $\mathcal{A}$ is weaker than Property $\mathcal{A}'$.

If in Theorem A we require f to satisfy $\mathcal{A}'$ instead of $\mathcal{A}$ then the induction procedure that we use (induction step 1) is much simpler. Proof of the induction step 1 assuming only Property $\mathcal{A}$ requires much more lengthy and delicate argument. It is not clear to us what is the relation between Properties $\mathcal{A}$ and $\mathcal{A}'$. They may happen to be equivalent. Thus first we provide a proof of Theorem A assuming that

[1]We would like to thank the referee for pointing out this fact.

f has Property $\mathcal{A}'$. Then we present a separate proof of induction step 1 (namely Lemma 6.6) that uses only Property $\mathcal{A}$.

Minimality of a foliation can be characterized similarly to tubular transitivity.

Proposition 5. *Foliation $\mathcal{F}$ is minimal if and only if for every x and every open ball $B \ni x$*

$$\bigcup_{y\in B} \mathcal{F}(y) = M.$$

The proof is simple so we omit it. As a corollary we get that foliation $\mathcal{F}$ is minimal if and only if for every x and every open ball $B \ni x$ there exists a number R such that

$$\bigcup_{y\in B} \mathcal{F}(y, R) = M. \tag{20}$$

This is the property which we will actually use in the proof of the induction step 1.

4.2. **Examples of diffeomorphisms that satisfy Property $\mathcal{A}$.**

Proposition 6. *Assume that L is irreducible. Then foliations U_j^L, V_i^L, $j = 1 \dots l$, $i = 1 \dots k$ are minimal.*

Proof. Denote by $\mathcal{F}$ one of the foliations under consideration. Since $\mathcal{F}$ is a foliation by straight lines the closure of a leaf $\mathcal{F}(x)$ is a subtorus of $\mathbb{T}^d$. This subtorus lifts to a rational invariant subspace of $\mathbb{R}^d$. The invariant subspace corresponds to a rational factor of the characteristic polynomial of L while we have assumed that it is irreducible over $\mathbb{Q}$. Hence the invariant subspace is the whole $\mathbb{R}^d$ and the subtorus is the whole $\mathbb{T}^d$. □

Hence the conclusion of Theorem A holds at least for $f = L$.

We will see in Section 6.1 that for any $f \in \mathcal{U}$ foliations U_1^f and V_1^f are minimal. Hence the conclusion of Theorem A holds for any $f \in \mathcal{U}$ if $\max(k, l) \le 2$.

It is easy to construct $f \ne L$ that satisfies $\mathcal{A}$ when $k = 3$ and $l = 2$ since we only have to worry about the foliation V_2^f. We let $f = s \circ L$ where s is any small shift along V_2^f. Clearly $V_2^f = V_2^L$ and hence f satisfies $\mathcal{A}$.

Question about robust minimality of foliations U_{l-1}^f, U_{l-2}^f,... U_1^f, V_1^f, V_2^f,... V_{k-1}^f arises naturally. Robust minimality of strong stable and strong unstable foliations of partially hyperbolic systems received some attention in the literature due to its intimate connection with robust transitivity. See [Ma78] and more recent papers [BDU02], [PS06], where robust minimality of the *full* expanding foliation is established under some assumptions. We do not have this luxury in our setting: expanding foliations that we are intrested in subfoliate full unstable foliation. A representative problem here is the following.

Question 2. *Let $L : \mathbb{T}^3 \to \mathbb{T}^3$ be a hyperbolic linear automorphism with real spectrum $\lambda_1 < 1 < \lambda_2 < \lambda_3$. Consider one dimensional strong unstable foliation. Is it true that this foliation is robustly minimal? In other words, is it true that for any f sufficiently C^1-close to L the strong unstable foliation of f is minimal?*

In addition to the simple examples above we construct a C^1-open set of diffeomorphisms that possess Property $\mathcal{A}$ in the next section. The following statement can be obtained by applying the construction and the arguments of the next section in the setup of Question 2.

Proposition 7. *Let L be as in Question 2. Then there exists a C^1-open set $\mathcal{U}$ C^1-close to L such that for every $f\in\mathcal{U}$ the strong unstable foliation of f is transitive.*

5. An example of an open set of diffeomorphisms that possess Property $\mathcal{A}$

Let $L:\mathbb{T}^5\to\mathbb{T}^5$ be a hyperbolic automorphism as in Theorem A, $l=2, k=3$, and let $\mathcal{U}$ be a C^1-neighborhood of L chosen as in Section 6.1.

Recall that D_f^{wu} stands for the derivative of $f\in\mathcal{U}$ along V_1^f. Choose $f\in\mathcal{U}$ in such a way that

$$\forall x\neq x_0 \quad D_f^{wu}(x)>D_f^{wu}(x_0), \tag{21}$$

where x_0 is a fixed point of f.

Proposition 8. *There exists a C^1-neighborhood $\tilde{\mathcal{U}}$ of f such that any diffeomorphism $g\in\tilde{\mathcal{U}}$ has Property $\mathcal{A}$.*

Remark. Similar example can be constructed on $\mathbb{T}^6$ with $l=3$, $k=3$. We only need to do the trick described below for both stable and unstable manifolds of the fixed point x_0.

Before proving the proposition let us briefly explain the idea behind the proof. We know that U_1^g and V_1^g are minimal. Hence we only need to show that foliation V_2^g is tubularly minimal i. e. for every $x\in\mathbb{T}^5$ and every open ball $B\ni x$

$$\overline{\bigcup_{y\in B} V_2^g(y)}=\mathbb{T}^5. \tag{22}$$

To illustrate the idea we take $g=f$ and $x=x_0$. We work on the universal cover $\mathbb{R}^5$ with lifted foliations. Let

$$\mathcal{T}\overset{\text{def}}{=}\bigcup_{y\in B} V_2^f(y)\subset\mathbb{R}^5, \tag{23}$$

which is an open tube.

We show that $\mathcal{T}$ contains arbitrarily long connected pieces of the leaves of V_1^f as shown on Figure 2. It would follow that $\mathcal{T}$ is dense in $\mathbb{T}^5$. Indeed, foliation V_1^f is not just minimal but uniformly minimal: for any $\varepsilon>0$ there exists $R>0$ such that $\forall z\in\mathbb{T}^5$ $V_1^f(z,R)$ is ε-dense in $\mathbb{T}^5$. This property follows from the fact that V_1^f is conjugate to the linear foliation V_1^L.

Pick $y_0\in B\cap V_1^f(x_0)$ close to x_0. Let $x\in V_2^f(x_0)$ be a point far away in the tube $\mathcal{T}$ and $y=V_1^f(x)\cap V_2^f(y_0)$. To show that $\mathcal{T}$ contains arbitrarily long pieces of leaves of V_1^f we prove that $d_1^f(x,y)$ (recall that d_i^f is the Riemannian distance along V_i^f) is unbounded function of x.

We make use of the affine structure on V_1^f. We refer to [GG08] for the definition of affine distance-like function $\tilde{d}_1$. Recall crucial properties of $\tilde{d}_1$

(D1) $\tilde{d}_1(x,y)=d_1^f(x,y)+o(d_1^f(x,y))$,
(D2) $\tilde{d}_1(f(x),f(y))=D_f^{wu}(x)\tilde{d}_1(x,y)$,
(D3) $\forall K>0\ \exists C>0$ such that

$$\frac{1}{C}\tilde{d}_1(x,y)\le d_1^f(x,y)\le C\tilde{d}_1(x,y)$$

whenever $d_1(x,y)<K$.

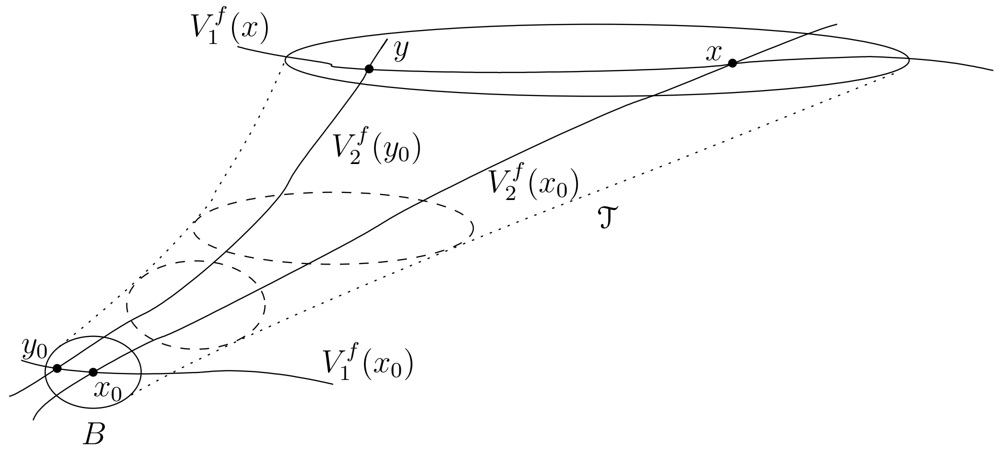

FIGURE 2. Tube $\mathcal{T}$ contains arbitrarily long pieces of leaves of V_1^f.

By property (D3) it is enough to show that $\tilde{d}_1(x,y)$ is unbounded.

Given x as above pick N large so that the ratio $\tilde{d}_1(f^{-N}(x), f^{-N}(y))/\tilde{d}_1(x_0, f^{-N}(y_0))$ is close to 1 as shown on the Figure 3. It is possible since V_2^f contracts exponentially faster than V_1^f under the action of f^{-1}.

It is not hard to see that given a large number n we can pick x (and N correspondingly) far enough from x_0 so that at least n points from the orbit $\{x, f^{-1}(x), \dots f^{-N}(x)\}$ lie outside of B. For such a point $z = f^{-i}(x)$ that is not in B

$$D_f^{wu}(z) \geq D_f^{wu}(x_0) + \delta,$$

where $\delta > 0$ depends only on the size of B.

Using (D2) we get

$$\begin{aligned}\frac{\tilde{d}_1(x,y)}{\tilde{d}_1(x_0,y_0)} &= \prod_{i=1}^{N} \frac{D_f^{wu}(f^{-i}(x))}{D_f^{wu}(x_0)} \cdot \frac{\tilde{d}_1(f^{-N}(x), f^{-N}(y))}{\tilde{d}_1(x_0, f^{-N}(y_0))} \\ &\geq \left(\frac{D_f^{wu}(x_0)+\delta}{D_f^{wu}(x_0)}\right)^n \cdot \frac{\tilde{d}_1(f^{-N}(x), f^{-N}(y))}{\tilde{d}_1(x_0, f^{-N}(y_0))} \qquad (24)\end{aligned}$$

which is an arbitrary large number. Hence $\tilde{d}_1(x,y)$ is arbitrarily large and we are done.

Remark. Although Proposition 8 deals with a pretty special situation we believe that the picture on Figure 2 is generic. To be more precise we think that for any $g \in \mathcal{U}$ the following alternative holds. Either V_2^g is conjugate to the linear foliation V_2^L or there exist a dense set Λ such that for any $x \in \Lambda$ and any $B \ni x$ the tube

$$\bigcup_{y \in B} V_2^f(y) \subset \mathbb{R}^5$$

contains arbitrarily long connected pieces of the leaves of V_1^g.

Proof of Proposition 8. The argument is more delicate than the one presented above since we do not know that the minimum of the derivative is achieved at x_0.

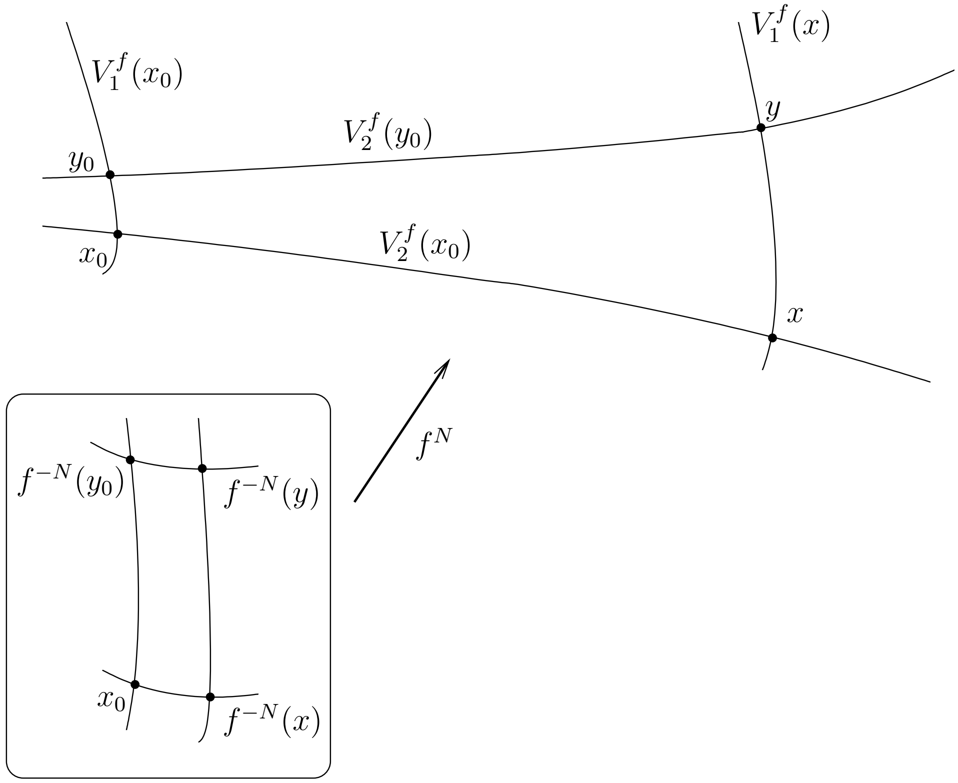

FIGURE 3. Illustration to the argument. Quadrilateral in the box is much smaller then the one outside.

Let B_0 be a small ball around x_0 and $B_1 \supset B_0$ be a bigger ball. Condition (21) guarantees that we can choose them in such a way that

$$m_0 < D_f^{wu}(x_0) < \sup_{x\in B_0} D_f^{wu}(x) < m_1 < M < \min_{x\notin B_1} D_f^{wu}(x)$$

with m_0, m_1 and M satisfying

$$\frac{Mm_0^{q-1}}{m_1^q} > 1, \tag{25}$$

where q is an integer that depends only on the size of $\mathcal{U}$ and the size of B_1. After that we choose $\tilde{\mathcal{U}} \subset \mathcal{U}$ so the fixed point of g (that corresponds to x_0) is inside of B_0 and the property above persists. Namely,

$$\forall g \in \tilde{\mathcal{U}} \quad m_0 < \inf_{x\in B_0} D_g^{wu}(x) < \sup_{x\in B_0} D_g^{wu}(x) < m_1 < M < \min_{x\notin B_1} D_f^{wu}(x). \tag{26}$$

Note that provided that f is sufficiently C^1-close to L and the ball B_1 is small enough any piece of a leaf of V_2^g outside of B_1 that starts and ends on the boundary of B_1 cannot be homotoped into a point keeping the endpoints on the boundary. This is a minor technical detail that makes sure that the picture shown on Figure 4a does not occur. Thus there is a lower bound R on the lengths of pieces of leaves of V_2^g outside of B_1 with endpoints on the boundary of B_1. Obviously, there is also an upper bound r on the lengths of pieces of leaves of V_2^g inside B_1.

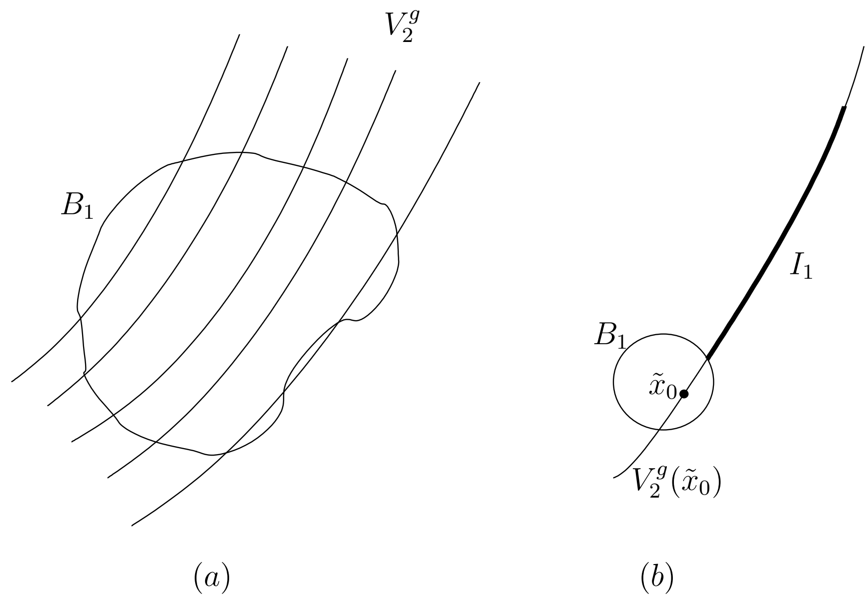

FIGURE 4. (a) does not occur if B is sufficiently small; (b) choice of I_1.

It is enough to check (22) for a dense set Λ of points $x \in \mathbb{T}^5$. We take Λ to be a subset of the set of periodic points of g

$$\Lambda = \{p : \quad D^{wu}_{f^{n(p)}}(p) \le m_1^{n(p)}\}, \tag{27}$$

where $n(p)$ stands for the period of p. Set Λ consists of periodic points that spend large but fixed percentage of time inside of B_0. It is fairly easy to show that Λ is dense in $\mathbb{T}^5$. The proof is a trivial corollary of specification property (e. g. see [KH95]).

So we fix $\tilde{x}_0 \in \Lambda$, a small ball B centered at $\tilde{x}_0$ and $y_0 \in B \cap V_1^g(x_0)$ close to $\tilde{x}_0$. Our goal now is to find $x \in V_2^g(\tilde{x}_0)$ far in the tube $\mathcal{T}$ defined by (23) for which we can carry out estimates similar to (24).

