

HOW TYPICAL ARE PATHOLOGICAL FOLIATIONS IN PARTIALLY HYPERBOLIC DYNAMICS: AN EXAMPLE

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ABSTRACT. We show that for a large space of volume preserving partially hyperbolic diffeomorphisms of the 3-torus with non-compact central leaves the central foliation generically is non-absolutely continuous.

1. INTRODUCTION

Let M be a smooth Riemannian manifold. In this paper we will consider continuous foliations of M with smooth leaves. A *continuous foliation \mathcal{W} with smooth leaves* $\mathcal{W}(x)$, $x \in M$, is a foliation given by continuous charts whose leaves are smoothly immersed and whose tangent distribution $T\mathcal{W}$ is continuous on M . Riemannian metric induces volume m on M as well as volume on the leaves of \mathcal{W} . Following Shub and Wilkinson [SW00] we call such foliation \mathcal{W} *pathological* if there is a full volume set on M that meets every leaf of the foliation on a set of leaf-volume zero. According to Fubini Theorem, smooth foliations cannot be pathological, but continuous foliations might happen to be pathological. This phenomenon naturally appears for central foliations of partially hyperbolic diffeomorphisms and is also known as “Fubini’s nightmare.” A diffeomorphism f is called partially hyperbolic if the tangent bundle TM splits into a Df -invariant direct sum of an exponentially contracting stable bundle, an exponentially expanding unstable bundle and a central bundle of intermediate growth (precise definitions appear in the next section).

The first example of a pathological foliation was constructed by Katok and it has been circulating in dynamics community since the eighties. Katok suggested to consider one parameter family $\{A_t, t \in \mathbb{R}/\mathbb{Z}\}$ of area-preserving Anosov diffeomorphisms C^1 -close to a hyperbolic automorphism A of the 2-torus. By Hirsch-Pugh-Shub Theorem diffeomorphism $F(x, t) = (A_t(x), t)$ is partially hyperbolic with uniquely integrable central distribution. Then, under certain generic conditions (the metric entropy or periodic eigendata of A_t should vary with t) on path A_t , one can show that the central foliation by embedded circles is pathological. See [Pes04], Section 7.4, or [HassP06], Section 6, for detailed constructions with proofs.

A version of above construction on the square appeared in expository paper by Milnor [Mil97]. Milnor remarks that a different version of the construction, based on tent maps, has also been given by Yorke.

Shub and Wilkinson [SW00] came across the same phenomenon when looking for volume preserving non-uniformly hyperbolic systems in the neighborhood of $F_0: (x, t) \mapsto (A_0(x), t)$. They have showed existence of C^1 -open set of diffeomorphisms in the C^1 -neighborhood of F_0 with non-zero central exponent. Then one can argue that the central foliation is pathological using the following “Mañé’s argument”. By Oseledets’ Theorem the set of Lyapunov regular points has full volume.

If any central leaf intersected the set of regular points by a set of positive Lebesgue measure, then it would increase exponentially in length under the dynamics. But the lengths of central leaves are uniformly bounded.

Work [SW00] was further generalized by Ruelle [Ru03]. Ruelle and Wilkinson [RW01] also showed that conditional measures are in fact atomic. Case of higher dimensional central leaves was considered by Hirayama and Pesin [HirP07]. They showed that central foliation is not absolutely continuous if it has compact leaves and the sum of the central exponents is nonzero on a set of positive measure.

This work is devoted to the study of pathological foliations with one-dimensional non-compact leaves. Consider a hyperbolic automorphism L of the 3-torus \mathbb{T}^3 with eigenvalues ν , μ and λ such that $\nu < 1 < \mu < \lambda$. One can view L as a partially hyperbolic diffeomorphism. It was noted in [GG08] and independently in [SX08] that for a small C^1 -open set in the neighborhood of L “Mañé’s argument” can be applied to show that corresponding central foliations are pathological. In this paper we apply a completely different approach to show that *there is an open and dense set \mathcal{U} of a large C^1 -neighborhood of L in the space of volume preserving partially hyperbolic diffeomorphisms such that all diffeomorphisms from \mathcal{U} have pathological central foliations.* This result confirms a conjecture from [HirP07].

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2. PRELIMINARIES

Here we introduce all necessary notions and some standard tools that we need for precise formulation of the result and the proof. The reader may consult [Pes04] for an introduction on partially hyperbolic dynamics.

