SMOOTH RIGIDITY FOR VERY NON-ALGEBRAIC
EXPANDING MAPS

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Abstract. We show that the space of expanding maps contains an open and dense set where smooth conjugacy classes of expanding maps are characterized by the values of the Jacobians of return maps at periodic points.

1. Introduction

Let $M$ be a smooth closed manifold. Recall that a $C^r$, $r \geq 1$, map $f : M \to M$ is called expanding if
\[
\|Dfv\| > \|v\|
\]
for all non-zero $v \in TM$ and some choice of Riemannian metric on $M$. It is easy to see that an expanding map is necessarily a covering map.

Recall that expanding maps have been classified up to topological conjugacy. Shub [Sh69] proved that $M$ is covered by the Euclidean space and also that an expanding endomorphism of $M$ is topologically conjugate to an affine expanding endomorphism of an infranilmanifold if and only if the fundamental group $\pi_1(M)$ contains a nilpotent subgroup of finite index. Franks [Fr70] showed that if $M$ admits an expanding endomorphism then $\pi_1(M)$ has polynomial growth. Finally, in 1981, Gromov [Gr81] completed classification by showing that any finitely generated group of polynomial growth contains a nilpotent subgroup of finite index. Hence any expanding endomorphism is topologically conjugate to an affine expanding endomorphism of an infranilmanifold.

Let $f_i : M_i \to M_i$ be $C^r$ smooth, $r \geq 1$, expanding maps $i = 1, 2$. Also we will assume that $f_1$ and $f_2$ are conjugated via a homeomorphism $h : M_1 \to M_2$, i.e., $h \circ f_1 = f_2 \circ h$. For example, homotopic expanding maps on the same manifold are always conjugate.

It is well known that $h$ is necessarily bi-Hölder continuous. However, a priori $h$ is not $C^1$ smooth with obvious obstructions carried by the eigendata of periodic points. That is, when $h$ is $C^1$, the differential of the return map $Df^n_1(x)$ is conjugate to $Df^n_2(h(x))$ for $x = f^n_1(x)$. A weaker necessary assumption is coincidence of jacobians
\[
\text{Jac}(f^n_1)(x) = \text{Jac}(f^n_2)(h(x))
\]
for all periodic points $x = f^n_1(x)$.

The authors were partially supported by NSF grants DMS-1823150 and DMS-1500947 & DMS-1900778, respectively.

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In this paper we offer the following progress for higher dimensional expanding maps. For \( r \geq 2 \) there exists a \( C^r \)-dense and \( C^1 \)-open subset \( \mathcal{U} \) in the space of \( C^r \) expanding maps such that if \( f_1, f_2 \in \mathcal{U} \) and \( f_2 \) is an expanding map which is conjugate to \( f_1 \) and has the same Jacobian data then the conjugacy is \( C^{r-1} \). In the proof we use the fact that \( f_1 \) lives on an infranilmanifold. We give precise statements which, in particular, explicitly describe the set \( \mathcal{U} \) in the next section. Our proof of this result was partially inspired by the Embedding theorem (or Reconstruction theorem) of Takens [T81].

In dimension 1 smooth classification was already known. Indeed, Shub and Sullivan showed that for \( C^r \), \( r \geq 2 \), expanding maps of the circle \( S^1 \) the above condition on coincidence of Jacobians implies that the conjugacy \( h \) is \( C^r \) smooth [SS85]. In fact they proved a stronger result that an absolutely continuous conjugacy (which is not, a priori, even continuous) must be coincide a.e. with smooth conjugacy provided that the Jacobian of one of the expanding maps is not cohomologous to a constant.

The analogous “smooth conjugacy problem” in the setting of Anosov diffeomorphisms was completely resolved by de la Llave, Marco and Moriyón in dimension 2 [dIL87, dILM88, dIL92]. In higher dimensions there was a lot of partial progress, e.g., see [dIL04, G08, KS09] and references therein. However progress was made only for certain special classes of Anosov diffeomorphisms such as conformal or with a fine dominated splitting. When compared to this body of work, the current paper is very different. It relies on a fundamentally different approach — too examine matching functions rather than matching measures. And it yields a result on a large open set set rather than characterization of smooth conjugacy classes of certain special maps.

The next section contains the statement of our main technical result Theorem 2.1. Then we state a number of corollaries for smooth conjugacy problem and discuss necessity of various assumptions. Section 3 is devoted to preliminaries on properties of the transfer operator associated to an expanding map. Section 4 and 5 contain the proofs. In Section 6 we give a number of examples of expanding maps illustrating various features of our results and proofs. Finally, in Section 7 we state a generalized factor version of Theorem 2.1 and give an application.

Acknowledgement: The second author was spending his sabbatical year in Laboratoire Paul Painlevé, Université de Lille during this research, he wants to thank them and specially Livio Flaminio for their warm and generous hospitality. He also thanks Livio Flaminio for all the discussions.

2. The results.

We adopt the standard convention and call a map \( f: M \to M \) \( C^r \)-smooth, \( r \geq 0 \), if it is \( [r] \) times continuously differentiable and its \( C^{[r]} \)-differential is Hölder
continuous with exponent $r - |r|$. We also allow $r = \infty$ and $r = \omega$ (real analytic maps). One defines $C^r$ smooth functions on $M$ in the same way.

Recall that we denote by $f_i : M_i \to M_i$, $i = 1, 2$, $C^r$ smooth expanding maps $i = 1, 2$ and we assume that $f_1$ and $f_2$ are conjugated, $h \circ f_1 = f_2 \circ h$. Given functions $\phi_i : M_i \to \mathbb{R}$, $i = 1, 2$, we say that $(f_1, \phi_1)$ is equivalent to $(f_2, \phi_2)$ and write

$$(f_1, \phi_1) \sim (f_2, \phi_2)$$

if there exists a function $u : M_1 \to \mathbb{R}$ such that

$$\phi_1 - \phi_2 \circ h = u - u \circ f_1$$

It is well-known (see e.g., [Bow75]) that by the Livshits theorem $(f_1, \phi_1) \sim (f_2, \phi_2)$ if and only if for every periodic point $x \in \text{Fix}(f_1^n)$

$$\sum_{k=0}^{n-1} \phi_1(f_1^k(x)) = \sum_{k=0}^{n-1} \phi_2(f_2^k(h(x)))$$

Further, if $\phi_i$ are $C^r$ smooth then the transfer function $u$ is also $C^r$ smooth.

**Theorem 2.1.** Assume that $M_i$, $i = 1, 2$, are closed manifolds homeomorphic to a nilmanifold. Let $f_i : M_i \to M_i$, $i = 1, 2$, be $C^r$, $r \geq 1$, smooth expanding maps and assume they are conjugate via a homeomorphism $h : M_1 \to M_2$. Then there exist manifolds $\tilde{M}_i$ (which are homeomorphic to a nilmanifold) and $C^r$ fibrations (whose fibers are homeomorphic to a connected nilmanifold) $p_i : \tilde{M}_i \to \tilde{M}_i$, $i = 1, 2$, and $C^r$ expanding maps $\tilde{f}_i : \tilde{M}_i \to \tilde{M}_i$, such that $f_i$ fibers over $\tilde{f}_i$

$$p_i \circ f_i = \tilde{f}_i \circ p_i, \quad i = 1, 2$$

The conjugacy $h$ maps fibers to fibers i.e.,

$$p_2 \circ h = \tilde{h} \circ p_1$$

where the induced conjugacy $\tilde{h} : \tilde{M}_1 \to \tilde{M}_2$, $\tilde{h} \circ \tilde{f}_1 = \tilde{f}_2 \circ \tilde{h}$, is a $C^r$ diffeomorphism.

Further, the fibrations $p_i$, $i = 1, 2$, have the following property. If $\phi_i : M_i \to \mathbb{R}$, $i = 1, 2$, are $C^r$ smooth functions such that $(f_1, \phi_1) \sim (f_2, \phi_2)$ then there exist $C^r$ functions $\tilde{\phi}_i : \tilde{M}_i \to \mathbb{R}$, $i = 1, 2$, such that $\phi_i$ is $f_i$–cohomologous to $\tilde{\phi}_i \circ p_i$, $i = 1, 2$, and

$$\tilde{\phi}_2 \circ \tilde{h} = \tilde{\phi}_1$$

**Remark 2.2.** Manifold $\tilde{M}_1$ may be equal to $M_1$ or may be a point or some dimension in between. In the first case we obtain that $f_1$ and $f_2$ are $C^r$ smoothly conjugate and in the second case we obtain that the functions $\phi_1$ and $\phi_2$ are cohomologous to a constant.

Also notice that the regularity of the $f_i$’s and $\phi_1$’s may be different to start with. Then naturally one takes $r$ to be the minimal value. Moreover, for a given pair of $f_i$, $i = 1, 2$, but different choices of $r \geq 1$, the resulting fibrations $p_i$, $i = 1, 2$, may, in fact, depend on $r$. 
Remark 2.3. If one does not assume that $M_i$ are homeomorphic to a nilmanifold then, instead of fibrations, the construction in the proof of Theorem 2.1 yields compact foliations $F_i$ i.e., foliations with all leaves compact. Further, by improving the argument used to show that the leaves of $F_i$ are compact, one can check that these foliations are generalized Seifert fibrations. The argument for compactness and the Seifert property of the foliation is independent of classification of expanding maps. Klein bottle Example 6.7 shows that such foliations, indeed, can have singular leaves on infranilmanifolds, that is, they are not necessarily locally trivial fibrations. Hence the assumption that $M_i$ are homeomorphic to nilmanifolds is a necessary one. However, in practice, this assumption is not a big restriction. Indeed, by classification, any manifold which supports an expanding map is homeomorphic to an infranilmanifold. Hence, one can always lift given expanding maps to finite nilmanifold covers.

Remark 2.4. It will become clear from the proof of Theorem 2.1 that the fibrations $p_i$ are uniquely determined by $f_i$, $h$, and $r$. However, if one does not require the latter property in the statement, i.e., that “matching” functions $\varphi_i$ are cohomologous to $\bar{\varphi}_i \circ p_i$ then the choice of fibrations, in general, is not unique. For example, there is always the trivial fibration whose fibers are points. In general there are finitely or infinitely many distinct smooth fibrations for a given expanding map and the maximal number of possible fibrations occurs when $h$ is smooth. This maximal number of fibrations is determined by the linearization of $f_i$ (see also Remark 6.2). There is also a naturally defined partial order on the set of fibrations with the trivial one being subordinate to any other fibration and the one given by Theorem 2.1 being the maximal one.