We will be working with pieces of leaves of V_2^g. Given a piece I with endpoints z_1 and z_2 let $|I| = d_2^g(z_1, z_2)$. Let q be a number such that for any piece I, $|I| = R$, we have

$$|g^q(I)| > 2R + r. \tag{28}$$

Notice that q can be chosen to be independent of g and depends only on $\tilde{\beta}_2$, R and r.

Pick $I_1 \subset V_2^g(\tilde{x}_0)$, $|I_1| = R$, $I_1 \cap B_1 = \emptyset$, as close to $\tilde{x}_0$ as possible if $\tilde{x}_0 \in B_1$ (see Figure 4b) or passing through $\tilde{x}_0$ if $\tilde{x}_0 \notin B_1$. Given I_i, $i \ge 1$ we choose $I_{i+1} \subset f^q(I_i)$, $|I_{i+1}| = R$, $I_{i+1} \cap B_1 = \emptyset$. Condition (28) guarantees that such choice is possible.

We fix N large and take $x \in I_{Nq} \subset V_2^g(\tilde{x}_0)$. Let $y = V_1^g(x) \cap V_2^g(y_0)$ as before. Construction of the sequence $\{I_i, i \ge 1\}$ ensures that points $f^{-qi}(x)$, $i = 0, \dots N-1$, are outside B_1. This fact together with (26) and (27) allows to carry out the following estimate

$$\frac{\tilde{d}_1(x,y)}{\tilde{d}_1(\tilde{x}_0,y_0)} = \prod_{i=1}^{Nq} \frac{D_g^{wu}(g^{-i}(x))}{D_g^{wu}(g^{-i}(\tilde{x}_0))} \cdot \frac{\tilde{d}_1(f^{-Nq}(x), f^{-Nq}(y))}{\tilde{d}_1(\tilde{x}_0, f^{-Nq}(y_0))}$$
$$\geq \frac{M^N m_0^{N(q-1)}}{m_1^{Nq}} \cdot \frac{\tilde{d}_1(f^{-Nq}(x), f^{-Nq}(y))}{\tilde{d}_1(\tilde{x}_0, f^{-Nq}(y_0))}.$$

The affine-like distance ratio on the right is bounded away from 0 independently of N since $f^{-Nq}(x) \in I_1$ while the coefficient in front of it is arbitrarily large according to (25). Hence $\tilde{d}_1^g(x,y)$ is arbitrarily large and the projection of tube $\mathcal{T}$ is dense in $\mathbb{T}^5$. □

6. Proof of Theorem A

For reasons explained in Section 4 we first prove Theorem A assuming that f has Property $\mathcal{A}'$. The only place where we use $\mathcal{A}'$ is the proof of Lemma 6.6. In Section 6.6 we give another proof of Lemma 6.6 that uses Property $\mathcal{A}$ only.

6.1. **Scheme of the proof of Theorem A.** Recall the notation from 1.1 for the L-invariant splitting

$$T\mathbb{T}^d = F_l \oplus F_{l-1} \oplus \ldots \oplus F_1 \oplus E_1 \oplus E_2 \oplus \ldots \oplus E_k$$

along the eigendirections with corresponding eigenvalues

$$\mu_l < \mu_{l-1} < \ldots < \mu_1 < 1 < \lambda_1 < \lambda_2 < \ldots < \lambda_k.$$

We choose neighborhood $\mathcal{U}$ in such a way that for any f in $\mathcal{U}$ the invariant splitting survives

$$T\mathbb{T}^d = F_l^f \oplus F_{l-1}^f \oplus \ldots \oplus F_1^f \oplus E_1^f \oplus E_2^f \oplus \ldots \oplus E_k^f,$$

with

$$\measuredangle(F_i, F_i^f) < \frac{\pi}{2},\ \measuredangle(E_j, E_j^f) < \frac{\pi}{2},\quad i = 1, \ldots l,\ j = 1, \ldots k \tag{29}$$

and f is partially hyperbolic in the strongest sense: there exist $C > 0$ and constants

$$\alpha_l < \tilde{\alpha}_{l-1} < \alpha_{l-1} < \ldots < \tilde{\alpha}_1 < \alpha_1 < 1 < \tilde{\beta}_1 < \beta_1 < \ldots < \tilde{\beta}_k$$

independent of the choice of f in $\mathcal{U}$ such that for $n > 0$

$$\|D(f^n)(x)(v)\| \leq C\alpha_l^n \|v\|, \quad v \in F_l^f(x),$$
$$\frac{1}{C}\tilde{\alpha}_{l-1}^n \|v\| \leq \|D(f^n)(x)(v)\| \leq C\alpha_{l-1}^n \|v\|, \quad v \in F_{l-1}^f(x),$$
$$\ldots$$
$$\frac{1}{C}\tilde{\alpha}_1^n \|v\| \leq \|D(f^n)(x)(v)\| \leq C\alpha_1^n \|v\|, \quad v \in F_1^f(x),$$
$$\frac{1}{C}\tilde{\beta}_1^n \|v\| \leq \|D(f^n)(x)(v)\| \leq C\beta_1^n \|v\|, \quad v \in E_1^f(x),$$
$$\ldots$$
$$\frac{1}{C}\tilde{\beta}_k^n \|v\| \leq \|D(f^n)(x)(v)\|, \quad v \in E_k^f(x). \tag{30}$$

Equivalently the Mather spectrum of f does not contain 1 and has d connected components.

We show that the choice of $\mathcal{U}$ guarantees unique integrability of intermediate distributions. From now on for the sake of concreteness we work with unstable distributions and foliations.

For a given $f \in \mathcal{U}$ let $E^f(i,j) = E_i^f \oplus E_{i+1}^f \oplus \ldots \oplus E_j^f$, $i \le j$.

Lemma 6.1. *For any f in $\mathcal{U}$ distribution $E^f(1,1), E^f(1,2), \ldots E^f(1,k)$ are uniquely integrable.*

This is a direct corollary of Hirsch, Pugh and Shub theorem but we will present a direct proof.

Let $W_1^f \subset W_2^f \subset \ldots \subset W_k^f$ be the corresponding flag of weak unstable foliations. The last foliation in the flag is the unstable foliation $W^f = W_k^f$.

Lemma 6.2. *For any f in $\mathcal{U}$ and $i \le j$ distribution $E(i,j)$ is uniquely integrable.*

Denote by $W^f(i,j)$, $i \le j$, the integral foliation of $E^f(i,j)$. Also recall that we denote by $V_1^f, V_2^f, \ldots V_k^f$ the integral foliations of $E_1^f, E_2^f, \ldots E_k^f$ correspondingly. Notice that $V_i^f = W^f(i,i)$ and $W_i^f = W^f(1,i)$, $i = 1, \ldots k$.

Now we consider f and g as in Theorem A, $h \circ f = g \circ h$. The conjugacy h maps unstable (stable) foliation of f into unstable (stable) foliation of g. Moreover, h preserves the whole flag of weak unstable (stable) foliations.

Lemma 6.3. *Fix an $i = 1, \ldots k$. Then $h(W_i^f) = W_i^g$.*

Remark. Proof of this lemma does not use the assumption on p. d. We only need f and g to be in $\mathcal{U}$.

Lemmas 6.1, 6.2 and 6.3 can be proved under a milder assumption. Instead of requiring f and g to be in $\mathcal{U}$ we can require an

Alternative assumption: f and g are partially hyperbolic in the strongest sence (30) with the rate constants satisfying

$$\mu_l < \alpha_l < \tilde{\alpha}_{l-1} < \mu_{l-1} < \alpha_{l-1} < \ldots < \tilde{\beta}_{k-1} < \lambda_{k-1} < \beta_{k-1} < \tilde{\beta}_k < \lambda_k. \qquad (\star)$$

We think that $(\star)$ is actually automatic from (30).

Remark. To carry out proofs of Lemmas above under the Alternative assumption one needs to transfer the picture to the linear model by the conjugacy and use inequalities $(\star)$ for growth arguments. This way one uses quasi-isometric foliations by straight lines of the linear model instead of foliations of f which are a priori not known to be quasi-isometric.

Conjecture 3. *Suppose that f is homotopic to L and partially hyperbolic in the strongest sense (30) then the rate constants satisfy $(\star)$.*

Remark. The proof of Lemmas 6.1, 6.2 and 6.3 is the only place where we really need f and g to be in $\mathcal{U}$. So in Theorem A the assumption that $f, g \in \mathcal{U}$ can be substituted by the alternative assumption.

Lemma 6.4. *A leaf $W_1^f(x)$ is dense in $\mathbb{T}^d$*

Proof. By Lemma 6.3 we have that the conjugacy between L and f takes the foliation W_1^L into the foliation W_1^f. According to Proposition 6 leaves of W_1^L are dense. Hence leaves of W_1^f are dense. □

Next we describe the inductive procedure which leads to smoothness of h along the unstable foliation.

Induction base. We know that h takes W_1^f into W_1^g.

Lemma 6.5. *Conjugacy h is $C^{1+\nu}$-differentiable along W_1^f i. e. restrictions of h to the leaves of W_1^f are differentiable and the derivative is C^ν function on $\mathbb{T}^d$.*

Provided that we have Lemma 6.4 the proof of Lemma 6.5 is the same as the proof of Lemma 5 from [GG08].

Induction step. The induction procedure is based on the following lemmas.

Lemma 6.6. *Assume that h is $C^{1+\nu}$-differentiable along W_{m-1}^f and $h(V_i^f) = h(V_i^g)$, $i = 1, \dots m-1$, $1 < m \le k$. Then $h(V_m^f) = V_m^g$.*

Lemma 6.7. *Assume that $h(V_m^f) = V_m^g$ for some $m = 1, \dots k$. Then h is $C^{1+\nu}$-differentiable along V_m^f.*

We also use a regularity result due to Journé.

Regularity Lemma ([J88])**.** *Let M_j be a manifold and W_j^s, W_j^u be continuous transverse foliations with uniformly smooth leaves, $j = 1, 2$. Suppose that $h : M_1 \to M_2$ is a homeomorphism that maps W_1^s into W_2^s and W_1^u into W_2^u. Moreover, assume that the restrictions of h to the leaves of these foliations are uniformly $C^{r+\nu}, r \in \mathbb{N}$, $0 < \nu < 1$. Then h is $C^{r+\nu}$.*

Remark. There are two more methods of proving analytical results of this flavor besides Journé's. One is due to de la Llave, Marco, Moriyón and the other one is due to Hurder and Katok (see [KN08] for a detailed discussion and proofs). We remark that we really need Journé's result since the alternative approaches require foliations to be absolutely continuous while we apply the Regularity Lemma to various foliations that do not have to be absolutely continuous.

Now the inductive scheme can be described as follows. Assume that h is $C^{1+\nu}$ along W_{m-1}^f for some $m \le k$ and $h(V_i^f) = h(V_i^g)$, $i = 1, \dots m-1$. By Lemma 6.6 we have that $h(V_m^f) = V_m^g$ and by Lemma 6.7 h is $C^{1+\nu}$ along V_m^f. Fix a leaf $W_m^f(x)$. Leaves of W_{m-1}^f and V_m^f subfoliate $W_m^f(x)$ and it is clear that the Regularity Lemma can be applied for $h : W_m^f(x) \to W_m^g(h(x))$. Hence we get that h is $C^{1+\nu}$ on every leaf of W_m^f. Hölder continuity of the derivative of h in the direction transverse to W_m^f is direct consequence of Hölder of the derivatives along W_{m-1}^f and V_m^f. We conclude that h is $C^{1+\nu}$-differentiable along W_m^f.

By induction we get that h is $C^{1+\nu}$-differentiable along the unstable foliation and analogously along the stable foliation. We finish the proof of the Theorem A by applying the Regularity Lemma to stable and unstable foliations.

6.2. **Proof of the integrability lemmas.** In the proofs of Lemmas 6.1 and 6.2 we work with lifts of maps, distributions and foliations to $\mathbb{R}^d$. We use the same notation for lifts as for the objects themselves.

Proof of Lemma 6.1. Fix $i < k$. We assume that the distribution $E^f(1, i)$ is not integrable or it is integrable but not uniquely. In any case it follows that we can find distinct points $a_0, a_1, \dots a_m$ such that

(1) $\{a_1, a_2, \dots a_m\} \subset W^f(a_0)$,
(2) there are smooth curves $\tau_j : [0,1] \to W^f(a_0)$, $j = 1, \dots m$, such that $\tau_j(0) = a_{j-1}$, $\tau_j(1) = a_j$ and $\dot\tau_j \subset E^f_{p(j)}$, where $p(j) \le i$,
(3) there is a smooth curve $\tau : [0,1] \to W^f(a_0)$ such that $\tau(0) = a_0$, $\tau(1) = a_m$ and $\dot\tau_j \subset E^f_q$ for some $q > i$.

Let $\tilde\tau$ be a piecewise smooth curve obtained by concatenating τ_1, τ_2,... τ_{m-1} and τ_m. From the second property above and (30) we get the following rough estimate

$$\forall n \ge 0 \quad \text{length}(f^n(\tilde\tau)) \le \beta_i^n \text{length}(\tilde\tau). \tag{31}$$

Similarly

$$\forall n \ge 0 \quad \text{length}(f^n(\tau)) \ge \tilde\beta_{i+1}^n \text{length}(\tau). \tag{32}$$

Denote by $d(\cdot,\cdot)$ the usual distance in $\mathbb{R}^d$. It follows from the assumption (29) that any curve $\gamma : [0,1] \to \mathbb{R}^d$ tangent to the distribution E^f_q is quasi-isometric:

$$\exists c > 0 \text{ such that } \quad \text{length}(\gamma) \le c\, d(\gamma(0), \gamma(1)).$$

In particular

$$\forall n \ge 0 \quad d(f^n(a_0), f^n(a_m)) \ge \frac{1}{c} \text{length}(f^n(\tau)). \tag{33}$$

Inequalities (31), (32) and (33) sum up to a contradiction. □

Proof of Lemma 6.2. The theory of partial hyperbolicity guarantees that distributions $E^f(i,k)$, $i = 1, \dots k$, integrate uniquely to foliations $W^f(i,k)$. Let us fix i and j, $i < j$, and define $W^f(i,j) = W^f(1,j) \cap W^f(i,k)$. Obviously $W^f(i,j)$ is an integral foliation for $E^f(i,j)$. Unique integrability of $E^f(i,j)$ is a direct consequence of the unique integrability of $E^f(1,j)$ and $E^f(i,k)$. □

6.3. **Weak unstable flag is preserved: proof of Lemma 6.3.**

Proof. We continue working on the universal cover. Pick two points a and b, $a \in W_i^f(b)$. Since

$$h_f(x + \vec m) = h_f(x) + \vec m, \quad \vec m \in \mathbb{Z}^d \tag{34}$$

we have that $d(h(x), h(y)) \le c_1 d(x,y)$ for any x and y such that $d(x,y) \ge 1$.

Hence for any $n > 0$

$$d(g^n(h(a)), g^n(h(b))) = d(h(f^n(a)), h(f^n(b))) \le c_2 d(f^n(a), f^n(b)) \le c_2 c_3 \beta_i^n,$$

where c_2 and c_3 depend on $d(a,b)$. This inequality guarantees that $h(a) \in W_i^g(h(b))$. Since the choice of a and b was arbitrary we conclude that $h(W_i^f) = W_i^g$. □

6.4. **Induction step 1: the conjugacy preserves foliation V_m.** We prove Lemma 6.6 which is the key ingredient in the proof of Theorem A. The proof is based on our idea from [GG08] but we take a rather different approach in order to deal with high dimension of W^f. We provide a complete proof almost without referring to [GG08]. Nevertheless we strongly encourage the reader to read Section 4.4 of [GG08] first.

The goal is to prove that $h(V_m^f) = V_m^g$. So we consider foliation $U = h^{-1}(V_m^g)$. As for usual foliations $U(x)$ stands for the leaf of U passing through x and $U(x,R)$ stands for the local leaf of size R. A priori, the leaves of U are just Hölder continuous curves. Hence the local leaf needs to be defined with certain care. One way is to consider the lift of U and define the lift of local leaf $U(x,R)$ as connected component of x of the intersection $U(x) \cap B(x,R)$. We prove Lemma 6.6 by induction.

Induction base.

We will be working on m-dimensional leaves of W^f_m. By Lemma 6.3 U subfoliate W^f_m. In other words for any $x \in \mathbb{T}^d$ $U(x) \subset W^f_m(x)$.

Induction step.

Suppose that U subfoliate $W^f(i,m)$ for some $i < m$. Then U subfoliate $W^f(i+1,m)$.

By induction we get that U subfoliate $W^f(m,m) = V^f_m$. Hence $U = V_m$.

First let us prove several auxiliary claims. Note that all foliations that we are dealing with are oriented and the orientation is preserved under the dynamics. Denote by d^f_j and d^g_j the induced distances on the leaves of V^f_j and V^g_j correspondingly, $j = 1, \dots k$.

Lemma 6.8. *Consider a point $a \in \mathbb{T}^d$. Pick a point $b \in U(a)$ and let $\tilde{b} = V^f_i(b) \cap W^f(i+1,m)(a)$. Assume that $\tilde{b} \neq b$. Pick a point $c \in V^f_i(a)$ and let $d = U(c) \cap W^f(i,m-1)(b)$, $\tilde{d} = V^f_i(d) \cap W^f(i+1,m)(c)$. Then $\tilde{d} \neq d$ and the orientations of the pairs $(b,\tilde{b})$ and $(d,\tilde{d})$ in V^f_i are the same.*

The statement of the lemma when $i = 1$ and $m = 3$ is illustrated on Figure 5.

Remark. Since by the induction hypothesis $h(W^f(i,m-1)) = W^g(i,m-1)$ we see that the leaf $U(a)$ intersects each leaf $W^f(i,m-1)(x)$, $x \in W^f(i,m)(a)$ exactly once.

Proof. Let $e = V^f_i(b) \cap W^f(i+1,m)(d)$ and $\tilde{e} = V^f_i(b) \cap W^f(i+1,m)(\tilde{d})$. Obviously $(e,\tilde{e})$ has the same orientation as $(d,\tilde{d})$ and also has advantage of lying on the leaf $V^f_i(b)$. Therefore we forget about $(d,\tilde{d})$ and work with $(e,\tilde{e})$.