Definition 1. A diffeomorphism f is called *Anosov* if there exists a Df -invariant splitting of the tangent bundle $TM = E_f^s \oplus E_f^u$ and constants $\lambda \in (0, 1)$ and $C > 0$ such that for $n > 0$

$$\|Df^n v\| \leq C\lambda^n \|v\|, \quad v \in E^s \quad \text{and} \quad \|Df^{-n} v\| \leq C\lambda^n \|v\|, \quad v \in E^u.$$

Definition 2. A diffeomorphism f is called *partially hyperbolic* if there exists a Df -invariant splitting of the tangent bundle $TM = E_f^s \oplus E_f^c \oplus E_f^u$ and positive constants $\nu_- < \nu_+ < \mu_- < \mu_+ < \lambda_- < \lambda_+$, $\nu_+ < 1 < \lambda_-$, and $C > 0$ such that for $n > 0$

$$\begin{aligned} \frac{1}{C}\nu_-^n \|v\| &\leq \|D(f^n)(x)v\| \leq C\nu_+^n \|v\|, & v \in E_f^s(x), \\ \frac{1}{C}\mu_-^n \|v\| &\leq \|D(f^n)(x)v\| \leq C\mu_+^n \|v\|, & v \in E_f^c(x), \\ \frac{1}{C}\lambda_-^n \|v\| &\leq \|D(f^n)(x)v\| \leq C\lambda_+^n \|v\|, & v \in E_f^u(x). \end{aligned}$$

The following definition is equivalent to the above one. We will switch between the definitions when convenient.

Definition 3. A diffeomorphism f is called *partially hyperbolic* if there exists a Riemannian metric on M , a Df -invariant splitting of the tangent bundle $TM =$

$E_f^s \oplus E_f^c \oplus E_f^u$ and positive constants $\nu_- < \nu_+ < \mu_- < \mu_+ < \lambda_- < \lambda_+$, $\nu_+ < 1 < \lambda_-$, such that

$$\begin{aligned} \nu_- \|v\| &\leq \|Df(x)v\| \leq \nu_+ \|v\|, & v \in E_f^s(x), \\ \mu_- \|v\| &\leq \|Df(x)v\| \leq \mu_+ \|v\|, & v \in E_f^c(x), \\ \lambda_- \|v\| &\leq \|Df(x)v\| \leq \lambda_+ \|v\|, & v \in E_f^u(x). \end{aligned}$$

The distributions E_f^s , E_f^c and E_f^u are continuous. Moreover, distributions E_f^s and E_f^u integrate uniquely to foliations \mathcal{W}_f^s and \mathcal{W}_f^u . When it does not lead to a confusion we drop dependence on the diffeomorphism. By $m_{\mathcal{W}^\sigma(\cdot)}$ or m_σ we denote induced Riemannian volume on the leaves of \mathcal{W}^σ , $\sigma = s, c, u$. Induced volume on other submanifolds such as transversals to a foliation will be denoted analogously with appropriate subscript.

We write d for the distance induced by the Riemannian metric and $d^\sigma(\cdot, \cdot)$ for the distance induced by the restriction of the Riemannian metric to $T\mathcal{W}^\sigma$. If expanding foliation \mathcal{W}^u is one-dimensional then it is convenient to work with the pseudo-distance $\tilde{d}^u(\cdot, \cdot)$ that is very well adapted to the dynamics. Let

$$D_f^u(x) = \|Df(x)|_{E_f^u(x)}\|$$

and

$$\rho_x(y) = \prod_{n \geq 1} \frac{D_f^u(f^{-n}(x))}{D_f^u(f^{-n}(y))}.$$

This infinite product converges and gives a continuous positive density $\rho_x(\cdot)$ on the leaf $\mathcal{W}^u(x)$. Define *pseudo-distance* \tilde{d}^u by integrating density $\rho_x(\cdot)$

$$\tilde{d}^u(x, y) = \int_x^y \rho_x(z) dm_{\mathcal{W}^u(x)}(z).$$

Obviously, pseudo-distance is not even symmetric, still it is useful for computations as it satisfies the formula

$$\tilde{d}^u(f(x), f(y)) = D_f^u(x) \tilde{d}^u(x, y)$$

verified by the following simple computation

$$\begin{aligned} \tilde{d}^u(f(x), f(y)) &= \int_{f(x)}^{f(y)} \rho_{f(x)}(z) dm_{\mathcal{W}^u(f(x))}(z) \\ &= \int_x^y \rho_{f(x)}(f(z)) D_f^u(z) dm_{\mathcal{W}^u(x)}(z) \\ &= \int_x^y \frac{D_f^u(x)}{D_f^u(z)} \rho_x(z) D_f^u(z) dm_{\mathcal{W}^u(x)}(z) = D_f^u(x) \tilde{d}^u(x, y). \end{aligned}$$

A compact domain inside a leaf $\mathcal{W}^\sigma(x)$ of a foliation \mathcal{W}^σ will be called *plaque* and will be denoted by \mathcal{P}^σ . We shall also write $\mathcal{P}^\sigma(x)$ when we need to indicate dependence on the point.

Given a transversal T to \mathcal{W} , consider a compact domain X which is a union of plaques of \mathcal{W} , that is, $X = \cup_{x \in T} \mathcal{P}(x)$. Then by Rokhlin's Theorem there exists

a unique system of conditional measures μ_x , $x \in T$, such that for any continuous function φ on X

$$\int_X \varphi dm_X = \int_T \int_{\mathcal{P}(x)} \varphi d\mu_x d\hat{m},$$

where \hat{m} is projection of m_X to T .