Remark 2.5. Recall that there exist expanding maps on exotic nilmanifolds, i.e., manifolds homeomorphic but not diffeomorphic (or even not $PL$-homeomorphic) to nilmanifolds [FJ78, FG14]. Our theorem applies to such examples. Moreover, by using Gromoll’s filtration, one can construct expanding map $f_1: M_1 \to M_1$, on a nilmanifold $M_1$ and an expanding map $f_2: M_2 \to M_2$ on an exotic nilmanifold $M_2$ in such a way that the fibrations $p_i: M_i \to \bar{M}_i$ are non-trivial, i.e., $\dim M_i > \dim \bar{M}_i > 0$. Also note that our theorem applies in the case when both $M_1$ and $M_2$ are exotic. We elaborate on this remark in Example 6.8.

A linear expanding endomorphism $L$ of a $d$-dimensional torus $M$ is called irreducible if the characteristic polynomial of the integral matrix defining $L$ is irreducible over $\mathbb{Z}$; equivalently $L$ does not have non-trivial invariant rational subspaces. Recall that any expanding map $f: M \to M$ is conjugate to an expanding endomorphism $L$. We will say $f$ is irreducible if $L$ is irreducible.

Corollary 2.6. Let $M_i$ be manifolds homeomorphic to the $d$-dimensional torus. Assume that $f_i: M_i \to M_i$ are $C^{r+1}$ smooth, $r \geq 1$, expanding maps. Assume that they are conjugate via $h$. Also assume that $f_1$ is irreducible and that the
entropy maximizing measure for $f_1$ is not absolutely continuous. If $\text{Jac}(f^n_1)(x) = \text{Jac}(f^n_2)(h(x))$ for every $x \in \text{Fix}(f^n_1)$ then $h$ is a $C^r$ diffeomorphism.

We make four remarks pertaining this corollary.

**Remark 2.7.** The condition on the measure of maximal entropy can be detected from a pair of periodic points. Hence the space of expanding maps which satisfy this assumption is $C^{r+1}$ dense and $C^1$ open in the space of expanding maps.

**Remark 2.8.** The analogue of Corollary 2.6 for non-abelian nilmanifolds is vacuous. This is because every linear expanding maps on a nilmanifolds leaves invariant the fibration given by the center subgroup of the corresponding nilpotent Lie group.

**Remark 2.9.** Recall that an infratorus $M$ is a closed manifold covered by the torus $T^d$. The Deck transformations of the covering $T^d \to M$ have the form $x \mapsto Qx + v$ and the linear parts $Q$ form so called holonomy group of $M$. We can define an expanding map $f: M \to M$ to be irreducible if its’ lift to $T^d$ is irreducible. Then Corollary 2.6 holds for such irreducible expanding maps of infratori by first passing to the torus cover and then arguing in the same way.

However, the supply of irreducible examples of expanding endomorphisms of infratori which are not tori is rather limited. Notice that any $Q \neq Id$ from the holonomy group has 1 for an eigenvalue. Indeed otherwise corresponding affine map of the torus $x \mapsto Qx + v$ would have a fixed point by the Lefschetz formula. Further $L$ acts on the holonomy group by conjugation. Hence, because the holonomy group is finite, for a sufficiently large $k$, $L^k$ and $Q$ commute and, hence, $L^k$ leaves invariant the non-trivial rational subspace — the eigenspace space of eigenvalue 1 for $Q$. Hence all irreducible examples must become reducible after passing to a finite power. Still such examples exist and we present one such example as Example 6.6.

**Remark 2.10.** Define the critical regularity $r_0$ by

$$r_0(f_1) = \min_{n \geq 1} \max_{x \in M_1} \log \frac{\|D_x f^n_1\|}{\log m(D_x f^n_1)}$$

where $m$ is the conorm. Then by the argument of de la Llave [dIL92, Section 6] one can rectify the loss of one derivative and bootstrap the regularity of the conjugacy. That is if $r > r_0$ then the $C^r$ conjugacy given by Corollary 2.6 is, in fact, $C^{r+1}$. Same observation applies to other statements in this section.

In fact $r_0(f_1)$ admits an alternative expression

$$r_0(f_1) = \max_{p \in \text{Per}(f_1)} \frac{\lambda^+(p)}{-\lambda^-(p)}$$

where $\lambda^+(p)$ is respectively the largest/smallest Lyapunov exponent for $f_1$ at $p$. Therefore $r_0(f_1)$ can be computed directly from Lyapunov exponents along periodic orbits. To see that the two formulae give the same value $r_0(f_1)$ one can pass to the invertible solenoid diffeomorphism and apply the approximation result [WW10].
Notice also that a priori it does not follow from the hypothesis of Corollary 2.6 that $r_0(f_1) = r_0(f_2)$, however a posteriori one obtains this equality from smoothness of the conjugacy.

We say that an expanding map $f: M \to M$ is \textit{very non-algebraic} if for every $\lambda \in \mathbb{Z}$ and for every $m, 1 \leq m \leq \dim(M)$, there exists a periodic point $x$ of period $n$ such that $\lambda^n$ is not an eigenvalue of the $m$-fold exterior power

$$\bigwedge^m D_x f^n$$

Notice that this condition is open and dense.

**Corollary 2.11.** Assume that $f_i: M_i \to M_i$ are $C^{r+1}$ smooth, $r \geq 1$, expanding maps. Assume that they are topologically conjugate and also assume that $f_1: M_1 \to M_1$ is very non-algebraic. Furthermore, assume that for every periodic point $x$ of $f_1$ of period $n$

$$\text{Jac}(f_1^n)(x) = \text{Jac}(f_2^n)(h(x))$$

Then $h$ is a $C^r$ diffeomorphism.

**Remark 2.12.** It will be clear from the proof that the very non-algebraic assumption can be weakened to asking that for $m = 1, 2, \ldots, \dim(M)$ if $\lambda \in \mathbb{Z}$ appears in the spectrum of

$$\bigwedge^m D_{f_1 \ast}$$

then $\lambda^n$ does not appear in the spectrum of

$$\bigwedge^m D_{x} f_1^n$$

for some periodic point $x, x = f_1^n x$. Here $f_1 \ast$ stands for the linear expanding automorphism induced by $f_1$ on the nilpotent Lie group and $Df_1 \ast$ is the corresponding Lie algebra automorphism.

Note that the very non-algebraic condition prevents $f_1$ from being linear.

Given two linear maps $D_i: \mathbb{R}^d \to \mathbb{R}^d, i = 1, 2$, we say that $D_1$ and $D_2$ have disjoint spectrum if for every $m = 1, \ldots, d$, the $m$-th exterior powers $\wedge^m D_1$ and $\wedge^m D_2$ do not share any real eigenvalues. Given two periodic points $x = f^k(x)$ and $y = f^l(y)$ we say that they have disjoint spectrum if the differentials $D_x f^{kl}$ and $D_y f^{kl}$ have disjoint spectrum.

**Corollary 2.13.** Assume that $f_i: M_i \to M_i$ are $C^{r+1}$ smooth, $r \geq 1$, expanding maps. Assume that they are conjugate and also assume that there exists $f_1$-periodic points $x$ and $y$ which have disjoint spectrum. If for every periodic point $x$ of $f_1$ of period $n$ the Jacobians $\text{Jac}(f_1^n)(x)$ and $\text{Jac}(f_2^n)(h(x))$ coincide then $f_1$ is $C^r$ conjugate to $f_2$.

Corollary 2.13 follows directly from Corollary 2.11 since the property of having two periodic points with disjoint spectrum implies the very non-algebraic property.
Corollary 2.14. Let \( r \geq 1 \). If two \( C^{r+1} \) very non-algebraic expanding maps which are conjugate via an absolutely continuous homeomorphism \( h \) then \( h \) is, in fact, \( C^r \) smooth.

Corollary 2.14 follows directly from Corollary 2.11. Indeed, by ergodicity \( h \) must map the smooth absolutely continuous measure of \( f_1 \) to the smooth absolutely continuous measure for \( f_2 \). It follows that the Jacobians at corresponding periodic points must be equal.

3. Krzyżewski-Sacksteder Theorem for expanding maps

Given a \( C^r \), \( r \geq 1 \), expanding map \( f: M \to M \) and a \( C^r \) potential \( \varphi: M \to \mathbb{R} \) the transfer operator \( \mathcal{L}_{\varphi,f}: C^k(M) \to C^k(M) \) given by

\[
\mathcal{L}_{\varphi,f} u(x) = \sum_{y \in f^{-1}x} e^{\varphi(y)} u(y)
\]

is defined for \( C^k \) functions \( u \), where \( k \leq r \). When no confusion is possible we abbreviate the notation for the transfer operator to \( \mathcal{L}_{\varphi} \).

Theorem 3.1 (Ruelle-Perron-Frobenius/Krzyżewski-Sacksteder). Let \( f: M \to M \) be a \( C^r \), \( r \geq 1 \), expanding map and let \( \varphi: K \to \mathbb{R} \) be a \( C^r \) potential; let \( 0 \leq k \leq r \). Then the transfer operator \( \mathcal{L}_{\varphi}: C^k(M) \to C^k(M) \) has a unique maximal positive eigenvalue \( e^c \)

\[
\mathcal{L}_{\varphi} e^u = e^{c+u}
\]

Corresponding eigenfunction \( e^u \) is positive and is unique up to scaling. The eigenvalue \( e^c \) and the eigenvalue \( e^u \) are independent of the choice of \( k \in [0,r] \). Further, \( e^u \) is \( C^r \) smooth.

Remark 3.2. Originally this theorem was established by Ruelle for a more general class of expanding maps and in Hölder regularity [Rue68, Rue76] (see also [Bow75, 1.7]). Sacksteder [Sac74] and Krzyżewski [Krz77] independently established regularity of the eigenfunction. Krzyżewski [Krz82] has done the analytic case as well. We note that both Sacksteder and Krzyżewski only considered the case when \( \varphi = -\log \text{Jac}(f) \) because they were interested in regularity of the smooth invariant measure for \( f \). However the proofs work equally well for arbitrary smooth potentials.

Note that the uniqueness of the eigenspace occurs already among continuous functions provided that the potential is at least Hölder.