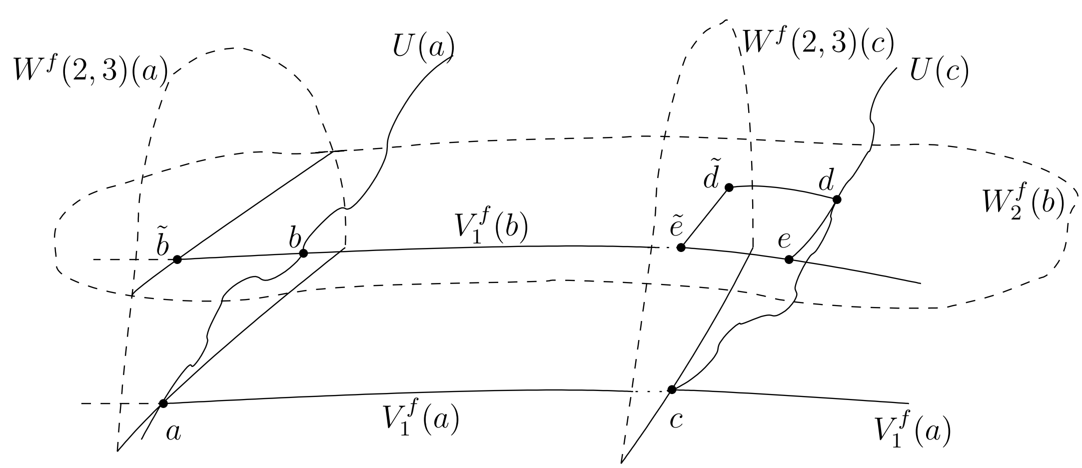

FIGURE 5. Illustration to Lemma 6.8 when $i = 1$ and $m = 3$.

We use affine structure on the expanding foliation V^f_i. Namely we work with affine distance-like function $\tilde{d}_i$. We refer to [GG08] for the definition. There we define affine distance-like function on weak unstable foliation. The definition for foliation V_i^f is the same with obvious modifications. Recall crucial properties of $\tilde{d}_i$

(D1) $\tilde{d}_i(x,y) = d_i^f(x,y) + o(d_i^f(x,y))$,
(D2) $\tilde{d}_i(f(x), f(y)) = D_f^i(x)\tilde{d}_i(x,y)$, where D_f^i is the derivative of f along V_i^f.
(D3) $\forall K > 0\ \exists C > 0$ such that

$$\frac{1}{C}\tilde{d}_i(x,y) \le d_i^f(x,y) \le C\tilde{d}_i(x,y)$$

whenever $d_i(x,y) < K$.

Assume that $(e,\tilde{e})$ has orientation opposite to $(b,\tilde{b})$ or $e = \tilde{e}$. For the sake of concreteness we assume that these points lie on $V_i^f(b)$ in the order $b, \tilde{b}, \tilde{e}, e$. All other cases can be treated similarly. Then

$$\tilde{d}_i(b,e) \ge \tilde{d}_i(b,\tilde{e}) > \tilde{d}_i(b,\tilde{e}) - \tilde{d}_i(b,\tilde{b}).$$

Remark. Notice that $\tilde{d}_i(b,\tilde{e}) - \tilde{d}_i(b,\tilde{b}) \ne \tilde{d}_i(\tilde{b},\tilde{e})$ since $\tilde{d}_i$ is neither symmetric nor additive. Distance $\tilde{d}_i$ is given by an integral of a certain density with normalization defined by the first argument. As long as the first argument (point b in the above inequality) is the same all natural inequalities hold.

Applying (D2) we get that

$$\forall n > 0 \quad \frac{\tilde{d}_i(f^{-n}(b), f^{-n}(e))}{\tilde{d}_i(f^{-n}(b), f^{-n}(\tilde{e})) - \tilde{d}_i(f^{-n}(b), f^{-n}(\tilde{b}))} = c_1 > 1$$

where c_1 does not depend on n. By property (D1) we can switch to the usual distance

$$\exists N :\ \forall n > N \quad \frac{d_i^f(f^{-n}(b), f^{-n}(e))}{d_i^f(f^{-n}(\tilde{b}), f^{-n}(\tilde{e}))} > c_2 > 1 \tag{35}$$

where c_2 does not depend on n.

Under the action of f^{-1} strong unstable leaves of $W^f(i+1,m)$ contract exponentially faster then weak unstable leaves of V_i^f. Thus we get that

$$\forall \varepsilon > 0\ \exists N :\ \forall n > N \quad \left| \frac{d_i^f(f^{-n}(a), f^{-n}(c))}{d_i^f(f^{-n}(\tilde{b}), f^{-n}(\tilde{e}))} - 1 \right| < \varepsilon. \tag{36}$$

Point $h(e) \in W^g(i+1,m)(h(c))$. Indeed, notice that $e = V_i^f(b) \cap W^f(i+1,m)(d) = V_i^f(b) \cap W^f(i+1,m-1)(d)$ (if $i = m-1$ than we have $e = d$). Thus

$$\begin{aligned} h(e) = h(V_i^f(b) \cap W^f(i+1,m-1)(d)) &= V_i^g(h(b)) \cap W^g(i+1,m-1)(h(d)) \\ = V_i^g(h(b)) \cap W^g(i+1,m)(h(d)) &= V_i^g(h(b)) \cap W^g(i+1,m)(h(c)), \end{aligned}$$

where the last equality is justified by the fact that $h(d) \in V_m^g(h(c))$. We know also that $h(b) \in W^g(i+1,m)(h(a))$. Hence, analogously to (36), we have

$$\forall \varepsilon > 0\ \exists N :\ \forall n > N \quad \left| \frac{d_i^g(g^{-n}(h(a)), g^{-n}(h(c)))}{d_i^g(g^{-n}(h(b)), g^{-n}(h(e)))} - 1 \right| < \varepsilon. \tag{37}$$

On the other hand, we know that h is continuously differentiable along V_i^f. Hence

$$\forall\varepsilon>0\ \exists N:\ \forall n>N \qquad \left|\frac{d_i^g\,(g^{-n}(h(a)),g^{-n}(h(c)))}{d_i^f(f^{-n}(a),f^{-n}(c))}-D_h^i(f^{-n}(a))\right|<\varepsilon$$
$$\text{and}\ \left|\frac{d_i^g\,(g^{-n}(h(b)),g^{-n}(h(e)))}{d_i^g(f^{-n}(b),f^{-n}(e))}-D_h^i(f^{-n}(a))\right|<\varepsilon. \quad (38)$$

Therefore from (37) and (38) we have

$$\forall\varepsilon>0\ \exists N:\ \forall n>N \quad \left|\frac{d_i^f(f^{-n}(a),f^{-n}(c))}{d_i^f(f^{-n}(b),f^{-n}(e))}-1\right|<\varepsilon,$$

which we combine with (36) to get

$$\forall\varepsilon>0\ \exists N:\ \forall n>N \quad \left|\frac{d_i^f(f^{-n}(b),f^{-n}(e))}{d_i^f(f^{-n}(\tilde b),f^{-n}(\tilde e))}-1\right|<\varepsilon.$$

We have reached a contradiction with (35) □

Remark. By the same argument one can prove that if $b=\tilde b$ then $d=\tilde d$.

Lemma 6.9. *Consider a weak unstable leaf $W^f_{m-1}(a)$ and $b\in V^f_m(a)$, $b\neq a$. For any $y\in W^f_{m-1}(a)$ let $y'=W^f_{m-1}(b)\cap V^f_m(y)$. Then $\exists c_1,c_2>0$ such that $\forall y\in W^f_{m-1}(a) \quad c_1>d^f_m(y,y')>c_2$.*

Proof. We will be working on the universal cover $\mathbb{R}^d$. We abuse the notation slightly by using the same notation for the lifted objects. Note that the leaves on $\mathbb{R}^d$ are connected components of preimages by the projection map of the leaves on $\mathbb{T}^d$.

Let h_f be the conjugacy with the linear model, $h_f\circ f=L\circ h_f$. Lemma 6.3 holds for h_f: $h_f(W^f_{m-1})=W^L_{m-1}$. Leaves $W^L_{m-1}(h_f(a))$ and $W^L_{m-1}(h_f(b))$ are parallel hyperplanes. Thus the lower bound follows from the uniform continuity of h_f.

It follows from (34) that $h_f^{-1}-Id$ is bounded. Hence we can find positive R that depends only on size of $\mathcal{U}$ such that

$$W^f_{m-1}(a)\subset Tube_a \overset{\text{def}}{=} \cup_{x\in B(a,R)}W^L_{m-1}(x)$$

and

$$W^f_{m-1}(b)\subset Tube_b \overset{\text{def}}{=} \cup_{x\in B(b,R)}W^L_{m-1}(x).$$

Then, obviously,

$$d^f_m(y,y')\le\sup\{d^f_m(x,x')\mid x\in Tube_a, x'\in Tube_b\cap V^f_m(x)\}.$$

Assumption (29) guarantees that E^f_m is uniformly transversal to $TW^L_{m-1}=E^L_1\oplus E^L_2\oplus\ldots\oplus E^L_{m-1}$. Thus the supremum above is finite. □

Remark. Given two points $a,b\in\mathbb{R}^d$ let $\hat d(a,b)=distance(W^L_{m-1}(h_f(a)),W^L_{m-1}(h_f(b)))$. It is clear from the proof that constants c_1 and c_2 can be chosen in such a way that they depend only on $\hat d(a,b)$.

Remark. In the proof above we do not use the fact that both W^f_{m-1} and V^f_m are expanding. We only need them to be transversal. Thus, if we substitute weak unstable foliation W^f_{m-1} by some weak stable foliation $\mathcal{F}$, the statement still holds.

Remark. As mentioned earlier the assumption (29) is crucial only for Lemmas 6.1, 6.2 and 6.3. We used this assumption in the proof above only for convenience. Slightly more delicate argument goes through without using assumption (29).

Proof of the induction step. We will be working inside of the leaves of $W^f(i,m)$. Assume that U does not subfoliate $W^f(i+1,m)$. Then there exists a point x_0 and $x_1 \in U(x_0)$ close to x_0 such that $x_1 \notin W^f(i+1,m)(x_0)$.

We fix orientation $\mathcal{O}$ of U and V_i^f that is defined on pairs of points (x,y), $y \in U(x)$ and (x,y), $y \in V_i^f(x)$. Although we denote these orientations by the same symbol it will not cause any confusion since U and V_i^f are topologically transverse.

For every (x,y), $y \in U(x)$ with $\mathcal{O}(x,y) = \mathcal{O}(x_0,x_1)$, define $[x,y] = W^f(i+1,m)(x) \cap V_i^f(y)$. For instance in Lemma 6.8 $\tilde{b} = [a,b]$, $\tilde{d} = [c,d]$.

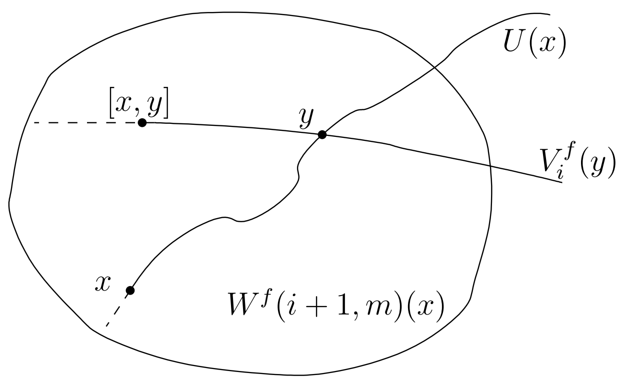

FIGURE 6. Definition of $[x,y]$.

Lemma 6.10. *For every (x,y) as above either $[x,y] = y$ or $\mathcal{O}([x,y],y) = O^+ \stackrel{\text{def}}{=} \mathcal{O}([x_0,x_1],x_1)$.*

Proof. Let $a_0 = \hat{d}(x_0,x_1)$ (for definition of $\hat{d}$ see the remark after the proof of Lemma 6.9). Number a_0 is positive since $U(x)$ is transverse to W^f_{m-1}.

For any $y \in \mathbb{T}^d$ there is a unique point $sh(y) \in U(y)$ such that $\hat{d}(y, sh(y)) = a_0$ and $\mathcal{O}(y, sh(y)) = \mathcal{O}(x_0,x_1)$.

The leaves of all foliations that we consider depend continuously on the point. Therefore we can find a small ball B centered at x_0 such that $\forall y \in B$ $[y, sh(y)] \neq sh(y)$ and $\mathcal{O}([y, sh(y)], sh(y)) = O^+$.

Next, let us fix $y \in B$ and choose any $z \in V_i^f(y)$. Apply Lemma 6.8 for $a = y$, $b = sh(y)$, $c = z$, $d = sh(z)$ to get that $[z, sh(z)] \neq sh(z)$ and $\mathcal{O}([z, sh(z)], sh(z)) = \mathcal{O}([y, sh(y)], sh(y)) = O^+$ as shown on the Figure 7.

By Property $\mathcal{A}$

$$\bigcup_{y \in B} V_i^f = \mathbb{T}^d.$$

Thus

$$\forall z \in \mathbb{T}^d \quad [z, sh(z)] \neq sh(z) \text{ and } \mathcal{O}([z, sh(z)], sh(z)) = O^+. \tag{39}$$

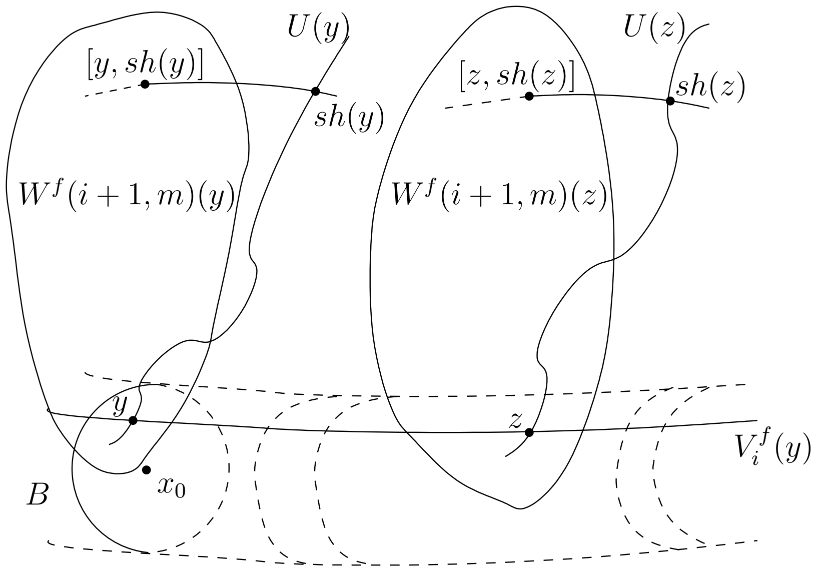

FIGURE 7. Orientation of $([z,sh(z)],sh(z))$ is positive for any z in the V_i^f-tube through the ball B. Foliation $W^f(i+1,m)$ is two dimensional on the picture.

Now let us assume contrary to the statement of the lemma. Namely, assume that there exists $\tilde{x}_0$ and $\tilde{x}_1$, $\tilde{x}_1 \in U(\tilde{x}_0)$, $\mathcal{O}(\tilde{x}_0,\tilde{x}_1) = \mathcal{O}(x_0,x_1)$, such that $[\tilde{x}_0,\tilde{x}_1] \neq \tilde{x}_1$ and $\mathcal{O}([\tilde{x}_0,\tilde{x}_1],\tilde{x}_1) \overset{\text{def}}{=} O^- \neq O^+$. By tinkering $\tilde{x}_1$ infinitesimally along $U(\tilde{x}_0)$ we can ensure that $N_1 a_0 = N_2 \hat{d}(\tilde{x}_0,\tilde{x}_1)$, where N_1 and N_2 are some large integer numbers.

For any $y \in \mathbb{T}^d$ there is a unique point $\tilde{sh}(y) \in U(y)$ such that $\hat{d}(y,\tilde{sh}(y)) = \hat{d}(\tilde{x}_0,\tilde{x}_1)$ and $\mathcal{O}(y,\tilde{sh}(y)) = \mathcal{O}(\tilde{x}_0,\tilde{x}_1)$. Then by the same argument we show an analogue of (39):

$$\forall z \in \mathbb{T}^d \quad [z,\tilde{sh}(z)] \neq \tilde{sh}(z) \ \text{ and } \ \mathcal{O}([z,\tilde{sh}(z)],\tilde{sh}(z)) = O^-. \tag{40}$$

Pick a point $x \in \mathbb{T}^d$ and $y,z \in U(x)$, $\mathcal{O}(x,y) = \mathcal{O}(y,z)$. Assume that $\mathcal{O}([x,y],y) = \mathcal{O}([y,z],z)$. Then $\mathcal{O}([x,z],z) = \mathcal{O}([x,y],y)$. This obvious property allows us to "iterate" sh and $\tilde{sh}$.

Choose any z and "iterate" (39) and (40) N_1 and N_2 times correspondingly as shown on the Figure 8.

