LOCAL RIGIDITY FOR ANOSOV AUTOMORPHISMS

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WITH AN APPENDIX BY RAFAEL DE LA LLAVE

ABSTRACT. We consider an irreducible Anosov automorphism L of a torus $\mathbb{T}^d$ such that no three eigenvalues have the same modulus. We show that L is locally rigid, that is, L is $C^{1+\text{Hölder}}$ conjugate to any C^1-small perturbation f such that the derivative $D_p f^n$ is conjugate to L^n whenever $f^n p = p$. We also prove that toral automorphisms satisfying these assumptions are generic in $SL(d, \mathbb{Z})$. Examples constructed in the Appendix show importance of the assumption on the eigenvalues.

1. INTRODUCTION

Hyperbolic dynamical systems have been one of the main objects of study in smooth dynamics. Basic examples of such systems are given by *Anosov automorphisms* of tori: for a hyperbolic matrix F in $SL(d, \mathbb{Z})$ the map $F : \mathbb{R}^d \to \mathbb{R}^d$ projects to an automorphism of the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. More generally, a diffeomorphism f of a compact Riemannian manifold $\mathcal{M}$ is called *Anosov* if there exist a decomposition of the tangent bundle $T\mathcal{M}$ into two f-invariant continuous distributions $E^{s,f}$ and $E^{u,f}$, and constants $C > 0$, $\lambda > 0$, such that for all $n \in \mathbb{N}$,

$$\|Df^n(v)\| \leq Ce^{-\lambda n}\|v\| \text{ for all } v \in E^{s,f},$$
$$\|Df^{-n}(v)\| \leq Ce^{-\lambda n}\|v\| \text{ for all } v \in E^{u,f}.$$

The distributions $E^{s,f}$ and $E^{u,f}$ are called the stable and unstable distributions of f.

Structural stability is a fundamental property of hyperbolic systems. If g is an Anosov diffeomorphism and f is sufficiently C^1 close to g, then f is also Anosov and is topologically conjugate to g, i.e. there is a homeomorphism h of $\mathcal{M}$ such that

$$g = h^{-1} \circ f \circ h.$$

In this paper we study regularity of the conjugacy h. It is well known that in general h is only Hölder continuous. A necessary condition for it to be C^1 is that the derivatives of the return maps of f and g at the corresponding periodic points

are conjugate. Indeed, differentiating $g^n = h^{-1} \circ f^n \circ h$ at a periodic point $p = f^n(p)$ gives

$$D_p g^n = (D_p h)^{-1} \circ D_{h(p)} f^n \circ D_p h.$$

A diffeomorphism g is said to be *locally rigid* if for any C^1-small perturbation f this condition is also sufficient for the conjugacy to be a C^1 diffeomorphism. The problem of local rigidity has been extensively studied and Anosov systems with one-dimensional stable and unstable distributions were shown to be locally rigid [dlL87, dlLM88, dlL92, P90].

Local rigidity problem in higher dimensions is much less understood. Examples where the periodic condition is not sufficient were constructed by R. de la Llave [dlL92, dlL02]. However, the one-dimensional results were extended in two directions. In the case when g is conformal on the full stable and unstable distributions, local rigidity was established for some classes of systems [dlL02, KS03, dlL04, KS09].

In a different direction, local rigidity was proved in [G08] for an irreducible Anosov toral automorphism $L : \mathbb{T}^d \to \mathbb{T}^d$ with real eigenvalues of distinct moduli, as well as for some nonlinear systems with similar structure. We recall that L is said to be *irreducible* if it has no rational invariant subspaces, or equivalently if its characteristic polynomial is irreducible over $\mathbb{Q}$. It follows that all eigenvalues of L are simple. An important feature of this case is that $\mathbb{R}^d$ splits into a direct sum of one-dimensional L-invariant subspaces. This splitting gives rise to the corresponding linear foliations on $\mathbb{T}^d$ which are expanded or contracted by L at different rates. Such a splitting persist for C^1-small perturbations of L and provides a framework for studying regularity of the conjugacy.

Examples in [G08] show that irreducibility of L is a necessary assumption for local rigidity except when L is conformal on the stable and unstable distributions. The main result of this paper is the following theorem which establishes local rigidity for a broad class of irreducible toral automorphisms. We give a concise proof that uses techniques from [G08, dlL02, KS09] along with some new results on conformality of cocycles from [KS10].