Corollary 3.3. Let \( f \) and \( \varphi \) be the same as in Theorem 3.1. Then there exists a unique \( C^r \) smooth function \( \hat{\varphi}: M \to \mathbb{R} \) and a unique constant \( c \) given by Theorem 3.1 such that

1. \( \hat{\varphi} + c \) is cohomologous to \( \varphi \);
2. \( 1 \) is the maximal eigenvalue of the transfer operator \( \mathcal{L}_{\hat{\varphi}} \);
3. \( \mathcal{L}_{\hat{\varphi}} 1 = 1 \)
Proof. Let $e^c$ be the maximal eigenvalue with eigenfunction $e^u$ for $L_\varphi$ given by Theorem 3.1

$$L_\varphi e^u = e^{c+u}$$

Let $\hat{\varphi} = \varphi - c + u - u \circ f$ then

$$L_{\hat{\varphi}} 1 = 1$$

It is also clear that 1 is the maximal eigenvalue of $L_{\hat{\varphi}}$ since otherwise $e^c$ would not be maximal positive eigenvalue for $L_\varphi$.

Further, assume that $c' \in \mathbb{R}$ and $\varphi'$ continuous also satisfy the conclusion of the corollary with

$$\varphi' = \varphi - c' + u' - u' \circ f$$

Then by the same calculation we have

$$L_\varphi e^{u'} = e^{c'+u'}$$

with $e^{c'}$ being the maximal positive eigenvalue. Hence, by the uniqueness part of Theorem 3.1 we obtain that $c = c'$ and $u = u'$.

Such normalized potentials $\hat{\varphi}$ have been recently studied in the context of thermodynamical formalism [GKLM18].

Remark 3.4. Constant $c$ equals to topological pressure $P(\varphi)$. It follows that if $(f_1, \varphi_1) \sim (f_2, \varphi_2)$ then the maximal eigenvalue is the same for corresponding operators and hence $(f_1, \hat{\varphi}_1) \sim (f_2, \hat{\varphi}_2)$. (But we won’t use this fact.)

Remark 3.5. Let $e^c$ be the maximal positive eigenvalue for $L_\varphi$ with eigenfunction $e^u$ and assume that $e^w$ is another positive continuous eigenfunction for $L_\varphi$, i.e., $L_\varphi e^w = \sigma e^w$ for some $\sigma \in \mathbb{R}$, then $w = u + k$ for some $k \in \mathbb{R}$ and $\sigma$ is the maximal eigenvalue. Notice that it follows that condition 2 of Corollary 3.3 is automatic from condition 3 because a positive eigenfunction necessarily corresponds to the maximal eigenvalue. (But we won’t use this fact.)

4. Proofs

4.1. Foliations. We begin by explaining the construction of fibrations $p_i$, $i = 1, 2$, which appear in Theorem 2.1.

Recall that $h \circ f_1 = f_2 \circ h$ and consider the following space of pairs of smooth functions

$$V^r = \{ (\psi_1, \psi_2) \in C^r(M_1) \times C^r(M_2) : \psi_1 = \psi_2 \circ h \}$$

This is a closed subspace of $C^r(M_1) \times C^r(M_2)$. Note that if $(\psi_1, \psi_2) \in V^r$ then $(\psi_1 \circ f_1, \psi_2 \circ f_2) \in V^r$. Also note that $V^r$ always contains constants $(c, c)$ and is an algebra. Denote by $V_i^r$ the projection of $V^r$ on $C^r(M_i)$, $i = 1, 2$.

Define the subspaces $E_i(x) \subset T_x M_i$ by

$$E_i(x) = \bigcap_{\psi_i \in V_i} \ker d_x \psi_i$$
Notice that if \( x_n \to x, \ n \to \infty \), then \( \limsup E_i(x_n) \subset E_i(x) \). This property implies that the function \( d_i \) given by \( d_i(x) = \dim E_i(x) \) is upper semicontinuous. Let \( m_i = \min_{x \in M_i} d_i(x) \), then upper semicontinuity implies that the set

\[ U_i = \{ x \in M_i : d_i(x) = m_i \} \]

is open. It is clear from the definition that \( D_x f_i E_i(x) \subset E_i(f_i(x)) \). Hence if \( f_i(x) \in U_i \) then \( x \in U_i \), i.e.,

\[ f_i^{-1}(U_i) \subset U_i \]

Since \( U_i \) is a non-empty open set and \( f_i \) is expanding, we obtain that \( U_i = M_i \).

It is easy to see now that the distributions \( E_i \) integrate to \( C^r \) foliation \( F_i \). Indeed, for every \( x \in M_i \) there exist finitely many functions \( \psi_1, \ldots, \psi_{d-m_i} \in V_i^r \) such that

\[ E_i(x) = \bigcap_{j=1}^{d-m_i} \ker d_x \psi_j \]

Indeed, just take \( \psi_j \) such that \( \{d_x \psi_j\}_j \) is a maximal linearly independent set of \( \{d_x \psi\}_{\psi \in V_i^r} \).

By continuity of \( d\psi \) and since \( E_i \) has constant dimension, the same formula holds on a small neighborhood of \( x \). That is, there exists a neighborhood of \( U_{i,x} \) of \( x \) such that

\[ E_i(y) = \bigcap_{j=1}^{d-m_i} \ker d_y \psi_j \]

for all \( y \in U_{i,x} \). Therefore, by the implicit function theorem, we have that the maps

\[ \Psi_{i,x} : U_{i,x} \to \mathbb{R}^{d-m_i}, \]

\[ \Psi_{i,x}(y) = (\psi_1(y), \ldots, \psi_{d-m_i}(y)) \]

define a foliation atlas of a \( C^r \) foliation which is tangent to \( E_i \). We denote these foliations by \( F_i, \ i = 1, 2 \).

**Lemma 4.1.** The leaves of \( F_i \) are compact. In fact, the leaf \( F_i(x) \) for \( x \in M_i \), is the connected component of \( x \) of the intersection

\[ \bigcap_{\psi \in V_i^r} \psi^{-1}(\psi(x)) \]

Moreover, for each leaf \( F(x) \) one can pick finitely many functions \( \psi_1, \psi_2, \ldots, \psi_N \in V_i^r \) such that \( F_i(x) \) is the connected component of \( x \) of the intersection

\[ \bigcap_{j=1}^{N} \psi_j^{-1}(\psi_j(x)) \]

**Proof.** Let \( \psi \) be a function in \( V_i^r \) and let \( x \in M_i \). Then by the definition

\[ E_i(y) \subset \ker d_y \psi \]

for every \( y \in F_i(x) \). Hence \( \psi \) is constant on \( F_i(x) \) and \( F_i(x) \subset \psi^{-1}(\psi(x)) \) and hence

\[ F_i(x) \subset \bigcap_{\psi \in V_i^r} \psi^{-1}(\psi(x)) \]
On the other hand, recall that, locally, for sufficiently small $U_{i,x} \ni x$ we have the foliation chart and hence

$$\mathcal{F}_i(x) \cap U_{i,x} = \Psi_{i,x}^{-1}(\Psi_{i,x}(x)) = \bigcap_{j=1}^{d-m_i} (\psi_j^{i})^{-1}(\psi_j^{i}(x)) \cap U_{i,x} \supset \bigcap_{\psi \in V_i} \psi^{-1}(\psi(x)) \cap U_{i,x}$$

and the main claim of the lemma follows. Finally to see the last claim of the lemma we can use compactness of $\mathcal{F}_i(x)$ and cover $\mathcal{F}_i(x)$ by finitely many small neighborhoods such that each neighborhood one needs $d - m_i$ functions. \hfill $\square$

Recall that for every function $\psi_1 \in V_i^r$ there is $\psi_2 \in V_i^r$ such that $\psi_2 \circ h = \psi_1$ and vice versa. This implies that $h(\mathcal{F}_1(x)) = \mathcal{F}_2(h(x))$ for every $x \in M_1$. Hence by the invariance of domain theorem we obtain $m_1 = m_2$, i.e., the dimensions of foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are the same.

When $M_i$ are nilmanifolds the foliations $\mathcal{F}_i$ are actually fibrations $p_i: M_i \to \hat{M}_i$, whose fibers and the base spaces $\hat{M}_i$ are homeomorphic to nilmanifolds. We postpone this argument of a different nature until Corollary 5.5 of the next section and complete the proof of Theorem 2.1 first.

Because $h$ sends $\mathcal{F}_1$ to $\mathcal{F}_2$ it induces a homeomorphism $\tilde{h}: \hat{M}_1 \to \hat{M}_2$. To see that $\tilde{h}$ is smooth consider foliations charts around $x$ and $h(x)$, $x \in M_1$, given by

$$\Psi_{1,x}(y) = (\psi_{1,x}^1(y), \ldots, \psi_{1,x}^{d-m_1}(y)),$$

and

$$\Psi_{2,h(x)}(y) = (\psi_{2,h(x)}^1(y), \ldots, \psi_{2,h(x)}^{d-m_2}(y))$$

respectively. In these local coordinates $\tilde{h}$ is given by $\tilde{h}(\Psi_{1,x}(y)) = \Psi_{2,h(x)}(h(y))$. However, by definition, we know that there exist $C^r$ functions $\tilde{\psi}_{j, h(x)})$ which satisfy

$$\tilde{\psi}_{j, h(x)} = \psi_{j, h(x)} \circ h, j = 1, \ldots, d - m_2.$$ Hence, $\tilde{h}$ is given by

$$\tilde{h}(\psi_{1,x}^1(y), \ldots, \psi_{1,x}^{d-m_1}(y)) = (\psi_{1,h(x)}^1(y), \ldots, \psi_{1,h(x)}^{d-m_2}(y))$$

and since $\Psi_{1,x}$ is a $C^r$ submersion on a neighborhood of $x$ we conclude that $\tilde{h}$ is $C^r$ on a neighborhood of $p_1(x)$. Symmetric argument proves that $\tilde{h}^{-1}$ is $C^r$.

4.2. End of the proof of Theorem 2.1. Finally we need to show that given $(f_1, \varphi_1) \sim (f_2, \varphi_2)$ we have that $\varphi_i$ are cohomologous to functions in $V_i$.