Definition 4. Foliation \mathcal{W} is called *absolutely continuous* with respect to the volume m if for any T and X as above the conditional measures μ_x have L^1 densities with respect to the volume $m_{\mathcal{P}(x)}$ for \hat{m} a. e. x .

Now consider a compact domain X as above and two transversal T_1 and T_2 so that $X = \cup_{x \in T_1} \mathcal{P}(x) = \cup_{x \in T_2} \mathcal{P}(x)$ with the same system of plaques. Then the holonomy map $p: T_1 \rightarrow T_2$ along \mathcal{W} is a homeomorphism.

Definition 5. Foliation \mathcal{W} is called *transversally absolutely continuous* if any holonomy map p as above is absolutely continuous, that is, $p_*m_{T_1}$ is absolutely continuous with respect to m_{T_2} .

Transverse absolute continuity is a stronger property than absolute continuity. Stable and unstable foliations of Anosov and partially hyperbolic diffeomorphisms are known to be transversally absolutely continuous.

3. FORMULATION OF THE RESULT

Let L be a hyperbolic automorphism of 3-torus \mathbb{T}^3 with positive real eigenvalues ν , μ and λ , $\nu < 1 < \mu < \lambda$. Observe that L can be viewed as a partially hyperbolic diffeomorphism with L -invariant splitting $T\mathbb{T}^3 = E_L^s \oplus E_L^{wu} \oplus E_L^{su}$, where “wu” and “su” stand for “weak unstable” and “strong unstable”.

Consider the space $Diff_m^r(\mathbb{T}^3)$ of C^r , $r \geq 2$, diffeomorphisms of \mathbb{T}^3 that preserve volume m . Let $\mathcal{U} \subset Diff_m^r(\mathbb{T}^3)$ be the set of Anosov diffeomorphisms conjugate to L via a conjugacy homotopic to identity and also partially hyperbolic. It is known that \mathcal{U} is C^1 -open (e.g., see [Pes04], Theorem 3.6). Given f in \mathcal{U} denote by $E_f^s \oplus E_f^{wu} \oplus E_f^{su}$ corresponding f -invariant splitting. According to [BBI09] distributions E_f^{wu} , $E_f^s \oplus E_f^{wu}$ and $E_f^u = E_f^{wu} \oplus E_f^{su}$ integrate uniquely to invariant foliations \mathcal{W}^{wu} , $\mathcal{W}^{s \oplus wu}$ and \mathcal{W}^u . It is known that \mathcal{W}^s and \mathcal{W}^u are C^1 and \mathcal{W}^{su} is C^1 when restricted to the leaves of \mathcal{W}^u (see, e.g., [Hass94, PSW97]). We shall need the following statement that shows that the structure of weak unstable foliation is essentially linear.

Proposition 1. *Let $f \in \mathcal{U}$ and let h_f be the conjugacy to the linear automorphism— $h_f \circ f = L \circ h_f$. Then $h_f(\mathcal{W}_f^{wu}) = \mathcal{W}_L^{wu}$.*

The proof will be given in the appendix.

Theorem A. *There is a C^1 -open and C^r -dense set $\mathcal{V} \subset \mathcal{U}$ such that $f \in \mathcal{V}$ if and only if the central foliation \mathcal{W}^{wu} is non-absolutely continuous with respect to the volume m .*

Remark. Since we know that \mathcal{W}^u is C^1 the latter is equivalent to \mathcal{W}^{wu} being non-absolutely continuous on almost every plaque of \mathcal{W}^u with respect to the induced volume on the plaque.

Now we describe set \mathcal{V} . Given $f \in \mathcal{U}$ and given a periodic point x of period p let

$$\lambda^{su}(x) = \|Df^p(x)|_{E_f^{su}(x)}\|^{1/p}.$$

Then set \mathcal{V} can be characterized as follows.

$$\mathcal{V} = \{f \in \mathcal{U} : \text{there exist periodic points } x \text{ and } y \text{ with } \lambda^{su}(x) \neq \lambda^{su}(y)\}.$$

Proposition 2.

$$\mathcal{U} \setminus \mathcal{V} = \{f \in \mathcal{U} : \text{for any periodic point } x \text{ } \lambda^{su}(x) = \lambda\}.$$

We defer the proof to the appendix.

4. RELATED QUESTIONS

Our result does not give any information about the structure of singular conditional measures.

Question 1. *Given $f \in \mathcal{V}$, what can one say about singular conditional measures on \mathcal{W}^{wu} ? Are they atomic? What can be said about Hausdorff dimension of conditional measures?*

It seems that our method can be generalized for analysis of central foliation of partially hyperbolic diffeomorphisms in a C^1 neighborhood of $F_0: (x, t) \mapsto (A_0(x), t)$.