We get that

$$\mathcal{O}([z,sh^{N_1}(z)],sh^{N_1}(z)) = O^+ \ \text{ and } \ \mathcal{O}([z,\tilde{sh}^{N_2}(z)],\tilde{sh}^{N_2}(z)) = O^-.$$

To get a contradiction it remains to notice that $sh^{N_1} = \tilde{sh}^{N_2}$. Hence the lemma is proved. □

From (39) we see that for any $z \in \mathbb{T}^d$ $d_i^f([z,sh(z)],sh(z)) > 0$. Hence, due to compactness and continuity of function $d_i^f([\cdot,sh(\cdot)],sh(\cdot))$, we have $\delta < d_i^f([z,sh(z)],sh(z)) <$

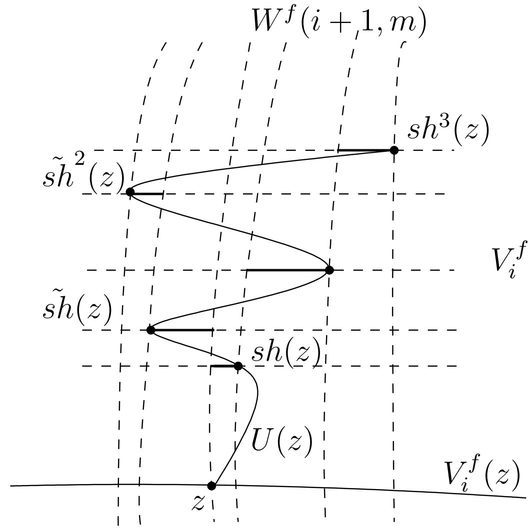

FIGURE 8. Illustration to the argument with shifts along $U(z)$. Foliation $W^f(i+1,m)$ is one dimensional here, $N_1=3$, $N_2=2$. Black segments of V_i^f carry known information about orientation of $([\cdot, sh(\cdot)], sh(\cdot))$ and $([\cdot, \tilde{sh}(\cdot)], \tilde{sh}(\cdot))$. This picture is clearly impossible if $sh^{N_1}=\tilde{sh}^{N_2}$.

Δ for some positive δ and Δ. Lemma 6.10 guarantees even more,

$$\forall x\in\mathbb{T}^d \text{ and } y\in U(x), \mathcal{O}(x,y)=\mathcal{O}(x_0,x_1), \text{ such that } \hat{d}(x,y)\le a_0 \quad \text{we have } d_i^f([x,y],y)<\Delta. \quad (41)$$

From now on it is more convenient to work on the universal cover. Although formally we do not have to do it since we are working inside of the leaves of $W^f(i,m)$ which are isometric to their lifts.

Let $x_n=sh^n(x_0)$, $n>0$. For every $n\ge 0$ $\mathcal{O}([x_n,x_{n+1}],x_{n+1})=O^+$ and $d_i^f([x_n,x_{n+1}],x_{n+1})>\delta$. Lemma 6.10 also tells us that U is monotone with respect to $W^f(i+1,m)$. Namely, for any $x\in\mathbb{T}^d$ the intersection $U(x)\cap W^f(i+1,m)(x)$ is a connected piece of $U(x)$.

Denote by $\overline{x_n,x_{n+1}}$ the piece of $U(x_0)$ that lies between x_n and x_{n+1}. We know that for any $n\ge 0$ $\overline{x_n,x_{n+1}}$ is confined between $W^f(i,m-1)(x_n)$ and $W^f(i,m-1)(x_{n+1})$. Lemma 6.10 guarantees that $\overline{x_n,x_{n+1}}$ is also confined between $W^f(i+1,m)(x_n)$ and $W^f(i+1,m)(x_{n+1})$ as shown on Figure 9. Thus, it makes sense to measure two different "dimensions" of $\overline{x_n,x_{n+1}}$. Namely, let $a_n=\hat{d}(x_n,x_{n+1})$ and $b_n=d_i^f([x_n,x_{n+1}],x_{n+1})$. As we have remarked earlier $b_n>\delta>0$ and $a_n=a_0$ by the definition of $\hat{d}$ and sh.

This "dimensions" behave nicely under the dynamics. Namely,

$$\forall N>0\ \ (f_*)^{-N}(b_n)\stackrel{\text{def}}{=}d_i^f([f^{-N}(x_n),f^{-N}(x_{n+1})],f^{-N}(x_{n+1}))\ge\delta\beta_i^{-N}$$

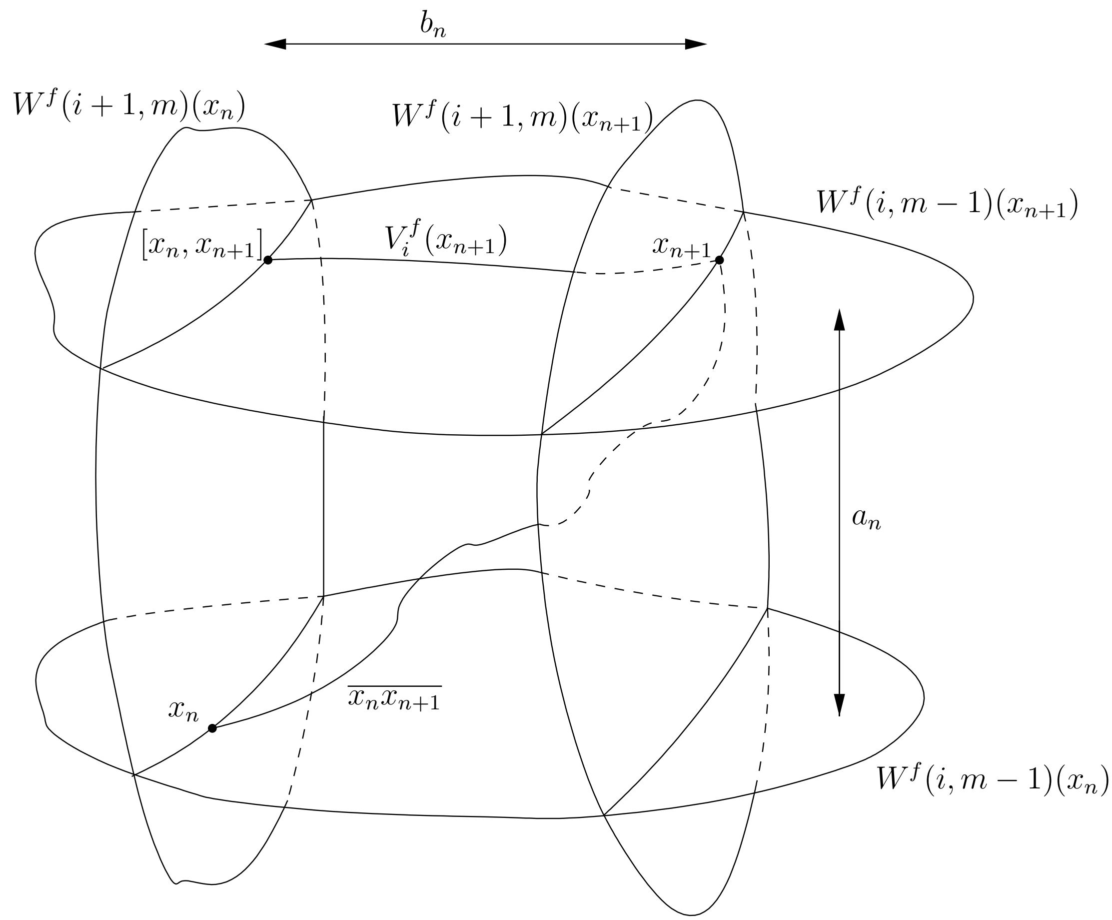

FIGURE 9. Piece $\overline{x_n x_{n+1}}$ is "monotone" with respect to foliation $W^f(i,m-1)$. By Lemma 6.10 $\overline{x_n x_{n+1}}$ is also "monotone" with respect to $W^f(i+1,m)$: the intersections of $\overline{x_n x_{n+1}}$ with local leaves of $W^f(i+1,m)$ are points or connected components of $\overline{x_n x_{n+1}}$. On this picture foliations $W^f(i,m-1)$ and $W^f(i+1,m)$ are two dimensional.

and

$$\forall N > 0 \ \ (f_*)^{-N}(a_n) \stackrel{\text{def}}{=} \hat{d}(f^{-N}(x_n), f^{-N}(x_{n+1})) = a_0 \lambda_m^{-N}.$$

Recall that $\lambda_m > \beta_i$.

The idea now is to show that the leaf $U(f^{-N}(x_0))$ is "too close" to $W^f(i,m-1)(x_0)$ for N large, which would lead to a contradiction.

Take N large and let $M = \lfloor \lambda_m^N \rfloor$. Then

$$\hat{d}(f^{-N}(x_0), f^{-N}(x_M)) = \sum_{j=0}^{M-1} \hat{d}(f^{-N}(x_j), f^{-N}(x_{j+1}))$$

$$= \sum_{j=0}^{M-1} (f_*)^{-N}(a_j) = Ma_0\lambda_m^{-N} \leq a_0. \quad (42)$$

The first equality holds since the holonomy along $W^f(i, m-1)$ is isometric with respect to $\hat{d}$.

To estimate $d_i^f([f^{-N}(x_0), f^{-N}(x_M)], f^{-N}(x_M))$ in the similar way we need to have control over holonomies along $W^f(i+1, m)$.

Fix two small one dimensional transversals $T(x) \subset V_i^f(x)$ and $T(y) \subset V_i^f(y)$, $y \in U(x)$ with $\hat{d}(x, y) \leq a_0$. This condition ensures that the distance between x and y along $W^f(i, m)(x)$ is uniformly bounded from above. To see this we only need to bound the distance between $h(x)$ and $h(y)$ along $W^g(i, m)(h(x))$. This, in turn, is a direct consequence of Lemma 6.9 applied for g since $h(y) \in V_m^g(h(x))$.

Consider holonomy map along $W^f(i+1, m)$ $H : T(x) \to T(y)$. This holonomy can be viewed as holonomy along $W^f(i+1, k)$. Recall that $W^f(i+1, k)$ is the fast unstable foliation. Since f is at least C^2-differentiable $W^f(i+1, k)$ is Lipschitz inside of $W^f(i, k)$. Moreover, since the distance between x and y is bounded from above, the Lipschitz constant C_{Hol} of H is uniform in x and y. For proof see [LY85], Section 4.2. They proof that the unstable foliation is Lipschitz within center-unstable leaves but the proof goes through for $W^f(i+1, k)$ within the leaves of $W^f(i, k)$.

Let $\tilde{x}_j = W^f(i+1, m)(f^{-N}(x_j)) \cap V_i^f(f^{-N}(x_M))$, $j = 1, \dots M$. Then

$$d_i^f([f^{-N}(x_0), f^{-N}(x_M)], f^{-N}(x_M)) = \sum_{j=0}^{M-1} d_i^f(\tilde{x}_j, \tilde{x}_{j+1})$$

$$\geq C_{Hol} \sum_{j=0}^{M-1} d_i^f([f^{-N}(x_j), f^{-N}(x_{j+1})], f^{-N}(x_{j+1})) = C_{Hol} \sum_{j=0}^{M-1} (f_*)^{-N}(b_j)$$

$$\geq C_{Hol} M\delta\beta_i^{-N}.$$

The holonomy constant is uniform since

$$\hat{d}(f^{-N}(x_j), \tilde{x}_j) \leq \hat{d}(f^{-N}(x_0), \tilde{x}_j) = \hat{d}(f^{-N}(x_0), f^{-N}(x_M)) \leq a_0$$

by (42).

Notice that $C_{Hol} M\delta\beta_i^{-N}$ can be arbitrarily big when $N \to \infty$, while $d(f^{-N}(x_0), f^{-N}(x_M)) \leq a_0$ which contradicts to (41). Hence the induction step is established. □

6.5. **Induction step 2: proof of Lemma 6.7 by transitive point argument.** The proof of Lemma 6.7 is carried out in a way similar to the proofs of Lemmas 4 and 5 from [GG08]. Here we overview the scheme and deal with complications that arise due to higher dimension.

First using the assumption on p. d. we argue that h is uniformly Lipschitz along V_m^f, i. e., for any point x the restriction $h\big|_{V_m^f(x)} : V_m^f(x) \to V_m^g(x)$ is a Lipschitz map with a Lipschitz constant that does not depend on x. At this step the assumption on p. d. along V_m^f is used.

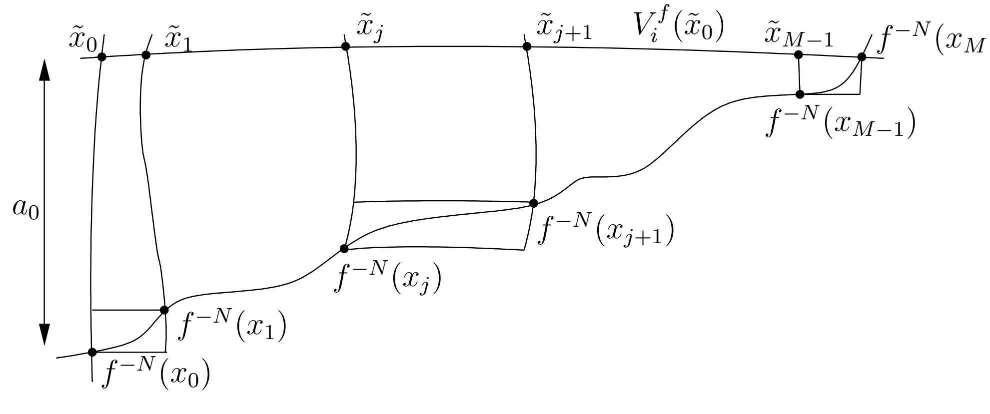

FIGURE 10. Small rectangles along leaf $U(f^{-N}(x_0))$ are very "flat" according to the estimates on $(f_*)^{-N}(b_n)$ and $(f_*)^{-N}(a_n)$. Together with Lipschitz property of foliation $W^f(i+1,m)$ this provides an estimate from below on the horizontal size $d_i^f(\tilde{x}_0, f^{-N}(x_M))$.

Lipschitz property implies differentiability at almost every point with respect to the Lebesgue measure on the leaves of V_m^f. The second step is to show that differentiability of h along V_m^f at a transitive point x implies $C^{1+\nu}$-differentiability along V_m^f. This is done by a direct approximation argument (see Step 1 in Section 4.3 in [GG08]). Transitive point x "spreads differentiability" all over the torus.

Last but not the least, we need to find such a transitive point x. For that we would like to find an ergodic measure μ with full support such that the foliation V_m^f is absolutely continuous with respect to μ. Then by the Birkhoff ergodic theorem almost every point is transitive. And since V_m^f is absolutely continuous we would have that almost every point with respect to the Lebesgue measure on the leaves is transitive. Hence we would find a full measure set of points that we are looking for.

Unfortunately we cannot carry out the scenario described above. The problem is that the foliation V_m^f is not absolutely continuous with respect to natural ergodic measures (see [GG08] for detailed discussion and [SX08] for in-depth analysis of this phenomenon). Instead we construct a measure μ such that almost every point is transitive and V_m^f is absolutely continuous with respect to μ. This is clearly sufficient.

The construction follows the lines of Pesin-Sinai [PS83] construction of u-Gibbs measures. Given a partially hyperbolic diffeomorphism they construct a measure such that the unstable foliation is absolutely continuous with respect to the measure. In fact this construction works well for any expanding foliation. We apply this construction to m-dimensional foliation W_m^f.

Construction is described as follows. Let x_0 be a fixed point of f. For any $y \in W_m^f(x_0)$ define

$$\rho(y) = \prod_{n \geq 0} \frac{J_m^f(f^{-n}(y))}{J_m^f(x_0)},$$

where $J^f_m = Jacobian(f\big|_{W^f_m})$.

Let $\mathcal{V}_0$ be an open bounded neighborhood of x_0 in $W^f_m(x_0)$. Consider a probability measure η_0 supported on $\mathcal{V}_0$ with density proportional to $\rho(\cdot)$. For $n>0$ define

$$\mathcal{V}_n = f^n(\mathcal{V}_0),\ \eta_n = (f^n)_*\eta_0.$$

Let

$$\mu_n = \frac{1}{n}\sum_{i=0}^{n-1}\eta_i.$$

By the Krylov-Bogoljubov theorem $\{\mu_n; n\geq 0\}$ is weakly compact and any of its limits is f-invariant. Let μ be an accumulation point of $\{\mu_n; n\geq 0\}$. This is the measure that we are looking for.

Foliation W^f_m is absolutely continuous with respect to μ. We refer to [PS83] or [GG08] for the proof. Proof of [GG08] requires some minimal modifications that are due to higher dimension of W^f_m.

Since foliation W^f_m is conjugate to the linear foliation W^L_m we have that for any open ball B

$$\exists R>0 \quad \bigcup_{y\in B} W^f_m(y,R) = \mathbb{T}^d,$$

where $W^f_m(y,R)$ is a ball of radius R inside of the leaf $W^f_m(y)$. Together with absolute continuity this guarantees that μ almost every point is transitive. See [GG08], Section 4.3, Step 3 for the proof. We stress that we do not need to know that μ has full support in that argument.

It is left to show that the conjugacy h is $C^{1+\nu}$-differentiable in the direction of V^f_m at μ almost every point. For this we need to argue that V^f_m is absolutely continuous with respect to μ.

Foliation $W^f(m,k)$ is Lipschitz inside of a leaf of W^f (again we refer to [LY85], Section 4.2). Hence $V^f_m = W^f(m,k)\cap W^f_m$ is Lipschitz inside of a leaf of $W^f_m = W^f\cap W^f_m$. So we have that V^f_m is absolutely continuous with respect to the Lebesgue measure on a leaf of W^f_m while W^f_m is absolutely continuous with respect to μ. Therefore V^f_m is absolutely continuous with respect to μ.

6.6. **Induction step 1 revisited.** To carry out proof of Lemma 6.6 assuming Property $\mathcal{A}$ only we shrink neighborhood $\mathcal{U}$ even more. In addition to (29) and (30) we require $f\in\mathcal{U}$ to have narrow spectrum. Namely,

$$\forall m,\ 1<m\leq k \quad \frac{\log\tilde\beta_m}{\log\beta_m} > \frac{\log\beta_{m-1}}{\log\tilde\beta_m}$$

and the analogous condition on the contraction rates α_j, $\tilde\alpha_j$. The following condition that we will actually use is obviously a consequence of the above one.

$$\forall i<k \quad \text{and} \quad \forall m,\ i<m\leq k \quad \rho \stackrel{\text{def}}{=} \frac{\log\tilde\beta_m}{\log\beta_m} > \frac{\log\beta_i}{\log\tilde\beta_m}. \tag{43}$$

This inequality can be achieved by shrinking the size of $\mathcal{U}$ since β_j and $\tilde\beta_j$ get arbitrarily close to λ_j, $j=1,\dots k$.

Remark. Condition (43) greatly simplifies the proof of Lemma 6.6. We have yet another longer proof (but based on the same idea) of Lemma 6.6 that works for any f with Property $\mathcal{A}$ in $\mathcal{U}$ as defined in Section 6.1. It will not appear here.