By Corollary 3.3 we have $C^r$ functions $\tilde{\varphi}_i$ and constants $c_i \in \mathbb{R}$ such that $\varphi_i$ is $f_i$-cohomologous to $\tilde{\varphi}_i + c_i$ and we also have $\mathcal{L}_{\tilde{\varphi}_i, f_i} 1 = 1$, $\mathcal{L}_{\tilde{\varphi}_2, f_2} 1 = 1$. Moreover, $\tilde{\varphi}_i$ are unique among the functions cohomologous to $\varphi_i$ up to a constant with this property. We know that $\tilde{\varphi}_2 \circ h$ is cohomologous to $\tilde{\varphi}_1 + c_2 - c_1$. In fact, we will show that

$$\tilde{\varphi}_2 \circ h = \tilde{\varphi}_1$$

By direct calculation, we have that

$$(\mathcal{L}_{\tilde{\varphi}_2, f_2} v) \circ h = \mathcal{L}_{\tilde{\varphi}_2 \circ h, f_1} (v \circ h)$$

for every function $v$. In particular, for the constant function $v = 1$ we have

$$1 \circ h = (\mathcal{L}_{\tilde{\varphi}_2, f_2} 1) \circ h = \mathcal{L}_{\tilde{\varphi}_2 \circ h, f_1} (1 \circ h) = \mathcal{L}_{\tilde{\varphi}_2 \circ h, f_1} (1)$$
Since \( \hat{\varphi}_2 \circ h \) is cohomologous to \( \varphi_1 \) up to a constant we get that \( \hat{\varphi}_2 \circ h = \hat{\varphi}_1 \). Hence \((\hat{\varphi}_1, \hat{\varphi}_2) \in V^r \) and, by the definition of foliations \( \mathcal{F}_i \), we conclude that \( \hat{\varphi}_i \) is constant on \( \mathcal{F}_i \), \( i = 1, 2 \). It remains to set \( \hat{\varphi}_i(p_i(x)) = \hat{\varphi}_i(x) + c_i \).

4.3. Proofs of Corollaries.

**Proof of Corollary 2.6.** We denote by \( L \) the linear endomorphism to which both \( f_1 \) and \( f_2 \) are conjugated.

By passing to the second iterate we may assume \( \text{Jac}(f_i) > 0 \), \( i = 1, 2 \). Let \( \varphi_i = -\log \text{Jac}(f_i) \). By Theorem 2.1 we have \( C^r \) fibrations (with connected fiber) \( p_i : M_i \to \tilde{M}_i \) and functions \( \hat{\varphi}_i : \tilde{M}_i \to \mathbb{R} \) such that \( \hat{\varphi}_i \circ p_i \) is cohomologous to \( \varphi_i \) and the induced conjugacy \( \hat{h} : \tilde{M}_1 \to \tilde{M}_2, \hat{h} \circ p_1 = p_2 \circ h \), is a \( C^r \) diffeomorphism.

If \( \dim \tilde{M}_1 = 1 \), then \( \varphi_1 \) is constant and, hence, \( \varphi_1 \) is cohomologous to a constant. Then the equilibrium state for \( \varphi_1 \), which is the absolutely continuous measure equals the equilibrium state for the constant function which is the entropy maximizing measure [Bow75], contradicting the assumption of the corollary.

If \( \dim \tilde{M}_1 = d \), then \( p_1 \) and \( p_2 \) are diffeomorphisms (in fact, identity diffeomorphisms) and, hence, \( h \) is a \( C^r \) diffeomorphism since \( h = p_2^{-1} \circ \hat{h} \circ p_1 \).

It remains to consider the case when \( 0 < \dim \tilde{M}_1 < d \). From now on we abbreviate \( M = M_1 \) and \( \tilde{M} = \tilde{M}_1 \). Let \( x \) be a fixed point of \( f_1 \) and let \( F \) be the fiber of \( p_1 \) which contains \( x \). Recall that, by Theorem 2.1, \( \tilde{M} \) supports an expanding map \( \hat{f}_1 \) and, hence, is aspherical. Therefore the fundamental groups fit into the short exact sequence

\[
0 \to \pi_1(F) \to \pi_1(M) \to \pi_1(\tilde{M}) \to 0
\]

Note that taking tensor product with \( \mathbb{R} \) leaves the sequence exact.

Because \( f_1(F) = F \) we have \( (f_1)_* (\pi_1(F)) = L_\#(\pi_1(F)) < \pi_1(F) \approx \mathbb{Z}^d \). Since \( \dim \tilde{M} < d \) we have that \( \text{dim } F > 0 \) and \( F \) is compact and also aspherical (because it supports the expanding map \( f_1|_F \)). It follows that \( \pi_1(F) \otimes \mathbb{R} \) gives a non-zero rational invariant subspace for \( L \). Because \( L \) is irreducible we conclude that \( \pi_1(F) \otimes \mathbb{R} = \mathbb{R}^d \). Hence \( \pi_1(M) \otimes \mathbb{R} = 0 \), i.e., \( \pi_1(M) \) is torsion finitely generated abelian group, hence, finite. But any closed aspherical manifold of dimension \( > 0 \) has an infinite fundamental group, a contradiction. \( \square \)

**Proof of Corollary 2.11.** By classification of expanding maps, manifolds \( M_i \) are homeomorphic to infranilmanifolds. Therefore we can pass to the nilmanifold covers and, accordingly, pass to the lifts of expanding maps. It is easy to see that the very non-algebraic assumption still holds for the lifted maps. From now on we assume that \( M_i \) are homeomorphic to nilmanifolds and, hence, Theorem 2.1 applies.

Recall that \( f_i \) are \( C^{r+1} \), \( r \geq 1 \), very non-algebraic expanding maps and \( h : M_1 \to M_2 \) is a conjugacy. We apply Theorem 2.1 to \( f_i \) and \( r \) (not \( r + 1 \)). Let \( p_i : M_i \to \tilde{M}_i, \hat{f}_i : \tilde{M}_i \to \tilde{M}_i, \hat{h} : \tilde{M}_1 \to \tilde{M}_2 \) be the \( C^r \) maps given by Theorem 2.1. We shall show that \( \dim(\tilde{M}_i) = \dim(M_i) \) and, then \( h = \hat{h} \) and by Theorem 2.1 the conjugacy is \( C^r \).
Assume that \( \dim(M_i) - \dim(M_i) = m > 0 \). Recall that by Theorem 2.1 the fibers \( F_{i,x} = p_i^{-1}(p_i(x)) \) are nilmanifolds and, hence, are orientable. Moreover the fibers can be simultaneously coherently oriented because the base space \( M_i \) is also an orientable nilmanifold. We fix a choice of orientation on fibers and on the base. The expanding map \( f_i \) does not necessarily preserve any of the orientations. (And we cannot pass to finite iterates because such operation would not preserve the “very non-algebraic condition.”) Let \( d \) be the absolute value of the degree of the map between the fibers

\[
d = |\deg(f_i|_{F_{i,x}} : F_{i,x} \to F_{i,f_i(x)})|
\]

Note that \( d \) is indeed independent of \( x \) by continuity and is independent of \( i \) because \( f_i \) are conjugate. Further, if \( \dim F_{i,x} > 0 \) then \( d > 1 \) because the expanding map on the fiber through a fixed point is a self cover of degree \( > 1 \).

In the rest of the proof write \( J \) to denote the absolute value of the Jacobian of a map — \( Jf := |\text{Jac}(f)| \). Let \( \psi_i = \log(Jf_i|_{\ker Dp_i}) \). We note that these functions are only \( C^{r-1} \) because the distributions \( \ker Dp_i \) are merely \( C^{r-1} \).

First we pick Riemannian metrics on \( M_i \), \( i = 1,2 \), so that \( h \) is volume preserving (e.g., an isometry) and, hence,

\[
\log J\bar{f}_1 = \log J\bar{f}_2 \circ h
\]

Then pick a smooth connections \( E_i \) for \( p_i \) (subbundles transverse to \( \ker Dp_i \)) and then lift the Riemannian metrics from \( \bar{M}_i \) to \( E_i \). Then consider Riemannian metrics on \( M_i \) which are direct sums of metrics on \( \ker Dp_i \) and the lifted metrics on \( E_i \). By construction, the differential \( Df_i \) have upper-triangular form and we have

\[
\psi_i = \log Jf_i - \log J\bar{f_i} \circ p_i
\]

The Livshits Theorem for expanding maps together with the assumption on Jacobians at periodic points imply that \( \log Jf_1 \) is cohomologous to \( \log Jf_2 \circ h \). Note that \( \log Jf_i \) are \( C^r \) functions. Hence, by the main property of \((p_i, f_i, \bar{h}) \) given by Theorem 2.1, we have that \( \log Jf_i \) is \( f_i \)-cohomologous to a \( C^r \) function which is constant on the fibers. Because \( \log J\bar{f}_i \circ p_i \) are also constant on the fibers, it follows that \( \psi_i \) are cohomologous to \( \bar{\psi}_i \circ p_i \).

In other words, there exist \( C^r \) function \( u_i \) such that

\[
\log(Jf_i|_{\ker Dp_i}) - u_i + u_i \circ f_i = \bar{\psi}_i \circ p_i
\]

Therefore by replacing the volume form \( \omega \) on the fibers \( F_{i,x} \) with the volume form

\[
\bar{\omega} = e^{u_i}\omega
\]

we can assume that the absolute value of the Jacobian of \( f_1|_{\ker Dp_1} \) equals to \( e^{\bar{\psi}_1 \circ p_1} \). Denote by \( \text{vol}(F_{1,x}) \) the total \( \bar{\omega} \)-volume of \( F_{1,x} \). For any \( x \in M_1 \) we have

\[
e^{\bar{\psi}_1 \circ p_1}(x) = \frac{1}{\text{vol}(F_{1,x})} \int_{F_{1,x}} e^{\bar{\psi}_1 \circ p_1} \bar{\omega} = \frac{1}{\text{vol}(F_{1,x})} \int_{F_{1,x}} Jf_1|_{\ker Dp_1} \bar{\omega} = d
\]
Hence for every periodic point $x$, $f^k x = x$ we have that $Jf^k|_{ker Dp_1} = d^k$. This means that either $d^k$ or $(-d)^k$ belongs to the spectrum of \[
bigcap_{i=1}^m Df^k_1(x)\] which contradicts to $f_1$ being very non-algebraic. We conclude that $m = 0$, i.e., $\dim(M_1) = \dim(M_1)$ and we are done. \hfill \square

Note that even though the regularity of $f_i$ is $r + 1$, we use Theorem 2.1 with regularity $r$ because we work with Jacobian of $f_i$.

5. Fibrations

In this section we show that compact invariant foliations on nilmanifolds are, in fact, fibrations whose fibers and base-space are also (homeomorphic to) nilmanifolds. We use $e$ to denote the identity element in a Lie group and we use $[e]$ to denote the coset of $e$.

**Lemma 5.1.** Let $N_i$ be simply connected nilpotent Lie groups and let $\Lambda_i \subset N_i$ be lattices, $i = 1, 2$. Let $L_i : N_i \to N_i$, $i = 1, 2$, be expanding linear automorphisms such that $L_i\Lambda_i \subset \Lambda_i$ and we still denote by $L_i : N_i / \Lambda_i \to N_i / \Lambda_i$ the induced expanding maps. If $h : N_1 / \Lambda_1 \to N_2 / \Lambda_2$ is a continuous map such that $h \circ L_1 = L_2 \circ h$ and $h([e]) = [e]$ then the lift of $h$, $\tilde{h} : N_1 \to N_2$, which maps identity element $e$ to $e$ is a homomorphism. In particular, $\tilde{h}(N_1)$ is a subgroup of $N_2$.