Question 2. *Can one give explicit conditions in terms of stable and unstable Lyapunov exponents of periodic central leaves for non-absolute continuity?*

It would be interesting to generalize Theorem A to the higher dimensional setting. Namely, let L be an Anosov automorphism that leaves invariant a partially hyperbolic splitting $E_L^s \oplus E_L^{wu} \oplus E_L^{su}$, where $E_L^{wu} \oplus E_L^{su}$ is the splitting of the unstable bundle into weak and strong unstable subbundles. Let n_1, n_2 and n_3 be the dimensions of E_L^s, E_L^{wu} and E_L^{su} respectively. Let \mathcal{U} be a small C^1 neighborhood of L in the space of volume preserving diffeomorphisms.

Question 3. *Is it possible to describe the set*

$$\{f \in \mathcal{U} : \mathcal{W}^{wu} \text{ is not absolutely continuous}\}$$

in terms of strong unstable spectra at periodic points in higher dimensional setting?

It will become clear from the discussion in the next section that the value of n_1 is not important. Also it seems likely that our approach works in the case when $n_2 > 1$ and $n_3 = 1$, and gives a result analogous to Theorem A (the author does not claim to have done this).

The picture gets much more complicated when $n_3 > 1$. In principle, it is possible that the major link in our argument

$$(\mathcal{W}^{wu} \text{ is Lipschitz inside } \mathcal{W}^u) \Leftrightarrow (\mathcal{W}^{wu} \text{ is absolutely continuous inside } \mathcal{W}^u)$$

is no longer valid in this setting. However it is not immediately clear how to construct a counterexample.

5. OUTLINE OF THE PROOF

Clearly \mathcal{V} is C^1 -open. Given a diffeomorphism $f \in \mathcal{U} \setminus \mathcal{V}$ we can compose it with a special diffeomorphism h that is C^r -close to identity and equal to identity outside a small neighborhood of a fixed point so that strong unstable eigenvalue of f and $h \circ f$ at the fixed point are different. This gives that \mathcal{V} is C^r -dense.

To show that weak unstable foliations of diffeomorphisms from \mathcal{V} are non-absolutely continuous we start with some simple observations. First, notice that due to ergodicity conditional measures cannot have absolutely continuous and singular components simultaneously. Next, it follows from the absolute continuity of \mathcal{W}^u and the uniqueness of the system of conditional measures of m that the conditional measures of m on the leaves of \mathcal{W}^{wu} are equivalent to the conditional measures of the induced volume on the leaves of \mathcal{W}^u . Therefore we only need to look at two dimensional plaques of \mathcal{W}^u foliated by plaques of \mathcal{W}^{wu} . It turns out that absolute continuity of \mathcal{W}^{wu} inside the leaves of \mathcal{W}^u is equivalent to \mathcal{W}^{wu} being Lipschitz inside \mathcal{W}^u . Lipschitz property, in turn, can be related to the periodic eigenvalue data along \mathcal{W}^{su} .

Pick a plaque \mathcal{P}^u of \mathcal{W}^u and let $T_1 \subset \mathcal{P}^u$ and $T_2 \subset \mathcal{P}^u$ be two smooth compact transversals to \mathcal{W}^{wu} with holonomy map $p: T_1 \rightarrow T_2$. If p is Lipschitz for any choice of plaque and transversals then we say that \mathcal{W}^{wu} is *Lipshitz inside \mathcal{W}^u* .

Theorem A follows from the following lemmas

Lemma 3. *Foliation \mathcal{W}^{wu} is Lipschitz inside \mathcal{W}^u if and only if $f \in \mathcal{U} \setminus \mathcal{V}$.*

Lemma 4. *Foliation \mathcal{W}^{wu} is Lipschitz inside \mathcal{W}^u if and only if \mathcal{W}^{wu} is absolutely continuous inside \mathcal{W}^u .*

6. PROOFS

Let us begin with a useful observation. If one needs to show that \mathcal{W}^{wu} is Lipschitz in a plaque \mathcal{P}^u then it is sufficient to check Lipschitz property of the holonomy map for pairs of transversals that belong to a smooth family that foliates \mathcal{P}^u , *e.g.*, plaques of \mathcal{W}^{su} . Therefore we can always assume that the transversals are plaques of \mathcal{W}^{su} .

Proof of Lemma 3. First assume that $f \in \mathcal{U} \setminus \mathcal{V}$. Then Lipschitz property of \mathcal{W}^{wu} is shown below by a standard argument that uses Livshits Theorem.

Let T_1 and T_2 be two local leaves of \mathcal{W}^{su} in a plaque \mathcal{P}^u and let $p: T_1 \rightarrow T_2$ be the holonomy along \mathcal{W}^{wu} .