We start the proof as in Section 6.4. The first place where we use Property $\mathcal{A}'$ is the proof of Lemma 6.10. So we reprove induction step 1 with Property $\mathcal{A}$ only assuming that we have got everything that preceded Lemma 6.10. With Property $\mathcal{A}$ the proof of Lemma 6.10 still goes through. Although instead of (39) we get

$$\forall z \in \mathbb{T}^d \quad \text{either} \quad [z, sh(z)] = sh(z) \ \text{ or } \ \mathcal{O}([z, sh(z)], sh(z)) = O^+.$$

Thus we still have Lemma 6.10 and the upper bound (41) but not the lower bound $d_i^f([z, sh(z)], sh(z)) > \delta$. This is the reason why we cannot proceed with the proof of the induction step as at the end of Section 6.4.

Proof of the induction step. As before we need to show that U subfoliate $W^f(i+1, m)$.

Fix orientation $\mathcal{O}$ on V_m^f and V_i^f. Given $x \in \mathbb{T}^d$ and $\varepsilon > 0$ choose $\bar{x} \in V_m^f(x)$ such that $d_m^f(x, \bar{x}) = \varepsilon$ and $\mathcal{O}(x, \bar{x}) = O^+$. Let $\bar{y} = U(x) \cap W^f(i, m-1)(\bar{x})$ and $y = V_i^f(x) \cap W^f(i+1, m)(\bar{y})$. This way we define an ε-"rectangle" $\mathcal{R} = \mathcal{R}(x, \bar{x}, y, \bar{y})$ with the base point x, vertical size $d_m^f(x, \bar{x}) = \varepsilon$ and horizontal size $d_i^f(x, y) = \bar{\varepsilon}$.

Remark. Notice that we measure vertical size in a way different from one in 6.4.

It is clear that "rectangle" is uniquely defined by its "diagonal" $(x, \bar{y})$ (Figure 9 is the picture of "rectangle" with diagonal (x_n, x_{n+1}). Sometimes we will use notation $\mathcal{R}(x, \bar{y})$. Note that by Lemma 6.10 $\mathcal{O}(x, y)$ does not depend on x and ε. Also it guarantees that the piece of $U(x)$ between x and $\bar{y}$ lies "inside" of $\mathcal{R}(x, \bar{y})$. The horizontal size $\bar{\varepsilon}$ might happen to be equal to zero.

Next we define a set of base points X_ε such that $U(x)$, $x \in X_\varepsilon$, has big Hölder slope inside of corresponding ε-rectangle.

$$X_\varepsilon = \{x \in \mathbb{T}^d : \ \bar{\varepsilon} \le \varepsilon^\delta\}$$

with some δ satisfying inequality $\rho > \delta > \log \beta_i / \log \tilde{\beta}_m$.

Let μ be the measure constructed in Section 6.5. Recall that μ almost every point is transitive. Foliation $W^f(i, m)$ is absolutely continuous with respect to μ. The latter can be shown in the same way as absolute continuity of V_m^f is shown in Section 6.5.

We consider two cases.

Case 1. $\overline{\lim\limits_{\varepsilon \to 0}} \, \mu(X_\varepsilon) > 0$.

Then choose $\{X_{\varepsilon_n}, n \ge 1\}$, $\varepsilon_n \to 0$ as $n \to \infty$, such that $\overline{\lim}_{n\to\infty} \mu(X_{\varepsilon_n}) > 0$.

The idea now is to iterate a rectangle with base point in X_{ε_n} and vertical size ε_n until the vertical size is approximately 1. Since the Hölder slope of initial rectangle was big it will turn out that the horizontal size of the iterated rectangle is extremely small. This argument will show that for a set of base points of positive measure the horizontal size of rectangles is equal to zero. Hence the leaves of U lie inside of the leaves of $W^f(i+1, m)$.

Given n let $N = N(n)$ be the largest number such that $\frac{1}{C}\tilde{\beta}_m^N \varepsilon_n < 1$ (constant C here is from definition (30)). Take $x \in X_{\varepsilon_n}$ and corresponding ε_n-rectangle $\mathcal{R}(x, y, \bar{x}, \bar{y})$ and consider its image $\mathcal{R}(f^N(x), f^N(y), f^N(\bar{x}), f^N(\bar{y}))$. Choice of N provides lower bound on the vertical size

$$VS\left(\mathcal{R}(f^N(x), f^N(y), f^N(\bar{x}), f^N(\bar{y}))\right) = d_m^f(f^N(x), f^N(\bar{x})) \ge \frac{1}{\beta_m}.$$

While the horizontal size can be estimated as follows

$$HS\left(\mathcal{R}(f^N(x), f^N(y), f^N(\bar{x}), f^N(\bar{y}))\right) = d_i^f(f^N(x), f^N(y))$$
$$\leq C\beta_i^N \bar{\varepsilon} \leq C\beta_i^N \varepsilon^\delta \leq C\beta_i^N \left(\frac{C}{\tilde{\beta}_m^N}\right)^\delta = C^{1+\delta}\left(\frac{\beta_i}{\tilde{\beta}_m^\delta}\right)^N.$$

Instead of looking at rectangle $\mathcal{R}(f^N(x), f^N(y), f^N(\bar{x}), f^N(\bar{y}))$ let us look at the rectangle $\tilde{\mathcal{R}}(f^N(x))$ with base point $f^N(x)$ and fixed vertical size $1/\beta_m$. Lemma 6.10 together with the estimate above on the vertical size of $\mathcal{R}(f^N(x), f^N(y), f^N(\bar{x}), f^N(\bar{y}))$ guarantees that horizontal size of $\tilde{\mathcal{R}}(f^N(x))$ is less than $C^{1+\delta}\left(\beta_i/\tilde{\beta}_m^\delta\right)^N$ as well.

Thus for every $x \in f^N(X_{\varepsilon_n})$ the horizontal size of $\tilde{\mathcal{R}}(x) = \tilde{\mathcal{R}}(x, z, \tilde{x}, \tilde{z})$ is less than $C^{1+\delta}\left(\beta_i/\tilde{\beta}_m^\delta\right)^N$. Note that $\left(\beta_i/\tilde{\beta}_m^\delta\right)^N \to 0$ as $n \to \infty$ since $\beta_i/\tilde{\beta}_m^\delta < 1$ and $N \to \infty$ as $n \to \infty$.

Let $X = \overline{\lim}_{n\to\infty} f^N(X_{\varepsilon_n})$. Since any $x \in X$ also belong to $f^N(X_{\varepsilon_n})$ with arbitrarily large N we conclude that $\tilde{\mathcal{R}}(x)$ has zero horizontal size i. e. $x = z$. Hence by Lemma 6.10 we conclude that the piece of $U(x)$ from x to $\tilde{z}$ lies inside of $W^f(i+1, m)(x)$.

It is a simple exercise in measure theory to show that

$$\mu(X) \geq \overline{\lim_{n\to\infty}}\, \mu(f^N(X_{\varepsilon_n})) = \overline{\lim_{n\to\infty}}\, \mu(X_{\varepsilon_n}) > 0.$$

Finally recall that μ almost every point is transitive ($\overline{\{f^j(x),\ j \geq 1\}} = \mathbb{T}^d$). Hence by taking a transitive point $x \in X$ and applying straightforward approximation argument we get that $\forall y\ \ U(y) \subset W^f(i+1, m)(y)$.

Case 2. $\overline{\lim_{\varepsilon\to 0}}\, \mu(X_\varepsilon) = 0$.

In this case the idea is to use the assumption above to find a leaf $U(x)$ which is "flat" i. e. arbitrarily close to $W^f(i, m-1)(x)$. Since the leaf $U(x)$ has to "feel" measure μ we need to take it together with a small neighborhood. Choice of this neighborhood is done by multiple application of pigeonhole principle.

Given a point $\bar{y} \in U(x)$ denote by $U_{x\bar{y}}$ the piece of $U(x)$ between x and $\bar{y}$. As before by $\mathcal{R}(x, \bar{y})$ we denote the rectangle spanned by x and $\bar{y}$. Recall that $HS(\mathcal{R}(x, \bar{y}))$ and $VS(\mathcal{R}(x, \bar{y}))$ stand for horizontal and vertical sizes of $\mathcal{R}(x, \bar{y})$. Also we will need to measure sizes of $U_{x\bar{y}}$. Simply let $HS(U_{x\bar{y}}) = HS(\mathcal{R}(x, \bar{y}))$ and $VS(U_{x\bar{y}}) = VS(\mathcal{R}(x, \bar{y}))$.

Iterating Pigeonhole Principle. Divide $\mathbb{T}^d$ into a finite number of tubes $\mathcal{T}_1, \mathcal{T}_2, \ldots \mathcal{T}_q$ foliated by U such that any connected component of $U(x) \cap \mathcal{T}_j$, $j = 1, \ldots q$, has vertical size between S_0 and S_1. Numbers S_0 and S_1 are fixed, $0 < S_0 < S_1$. We also require every tube $\mathcal{T}_j$ to be $W^f(i, m-1)$-foliated so that it can be represented as

$$\mathcal{T}_j = \bigcup_{y \in Transv} Plaque(y),$$

where $Transv$ is a plaque of U and $Plaque(y)$ are plaques of $W^f(i, m-1)$.

Given a small number $\tau > 0$ we can find an $\varepsilon > 0$ such that $\mu(X_\varepsilon) < \tau$. Then by the pigeonhole principle we can choose a tube $\mathcal{T}_j$ such that $\mu(\mathcal{T}_j) \neq 0$ and

$$\frac{\mu(\mathcal{T}_j \cap X_\varepsilon)}{\mu(\mathcal{T}_j)} < \tau.$$

Tube $\mathcal{T}_j$ can be represented as $\mathcal{T}_j = \bigcup_{z\in\hat{\mathcal{T}}_j} W(z)$, where $\hat{\mathcal{T}}_j$ is a transversal to $W^f(i,m)$ and $W(z)$, $z \in \hat{\mathcal{T}}_j$, are connected plaques of $W^f(i,m)$. By absolute continuity

$$\mu(\mathcal{T}_j) = \int\limits_{\hat{\mathcal{T}}_j} d\hat{\mu}(z) \int\limits_{W(z)} d\mu_{W(z)},$$

where $\hat{\mu}$ is the factor measure on $\hat{\mathcal{T}}_j$ and $\mu_{W(z)}$ is the conditional measure on $W(z)$.

Apply pigeonhole principle again to choose $W = W(z)$ such that

$$\mu_W(W \cap X_\varepsilon) < \tau.$$

Recall that $\mu_W(W) = 1$ by the definition of conditional measure and μ_W is equivalent to the induced Riemannian volume on W by absolute continuity of $W^f(i,m)$.

Plaque W is subfoliated by plaques of U of sizes between S_0 and S_1. Unfortunately we do not know if U is absolutely continuous with respect to μ_W. So we construct a finite partition of W into smaller plaques of $W^f(i,m)$ which are very thin U-foliated tubes.

To construct this partition we switch to $h(W)$ which is a plaque of $W^g(i,m)$ subfoliated by the plaques of $h(U) = V^g_m$. The partition $\{\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2, \dots \tilde{\mathcal{T}}_p\}$ will consist of V^g_m-tubes inside of $h(W)$ that can be represented as

$$\tilde{\mathcal{T}}_j = \bigcup_{z\in\hat{\mathcal{T}}_j} V(z), \ \ j = 1, \dots p,$$

where $\hat{\mathcal{T}}_j$ is a transversal to V^g_m inside of $h(W)$ and $V(z)$ are plaques of V^g_m. For every $j = 1, \dots p$ choose $z_j \in \hat{\mathcal{T}}$. Then the tube $\tilde{\mathcal{T}}_j$ can also be represented as

$$\tilde{\mathcal{T}}_j = \bigcup_{y\in V(z_j)} \tilde{P}_j(y),$$

where $\tilde{P}_j(y) \subset W^g(i, m-1)(y)$ are connected plaques.

Recall that V^g_m is Lipschitz inside of $W^g(i,m)$. Hence for any $\xi > 0$ it is possible to find a partition $\{\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2, \dots \tilde{\mathcal{T}}_p\}$, $p = p(\xi)$, such that

$$\forall j = 1, \dots p \ \ \forall y \in V(z_j) \ \ \ \exists B_j(\tilde{C}_1\xi), B_j(\tilde{C}_2\xi) \subset W^g(i, m-1)(y)$$
$$\text{such that} \quad B_j(\tilde{C}_1\xi) \subset \tilde{P}_j(y) \subset B_j(\tilde{C}_2\xi), \tag{44}$$

where $B_j(\tilde{C}_1\xi)$ and $B_j(\tilde{C}_2\xi)$ are balls inside of ($W^g(i, m-1)(y)$, induced Riemannian distance) of radii $\tilde{C}_1\xi$ and $\tilde{C}_2\xi$ respectively. Constants $\tilde{C}_1$ and $\tilde{C}_2$ are independent of ξ. Since we are working in a bounded plaque $h(W)$ they also do not depend on any other choices but S_1.

In the sequel we will need to take ξ to be much smaller than ε.

Now we pool this partition back into a partition of W.

$$\{\mathcal{T}_1, \mathcal{T}_2, \dots \mathcal{T}_p\} = \{h^{-1}(\tilde{\mathcal{T}}_1), h^{-1}(\tilde{\mathcal{T}}_2), \dots h^{-1}(\tilde{\mathcal{T}}_p)\}.$$

Although we use the same notation for this partition it is clearly different from the initial partition of $\mathbb{T}^d$.

Each tube $\mathcal{T}_j$ can be represented as

$$\mathcal{T}_j = \bigcup_{y\in U(h^{-1}(z_j))} P_j(y), \tag{45}$$

where $P_j(y) = h^{-1}(\tilde{P}_j(y)) \subset W^f(i, m-1)(y)$.

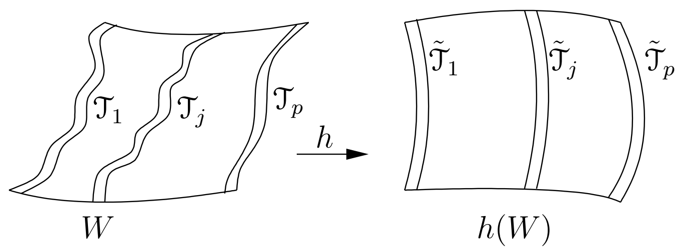

FIGURE 11. We construct partition $\{\mathcal{T}_1, \mathcal{T}_2, \dots \mathcal{T}_p\}$ as a pullback of partition of $h(W)$ by V_m^g-tubes. Foliation V_m^g is Lipschitz and h is continuously differentiable along $W^f(i, m-1)$. This guarantees that the "width" of a tube $\mathcal{T}_j$ is of the same order as we move along $\mathcal{T}_j$ (46). Hence μ_W is "uniformly distributed" along $\mathcal{T}_j$.

By Lemma 6.7 h is continuously differentiable along $W^f(i, m-1)$. Moreover, the derivative depend continuously on the point in W. Hence property (44) persists

$$\forall j = 1, \dots p \ \ \forall y \in U(h^{-1}(z_j)) \quad \exists B_j(C_1\xi), B_j(C_2\xi) \subset W^f(i, m-1)(y)$$
$$\text{such that} \quad B_j(C_1\xi) \subset P_j(y) \subset B_j(C_2\xi). \tag{46}$$

Constants C_1 and C_2 differ from $\tilde{C}_1$ and $\tilde{C}_2$ by a finite factor due to the bounded distortion along $W^f(i, m-1)$ by the differential of h.

Apply pigeonhole principle for the last time to find $\mathcal{T} \in \{\mathcal{T}_1, \mathcal{T}_2, \dots \mathcal{T}_p\}$ such that

$$\frac{\mu_W(\mathcal{T} \cap X_\varepsilon)}{\mu_W(\mathcal{T})} < \tau. \tag{47}$$

Take a plaque $U_{x\bar{y}}$ inside of $\mathcal{T}$. By the construction

$$S_0 < VS(U_{x\bar{y}}) < S_1.$$

Estimating horizontal size of $U_{x\bar{y}}$ from below. We have constructed $U_{x\bar{y}}$ so that a lot of points in the neighborhood of $U_{x\bar{y}}$ $\mathcal{T}$ lie outside of X_ε. Corresponding ε-rectangles $\mathcal{R}(x)$ have vertical size greater than ε^δ. It is clear that we can use this fact to show that $VS(U_{x\bar{y}})$ is large.

Choose a sequence $\{x_0 = x, x_1, \dots x_N\} \subset U_{x\bar{y}}$ such that

$$VS(\mathcal{R}(x_0, x_N)) \geq S_0 \quad \text{and} \quad VS(\mathcal{R}(x_j, x_{j+1})) = \varepsilon, \ j = 0, \dots N-1.$$

First we estimate the number of rectangles N.

Lemma 6.11. *The holonomy map $Hol : T(a) \to T(b)$, $b \in W^f(i, m)(a)$, $T(a) \subset V_m^f(a)$, $T(b) \subset V_m^f(b)$, along $W^f(i, m-1)$ is Hölder continuous with exponent*

$$\rho \overset{\text{def}}{=} \frac{\log \tilde{\beta}_m}{\log \beta_m}.$$

We postpone the proof until the end of the current section.