**Proof.** Let $h_\# : \Lambda_1 \to \Lambda_2$ be the homomorphism of fundamental groups induced by $h$. By work Mal’cev [Mal49], there exists a unique homomorphism $H : N_1 \to N_2$ which extends $h_\#$, i.e., $H|_{\Lambda_1} = h_\#$. Since $h \circ L_1 = L_2 \circ h$ and $\tilde{h}$ maps identity element to identity element, we have that $\tilde{h} \circ L_1 = L_2 \circ \tilde{h}$. We also have that $h_\# \circ L_1 = L_2 \circ h_\#$ and hence $H \circ L_1 = L_2 \circ H$.

Note also that by the uniqueness of the lifting property we have that for every $\lambda_1 \in \Lambda_1$,

$$\tilde{h}(\lambda_1) = h_\#(\lambda_1) = H(\lambda_1)$$

Both $\tilde{h}$ and $H$ conjugate $L_1$ and $L_2$. Hence for every $\lambda_1 \in \Lambda_1$ and any $n_1 = L_1^{-k}\lambda_1$, $k \geq 0$ we have

$$L_2^k\tilde{h}(n_1) = \tilde{h}(L_1^k n_1) = \tilde{h}(\lambda_1) = H(\lambda_1) = H(L_1^k n_1) = L_2^k H(n_1)$$

Since $L_2$ is invertible we have $\tilde{h}(n_1) = H(n_1)$. Finally, since $L_1$ is expanding, the $n_1 \in N_1$ such that $L^k n_1 = \lambda_1$ for some $k \geq 0$ and $\lambda_1 \in \Lambda_1$ are dense in $N_1$ and hence $\tilde{h} = H$. \hfill \square

**Lemma 5.2.** Let $f : M \to M$ be an expanding map and let $K \subset M$ be a $C^1$ smooth, compact, connected $f$-invariant submanifold. Let $N$ be a simply connected nilpotent Lie group, let $\Lambda < N$ be a lattice and let $L : N \to N$ be an expanding automorphism such that $L\Lambda \subset \Lambda$. Assume that there is a homeomorphism $h : M \to N / \Lambda$ such that
\( h \circ f = L \circ h \). Then there exists a connected Lie subgroup \( F < N \) and \( g \in N \) such that \( h(K) = gF\Lambda \).

**Proof.** Since the restriction \( f|_K \) is an expanding map, it has a fixed point \( x_0 \). By post-compounding \( h \) with a left translation we may assume that \( h(x_0) = [e] \) and we shall show that \( h(K) = F\Lambda \) for some connected Lie subgroup \( F \), i.e., \( g = e \).

We can apply the classification theorem [Sh69, Gr81] to \( f|_K \). There exists a simply connected nilpotent Lie group \( N_K \), a lattice \( \Lambda_K < N_K \) and an expanding automorphism \( L_K : N_K \to N_K \) such that \( L_K \Lambda_K \subset \Lambda_K \) and a finite-to-one covering map \( h_K : N_K/\Lambda_K \to K \) such that \( h_K \circ L_K = f \circ h_K \), where we also denote by \( L_K : N_K/\Lambda_K \to N_K/\Lambda_K \) the induced expanding map. We may assume that \( h_K([e]) = x_0 \).

Consider \( h \circ h_K : N_K/\Lambda_K \to N/\Lambda \) and notice that \( h \circ h_K([e]) = [e] \) and \( L \circ (h \circ h_K) = (h \circ h_K) \circ L_K \). Then, by Lemma 5.1, we have that \( h \circ h_K \) lifts to a homomorphism \( H : N_K \to N \). The image of \( H \) is a connected Lie subgroup of \( N \) which we denote by \( F, F = H(N_K) \). Then

\[
 h(K) = h \circ h_K(N_K/\Lambda_K) = H(N_K)\Lambda = F\Lambda
\]

\( \square \)

**Remark 5.3.** Note that a priori \( K \) could be infranilmanifold, however, we have proved that \( K \) is homeomorphic to a nilmanifold.

**Lemma 5.4.** Let \( M = N/\Lambda \) be a nilmanifold and let \( F < N \) be a subgroup. Assume that the foliation induced by the left action of \( F \) on \( M \) has all leaves compact. Then \( F \) is normal in \( N \) and the natural projection \( \pi : N \to F\backslash N \) induce a fibration \( \pi : N/\Lambda \to F\backslash N/\Lambda \) whose fibers and the base are nilmanifolds.

Before starting the proof let us introduce some basic notions. Since \( N \) admits a lattice, there is a basis of \( \text{Lie}(N) = \mathfrak{n} \) with rational structure constants [Mal49]. Thus, we may identify \( \mathfrak{n} \) with \( \mathbb{R}^d \) in such a way that \( \exp^{-1}(\Lambda) \subset \mathbb{Q}^d \). Connected Lie subgroups of \( N \) are in one-to-one correspondence with Lie sub-algebras via the exponential map. We say a Lie sub-algebra \( \mathfrak{f} < \mathfrak{n} \) is rational if \( \mathfrak{f} \cap \mathbb{Q}^d \) spans \( \mathfrak{f} \), and, accordingly, we say that a Lie subgroup is rational if its Lie algebra is rational. Notice that a Lie subgroup \( F \) is rational if and only if \( F \cap \Lambda \) is a lattice in \( F \), if and only if \( F\Lambda \) is compact.

**Proof of Lemma 5.4.** Let \( k = \dim F \) and notice that the subset of the grassmanian, \( \text{Rat}(k,d) \subset \text{Gr}(k,d) \), of \( k \)-planes in \( \mathbb{R}^d \) that are generated by rational vectors is totally disconnected (as is any countable Hausdorff metric space). Let \( \mathfrak{f} := \text{Lie}(F) \) and for \( n \in N \) let \( \mathfrak{f}_n := \text{Lie}(n^{-1}F_n) = \text{Ad}_n(\mathfrak{f}) \). The map \( N \ni n \to \mathfrak{f}_n \in \text{Gr}(k,d) \) is continuous. On the other hand, since \( F_n\Lambda \) is compact for every \( n \in N \), we obtain that the conjugate subgroup \( n^{-1}F_n \) is rational for every \( n \) and hence \( \mathfrak{f}_n \in \text{Rat}(k,d) \) for every \( n \in N \). So, since \( n \to \mathfrak{f}_n \) varies continuously and takes values on the
Corollary 5.5. Assume \( F \) is an \( f \)-invariant foliation with all leaves compact and \( C^1 \), where \( f : M \to M \) is an expanding map on a manifold homeomorphic to a nilmanifold. Then \( F \) is a fibration conjugate to a linear fibration for the associated linear map, whose fiber and base are homeomorphic to nilmanifolds.

Proof of Corollary. Let \( h : M \to N/\Lambda \) be the conjugacy to the linear map \( L : N/\Lambda \to N/\Lambda \). Let \( \hat{F} = h(F) \) be the corresponding topological foliation and let \( x_0 = h^{-1}(e) \).

Since \( F(x_0) \) is a compact connected \( C^1 \) invariant submanifold, Lemma 5.2 gives \( F < N \) a subgroup such that \( h(F(x_0)) = \hat{F}[e] = FA \). Observe that \( L(F) = F \) and let \( G \) be the foliation on \( N/\Lambda \) induced by left multiplication by \( F \). It is an \( L \)-invariant foliation and \( \hat{F}[e] = G[e] = FA \) is compact. Since both \( \hat{F} \) and \( G \) are \( L \)-invariant we have that if \( L^n[g] = [e] \) then \( \hat{F}[g] = G[g] \). And since preimages of \([e]\) are dense we have that \( \hat{F} = G \). In particular, the foliation \( G \) is by compact leaves.

Now Lemma 5.4 shows that \( F \) is normal and \( G = \hat{F} \) is a fibration and we obtain the Corollary.

Remark 5.6. Same proof shows that an invariant foliation by compact leaves in an infranilmanifold corresponds to the quotient of a “linear” fibration on a nilmanifold by the deck group.

6. Examples

Example 6.1 (Basic example). Here we give an explicit example where non-trivial fibrations \( p_i : M_i \to \bar{M}_i, i = 1, 2 \), with \( \dim \bar{M}_i \neq 0, \dim M_i \) appear. Consider the expanding maps \( L, f : \mathbb{T}^2 \to \mathbb{T}^2 \) given by \( L(x, y) = (2x, 2y) \) and \( f(x, y) = (g(x), 2y) \), where \( g \) is conjugate to \( \times 2 \) map via nowhere differentiable conjugacy \( h_0, h_0 \circ g = 2h_0 \). For simplicity we may assume that \( g(0) = 0 \) and \( g'(0) < 2 \). Then \( h = (h_0, id_{S^1}) \) is the conjugacy between \( f \) and \( L \). Recall that fibrations \( p_i \) arise from the space of pair of \( C^r \) functions \( (\psi_1, \psi_2) \) which satisfy \( \varphi_1 = \psi_2 \circ h, i.e., \)

\[
\psi_1(x, y) = \psi_2(h_0(x), y)
\]

Clearly any \( C^r \) function \( \psi_1(x, y) = \psi(y) \) belong to this space. We will show that these are the only functions which could appear. Then, it immediately follows that \( p_1(x, y) = p_2(x, y) = y \). That is, \( p_i \) are circle fibrations over \( S^1 \).

Denote by \( \bar{c}_{inf}h_0 \) the lower derivative of \( h_0 \) defined via \( \liminf \). All periodic points which spend sufficiently large proportion of time near 0 have Lyapunov exponent \( < \log 2 \). Such periodic points \( x \) are dense in \( S^1 \) and it is easy to see that
$\partial_{in} h_0(x) = 0$ for any such $x$. Hence differentiating the relation between $\psi_1$ and $\psi_2$ with respect to $x$ yields
\[
\frac{\partial}{\partial x} \psi_1(x, y) = \frac{\partial}{\partial x} \psi_2(h_0(x), y) \partial_{in} h_0(x) = 0
\]
for a dense set of $x$. Hence, indeed, $\psi_1$ and $\psi_2$ are functions of $y$ only.

**Remark 6.2.** Any primitive vector $(m, n) \in \mathbb{Z}^2$ yields a fibration $S^1 \to \mathbb{T}^2 \to S^1$ whose fibers in the universal cover $\mathbb{R}^2$ are lines parallel to the vector $(m, n)$. This gives infinitely different fibrations each of which is preserved by the conformal map $L$ from Example 6.1. Further, similarly to the construction of Example 6.1, one can construct perturbations $f_{(m,n)}$ such that the fibration given by the Theorem 2.1 is precisely the fibration coming from $(m,n)$.