For $x, y \in T_1$ with $d^{su}(x, y) \geq 1$ the Lipschitz property

$$d^{su}(p(x), p(y)) \leq C d^{su}(x, y), \quad d^{su}(x, y) \geq 1, \tag{1}$$

follows from compactness for uniformly bounded plaques \mathcal{P}^u . It might happen that $f^n(x)$ and $f^n(p(x))$ are far from each other on $\mathcal{W}^{wu}(f^n(x))$. Hence we need (1) with uniform C not only on plaques \mathcal{P}^u of bounded size but also on plaques that are long in the weak unstable direction. In this case (1) cannot be guaranteed solely by compactness but easily follows from Proposition 1.

For x and y close to each other we may use \tilde{d}^{su} rather than d^{su} since \tilde{d}^{su} is given by an integral of a continuous density. Then

$$\frac{\tilde{d}^{su}(p(x), p(y))}{\tilde{d}^{su}(x, y)} = \prod_{i=0}^{n-1} \frac{D_f^{su}(f^i(p(x)))}{D_f^{su}(f^i(x))} \cdot \frac{\tilde{d}^{su}(f^n(p(x)), f^n(p(y)))}{\tilde{d}^{su}(f^n(x), f^n(y))},$$

where n is chosen so that $d^{su}(f^{n-1}(x), f^{n-1}(y)) < 1 \leq d^{su}(f^n(x), f^n(y))$. The Lipschitz estimate follows since according to the Livshits Theorem D_f^{su} is cohomologous to λ and therefore the product term equals to $F(f^n(x))F(f^n(p(x)))(F(x)F(p(x)))^{-1}$ for some positive continuous transfer function F .

Now let us take f from \mathcal{V} . Specification implies that the closure of the set $\{\lambda^{su}(x) : x \text{ periodic}\}$ is an interval $[\lambda_-^{su}, \lambda_+^{su}]$. By applying Anosov Closing Lemma it is possible to change the Riemannian metric so that the constants λ_- and λ_+ from Definition 3 are equal to $\lambda_-^{su}/(1 + \delta)$ and $\lambda_+^{su}(1 + \delta)$ correspondingly. Here δ is an arbitrarily small number.

Next we choose periodic points a and b such that

$$\max \left\{ \frac{\lambda_+^{su}}{\lambda^{su}(a)}, \frac{\lambda^{su}(b)}{\lambda_-^{su}} \right\} \leq 1 + \delta \quad \text{and} \quad \frac{\lambda^{su}(b)}{\lambda^{su}(a)} \leq \frac{1}{(1 + \delta)^{2/\gamma}}.$$

This is possible if δ is small enough. From now on δ will be fixed. Constant γ does not depend on our choice of a and b and hence δ . It will be introduced later.

Denote by n_0 the least common period of a and b . Take $\tilde{a} \in \mathcal{W}^{su}(a)$ such that $d^{su}(a, \tilde{a}) = 1$. If one considers an arc of a leaf of W_L^{wu} of length D then it is easy to see that this arc is $const/\sqrt{D}$ -dense in \mathbb{T}^3 . Since conjugacy h_f between f and L is Hölder continuous, Proposition 1 implies that an arc of $\mathcal{W}^{wu}(a)$ of length D is C_1/D^α -dense in \mathbb{T}^3 , $\alpha > 0$. It follows that there exists a point $c \in \mathcal{W}^{wu}(a)$ such that $d^{wu}(a, c) \leq D$, $d(c, b) \leq C_1/D^\alpha$ and $W^s(b)$ intersects the arc of strong unstable leaf $\mathcal{W}^{su}(c)$ that connects c and $\tilde{c} = \mathcal{W}^{su}(c) \cap \mathcal{W}^{wu}(\tilde{a})$ at point \tilde{b} as shown on the Figure 1.

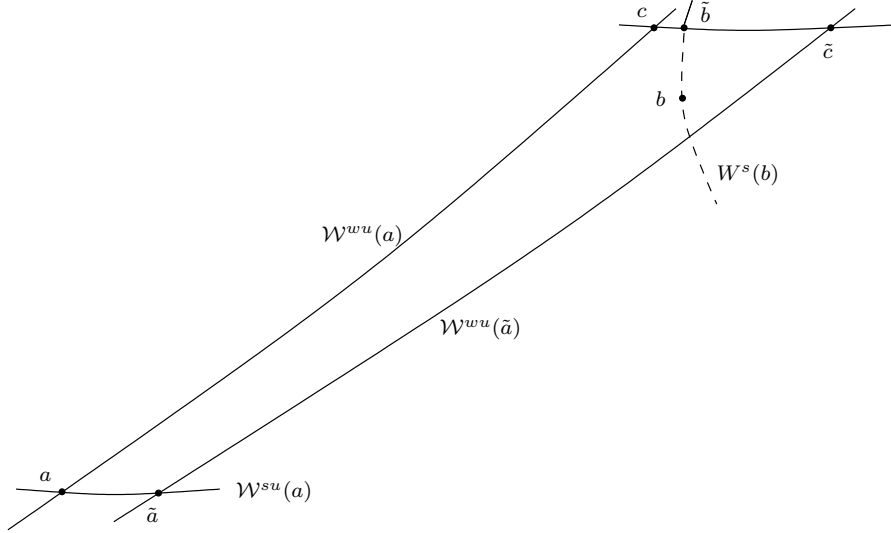


FIGURE 1

Take N such that $d^{wu}(a, f^{-n_0N}(c)) \leq 1 < d^{wu}(a, f^{-n_0(N-1)}(c))$. Now our goal is to show that the ratio