Let $\tilde{x}_j = W^f(i, m-1)(x_j) \cap V_m^f(x_0)$, $j = 0, \dots N$. Then according to the lemma above

$$d_m^f(\tilde{x}_{j-1}, \tilde{x}_j) \leq C_{Hol} VS(\mathcal{R}(x_{j-1}, x_j))^\rho = C_{Hol}\varepsilon^\rho, \quad j = 1, \dots N,$$

which allows to estimate N

$$S_0 \le VS(\mathcal{R}(x_0, x_N)) = \sum_{j=1}^{N} d_m^f(\tilde{x}_{j-1}, \tilde{x}_j) \le NC_{Hol}\varepsilon^\rho.$$

Hence

$$N \ge \frac{S_0}{C_{Hol}\varepsilon^\rho}. \tag{48}$$

Along with the rectangles $\mathcal{R}(x_j, x_{j+1})$ let us consider sets $A(x_j, x_{j+1}) \subset \mathcal{T}$, $j = 0, \dots N-1$ given by the formula

$$A(x_j, x_{j+1}) = \bigcup_{y \in U_{x_j x_{j+1}}} P(y),$$

where $P(y)$ are plaques of $W^f(i, m-1)$ from representation (45) for $\mathcal{T}$. Sets $A(x_j, x_{j+1})$ have the same vertical size. The following property of these sets is a direct consequence of (46) and the fact that μ_W is equivalent to the Riemannian volume on W.

$$\exists C_{univ} \quad \text{such that} \quad \forall j, \tilde{j} = 1, \dots N-1 \quad \frac{1}{C_{univ}} < \frac{\mu_W(A(x_j, x_{j+1}))}{\mu_W(A(x_{\tilde{j}}, x_{\tilde{j}+1}))} < C_{univ}. \tag{49}$$

Constant C_{univ} depends on C_1, C_2 and size of W, but independent of ε and ξ.

Let

$$A_1 = \bigcup_{\substack{j=1 \\ j \text{ is odd}}}^{N-1} A(x_j, x_{j+1}) \quad \text{and} \quad A_2 = \bigcup_{\substack{j=1 \\ j \text{ is even}}}^{N-1} A(x_j, x_{j+1})$$

It follows from (47) that we have that either

$$\frac{\mu_W(A_1 \cap X_\varepsilon)}{\mu_W(A_1)} < \tau \quad \text{or} \quad \frac{\mu_W(A_2 \cap X_\varepsilon)}{\mu_W(A_2)} < \tau.$$

For concreteness assume that the first holds.

Bounds (49) allow to estimate the number N_1 of sets $A(x_j, x_{j+1}) \subset A_1$ that have a point $q_j \in A(x_j, x_{j+1})$ such that $q_j \notin X_\varepsilon$.

$$N_1 \ge \left\lfloor \frac{N}{2} \right\rfloor - \lfloor C_{univ}\tau N \rfloor.$$

Here $\lfloor N/2 \rfloor$ is the total number of sets $A(x_j, x_{j+1})$ in A_1 and $\lfloor C_{univ}\tau N \rfloor$ is the maximal possible number of sets $A(x_j, x_{j+1})$ in $A_1 \cap X_\varepsilon$. Clearly we can choose τ and ε accordingly such that $N_1 \ge N/3$.

For every $A(x_j, x_{j+1})$ as above fix $q_j \in A(x_j, x_{j+1})$, $q_j \notin X_\varepsilon$, and consider rectangle $\mathcal{R}(q_j)$ of vertical size ε. Then

$$HS(\mathcal{R}(q_j)) \ge \varepsilon^\delta.$$

Consider two rectangles $\mathcal{R}(q_j)$ and $\mathcal{R}(q_{\tilde{j}})$ as above. Since $|j - \tilde{j}| \ge 2$ they do not "overlap" vertically if ξ is sufficiently small (although this is not important to us). They might happen to "overlap" horizontally as shown on the Figure 12 but the size of the overlap cannot exceed the diameter of the tube $\mathcal{T}$ which, according to (46), is bounded by $C_2\xi$.

Above considerations result in the following estimate

$$HS(U_{x\bar{y}}) \geq HS(U_{x_0x_N}) \geq \frac{1}{C_H}\sum_{j=1}^{N_1} HS(\mathcal{R}(q_j)) - C_H N_1 C_2 \xi$$
$$\geq \frac{1}{C_H} N_1 \varepsilon^\delta - C_H N C_2 \xi \geq \frac{N}{3C_H}\varepsilon^\delta - N C_H C_2 \xi \geq \frac{S_0}{3C_H C_{Hol}}\varepsilon^{\delta-\rho} - N C_H C_2 \xi, \tag{50}$$

where C_H is the Lipschitz constant of the holonomy map along $W^f(i+1,m)$. We used estimate on N_1 and estimate (48) on N.

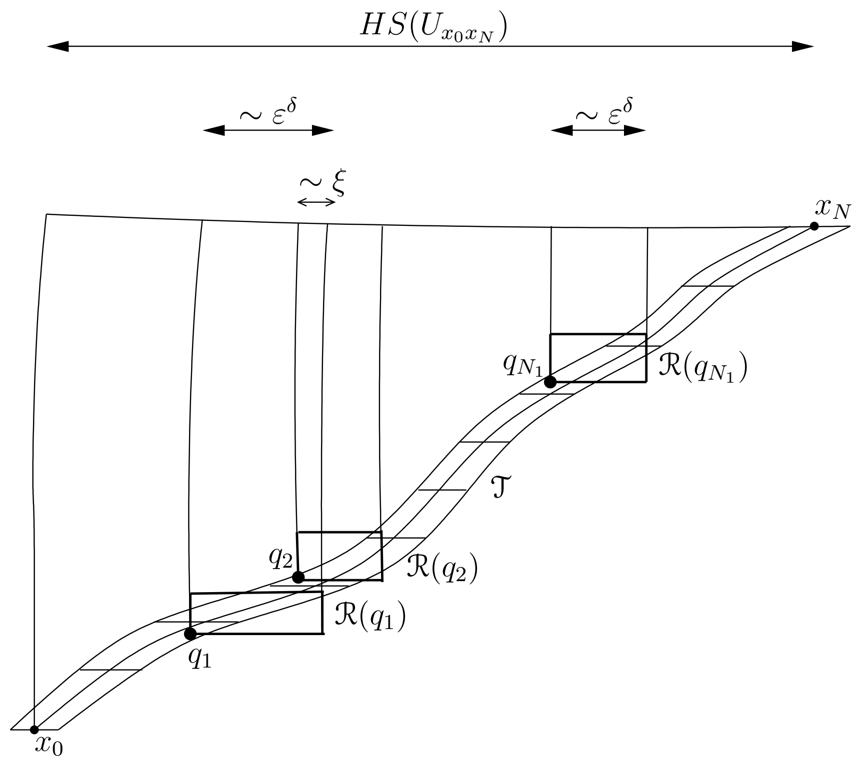

FIGURE 12. This picture illustrates the key estimate (50). Since the holonomy along $W^f(i+1,m)$ is Lipschitz the horizontal size of $U_{x_0x_N}$ can be estimated from below by the sum of horizontal sizes of "flat" rectangles with base points $q_j \in A_1 \subset \mathcal{T}$, $j = 1, \dots N_1$. They might overlap horizontally as shown but the overlap is of order $\xi \lll \varepsilon$.

Finally recall that $\delta-\rho < 0$ while ξ can be chosen arbitrarily small independently of ε (and hence N). Hence by choosing ε small we can find $U_{x\bar{y}}$ with arbitrarily big horizontal size that contradicts to the uniform upper bound (41) that follows from compactness. Hence Case 2 is impossible. □

Remark. Note that we do not need to take τ arbitrarily small. Constant τ just need to be small enough to provide the estimate on N_1.

Proof of Lemma 6.11. Take points x and $y\in V^f_m(x)$ such that

$$1\le d^f_m(x,y)\le C\beta_m \tag{51}$$

By Lemma 6.9 there exist constants c_1 and c_2 such that

$$\forall \tilde{x},\tilde{y},\ \ \tilde{y}\in V^f_m(\tilde{x}),\ \ \tilde{x}\in W^f(i,m-1)(x),\ \ \tilde{y}\in W^f(i,m-1)(y) \qquad c_1<d^f_m(\tilde{x},\tilde{y})<c_2. \tag{52}$$

Moreover, since c_1 and c_2 depend only on $\hat{d}(x,y)$ (see remark after the proof of Lemma 6.9) they can be chosen independently of x and y as long as x and y satisfy (51).

Take $x,y\in T(a)$ close to each other. Let N be the smallest integer such that $d^f_m(f^N(x),f^N(y))\ge 1$. Then

$$d^f_m(f^N(x),f^N(y))\ge \frac{1}{C}\tilde{\beta}^N_m d^f_m(x,y) \tag{53}$$

and, obviously,

$$d^f_m(f^N(x),f^N(y))\le C\beta_m. \tag{54}$$

Hence by taking in (52) $\tilde{x}=f^N(Hol(x))$ and $\tilde{y}=f^N(Hol(y))$ we get

$$d^f_m\left(f^N(Hol(x)),f^N(Hol(y))\right)>c_1. \tag{55}$$

On the other hand

$$d^f_m\left(f^N(Hol(x)),f^N(Hol(y))\right)\le C\beta^N_m d^f_m(Hol(x),Hol(y)). \tag{56}$$

Combining (53), (54), (55) and (56) we finish the proof

$$\begin{aligned}
&d^f_m(x,y)\le \frac{C}{\tilde{\beta}^N_m}d^f_m(f^N(x),f^N(y))\le \frac{C^2\beta_m}{c_1^\rho\tilde{\beta}^N_m}\cdot c_1^\rho\\
&<\frac{C^2\beta_m}{c_1^\rho}\cdot\frac{1}{\tilde{\beta}^N_m}d^f_m\left(f^N(Hol(x)),f^N(Hol(y))\right)^\rho\le C_{Hol}\frac{\beta^{\rho N}_m}{\tilde{\beta}^N_m}d^f_m(Hol(x),Hol(y))^\rho\\
&\qquad\qquad\qquad\qquad = C_{Hol}d^f_m(Hol(x),Hol(y))^\rho.
\end{aligned}$$

We used (43) for the last equality. □

7. Proof of Theorem C

7.1. **Scheme of the proof of Theorem C.** The way to choose neighborhood $\mathcal{U}$ is the same as in Theorem A. We look at the L-invariant splitting

$$T\mathbb{T}^4=E^{ss}_L\oplus E^{ws}_L\oplus E^{wu}_L\oplus E^{su}_L,$$

where E^{ws}_L, E^{wu}_L are eigendirections with eigenvalues $\lambda^{-1}<\lambda$ and $E^{ss}_L\oplus E^{su}_L$ is the Anosov splitting of g. We choose $\mathcal{U}$ in such a way that for any $f\in\mathcal{U}$ the invariant splitting survives

$$T\mathbb{T}^4=E^{ss}_f\oplus E^{ws}_f\oplus E^{wu}_f\oplus E^{su}_f \tag{57}$$

with

$$\max_{x\in\mathbb{T}^4,\sigma=ss,ws,wu,su}\left(\measuredangle(E^\sigma_f(x),E^\sigma_L(x))\right)<\frac{\pi}{2} \tag{58}$$

and f is partially hyperbolic in the strongest sense (30) with respect to the splitting (57).

Lemma 6.1 works for $f \in \mathcal{U}$. Hence the distributions E_f^{ss}, E_f^{ws}, E_f^{wu} and E_f^{su} integrate uniquely to foliations W_f^{ss}, W_f^{ws}, W_f^{wu} and W_f^{su}. Also, as usually, W_f^s and W_f^u stand for two dimensional stable and unstable foliations.

Fix $f \in \mathcal{U}$ and let H be the conjugacy with the model, $H \circ f = L \circ H$. Distribution $E_L^{ws} \oplus E_L^{wu}$ obviously integrate to foliation W_L^c which is subfoliated by W_L^{ws} and W_L^{wu}. Applying Lemma 6.3 to the weak foliations we get that $H(W_f^{ws}) = W_L^{ws}$ and $H(W_f^{wu}) = W_L^{wu}$. Hence distribution $E_f^{ws} \oplus E_f^{wu}$ integrates to foliation W_f^c which is subfoliated by W_f^{ws} and W_f^{wu}.

Note that the leaves of W_f^c are embedded two dimensional tori.

Lemma 7.1. *Conjugacy H is $C^{1+\nu}$ along W_f^{ws} and W_f^{wu}. Hence, by the Regularity Lemma, H is $C^{1+\nu}$ along W_f^c.*

Proposition 10 is a more general statement which we prove in Section 8. So we omit the proof of Lemma 7.1 here.

We establish smoothness of central holonomies.

Lemma 7.2. *Let T_1 and T_2 be open $C^{1+\nu}$-disks transversal to W_f^c. Then the holonomy map along W_f^c, $H_f^c : T_1 \to T_2$, is $C^{1+\nu}$-differentiable.*

Next we introduce distance on the leaves of W_f^{ws} and W_f^{wu} by simply letting $d^\sigma(x, y) = d^\sigma(H(x), H(y))$, $y \in W_f^\sigma(x)$, $\sigma = ws, wu$. Notice that by Lemma 7.1 d^{ws} and d^{wu} are induced by a Hölder continuous Riemannian metric — the pullback by $DH^{-1}|_{W_L^c}$ of the Riemannian metric on W_L^c.

Let x_0 be the fixed point of f and let S_0 be the two dimensional torus passing through x_0 and tangent to $E_L^{ss} \oplus E_L^{su}$. Assumption (58) guarantees that S_0 is transversal to W_f^c.

Now we construct foliation S that is transversal to W_f^c. For any point $x \in \mathbb{T}^4$ let $x_1 = W_f^c(x) \cap S_0$ and x_2 be some point of intersection of $W_f^{ws}(x_1)$ and $W_f^{wu}(x)$. Fix $\tilde{x} \in \mathbb{T}^4$ and define

$$
\begin{aligned}
S(\tilde{x}) = \{x : \text{such that}\\
&(x_1, x_2) \text{ and } (\tilde{x}_1, \tilde{x}_2) \text{ have the same orientation in } W_f^{ws};\\
&(x_2, x) \text{ and } (\tilde{x}_2, x) \text{ have the same orientation in } W_f^{wu};\\
&d^{ws}(x_1, x_2) = d^{ws}(\tilde{x}_1, \tilde{x}_2);\ \ d^{wu}(x_2, x) = d^{wu}(\tilde{x}_2, \tilde{x})\}.
\end{aligned}
$$

According to this definition $S(\tilde{x})$ intersects each leaf of W_f^c exactly once. Also note that since the distances came from the model L the definition above does not depend on the choice of $\tilde{x}_2$. It is clear that S is a topological foliation into topological two dimensional tori. We show that these tori are in fact regular.

Lemma 7.3. *Leaves of S are $C^{1+\nu}$ embedded two dimensional tori.*

Let $f_0 : S_0 \to S_0$ be the factor map of f, $f_0(x) = W_f^c(f(x)) \cap S_0$. Lemma 7.2 guarantees that f_0 is a $C^{1+\nu}$-diffeomorphism. Every periodic point of f_0 lifts to a periodic point of f. Applying Lemma 7.2 again we see that p. d. of f_0 are the same as strong stable and unstable p. d. of f which is the same as p. d. of g. Hence there is a $C^{1+\nu}$-diffeomorphism h_0 homotopic to identity such that $h_0 \circ f_0 = g \circ h_0$.

Let $f_c : W_f^c(x_0) \to W_f^c(x_0)$ be the restriction of f to $W_f^c(x_0)$. Obviously p. d. of f_c and A are the same. Hence there is a $C^{1+\nu}$ diffeomorphism h_c homotopic to identity such that $h_c \circ f_c = A \circ h_c$.

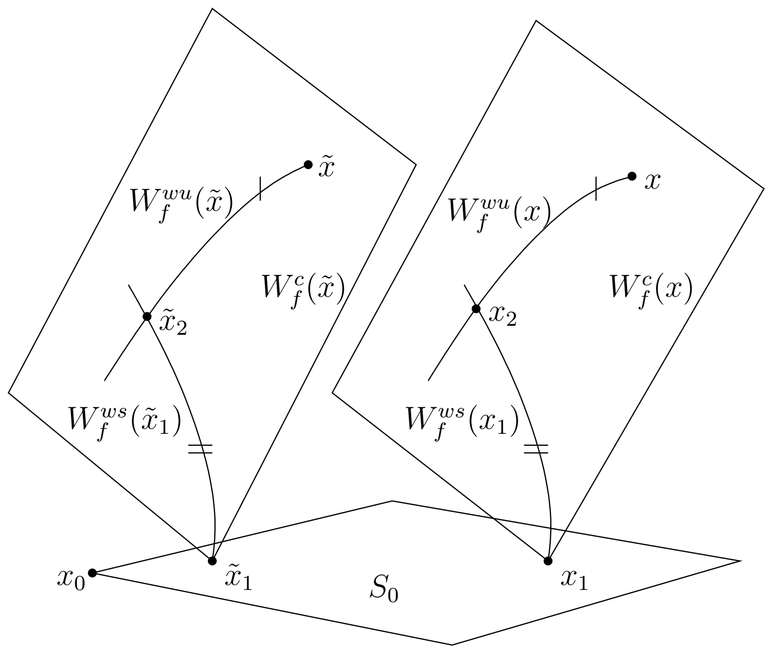

FIGURE 13. Definition of S. Point $x \in S(\tilde{x})$.

We are ready to construct the conjugacy $h : \mathbb{T}^4 \to \mathbb{T}^2 \times \mathbb{T}^2$

$$h(x) = \left(h_c(S(x) \cap W_f^c(x_0)), h_0(W_f^c(x) \cap S_0)\right).$$

Homeomorphism h maps central foliation into vertical foliation and foliation S into horizontal foliation.

Remark. Notice that at this point we do not know if h is $C^{1+\nu}$ diffeomorphism although h_c and h_0 are $C^{1+\nu}$ differentiable.

Lemma 7.4. *Homeomorphism h is $C^{1+\nu}$-differentiable along W_f^c.*

Proof. The projection $x \mapsto S(x) \cap W_f^c(x_0) \stackrel{\text{def}}{=} pr(x)$ projects weak stable leaf $W_f^{ws}(x)$ into $W_f^{ws}(pr(x))$. Moreover, it is clear from the definition of S that the restriction of this projection to $W_f^{ws}(x)$ is an isometry with respect to distance d^{ws}. According to the formula for the first component of h we compose this projection with h_c which is an isometry when restricted to the leaf $W_f^{ws}(pr(x))$ by the definition of d^{ws}. Diffeomorphism h_c straightens weak stable foliation into foliation by straight lines W_L^{ws}. Hence $h(W_f^{ws}) = W_L^{ws}$ and h is an isometry as a map $(W_f^{ws}(x), d^{ws}) \mapsto (W_L^{ws}(h(x))$, Riemannian metric). Thus h is $C^{1+\nu}$ along W_f^{ws}.