**Example 6.3** (de la Llave example). Non-trivial fibration may appear in a more subtle way when Jacobians full periodic data match. Of course, this can only happen for expanding maps which are not very non-algebraic. The example presented here is due to de la Llave [dlL92].

Consider the maps
\[ L(x, y) = (dx, ay), \quad d \geq 2, a \geq 2, \]
and
\[ f(x, y) = (dx + \alpha(y), ay) \]
Then the conjugacy between $L$ and $f$ has the form
\[ h(x, y) = (x + \beta(y), y) \]
where $\beta$ can be expressed explicitly as the series [dlL92]
\[
\beta(y) = \frac{1}{d} \sum_{i \geq 0} \frac{1}{d^i} \alpha(a^i y)
\]
Notice that $\beta$ is a Weierstrass function. Let
\[ r_0 = \frac{\log d}{\log a} \]
and let $r_0 = n + \theta$ where $r_0 \in \mathbb{N}_0$ and $\theta \in (0, 1]$.

To analyze the regularity of $\beta$ there are several cases to consider which give different answers.

**Lemma 6.4.** Assume that $\alpha \in C^r$, $r = k + \delta$, $k \in \mathbb{N}_0$, $\delta \in [0, 1)$ and let $r_0 = n + \theta$, as above $n \in \mathbb{N}_0$ and $\theta \in (0, 1]$, then
1. Case I: $r < r_0$ then $\beta \in C^r$;
2. Case II: $r > r_0$, $r_0 \notin \mathbb{N}$ then $\beta \in C^{r_0}$;
3. Case III: $r > r_0$, $r_0 = n + 1 \in \mathbb{N}$ then $\beta \in C^{n+x|\log x|}$;
4. Case IV: $r = r_0$ then $\beta \in C^{n+x^\theta|\log x|}$;

In all cases, there is a generic set of $\alpha \in C^r$ where the regularity is optimal, in particular for such $\alpha$, $\beta \notin C^{r_0+\epsilon}$ for any $\epsilon > 0$. 
Proof. We give the proof for Case IV, all other cases being analogous.

By term-wise differentiation we have that
\[
\beta^{(n)}(y) = \frac{1}{d} \sum_{i=0}^{n-1} \left( \frac{a^n}{d} \right)^i \alpha^{(i)}(a^i y)
\]
which is convergent because \(r_0 > n\). Comparing the series for \(\beta\) and \(\beta^{(n)}\), clearly we can assume that \(n = 0\) because the argument for \(n > 0\) would be the same with \(\beta^{(n)}\) in place of \(\beta\).

Let \(A = \max |\alpha|\) and \(C\) the \(\theta\)-Hölder constant for \(\alpha\). Take \(x \neq y\) and let \(N\) be such that
\[
\frac{1}{a^{N-1}} \leq |x - y| \leq \frac{1}{a^{N}}
\]
Then
\[
|\beta(x) - \beta(y)| \leq \sum_{k=0}^{N-1} \frac{1}{a^{\theta k}} |\alpha(a^k x) - \alpha(a^k y)| + \sum_{k \geq N} \frac{1}{a^{\theta k}} |\alpha(a^k x) - \alpha(a^k y)|
\]
The first summand is smaller than
\[
C \sum_{k=0}^{N-1} \frac{1}{a^{\theta k}} a^{\theta k} |x - y|^\theta \leq CN |x - y|^\theta \leq C|x - y|^\theta \|\log |x - y|\|
\]
The second summand is smaller than
\[
\frac{2A}{1 - a^\theta} \frac{1}{a^{N\theta}} \leq C|x - y|^\theta
\]
Hence we obtain the posited \(x^\theta |\log |x||\) modulus of continuity for \(\beta\).

On the other hand, if we assume, to simplify notation, that \(\alpha(0) = 0\), and say \(\alpha(x) > 0\) for \(x > 0\), and that \(\liminf_{x \to 0} \frac{\alpha(x)}{|x|^\theta} > 0\), then, taking \(x > 0\) very close to \(0\) and \(N > 0\) first such that \(a^N x \geq \epsilon_0\), we obtain
\[
|\beta(x) - \beta(0)| \geq \sum_{k=0}^{N-1} \frac{1}{a^{\theta k}} \alpha(a^k x) - \sum_{k \geq N} \frac{1}{a^{\theta k}} \alpha(a^k x)
\]
\[
\geq \left( KN - \frac{2A}{1 - a^\theta} \right) |x|^\theta \geq \left( KC_{\epsilon_0} \log |x|| - \frac{2A}{1 - a^\theta} \right) |x|^\theta
\]
where \(K = \inf_{|x| < \epsilon_0} \frac{\alpha(x)}{|x|^\theta}\). So, by taking \(x\) close enough to \(0\) we see that \(\beta\) is not \(C^\theta\) at \(0\).

Now notice that
\[
\beta(x) = \frac{1}{d} \beta(ax) + \frac{1}{d} \alpha(x)
\]
and \(\alpha \in C^{r_0}\). Let \(A\) be the set of \(x\) such that \(\beta\) is not \(C^{r_0}\) at \(x\). Then, from the above equation, if \(ax \in A\) then \(x \in A\), i.e., \(A\) is backward invariant (and non empty) and hence dense. \(\square\)

Remark 6.5. Notice that if \(r_0 \in \mathbb{N}\), then \(d = a^{r_0}\). And we have that if \(\beta \in C^{r_0}\) then we can differentiate the above equation and obtain that \(\beta^{(r_0)}\) solves the equation
\[
d^{r_0} \beta^{(r_0)}(x) - a^{r_0} \beta^{(r_0)}(ax) = \alpha^{(r_0)}(x)
\]
meaning that \(\frac{\alpha^{(r_0)}}{a}\) is cohomologous to 0, which does not happen for generic \(\alpha\).
Hence if we apply Theorem 2.1 to \( L, f \) and \( r < r_0 \) then the fibrations \( p_i \) are trivial with point fibers.

If \( r \geq r_0 \) we would have \( p_1(x, y) = p_2(x, y) = y \), i.e., fibrations with circle fiber. Let us show this fact.

Differentiating \( \psi_1(x, y) = \psi_2(x + \beta(y), y) \) with respect to \( y \) variable yields

\[
\partial_y \psi_1(x, y) = \partial_y \psi_2(x + \beta(y), y) + \partial_x \psi_2(x + \beta(y), y) \beta'(y)
\]

Notice that \( \partial_x \psi_2(x + \beta(y), y) \in C^{r_0-1} \) and, in particular, it is continuous. Let

\[
U = \{ y : \partial_x \psi_2(x + \beta(y), y) \neq 0 \text{ for some } x \},
\]

then \( U \) is open and for \( y \in U \) and the appropriate \( x \),

\[
\beta'(y) = \frac{\partial_y \psi_1(x, y) - \partial_y \psi_2(x + \beta(y), y)}{\partial_x \psi_2(x + \beta(y), y)}.
\]

So, for \( y \in U \) we obtain that the right hand side is locally in \( C^{r_0-1} \) and hence \( \beta' \) is locally in \( C^{r_0-1} \) for points in \( U \). Hence by Lemma 6.4, \( U \) is empty and hence \( \frac{\partial_x \psi_2(x + \beta(y), y)}{\partial_x \psi_2(x + \beta(y), y)} = 0 \) for all \( x \) and a dense set of \( y \in S^1 \). We conclude that \( \psi_2 \) (and similarly, \( \psi_1 \)) depends solely on the \( y \)-coordinate.

**Example 6.6 (Irreducible automorphism of an infratorus).** We have explained in Remark 2.9 that (non-trivial) infratori do not support totally irreducible affine automorphisms. Here we show that one can still construct irreducible examples (which become reducible after passing to a finite iterate).

Define the expanding endomorphism by

\[
L = \begin{pmatrix} 0 & 0 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\]

Note that \( L^3 \) is diagonal. Define the holonomy group \( \{ id, \gamma_1, \gamma_2, \gamma_3 \} \) as follows

\[
\gamma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

Finally let

\[
v_1 = \begin{pmatrix} 1, 0, 1/2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1, 1/2, 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0, 1, 1/2 \end{pmatrix}
\]

and \( T_i(x) = \gamma_i(x) + v_i \), \( i = 1, 2, 3 \). We let \( \Gamma \) be the group of affine diffeomorphisms of \( T^3 \) generated by the \( T_i \)'s. It is easy to see that, in fact, \( \Gamma = \{ Id_{T^3}, T_1, T_2, T_3 \} \) and, hence, \( \Gamma \) acts freely on \( T^3 \).

Finally \( L \Gamma L^{-1} \subset \Gamma \) and, hence, induces an expanding endomorphism of the infratorus \( T^3/\Gamma \). Indeed, \( L \circ T_1 \circ L^{-1} = T_2 + (1, 0, 0) \), \( L \circ T_2 \circ L^{-1} = T_3 \) and \( L \circ T_3 \circ L^{-1} = T_1 + (1, 0, 0) \).

**Example 6.7 (Seifert fibration).** Recall that in Theorem 2.1 we assume that manifolds \( M_i \) are homeomorphic to nilmanifolds. If \( M_i \) are not homeomorphic to nilmanifolds then the construction of compact foliations in the proof of Theorem 2.1
still works, but these foliations might fail to be fibrations. The example below illustrates this point.

Consider the Klein bottle \( K \) given as a quotient of the torus \( T^2 = \mathbb{R}^2 / \mathbb{Z}^2 \) by the involution \( T(x, y) = (x + \frac{1}{2}, -y) \). We can also model \( K \) as the rectangle \( [0, \frac{1}{2}] \times [-1/2, 1/2] \) where the sides are identified by \( (x, y) \rightarrow (x - 1/2, -y) \) and \( (x, y) \rightarrow (x, y + 1) \). One can easily check that the expanding linear map
\[
L = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}
\]
induces an expanding map \( L: K \rightarrow K \) be an expanding map. We foliate \( K \) the foliation consisting of horizontal curves \( \{ y = \text{const} \} \). More precisely, for every \( y \in [-1/2, 1/2] \) define the circles
\[
C_y = \left\{ (t, y) : t \in \left[ 0, \frac{1}{2} \right] \right\} \cup \left\{ (t, -y) : t \in \left[ 0, \frac{1}{2} \right] \right\}
\]
Notice that if \( y \neq 0, \frac{1}{2} \) then \( C_y \) consists of two segments on the rectangle. For \( y = 0, C_0 \) is a singular curve that consists on only one segment \( [0, 1/2] \times \{ 0 \} \) and hence has half of the length of the other leaves. The same happens for \( y = \frac{1}{2} \), \( C_{\frac{1}{2}} \) is a singular curve that consist on only one segment \( [0, 1/2] \times \{ 1/2 \} \sim [0, 1/2] \times \{-1/2\} \). Moreover, notice that \( C_y = C_{-y} \).