$$\frac{\tilde{d}^{su}(a, f^{-n_0N}(\tilde{a}))}{\tilde{d}^{su}(f^{-n_0N}(c), f^{-n_0N}(\tilde{c}))}$$

can be arbitrarily small which would imply that \mathcal{W}^{su} is not Lipschitz. Note that we cannot take a smaller N since f^{-1} -orbit of c has to come to a local plaque about a .

Remark. We use \tilde{d}^{su} for convenience. Somewhat messier estimates go through if one uses d^{su} directly.

To estimate the denominator we split the orbit $\{c, f^{-1}(c), \dots, f^{-n_0N}(c)\}$ into two segments of lengths N_1 and N_2 , $N_1 + N_2 = n_0N$. Choose N_1 so that $d(f^{-N_1}(b), f^{-N_1}(c))$ is still small enough to provide the estimate on the strong unstable derivative

$$D_f^{su}(f^{-i}(c)) \leq (1 + \delta)\lambda^{su}(b), \quad i = 1, \dots, N_1 + 1.$$

The remaining derivatives will be estimated boldly

$$D_f^{su}(\cdot) \leq \lambda_+.$$

Since b and \tilde{b} are exponentially close — $d^s(b, \tilde{b}) \leq C_1/D^\alpha \leq \text{const} \cdot \mu_-^{-n_0N}$ — we see that there exists $\beta = \beta(\alpha, \nu_-, \mu_-)$ which is independent of N such that $N_1 > \beta N_2$.

Proposition 1 implies that the ratio $\tilde{d}^{su}(a, \tilde{a})/\tilde{d}^{su}(c, \tilde{c})$ is bounded independently of D (and N) by a constant C_2 . We are ready to proceed with the main estimate.

$$\begin{aligned} \frac{\tilde{d}^{su}(a, f^{-n_0N}(\tilde{a}))}{\tilde{d}^{su}(f^{-n_0N}(c), f^{-n_0N}(\tilde{c}))} &= \prod_{i=1}^{n_0N+1} \frac{D_f^{su}(f^i(c))}{(\lambda^{su})^{n_0N}} \cdot \frac{\tilde{d}^{su}(a, \tilde{a})}{\tilde{d}^{su}(c, \tilde{c})} \\ &\leq (\lambda^{su}(a))^{-n_0N} (1 + \delta)^{N_1} (\lambda^{su}(b))^{N_1} (1 + \delta)^{N_2} (\lambda_+^{su})^{N_2} C_2 \\ &\leq (1 + \delta)^{N_1 + N_2} \left(\frac{\lambda^{su}(b)}{\lambda^{su}(a)} \right)^{N_1} \left(\frac{\lambda_+^{su}}{\lambda^{su}(a)} \right)^{N_2} C_2 \\ &\leq (1 + \delta)^{N_1 + 2N_2} \left(\frac{\lambda^{su}(b)}{\lambda^{su}(a)} \right)^{\gamma(N_1 + 2N_2)} C_2 \leq \left(\frac{1}{1 + \delta} \right)^{n_0N + N_2} C_2, \end{aligned}$$

where $\gamma = \beta/\beta + 2$ so that $N_1 \geq \gamma(N_1 + 2N_2)$. The last expression goes to zero as $D \rightarrow \infty, N \rightarrow \infty$. Thus \mathcal{W}^{wu} is not Lipschitz. \square

Proof of Lemma 4. Obviously \mathcal{W}^{wu} being Lipschitz implies transverse absolute continuity property and hence absolute continuity. We have to establish the other implication.

Assume that \mathcal{W}^{wu} is absolutely continuous in the sense of Definition 4. A priori, conditional densities are only L^1 -functions. Our goal is to show that the densities are continuous. Moreover, for m almost every x the density $\rho_x(y)$ on a plaque \mathcal{P}^{wu} satisfies the equation

$$\frac{\rho_x(y)}{\rho_x(x)} = \prod_{n \geq 1} \frac{D_f^{wu}(f^{-n}(x))}{D_f^{wu}(f^{-n}(y))}, \quad (2)$$

where $D_f^{wu}(z) = \|Df|_{E_f^{wu}(z)}(z)\|$. The expression on the right hand side of the formula is a positive continuous function in y .