Everything above can be repeated for weak unstable foliation. Applying the Regularity Lemma we get the desired statement. □

Lemma 7.5. *Homeomorphism h is $C^{1+\nu}$-differentiable along S.*

Proof. Restriction of h to S_0 is just h_0. Restriction of h to some other leaf $S(x)$ can be viewed as composition of holonomy H_f^c, h_0 and holonomy H_L^c. Hence this restriction is $C^{1+\nu}$-differentiable as well. We need to make sure that the derivative of h along S is Hölder continuous on $\mathbb{T}^4$. For this we only need to show that derivative of $H_f^c : S(x) \to S_0$ depends Hölder continuously on x. This assertion will become clear in the proof of Lemma 7.3. □

Hence, by the Regularity Lemma, we conclude that h is $C^{1+\nu}$ diffeomorphism.

Let $\tilde{L} = h \circ f \circ h^{-1}$. Clearly foliations W_L^{ws} and W_L^{wu} are $\tilde{L}$-invariant. By construction h and h^{-1} are isometries when restricted to the leaves of weak foliations. Recall that f stretches by factor λ distance d^{wu} on W_f^{wu} and contracts by factor λ^{-1} distance d^{ws} on W_f^{ws}. Hence if we consider restriction of $\tilde{L}$ on a fixed vertical two torus $W_L^c(x) \mapsto W_L^c(\tilde{L}(x))$ then it acts by hyperbolic automorphism A.

Also it is obvious from the construction of h that the factor map of $\tilde{L}$ on a horizontal two torus is g. These observations show that $\tilde{L}$ is of the form

$$\tilde{L} = (Ax + \vec{\varphi}(y), g(y)). \tag{4}$$

Note that we do not have to argue additionally that $\vec{\varphi}$ is smooth since we know that $\tilde{L}$ is $C^{1+\nu}$-diffeomorphism.

Remark. An observant reader would notice that our choice of h and hence $\tilde{L}$ is far from being unique. The starting point of the construction of h is the torus S_0. Although we have chosen a concrete S_0, in fact, the only thing we need from S_0 is transversality to W_f^c. This is not surprising. Many diffeomorphisms of type (4) are C^1-conjugate to each other. In the linear case this is controlled by invariants (11).

In the rest of this section we prove Lemmas 7.2 and 7.3.

7.2. **A technical Lemma.** Before we proceed with proofs of Lemmas 7.2 and 7.3 we establish a crucial technical lemma which is a corollary of Lemma 7.1.

Let $U^\sigma = H(W_f^\sigma)$, $\sigma = ss, su$. These are foliations by Hölder continuous curves.

Lemma 7.6. *Fix $x \in \mathbb{T}^4$ and $y \in W_L^c(x)$. Let $\vec{v}$ be a vector connecting x and y inside of $W_L^c(x)$. Then*

$$U^\sigma(y) = U^\sigma(x) + \vec{v}.$$

In other words foliation U^σ is invariant under translations along W_L^c, $\sigma = ss, su$.

Proof. For concreteness we take $\sigma = ss$. The proof in case $\sigma = su$ is the same.

First let us assume that $y \in W_L^{ws}(x)$. This allows to restrict our attention to the stable leaf $W_L^s(x)$ since $U^{ss}(x)$ and $U^{ss}(y)$ lie inside of $W_L^s(x)$. Pick a point $z \in U^{ss}(x)$ and let $\tilde{z} = W_L^{ws}(z) \cap U^{ss}(y)$. We only need to show that $d(x,y) = d(z,\tilde{z})$, where d is the Riemannian distance along weak stable leaves. Simple idea of the proof of Claim 1 from [GG08] works here. We briefly outline the argument.

Let $c = d(z,\tilde{z})/d(x,y)$. Obviously

$$\forall n \quad \frac{d(L^n(z), L^n(\tilde{z}))}{d(L^n(x), L^n(y))} = c. \tag{59}$$

Since $H^{-1}(z) \in W_f^{ss}(x)$, $H^{-1}(\tilde{z}) \in W_f^{ss}(y)$ and strong stable leaves contract exponentially faster than weak stable leaves we have

$$\forall \varepsilon > 0 \quad \exists N : \forall n > N : \quad \left| \frac{d\left(H^{-1}(L^n(z)), H^{-1}(L^n(\tilde{z}))\right)}{d\left(H^{-1}(L^n(x)), H^{-1}(L^n(y))\right)} - 1 \right| = \left| \frac{d\left(f^n(H^{-1}(z)), f^n(H^{-1}(\tilde{z}))\right)}{d\left(f^n(H^{-1}(x)), f^n(H^{-1}(y))\right)} - 1 \right| < \varepsilon. \quad (60)$$

On the other hand, since derivative of H along W_f^{ws} is continuous, the ratios

$$\frac{d(L^n(z), L^n(\tilde{z}))}{d\left(H^{-1}(L^n(z)), H^{-1}(L^n(\tilde{z}))\right)} \quad \text{and} \quad \frac{d(L^n(x), L^n(y))}{d\left(H^{-1}(L^n(x)), H^{-1}(L^n(y))\right)}$$

are arbitrarily close when $n \to +\infty$. Together with (60) this shows that constant c from (59) is arbitrarily close to 1. Hence $c = 1$.

Finally, recall that for any x leaf $W_L^{ws}(x)$ is dense in $W_L^c(x)$. Hence by continuity we get the statement of the lemma for any $y \in W_L^c(x)$. □

Lemma 7.6 leads to some non-trivial structural information about f which is of interest on its own.

Proposition 9. *Distributions $E_f^{wu} \oplus E_f^{ss}$ and $E_f^{ws} \oplus E_f^{su}$ are integrable.*

Proof. It follows from Lemma 7.6 that foliations W_L^{wu} and U^{ss} integrate together. Thus foliations W_f^{wu} and W_f^{ss} integrate to a foliation with tangent distribution $E_f^{wu} \oplus E_f^{ss}$. □

7.3. **Smoothness of central holonomies.** We assume that the holonomy map $H_f^c : T_1 \to T_2$ is a bijection. It can be represented as a composition of holonomies along W_f^{ws} and W_f^{wu}. Indeed, let us work on the universal cover and consider two open three dimensional submanifolds of $\mathbb{R}^4$ $M_1 = \bigcup_{x \in T_1} W_f^{wu}(x)$ and $M_2 = \bigcup_{x \in T_2} W_f^{ws}(x)$. Let $T_3 = M_1 \cap M_2$. Obviously T_3 is a smooth two dimensional open submanifold. Also it is easy to see that T_3 is connected since we are working on the universal cover. Then $H_f^c : T_1 \to T_2$ is the composition of $H_f^{wu} : T_1 \to T_3$ and $H_f^{ws} : T_3 \to T_2$.

So, it is sufficient to study holonomy map along W_f^{wu} $H_f^{wu} : T_1 \to T_2$. The study of holonomies along W_f^{ws} is the same.

First we make a reduction that allows to work with one dimensional transversals instead of two dimensional transversals. Let $\tilde{W}_f$ and $\tilde{W}_L$ be the integral foliations of $E_f^{ws} \oplus E_f^{wu} \oplus E_f^{su}$ and $E_L^{ws} \oplus E_L^{wu} \oplus E_L^{su}$ respectively. Also let $\bar{W}_f$ and $\bar{W}_L$ be the integral foliations of $E_f^{ss} \oplus E_f^{ws} \oplus E_f^{wu}$ and $E_L^{ss} \oplus E_L^{ws} \oplus E_L^{wu}$ respectively.

Any transversal T to W_f^c can be foliated by connected components of intersections with leaves of $\tilde{W}_f$. Call this foliation $\tilde{T}$. This is a well-defined one dimensional foliation since T is two dimensional while the leaves of $\tilde{W}_f$ are three dimensional and both T and $\tilde{W}_f$ are transversal to W_f^{ss}. The holonomy map $H_f^{wu} : T_1 \to T_2$ maps $\tilde{T}_1$ into $\tilde{T}_2$ since W_f^{wu} subfoliate $\tilde{W}_f$.

Analogously any transversal T can be foliated by connected components of intersections with leaves of $\bar{W}_f$. Call this foliation $\bar{T}$. Then $H_f^{wu}(\bar{T}_1) = \bar{T}_2$ since W_f^{wu} subfoliate $\bar{W}_f$.

Hence we can consider restrictions of H_f^{wu} to the leaves of $\tilde{T}_1$ and $\bar{T}_1$.

Lemma 7.7. *Restriction of holonomy H_f^{wu} to a leaf of $\tilde{T}_1$, $H_f^{wu}: \tilde{T}_1(x) \to \tilde{T}_2(H_f^{wu}(x))$ is $C^{1+\nu}$-differentiable.*

Lemma 7.8. *Restriction of holonomy H_f^{wu} to a leaf of $\bar{T}_1$, $H_f^{wu}: \bar{T}_1(x) \to \bar{T}_2(H_f^{wu}(x))$ is $C^{1+\nu}$-differentiable.*

Note that $\tilde{T}_i$ and $\bar{T}_i$ are transverse since T_i is transverse to W_f^c, $i=1,2$. Hence, by the Regularity Lemma, the holonomy $H_f^{wu}: T_1 \to T_2$ is $C^{1+\nu}$-differentiable.

To prove Lemmas 7.7 and 7.8 we need to establish regularity of H in strong unstable direction.

Given $x \in \mathbb{T}^4$ define $H_x : W_f^{su}(x) \to W_L^{su}(H(x))$ by the following composition.

$$W_f^{su}(x) \xrightarrow{H} U^{su}(H(x)) \xrightarrow{H_L^{wu}} W_L^{su}(H(x)).$$

First we map $W_f^{su}(x)$ into a Hölder continuous curve $U^{su}(H(x)) \subset W_L^u(H(x))$ and then we project it on $W_L^{su}(H(x))$ along the linear foliation W_L^{wu} as shown on the Figure 14.

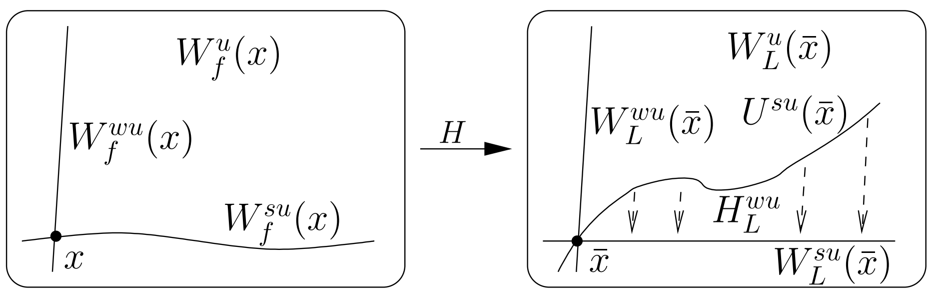

FIGURE 14. Definition of H_x. Here $\bar{x} \overset{def}{=} H(x)$.

Lemma 7.9. *For any $x \in \mathbb{T}^4$ the map H_x is $C^{1+\nu}$-differentiable.*

Proof. Let us show first that H_x is uniformly Lipshitz with a constant that does not depend on x. Denote by d, d_f^{su}, d_L^u and d_L^{su} Riemannian distances on the universal cover $\mathbb{R}^4$, along the leaves of W_f^{su}, along the leaves of W_L^u and along the leaves of W_L^{su} respectively. First we show that H_x is Lipshitz if the points are far enough. Assume that $y, z \in W_f^{su}(x)$ and $d_f^{su}(y,z) \geq 1$. Then on the universal cover

$$\begin{aligned}
&d_L^{su}(H_x(y), H_x(z)) \overset{1}{\leq} c_1 d_L^u(H_x(y), H_x(z)) \\
&\overset{2}{\leq} c_1c_2 \inf\{d_L^u(\tilde{y}, \tilde{z}) : \tilde{y} \in W_L^{wu}(H_x(y)), \tilde{z} \in W_L^{wu}(H_x(z))\} \overset{3}{\leq} c_1c_2 d_L^u(H(x), H(y)) \\
&\qquad \overset{4}{\leq} c_1c_2c_3 d(H(x), H(y)) \overset{5}{\leq} c_1c_2c_3c_4 d(y,z) \overset{6}{\leq} c_1c_2c_3c_4 d_f^{su}(y,z).
\end{aligned}$$

First and fourth inequality hold since W_L^{su} and W_L^u are quasi-isometric. Second inequality holds with universal constant c_2 due to uniform transversality of W_L^{wu} and W_f^{su}. Inequalities 3 and 6 are obvious. Fifth inequality holds since $d_f^{su}(y,z) \geq 1$ and the lift of the conjugacy satisfies

$$H(x + \vec{m}) = H(x) + \vec{m}, \quad x \in \mathbb{R}^4, \quad \vec{m} \in \mathbb{Z}^4.$$

Here we slightly abuse notation by denoting the lift and the map itself by the same letter.

Now we need to show that H_x is Lipschitz if y and z are close on the leaf. Notice that H_x is composition of H_y and holonomy $H_L^{wu} : W_L^{su}(H(y)) \to W_L^{su}(H(x))$ which is just a translation. Hence to show that H_x is Lipschitz at y we only need to show that H_y is Lipshitz at y.

So we fix x and y on $W_L^{su}(x)$ close to x and show that $d_L^{su}(H_x(x), H_x(y)) \le c\, d_f^{su}(x,y)$. The argument here is an adapted argument from the proof of Lemma 4 from [GG08]. Two major tools here are the Livshitz theorem and affine distance-like functions $\tilde{d}_f^{su}$ and $\tilde{d}_L^{su}$ on W_f^{su} and W_L^{su} respectively. We used the same distance like function on foliation V_i^f in the proof of Lemma 6.8. Recall properties of $\tilde{d}_f^{su}$

(D1) $\tilde{d}_f^{su}(x,y) = d_f^{su}(x,y) + o(d_f^{su}(x,y))$,
(D2) $\tilde{d}_f^{su}(f(x), f(y)) = D_f^{su}(x)\tilde{d}_f^{su}(x,y)$,
(D3) $\forall K > 0\ \exists C > 0$ such that
$$\frac{1}{C}\tilde{d}_f^{su}(x,y) \le d_f^{su}(x,y) \le C\tilde{d}_f^{su}(x,y)$$
whenever $d_f^{su}(x,y) < K$.

Consider Hölder continuous functions $D_f^{su}(\cdot)$ and $D_L^{su}(H(\cdot))$. The assumption on p. d. of f and L guarantee that the products of these derivatives along periodic orbits coincide. Thus we can apply Livshitz theorem and get the Hölder continuous positive transfer function P such that
$$\forall n > 0 \qquad \prod_{i=0}^{n-1} \frac{D_L^{su}\left(H(f^i(x))\right)}{D_f^{su}(f^i(x))} = \frac{P(x)}{P(f^n(x))}.$$
Choose the smallest N such that $d_f^{su}(f^N(x), f^N(y)) \ge 1$. Then
$$\frac{\tilde{d}_L^{su}(H_x(x), H_x(y))}{\tilde{d}_f^{su}(x,y)} = \prod_{i=0}^{N-1} \frac{D_L^{su}\left(L^i(H_x(x))\right)}{D_f^{wu}(f^i(x))} \cdot \frac{\tilde{d}_L^{su}\left(L^N(H_x(x)), L^N(H_x(y))\right)}{\tilde{d}_f^{su}(f^N(x), f^N(y))}$$
$$= \frac{P(x)}{P(f^N(x))} \cdot \frac{\tilde{d}_L^{su}\left(H_{f^N(x)}(f^N(x)), H_{f^N(x)}(f^N(y))\right)}{\tilde{d}_f^{su}(f^N(x), f^N(y))} \le \frac{P(x)}{P(f^N(x))} \cdot c_1c_2c_3c_4.$$
Function P is uniformly bounded away from zero and infinity. Hence, together with (D3) this shows that H_x is Lipschitz at x uniformly in x and hence is uniformly Lipschitz.

Next we apply the transitive point argument. Consider SRB measure μ^u which is the equilibrium state for the potential minus the logarithm of the unstable jacobian of f. It is well known that W_f^u is absolutely continuous with respect to μ^u. On a fixed leaf of W_f^u foliation W_f^{su} is absolutely continuous with respect to the Lebesgue measure on the leaf (for proof see [LY85], Section 4.2, they proof that the unstable foliation is Lipschitz with center-unstable leaves, but the proof goes through for strong unstable foliation within unstable leaves). Hence W_f^{su} is absolutely continuous with respect to μ^u.

We know that H_x is Lipschitz and hence almost everywhere differentiable on $W_f^{su}(x)$. It is clear from the definition that H_x is differentiable at y if and only if H_y is differentiable at y. Thus it does make sense to speak about differentiability at a point on strong unstable leaf without referring to a particular map H_x. Absolute continuity of W_f^{su} allows to conclude that H_x is differentiable at x for μ^u almost every x.

Since μ^u is ergodic and has full support we can consider a transitive point $\bar{x}$ such that $H_{\bar{x}}$ is differentiable at $\bar{x}$. Now C^1-differentiability of H_x for any $x \in \mathbb{T}^4$ can be shown by an approximation argument: we approximate the target point by iterates of $\bar{x}$. The argument is the same as the proof of Step 1, Lemma 5 from [GG08] with minimal modifications. So we omit it. This argument shows even more. Namely,

$$D(H_x)(x) = \frac{P(x)}{P(\bar{x})} D(H_{\bar{x}})(\bar{x}).$$

Note that $D(H_x)(y) = D(H_y)(y)$. Hence H_x maps Lebesgue measure on the leaf $W_f^{su}(x)$ into absolutely continuous measure $dy \mapsto \frac{P(y)}{P(\bar{x})} d\text{Leb}$. Recall that P is Hölder continuous. Hence H_x is $C^{1+\nu}$-differentiable. □

Proof of Lemma 7.7. We work in a ball B inside of the leaf $\tilde{W}_f(x)$ that contains $\tilde{T}_1(x)$ and $\tilde{T}_2(H_f^{wu}(x))$. Recall that B is subfoliated by W_f^c and W_f^{su}. We apply the conjugacy map H to the ball B. It maps W_f^{su} and W_f^c into U^{su} and W_L^c respectively. We construct a shift map $sh : H(B) \to \tilde{W}_L(H(x))$ in such a way that for any z the leaf $W_L^c(z)$ is sh-invariant and the action of sh on the leaf is a rigid translation.