We have defined a foliation on \( K \) which is obviously not a fibration. Indeed, the quotient map \( \pi: K \rightarrow S^1/\{ y = -y \} \) yields an orbifold structure on \( S^1/\{ y = -y \} \).

Notice that \( L(C_y) = C_{(2y \mod 1)} \), hence, the foliation is \( L \)-invariant.

We now now define expanding maps \( f_i: K \rightarrow K, i = 1, 2 \). We let \( f_i(x, y) = (g_i(x), 2y) \), where \( g_i(x) = 3x + \alpha_i(x) \) with \( \alpha_i(0) = 0 \) and \( \alpha_i(x + \frac{1}{2}) = \alpha_i(x) \) for every \( x \in S^1 \). Such formulae define a maps on the Klein bottle which are homotopic to \( L \). Moreover, these maps are expanding provided that \( C^2 \) norms of \( \alpha_i \) are sufficiently small. Also notice that \( f_i \) preserve the foliation \( C \).

The conjugacy between \( f_1 \) and \( f_2 \), \( h \circ f_1 = f_2 \circ h \) has the form \( h(x, y) = (h_0(x), y) \), where \( h_0 \circ g_1 = g_2 \circ h_0 \). Notice that by the symmetries of \( f_i \), \( h_0(x+1/2) = h_0(x) + 1/2 \) and hence \( h \) is indeed the conjugacy on the Klein bottle. We can assume that \( \alpha_i \) are chosen so that \( h_0 \) and, hence, \( h \) is not \( C^1 \).

Take any \( \varphi_0 : \mathbb{R} \rightarrow \mathbb{R} \) such that \( \varphi_0 (y + 1) = \varphi_0 (y) \) and \( \varphi_0 (-y) = \varphi_0 (y) \).\ e.g., \( \varphi_0 (y) = \cos 2 \pi y \). Then \( \varphi(x, y) = \varphi_0 (y) \) defines a function on \( K \) and \( \varphi \circ h = \varphi \). On the other hand if \( \phi = \varphi_2 \circ h \) for some smooth functions \( \varphi_1 \) and \( \varphi_2 \) then both \( \varphi_1 \) and \( \varphi_2 \) must be constant on the leaves of \( C \) because \( h_0 \) is non-differentiable on a dense set of \( x \in S^1 \). So defining \( \varphi_i = \varphi \) for \( i = 1, 2 \) we are in the hypothesis of the Theorem 2.1. The conclude that \( C \) is precisely the compact foliation given by the construction in the proof of Theorem 2.1.

Example 6.8 (Exotic examples). Here we explain that the fiber bundle structure given by Theorem 2.1 could be non-trivial even in the case when the ambient manifold is an exotic torus. Examples of expanding maps on exotic tori were first constructed by Farrell and Jones [FJ78] in dimensions \( d \geq 7 \). We explain how, with
some extra care, the beautiful construction of Farrell-Jones can be adapted to our setting.

Let \( \Sigma^d \) be a \( d \)-dimensional, \( d \geq 7 \), homotopy sphere and let \( \mathbb{T}^d \) be the standard torus. A simple way of constructing an exotic torus is by taking the connected sum \( \mathbb{T}^d \# \Sigma^d \). If \( \Sigma^d \) is not homeomorphic to the standard sphere then \( \mathbb{T}^d \# \Sigma^d \) is not homeomorphic to \( \mathbb{T}^d \) [Wal70, §15A]. Further, it is well-known that for \( d \geq 7 \), one can realize \( \mathbb{T}^d \# \Sigma^d \) as \( \mathbb{T}^d \) with a disk \( D^d \) removed and then glued back in using an orientation-preserving “twist diffeomorphism” \( \varphi \in \text{Diff}(S^{d-1}) \).

\[
\mathbb{T}^d \# \Sigma^d = (\mathbb{T}^d \setminus D^d) \cup_\varphi D^d
\]

It is easy to check that if \( \varphi' \) is isotopic to \( \varphi \) then the corresponding exotic tori are diffeomorphic.

We view the sphere \( S^{d-1} = \partial D^d \) as the standard sphere in \( \mathbb{R}^d \)

\[
S^{d-1} = \{(x_1, x_2, \ldots, x_d) : \sum_i x_i^2 = 1\}
\]

Cerf [Cer61] showed that for every homotopy sphere \( \Sigma^d \) one can realize \( \mathbb{T}^d \# \Sigma^d \) using a diffeomorphism \( \varphi : S^{d-1} \to S^{d-1} \) which preserves the first coordinate, i.e., has the form

\[
\varphi(x_1, x_2, x_3 \ldots x_d) = (x_1, x_2', x_3' \ldots x_d')
\]

Then \( \varphi \) can viewed as a path of diffeomorphisms and gives a representative of an element of \( \pi_1(\text{Diff}(S^{d-2})) \). More generally, one can consider the space \( \text{Diff}_k(S^{d-1}) \) of orientation preserving diffeomorphisms which preserve first \( k \) coordinates \( x_1, x_2, \ldots x_k \) and, hence, give an element of \( \pi_k(\text{Diff}(S^{d-1-k})) \). Isotopy classes of such diffeomorphism form a subgroup \( \Gamma_{k+1}^d \) of the group of isotopy classes of all orientation diffeomorphisms \( \Theta_d \) (which is identified with the group of homotopy spheres equipped with the connected sum operation). It is known that \( \Gamma_{k+1}^d \) is non-trivial in a certain range of pairs \((k, d)\) [ABK70].

Now we formulate the extra property of \( \varphi \in \text{Diff}_k(S^{d-1}) \) which we will need (and which is not needed in the original Farrell-Jones construction). Consider the obvious homomorphism

\[
\gamma : \pi_k(\text{Diff}(S^{d-1-k})) \to \pi_0(\text{Diff}(S^{d-1})) \simeq \Theta_d
\]

**Lemma 6.9.** ([ABK72, Proposition 1.2.3; §1.3]) There exists pairs \((k, d)\)\(^1\) and a torsion element \([\varphi] \in \pi_k(\text{Diff}(S^{d-1-k}))\), \([\varphi^p] = 0\), whose image in \( \pi_0(\text{Diff}(S^{d-1})) \) non-trivial, i.e., \( \gamma[\varphi] \neq 0 \).

We proceed to briefly recall the Farrell-Jones construction [FJ78, Far96] and then explain how the above lemma allows to produce exotic example which admit invariant fibrations with \((d - k)\)-dimensional fibers. The construction yields a \( \times s \)-map on \( \pi_1(\mathbb{T}^d \# \Sigma^d) \) for a sufficiently large \( s \) which also must satisfy certain congruence arithmetic condition.

\(^1\) For specific arithmetic conditions see [ABK72, Corollary 1.3.6].
We pick a \( \varphi \in \text{Diff}_k(\mathbb{S}^{d-1}) \) given by Lemma 6.9 and realize \( \mathbb{T}^d \# \Sigma^d \) by removing a disk \( \mathbb{D}^d \) from \( \mathbb{T}^d \) and then attaching it back with a twist \( \cup \varphi \mathbb{D}^d \). Given an integer \( s \geq 2 \) consider the manifold \( M_s \) which is diffeomorphic to \( \mathbb{T}^d \# \Sigma^d \) and which is obtained by removing the conformally scaled disk \( \frac{1}{s} \mathbb{D}^d \) and then attaching it back with a twist \( \cup \varphi \frac{1}{s} \mathbb{D}^d \). Because of our choice of \( \varphi \) both manifolds are naturally total spaces of smooth torus bundles

\[
\mathbb{T}^{d-k} \to \mathbb{T}^d \# \Sigma^d \xrightarrow{p_1} \mathbb{T}^k, \quad \mathbb{T}^{d-k} \to M_s \to \mathbb{T}^k
\]

where the base space \( \mathbb{T}^d \) corresponds to the first \( k \) coordinates fixed by \( \varphi \).

Let \( N \to \mathbb{T}^d \# \Sigma^d \) be the locally isometric cover which induces \( \times s \) map on the fundamental group. And let \( N_s \) be a copy of \( N \) with the Riemannian metric conformally scaled by \( \frac{1}{s} \). Clearly both \( N_s \) and \( N \) smoothly fiber over \( \mathbb{T}^k \). Then the posited expanding map is the composition

\[
\mathbb{T}^d \# \Sigma^d \xrightarrow{E_s} M_s \xrightarrow{G_s} N_s \xrightarrow{\times s} N \to \mathbb{T}^d \# \Sigma^d
\]

The diffeomorphisms \( F_s \) and \( G_s \) are constructed with a uniform (in \( s \)) lower bound on minimal expansion. It immediately follows that for sufficiently large \( s \) the composite map \( f : \mathbb{T}^d \# \Sigma^d \to \mathbb{T}^d \# \Sigma^d \) is uniformly expanding.

The diffeomorphism \( F_s \) which “shrinks” the exotic sphere is constructed using the “commutator trick” and it is easy to check that \( F_s \) is fiber preserving and fibers over the identity map \( \text{id}_{\mathbb{T}^k} \). We claim that the same is true for the diffeomorphism \( G_s \). The purpose of \( G_s \) is to introduce a certain number of scaled exotic spheres \( \cup \varphi \frac{1}{s} \mathbb{D}^d \), and, thus, create \( N_s \). These exotic spheres are introduced in groups of size \( b \) which is divisible by the order of \( \varphi \) in \( \Theta_d \) [FJ78, Lemma 3]. Alternatively one can think of \( G_s^{-1} \) as a diffeomorphism which removes exotic spheres in groups of size \( b \). To remove one such group one uses diffeomorphism given by the isotopy between \( \varphi^b \) and \( \text{id}_{\mathbb{T}^{d-1}} \). A priori such an isotopy does not preserve the fibers. However, we can require \( b \) to be divisible by \( p \) which is given by Lemma 6.9. Then \( \varphi^b \) is isotopic to \( \text{id}_{\mathbb{T}^{d-1}} \) in the space \( \text{Diff}_k(\mathbb{S}^{d-1}) \) and hence the resulting diffeomorphism \( G_s : M_s \to N_s \) is fiber-preserving and fibers over \( \text{id}_{\mathbb{T}^k} \). Finally we notice that the covering map \( N \to \mathbb{T}^d \# \Sigma^d \) and the expanding map \( \times s : N_s \to N \) are fiber preserving as well. We conclude that the expanding map \( f \) fibers over \( \times s \) map on \( \mathbb{T}^k \).

\[
\begin{array}{ccc}
\mathbb{T}^d \# \Sigma^d & \xrightarrow{f} & \mathbb{T}^d \# \Sigma^d \\
\downarrow{p_1} & & \downarrow{p_1} \\
\mathbb{T}^k & \xrightarrow{\times s} & \mathbb{T}^k
\end{array}
\]

The same diagram holds for the standard \( \times s \) expanding map \( E_s : \mathbb{T}^d \to \mathbb{T}^d 

\[
\begin{array}{ccc}
\mathbb{T}^d & \xrightarrow{E_s} & \mathbb{T}^d \\
\downarrow{p_2} & & \downarrow{p_2} \\
\mathbb{T}^k & \xrightarrow{\times s} & \mathbb{T}^k
\end{array}
\]
and it is easy to see that the conjugacy \( h: \mathbb{T}^d \# \Sigma^d \to \mathbb{T}^d \), \( h \circ f = E_s \circ h \), maps fibers to fibers and the induced conjugacy on \( \mathbb{T}^k \) is identity, i.e., \( \overline{h} = \text{id}_{\mathbb{T}^k} \).