Consider a full volume set where positive ergodic averages coincide for all continuous functions. By absolute continuity this set should intersect a plaque \mathcal{P}^{wu} by a positive leaf-volume m_{wu} set Y . Denote by m_Y restriction of m_{wu} to Y . For any $y \in Y$ consider measures

$$\Delta_n(y) = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(y)}, \quad \mu_n = \int_Y \Delta_n(y) dm_{wu}(y).$$

Sequences $\{\Delta_n(y)\}$, $y \in Y$, converge weakly to m . Hence μ_n converges to m as well. Notice that

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \int_Y \delta_{f^i(y)} dm_{wu}(y) = \frac{1}{n} \sum_{i=0}^{n-1} (f^i)_*(m_Y).$$

In case when Y is a plaque of \mathcal{W}^{wu} the latter expression is known to converge to a measure with absolutely continuous conditional densities on \mathcal{W}^{wu} that satisfy (2). This was established in [PS83] in the context of u -Gibbs measures, however the proof works equally well for any uniformly expanding foliation such as \mathcal{W}^{wu} . For arbitrary measurable Y same conclusion holds. One needs to use Lebesgue density argument to reduce the problem to the case when Y is a finite union of plaques.

Remark. The argument presented above can also be found in [BDV05], Section 11.2.2, in the context of u -Gibbs measures.

Take an m -typical plaque \mathcal{P}^u whose boundaries are leaves of \mathcal{W}^{wu} and transversals T_1 and T_2 as shown on the Figure 2. Then plaque \mathcal{P}^u is foliated by the plaques $\mathcal{P}^{wu}(x)$, $x \in T_1$. As usual, denote by $p: T_1 \rightarrow T_2$ the holonomy map. Lipschitz property of p will be established by comparing volumes of small rectangles R_1 and R_2 built on corresponding segments of T_1 and T_2 .

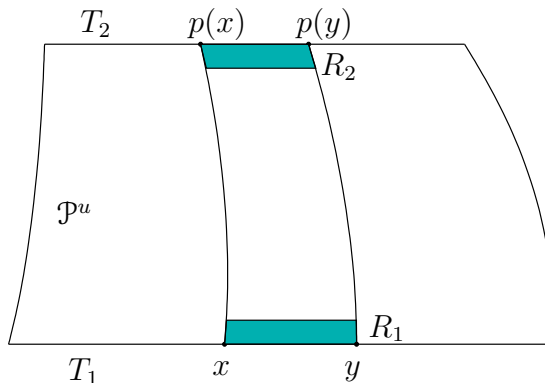


FIGURE 2

Denote by $\mu_{\mathcal{P}^u}$ the conditional measure on \mathcal{P}^u . The conditional densities $\rho_x(\cdot)$ of m on the plaques $\mathcal{P}^{wu}(x)$, $x \in T_1$, are the same as conditional densities with respect to $\mu_{\mathcal{P}^u}$.

Fix $x, y \in T_1$ and small $\varepsilon > 0$, $\varepsilon \ll m_{T_1}([x, y])$. Build rectangles R_1 and R_2 on the segments $[x, y]$ and $[p(x), p(y)]$ so that $m_{\mathcal{P}^{wu}(z)}(R_1 \cap \mathcal{P}^{wu}(z)) = m_{\mathcal{P}^{wu}(z)}(R_2 \cap \mathcal{P}^{wu}(z)) = \varepsilon$ for every $z \in [x, y]$. Then

$$\mu_{\mathcal{P}^u}(R_i) = \int_{[x, y]} d\hat{\mu}(z) \int_{\mathcal{P}^{wu}(z) \cap R_i} \rho_z(t) dm_{\mathcal{P}^{wu}(z)}(t), \quad i = 1, 2,$$

where $\hat{\mu}$ is projection of $\mu_{\mathcal{P}^u}$ to T_1 . These formulae together with uniform continuity of the conditional densities that is guaranteed by (2) imply that the ratio $\mu_{\mathcal{P}^u}(R_1)/\mu_{\mathcal{P}^u}(R_2)$ is bounded away from zero and infinity uniformly in x and y . Since $\mu_{\mathcal{P}^u}$ has positive continuous density with respect to $m_{\mathcal{P}^u}$ the same conclusion

holds for $m_{\mathcal{P}^u}(R_1)/m_{\mathcal{P}^u}(R_2)$ and therefore also for $m_{T_1}([x, y])/m_{T_2}([p(x), p(y)])$. \square

7. APPENDIX

Appendix is devoted to the proofs of Propositions 1 and 2. Both proofs rely on simple growth arguments and a result of Brin, Burago and Ivanov. We will work on the universal cover \mathbb{R}^3 and we will indicate this by using tilde sign for lifted objects. For example, the lift of foliation \mathcal{W}_f^{su} to \mathbb{R}^3 is denoted by $\widetilde{\mathcal{W}}_f^{su}$.