Given a point $z \in H(B)$ let $y(z) = W_L^c(H(x)) \cap U^{su}(z)$. Define

$$sh(z) = W_L^{su}(y(z)) \cap W_L^{wu}(z).$$

Clearly $sh(U^{su}(H(x))) = W_L^{su}(H(x))$. Moreover, by Lemma 7.6 $sh(U^{su}) = W_L^{su}$.

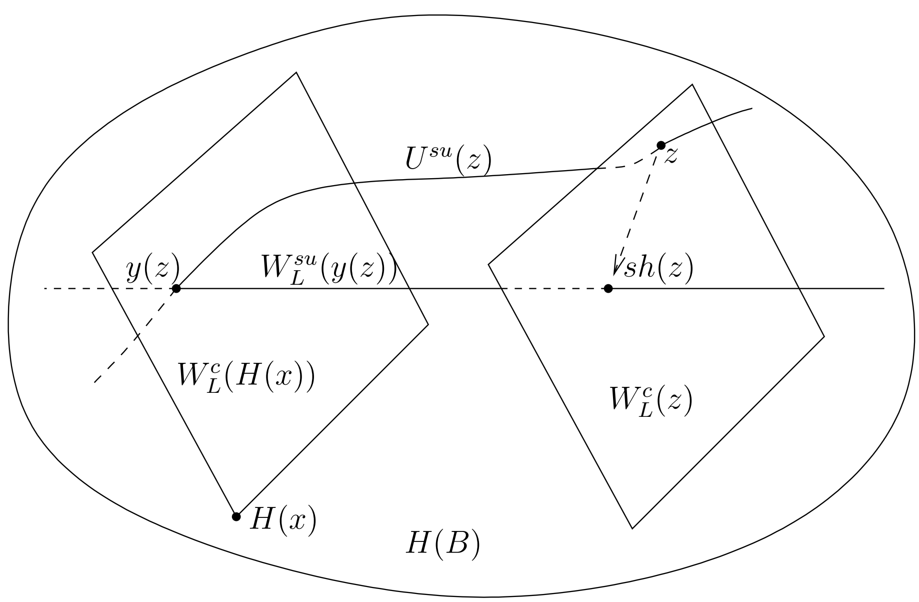

FIGURE 15. Definition of the shift.

The shift sh is designed so that the composition $sh \circ H$ maps foliation W_f^c into W_L^c and foliation W_f^{su} into W_L^{su}. According to Lemma 7.1 $sh \circ H$ is $C^{1+\nu}$-differentiable along W_f^c. Also notice that the restriction of $sh \circ H$ to a strong unstable leaf W_f^{su} is nothing but H_y composed with constant parallel transport along W_L^{wu}. Recall that H_y is $C^{1+\nu}$-differentiable by Lemma 7.9. Hence, by the Regularity Lemma, we conclude that $sh \circ H$ is $C^{1+\nu}$-diffeomorphism.

Therefore $\hat{T}_1 = sh \circ H(\tilde{T}_1(x))$ and $\hat{T}_2 = sh \circ H(\tilde{T}_2(H_f^{wu}(x)))$ are smooth curves inside of $H(B)$ and the holonomy map H_f^{wu} can be represented as a composition as shown on the commutative diagram

$$\begin{array}{ccc} \tilde{T}_1(x) & \xrightarrow{H_f^{wu}} & \tilde{T}_2(H_f^{wu}(x)) \\ {\scriptstyle sh\circ H}\downarrow & & {\scriptstyle sh\circ H}\downarrow \\ \hat{T}_1 & \xrightarrow{H_L^{wu}} & \hat{T}_2 \end{array}$$

Holonomy H_L^{wu} is smooth since W_L^{wu} is a foliation by straight lines. Hence H_f^{wu} is $C^{1+\nu}$-differentiable. □

Remark. Notice that this argument completely avoids dealing with geometry of transversals i. e. their relative position to the foliations.

Proof of Lemma 7.8. We use exactly the same argument as in the previous proof. Notice that the picture is not completely symmetric compared to the picture in Lemma 7.7 since we are dealing with weak unstable holonomy. Nevertheless the argument goes through by looking at transversals $\bar{T}_1(x)$ and $\bar{T}_2(H_f^{wu}(x))$ on the leaf of $\bar{W}_f$. The shift map must be constructed in such a way that it maps U^{ss} into W_L^{ss}. □

Proof of Lemma 7.3. In this proof we exploit the same idea of composing H with some shift map. We fix $S_1 = S(x_1) \in S$ which is, a priori, just an embedded topological torus. We assume that $x_1 \in W_f^{wu}(x_0)$. It is easy to see that this is not restrictive.

Foliate S_0 and S_1 by $\tilde{T}_0$, $\bar{T}_0$ and $\tilde{T}_1$, $\bar{T}_1$ respectively by taking intersections with leaves of $\tilde{W}_f$ and $\bar{W}_f$. To prove the lemma we only have to show that the leaves of $\tilde{T}_1$ and $\bar{T}_1$ are $C^{1+\nu}$-differentiable curves.

We restrict our attention to a leaf of $\tilde{W}_f$. Construct the shift map sh in the same way as in Lemma 7.7. Fix an $x \in S_0$ and let $\hat{T}_0 = sh \circ H(\tilde{T}_0(x))$, $\hat{T}_1 = sh \circ H(\tilde{T}_1(H_f^{wu}(x)))$.

$\hat{T}_0$ is a $C^{1+\nu}$-curve since $sh \circ H$ is $C^{1+\nu}$-diffeomorphism. By the definition of S_1

$$\forall y \in \tilde{T}_0 \quad d^{wu}(y, H_f^{wu}(y)) = d^{wu}(x, H_L^{wu}(x)).$$

Recall the definition of d^{wu} to see that conjugacy H acts as an isometry on a weak unstable leaf. Obviously sh is an isometry when restricted to a weak unstable leaf as well. Therefore

$$\forall y \in \hat{T}_0 \quad d(y, H_f^{wu}(y)) = d(sh \circ H(x), H_L^{wu}(sh \circ H(x))),$$

where d is the Riemannian distance along W_L^{wu}.

Hence $\hat{T}_1$ is smooth as a parallel translation of $\hat{T}_0$. We conclude that $\tilde{T}_1(H_f^{wu}(x)) = (sh \circ H)^{-1}(\hat{T}_1)$ is $C^{1+\nu}$-curve.

Repeating the same argument for $\bar{T}_0(x)$ and $\bar{T}_1(H_f^{wu}(x))$ we show that $\bar{T}_1(H_f^{wu}(x))$ is $C^{1+\nu}$-curve. Hence the lemma is proved. □

8. Proof of Theorem D

8.1. **Scheme of the proof of Theorem D.** We choose $\mathcal{U}$ in the same way as in 7.1. The only difference is that L is given by (1) not by (3).

Given $f \in \mathcal{U}$ we denote by W_f^c two dimensional central foliation. Take f and g in $\mathcal{U}$. Then they are conjugate, $h \circ f = g \circ h$.

Proposition 10. *Assume that f and g have the same p. d. Then $h(W_f^c) = W_g^c$ and the conjugacy h is $C^{1+\nu}$-differentiable along W_f^c.*

Remark. In the proof we only need coincidence of p. d. in the central direction.

After we have differentiability along the central foliation strong stable and unstable foliation moduli come into the picture.

Lemma 8.1. *Assume that f and g have the same p. d. and the same strong unstable foliation moduli. Then $h(W_f^{su}) = W_g^{su}$.*

Now the proof of Theorem D follows immediately. Coincidence of p. d. in strong unstable direction guarantees $C^{1+\nu}$-differentiability of h along W_f^{su}. This can be done by transitive point argument with SRB-measure in the same way as the proof of Lemma 6.7. Then we repeat everything for strong stable foliation. After this we apply Journé Regularity Lemma twice to conclude that h is $C^{1+\nu}$-differentiable.

In particular this argument shows that in the counterexample of de la Llave strong stable and unstable foliations are not preserved by the conjugacy. We can make use of this fact by extending the counterexample for the diffeomorphisms of the form $(x, y) \mapsto (Ax + \vec{\varphi}(y), g(y))$. Namely, take $L = (Ax, By)$ and $\tilde{L} = (Ax + \vec{\varphi}(y), By)$ as in (1) and (2) respectively. We know that strong foliations of L and $\tilde{L}$ do not match. Strong foliations depend continuously on the diffeomorphism in C^1 topology. Thus if we consider diffeomorphisms $L'(x, y) = (Ax, g(y))$ and $\tilde{L}'(x, y) = (Ax + \vec{\varphi}(y), g(y))$ with g being sufficiently C^1 close to B then strong foliations of L' and $\tilde{L}'$ do not much as well. Therefore L' and $\tilde{L}'$ are not C^1 conjugate.

We do not know how to show that the counterexample extends to the whole neighborhood $\mathcal{U}$.

Conjecture 4. *For any $f \in \mathcal{U}$ there exists $g \in \mathcal{U}$ with the same p. d. which is not C^1 conjugate to f.*

Proof of Lemma 8.1. Let $U = h^{-1}(W_g^{su})$. We need to show that $U = W_f^{su}$. The main tool is the following statement

Lemma 8.2. *Consider a point $a \in \mathbb{T}^4$. Suppose that there is a point $b \neq a$, $b \in W_f^{su}(a) \cap U(a)$. Let $c \in W_f^{wu}(a)$ and $d = W_f^{wu}(b) \cap W_f^{su}(c)$, $e = W_f^{wu}(b) \cap U(c)$. Then $d = e$.*

This means that the "intersection structure" of U and W_f^{su} is invariant under the shifts along W_f^{wu}. We refer to [GG08] for the proof. Claim 1 in [GG08] is exactly the same statement in the context of $\mathbb{T}^3$. The proof uses Proposition 10.

According to the definition of strong unstable foliation moduli we have to distinguish two cases.

First assume (7). It follows that there is a curve $\mathcal{C} \subset W_f^{su}(x)$ that corresponds to the interval I such that $\mathcal{C} \subset U$ as well. Let

$$\mathcal{S} = \bigcup_{a \in \mathcal{C}} W_f^{wu}(a).$$

Obviously $\mathcal{S} \subset W_f^u(x)$. It follows from Lemma 8.2 that $W_f^{su} = U$ when restricted to $\mathcal{S}$. Then $W_f^{su} = U$ when restricted to $f^n(\mathcal{S})$, $n > 0$ as well. It remains to notice that $\cup_{n>0} f^n(\mathcal{S})$ is dense in $\mathbb{T}^4$ since $length(f^n(\mathcal{C})) \to \infty$ as $n \to \infty$. Hence $W_f^{su} = U$.

Now let us consider the second case. Namely, assume (6). Let x_0 be a fixed point. Define $x_1 = \mathcal{I}^{su}(x_0)^{-1}(t)$. Then by (6) we have that $x_1 \in W_f^{su}(x_0) \cap U(x_0)$. We continue to define a sequence $\{x_k; k \geq 0\}$ inductively. Given x_k define $x_{k+1} = \mathcal{I}^{su}(x_k)^{-1}(t)$. Then for any k $x_{k+1} \in W_f^{su}(x_k) \cap U(x_k) = W_f^{su}(x_0) \cap U(x_0)$. Obviously $f^{-n}(x_k) \in W_f^{su}(x_0) \cap U(x_0)$ as well.

Map $\mathcal{I}^{su}(x_0)$ is an isometry, hence $d_f^{su}(x_k, x_{k+1})$ does not depend on k. Therefore the set $\{f^{-n}(x_k); n \geq 0, k \geq 0\}$ is dense in $W_f^{su}(x_0)$ which guarantees that $W_f^{su}(x_0) = U(x_0)$. We can proceed as in the first case now to conclude that $W_f^{su} = U$. □

8.2. **Smoothness along the central foliation: proof of Proposition 10.** We apply the transitive point argument as in the proof of Lemma 6.7. The technical difficulty that we have to deal with is that the leaves of W^c are not dense in $\mathbb{T}^4$.

Conjugacy h preserves weak stable and unstable foliations. By the Regularity Lemma we only need to show $C^{1+\nu}$-differentiability of h along these one dimensional foliations. For concreteness we work with weak unstable foliation W_f^{wu}.

For the transitive point argument to work we have to find an invariant measure μ such that μ a. e. point is transitive ($\overline{\{f^n(x); n \geq 0\}} = \mathbb{T}^4$) and W_f^{wu} is absolutely continuous with respect to μ. Provided that we have such a measure μ $C^{1+\nu}$-differentiability of h along W_f^{wu} is proved in the same way as Lemma 5 from [GG08].

We modify the construction from the proof of Lemma 6.7. Consider the space $\mathcal{T}$ of the leaves of W_f^c. Clearly this is a topological space homeomorphic to a two torus. Let $\tilde{f} : \mathcal{T} \to \mathcal{T}$ be the factor dynamics of f. Since the conjugacy to the linear model L maps the central leaves to the central leaves, $\tilde{f}$ is conjugate to the automorphism $B : \mathbb{T}^2 \to \mathbb{T}^2$, $\tilde{h} \circ B = \tilde{f} \circ \tilde{h}$. Then the measure $\tilde{\mu} = \tilde{h}_*(\text{Lebesgue})$ is $\tilde{f}$-invariant and ergodic.

Pick a point x_0 on a $\tilde{\mu}$ typical central leaf. Let $\mathcal{V}_0$ be an open bounded neighborhood of x_0 in $W_f^{wu}(x_0)$. Given x and $y \in W_f^{wu}(x)$ let

$$\rho(x, y) = \prod_{n \geq 0} \frac{D_f^{wu}(f^{-n}(y))}{D_f^{wu}(f^{-n}(x))}.$$

Consider a probability measure η_0 supported on $\mathcal{V}_0$ with density proportional to $\rho(x_0, \cdot)$. For $n > 0$ define

$$\mathcal{V}_n = f^n(\mathcal{V}_0), \ \eta_n = (f^n)_* \eta_0.$$

Let

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \eta_i.$$

An accumulation point of $\{\mu_n; n \geq 0\}$ is the measure μ that we are looking for.

By the choice of x_0 the projection of μ to $\mathcal{T}$ is $\tilde{\mu}$.

Foliation W_f^{wu} is absolutely continuous with respect to μ. We refer to [PS83] or [GG08] for the proof. In [GG08] x_0 is a fixed point but we do not use it in the proof of absolute continuity.

Now we have to argue that μ a. e. point is transitive. We fix a ball in $\mathbb{T}^4$ and we show that a. e. point visits the ball infinitely many times. Then to conclude transitivity we only need to cover $\mathbb{T}^4$ by a countable collection of balls such that every point is contained in an arbitrarily small ball.

So let us fix a ball B' and a slightly smaller ball B, $B \subset B'$. Let ψ be a non-negative continuous function supported on B' and equal to 1 on B. By Birkhoff ergodic theorem

$$E(\psi|\mathcal{I}) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi \circ f^i \tag{61}$$

where $\mathcal{I}$ is the σ-algebra of f-invariant sets.

Let $A = \{x : E(\psi|\mathcal{I})(x) = 0\}$. Then $\mu(A \cap B) = 0$ since $\int_A \psi d\mu = \int_A E(\psi|\mathcal{I}) d\mu = 0$. Hence

$$E(\psi|\mathcal{I})(x) > 0 \;\text{ for }\; \mu \text{ a. e. } \; x \in B.$$

Let $\tilde{B} \subset B$ be a slightly smaller ball and let $W^c(\tilde{B}) = \cup_{x\in\tilde{B}} W_f^c(x)$. Since weak unstable leaves are dense in corresponding central leaves it is possible to find $R > 0$ such that

$$W^c(\tilde{B}) \subset \bigcup_{x\in B} W_f^{wu}(x, R).$$

Applying the standard Hopf argument we get that for μ a. e. x the function $E(\psi|\mathcal{I})$ is constant on $W(x, R)$. Now absolute continuity of W_f^{wu} together with above observations show that

$$E(\psi|\mathcal{I})(x) > 0 \;\text{ for }\; \mu \text{ a. e. } \; x \in W^c(\tilde{B}).$$

Obviously

$$\forall n \quad E(\psi|\mathcal{I})(x) > 0 \;\text{ for }\; \mu \text{ a. e. } \; x \in f^n(B).$$

Repeat the same argument to get

$$\forall n \quad E(\psi|\mathcal{I})(x) > 0 \;\text{ for }\; \mu \text{ a. e. } \; x \in W^c(f^n(\tilde{B})).$$

Let $\mathcal{O}(\tilde{B}) = \cup_{n\in\mathbb{Z}} f^n(\tilde{B})$ and $W^c(\mathcal{O}(\tilde{B})) = \cap_{x\in\mathcal{O}(\tilde{B})} W_f^c(x)$. Then

$$E(\psi|\mathcal{I})(x) > 0 \;\text{ for }\; \mu \text{ a. e. } \; x \in W^c(\mathcal{O}(\tilde{B})).$$

Set $W^c(\mathcal{O}(\tilde{B}))$ is W_f^c-saturated. Hence $\mu(W^c(\mathcal{O}(\tilde{B})))$ is equal to $\tilde{\mu}$ measure of its projection $proj(W^c(\mathcal{O}(\tilde{B}))) = proj(\mathcal{O}(\tilde{B}))$ on $\mathcal{T}$. Set $proj(\mathcal{O}(\tilde{B}))$ is an open $\tilde{f}$-invariant set. By ergodicity of $\tilde{f}$ it has full measure. Hence $\mu(W^c(\mathcal{O}(\tilde{B}))) = 1$ and

$$E(\psi|\mathcal{I})(x) > 0 \;\text{ for }\; \mu \text{ a. e. } \; x \in \mathbb{T}^4.$$

According to (61) this means that μ a. e. x visits B' infinitely many times.

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