We claim that one can perturb \( f \) along the fibers so that the fibrations \( p_1 \) and \( p_2 \) are precisely the ones appearing in the Theorem 2.1. Indeed consider the restrictions of \( f_p: \mathbb{T}^{d-k} \to \mathbb{T}^{d-k} \) and \( E_s: \mathbb{T}^{d-k} \to \mathbb{T}^{d-k} \) to the fibers through the corresponding fixed points. Denote by \( p_i', i = 1, 2 \) the fibrations produced by Theorem 2.1 applied to \( f \) and \( E_s \). Then the fibers of \( p_i' \) refine the fibers of \( p_i \) and, hence, we can restrict \( p_1' \) and \( p_2' \) to \( \mathbb{T}^{d-k} \) and \( \mathbb{T}^{d-k} \), respectively. Denote by \( \ell \) the dimension of the base space for these restricted fibrations. Recall that the induced conjugacy on the base space is smooth. It follows that \( \ell' D f_p \) has \( s^\ell \) as an eigenvalue. Hence we perturb \( f \) in the neighborhood of \( p \) so that \( \ell' D f_p \) does not have \( s^\ell \) for an eigenvalue for all \( \ell = 1, 2, \ldots d - k \). Then we have \( \ell = 0 \) which means that \( p_i' = p_i \).

**Remark 6.10.** Similarly, one can perturb \( f \) along the fibers to an expanding map \( f_2: \mathbb{T}^d \# \Sigma^d \to\mathbb{T}^d \# \Sigma^d \) such that both \( p_i' \) given by Theorem 2.1 when applied to \( f \) and \( f_2 \) are equal to \( p_1 \)

**Remark 6.11.** An easier way of constructing an exotic expanding map with non-trivial fibration would be to take the product \( f \times L \) of an exotic expanding map \( f: \mathbb{T}^d \# \Sigma^d \to \mathbb{T}^d \# \Sigma^d \) and a linear expanding map \( L: \mathbb{T}^m \to \mathbb{T}^m \). Smoothing theory implies that \( \mathbb{T}^d \# \Sigma^d \times \mathbb{T}^m \) is not diffeomorphic to \( \mathbb{T}^{d+m} \). Then \( \mathbb{T}^d \# \Sigma^d \times \mathbb{T}^m \) fibers over \( \mathbb{T}^m \) and one can arrange this fibration to be the fibration given by Theorem 2.1 in a similar way. The example which we described above is more interesting because the smooth structure on \( \mathbb{T}^d \# \Sigma^d \) is irreducible, that is, \( \mathbb{T}^d \# \Sigma^d \) is not diffeomorphic to a smooth product of two lower dimensional smooth closed manifolds [FG12, Proposition 1.3].

### 7. Factor version

We formulate the following generalization of Theorem 2.1, where we replace the topological conjugacy by a continuous factor map. The proof follows the same lines with routine modifications.

**Theorem 7.1.** Assume that \( M_i, i = 1, 2 \), are closed manifolds homeomorphic to a nilmanifold. Let \( f_i: M_i \to M_i, i = 1, 2 \), be \( C^r, r \geq 1 \), smooth expanding maps and assume that \( f_2 \) is a topological factor of \( f_1 \), that is, there exists a continuous map \( h: M_1 \to M_2 \) such that \( h \circ f_1 = f_2 \circ h \).

Then there exist a \( C^r \) expanding map \( \hat{f}: \hat{M} \to \hat{M} \) where \( \hat{M} \) is homeomorphic to a nilmanifold, and \( C^r \) fibrations (with connected fiber homeomorphic to a nilmanifold) \( p_i: M_i \to \hat{M}, i = 1, 2 \), such that

\[
p_i \circ f_1 = \hat{f} \circ p_i, \quad i = 1, 2
\]

Further the conjugacy \( h \) maps fibers to fibers

\[
p_2 \circ h = p_1
\]
and the fibrations \( p_i, i = 1, 2, \) have the following property. If \( \phi_i : M_i \to \mathbb{R}, i = 1, 2, \)
are \( C^r \) smooth functions such that for every periodic point \( x \in \text{Fix}(f^n) \)
\[
\sum_{k=0}^{n-1} \phi_1(f^n_i(x)) = \sum_{k=0}^{n-1} \phi_2(f^n_i(h(x)))
\]
then there exist a \( C^r \) function \( \tilde{\phi} : \hat{M} \to \mathbb{R} \), such that \( \phi_i \) is \( f_i \)-cohomologous to \( \tilde{\phi} \circ p_i \).

Using Theorem 7.1 one can naturally study regularity properties of factors maps. Let \( M_1 = N \times M_2 \), where \( N \) and \( M_2 \) are nilmanifolds, and let \( L : M_1 \to M_1 \) be a product expanding map \( L = (A,B) \). Then \( L \) factors over \( B \). Hence if \( f_1 \) is an expanding map homotopic to \( L \) and \( f_2 \) is an expanding map homotopic to \( B \) then \( f_1 \) factors over \( f_2: h \circ f_1 = f_2 \circ h \).

To define nice invariants of smooth conjugacy we need to introduce a restriction on \( L \) and \( f_1 \). Namely, we assume that the maximal expansion of \( A \) is greater than the minimal expansion of \( B \). Then the “vertical foliation” \( N \times \{x\}, x \in M_2 \) is a weakly expanding foliation. It is easy to see, that for any sufficiently \( C^1 \) small perturbation \( f_1 \) of \( L \) the weakly expanding foliation survives as an \( f_1 \)-invariant foliation \( W_{wu} \).

**Corollary 7.2.** Consider \( L, f_1, f_2 \) are \( C^{r+1} \) expanding maps and \( h \) is the factor map, \( h \circ f_1 = f_2 \circ h \). Assume that \( f_1 \) belongs to a sufficiently small \( C^1 \) neighborhood of \( L \). Also assume that \( f_2 \) is very non-algebraic. If for any periodic point \( x = f_1^n(x) \)
\[
\frac{\text{Jac}(f_1^n(x))}{\text{Jac}(f_1^n|_{W_{wu}}(x))} = \text{Jac}(f_2^n(h(x)))
\]
then \( h \) is \( C^r \).

The proof is very similar to the proof of Corollary 2.11 and we merely provide a sketch. Also one can replace the very non-algebraic assumption on \( f_2 \) by asking \( f_2 \) to be an irreducible toral diffeomorphism and assuming that the entropy maximizing measure for \( f_2 \) is not absolutely continuous.

**Sketch of the proof.** Let \( p_i : M_i \to M_i \) be fibrations given by Theorem 7.1 when applied to \( f_i \) and \( r \). If \( \dim \hat{M} = \dim M_2 \) then \( p_2 \) is a diffeomorphism and hence we have that \( h = p_2^{-1} \circ p_1 \) is \( C^r \).

Hence we need to rule out the \( \dim \hat{M} < \dim M_2 \), i.e., the case when the fiber of \( p_2 \) has dimension \( \geq 1 \). In this case, following the proof of Corollary 2.11, we can apply Theorem 7.1 to (the logarithm of) the jacobian of \( f_2 \) to conclude that the jacobian of \( f_2|_{\ker(p_2)} \) is cohomologous to a function which is constant along the fibers of \( p_2 \) which yields a contradiction, again, similarly to the proof of Corollary 2.11.

One subtle detail, however, is that in order to apply Theorem 7.1 one needs to have a pair of \( C^r \) functions \( (\phi_1, \phi_2) \). We let \( \phi_2 = \log \text{Jac}(f_2) \) and \( \phi_1 = \log \text{Jac}(f_1) - \log \text{Jac}(f_1|_{W_{wu}}) \). (Assume for simplicity that \( f_1 \) are orientation preserving.) It clear from the assumption of the corollary that the sums of \( \phi_1 \) agree along the periodic
orbits and it is clear that $\varphi_2$ is $C^r$. However, one also need to argue that $\varphi_1$ is $C^r$ which is equivalent to $\log \text{Jac}(f_1|_{\tilde{W}^{uu}})$ being $C^r$.

Smoothness of $\log \text{Jac}(f_1|_{\tilde{W}^{uu}})$ can be established as follows. Pick a lift $\tilde{f}_1: \tilde{M}_1 \to \tilde{M}_1$ to the universal cover $\tilde{M}_1$. Foliation $W^{uu}$ lifts to $\tilde{W}^{uu}$. Because $\tilde{f}_1$ is invertible the fast foliation $\tilde{W}^{uu}$ is also well defined by the standard cone argument (but not equivariant under the Deck group). Then, by the usual application of the $C^r$ Section Theorem, we have that $\tilde{W}^{uu}$ is $C^r$ and hence $\log \text{Jac}(\tilde{f}_1|_{\tilde{W}^{uu}})$ is $C^r$. Finally, extending $\tilde{W}^{uu}$ to a smooth coordinate system, we have that $D\tilde{f}_1$ has an upper-triangular form and hence

$$\log \text{Jac}(\tilde{f}_1) = \log \text{Jac}(\tilde{f}_1|_{\tilde{W}^{uu}}) + \log \text{Jac}(\tilde{f}_1|_{\tilde{W}^{uu}})$$

which implies that $\log \text{Jac}(f_1|_{\tilde{W}^{uu}})$, and hence $\log \text{Jac}(f_1|_{\tilde{W}^{uu}})$ is $C^r$. □

References