The result of Brin, Burago and Ivanov [BBI09] says that lifts of leaves of strong unstable foliation are quasi-isometric. Namely, if d is the usual distance then

$$\exists C > 0 : \forall x, y \text{ with } y \in \widetilde{\mathcal{W}}^{su}(x), \quad d^{su}(x, y) \leq Cd(x, y).$$

Proof of Proposition 1. We argue by contradiction. Assume that $\tilde{h}_f(\widetilde{\mathcal{W}}_f^{wu}) \neq \widetilde{\mathcal{W}}_L^{wu}$. Then we can find points a, b and c with the following properties

$$b \in \widetilde{\mathcal{W}}_L^{wu}(a), c \notin \widetilde{\mathcal{W}}_L^{wu}(a), h_f^{-1}(c) = \widetilde{\mathcal{W}}_f^{wu}(\tilde{h}_f^{-1}(a)) \cap \widetilde{\mathcal{W}}_f^{su}(\tilde{h}_f^{-1}(b)).$$

We iterate automorphism L and look at the asymptotic growth of the distance between these points. Obviously, distance between images of a and b grows as μ^n , meanwhile distance between images of a and c , and images of b and c grows as λ^n .

Since conjugacy \tilde{h}_f is C^0 -close to Id we have the same growth rates for the triple $\tilde{h}_f^{-1}(a), \tilde{h}_f^{-1}(b)$ and $\tilde{h}_f^{-1}(c)$ as we iterate dynamics \tilde{f} . Points $\tilde{h}_f^{-1}(a)$ and $\tilde{h}_f^{-1}(c)$ lie on the same weak unstable manifold, therefore, constant μ_+ from the Definition 2 is not less than λ . Then, obviously, $\lambda_- > \lambda$. Since $\widetilde{\mathcal{W}}_f^{su}$ is quasi-isometric

$$d(\tilde{f}^n(\tilde{h}_f^{-1}(c)), \tilde{f}^n(\tilde{h}_f^{-1}(b))) \approx d^{su}(\tilde{f}^n(\tilde{h}_f^{-1}(c)), \tilde{f}^n(\tilde{h}_f^{-1}(b))) \gtrsim \lambda_-^n, \quad n \rightarrow \infty.$$

On the other hand, we have already established that the distance between images of $\tilde{h}_f^{-1}(c)$ and $\tilde{h}_f^{-1}(b)$ diverges as λ^n . This gives us a contradiction. \square

Proof of Proposition 2. We argue by contradiction. Assume that $f \in \mathcal{U} \setminus \mathcal{V}$. Then for every periodic point x , $\lambda^{su}(x) = \bar{\lambda} \neq \lambda$.

First assume that $\bar{\lambda} < \lambda$. Then constant λ_+ from Definition 2 can be taken to be equal to $\frac{1}{2}(\lambda + \bar{\lambda})$. Pick points a and b , $b \in \widetilde{\mathcal{W}}^{su}(a)$. Then

$$d(\tilde{f}^n(a), \tilde{f}^n(b)) \leq d^{su}(\tilde{f}^n(a), \tilde{f}^n(b)) \lesssim \lambda_+^n, \quad n \rightarrow \infty.$$

By Proposition 1 $\tilde{h}_f(b) \notin \widetilde{\mathcal{W}}^{wu}(\tilde{h}_f(a))$. Therefore,

$$d(\tilde{L}^n(\tilde{h}_f(a)), \tilde{L}^n(\tilde{h}_f(b))) \gtrsim \lambda^n, \quad n \rightarrow \infty.$$

On the other hand,

$$d(\tilde{L}^n(\tilde{h}_f(a)), \tilde{L}^n(\tilde{h}_f(b))) = d(\tilde{h}_f(\tilde{f}^n(a)), \tilde{h}_f(\tilde{f}^n(b))) \lesssim \lambda_+^n, \quad n \rightarrow \infty,$$

since \tilde{h}_f is C^0 -close to Id . The last two asymptotic inequalities contradict to each other.

Now let us assume that $\bar{\lambda} > \lambda$. In this case we can take λ_- from Definition 2 to be equal to $\frac{1}{2}(\lambda + \bar{\lambda})$. Take a and b as before. Since $\widetilde{\mathcal{W}}^{su}$ is quasi-isometric

$$d(\tilde{f}^n(a), \tilde{f}^n(b)) \gtrsim d^{su}(\tilde{f}^n(a), \tilde{f}^n(b)) \gtrsim \lambda_-^n, \quad n \rightarrow \infty.$$

On the other hand,

$$d(\tilde{f}^n(a), \tilde{f}^n(b)) \approx d(\tilde{h}_f(\tilde{f}^n(a)), \tilde{h}_f(\tilde{f}^n(b))) = d(\tilde{L}^n(\tilde{h}_f(a)), \tilde{L}^n(\tilde{h}_f(b))) \lesssim \lambda^n, \quad n \rightarrow \infty,$$

which gives us a contradiction in this case as well. \square

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