Theorem 1.1. *Let $L : \mathbb{T}^d \to \mathbb{T}^d$ be an irreducible Anosov automorphism such that no three of its eigenvalues have the same modulus. Let f be a C^1-small perturbation of L such that the derivative $D_p f^n$ is conjugate to L^n whenever $p = f^n p$. Then f is $C^{1+H\ddot{o}lder}$ conjugate to L.*

We note that irreducibility of L implies that it is diagonalizable over $\mathbb{C}$. Hence assuming that $D_p f^n$ is conjugate to L^n is equivalent to assuming that $D_p f^n$ is also diagonalizable over $\mathbb{C}$ and has the same eigenvalues as L^n. The only extra assumption in the theorem ensures that the dimensions of the subspaces in the splitting by rates of expansion/contraction are not higher than two. It allows L to have pairs of complex conjugate eigenvalues as well as pairs $\lambda, -\lambda$. We prove this theorem in Section 2.

In Section 3, we show that toral automorphisms satisfying the assumptions of the theorem are generic in the following sense. Consider the set of matrices in $SL(d, \mathbb{Z})$ of norm at most T. Then the proportion of matrices corresponding to automorphisms that do not satisfy our assumptions goes to zero as $T \to \infty$. Moreover, it can be estimated by $c\,T^{-\delta}$ for some $\delta > 0$.

Example A.3 in the Appendix yields an Anosov toral automorphism conformal on a three-dimensional invariant subspace and a perturbation with conjugate periodic data whose derivative is not uniformly quasi-conformal on the corresponding three-dimensional invariant distribution. This, in particular, precludes smoothness of the conjugacy. The automorphism is reducible, so the example does not prove that the extra assumption is indeed necessary for our theorem. However, it clearly shows that current methods cannot be pushed further to give the result without this assumption.

2. Proof of Theorem 1.1

2.1. **Notation and outline of the proof.** We denote by $E^{s,L}$ and $E^{u,L}$ the stable and unstable distributions of L. Since f is C^1 close to L, f is also Anosov and we denote its stable and unstable distributions by $E^{s,f}$ and $E^{u,f}$. They are tangent to the stable and unstable foliations $W^{s,f}$ and $W^{u,f}$ respectively (see, e.g. [KH95]). The leaves of these foliations are C^∞ smooth, but in general the distributions $E^{s,f}$ and $E^{u,f}$ are only Hölder continuous transversally to the corresponding foliations.

Let $1 < \rho_1 < \rho_2 < \cdots < \rho_l$ be the distinct moduli of the unstable eigenvalues of L, and let

$$E^{u,L} = E_1^L \oplus E_2^L \oplus \cdots \oplus E_l^L$$

be the corresponding splitting of the unstable distribution.

By the assumption, the distributions E_k^L, $k = 1, \ldots, l$, are either one- or two-dimensional. As f is C^1-close to L, the unstable distribution $E^{u,f}$ splits into a direct sum of l invariant Hölder continuous distributions close to the corresponding distributions for L:

$$E^{u,f} = E_1^f \oplus E_2^f \oplus \cdots \oplus E_l^f$$

(see, e.g. [Pes04, Section 3.3]). We also consider the distributions

$$E_{(i,j)}^f = E_i^f \oplus E_{i+1}^f \oplus \ldots \oplus E_j^f.$$

For any $1 < k \le l$, the distribution $E_{(k,l)}^f$ is a fast part of the unstable distribution and thus it integrates to a Hölder foliation $W_{(k,l)}^f$ with C^∞ smooth leaves (see, e.g. [Pes04, Section 3.3]). Moreover, the leaves $W_{(k,l)}^f(x)$ depend C^∞ smoothly on x along the unstable leaves $W^{u,f}$ (see, e.g. [KS07, Proposition 3.9]).

Notation. We say that an object is C^{1+} if it is C^1 and its differential is Hölder continuous with some positive exponent. We say that a homeomorphism h is C^{1+}

along a foliation $\mathcal{F}$ if the restrictions of h to the leaves of $\mathcal{F}$ is C^{1+} and the derivative $Dh|_{\mathcal{F}}$ is Hölder continuous on the manifold.

For any $1 \leq k < l$ the distribution $E^f_{(1,k)}$ is a slow part of the unstable distribution. It also integrates to an f-invariant foliation $W^f_{(1,k)}$ with C^{1+} smooth leaves. One way to see this is to view L as a partially hyperbolic automorphism with the splitting $E^{s,L} \oplus E^L_{(1,k)} \oplus E^l_{(k+1,l)}$. It follows from the structural stability of partially hyperbolic systems [HPS77, Theorem 7.1] that for a C^1-small perturbation f the "central" foliation survives; that is, $E^f_{(1,k)}$ integrates to a foliation $W^f_{(1,k)}$. For an alternative simple and short proof that uses specifics of our setup and also gives unique integrability, (as opposed to existence of some foliation tangent to $E^f_{(1,k)}$) see [G08, Lemma 6.1].

Thus within the unstable distribution $E^{u,f}$ there are flags of weak and strong distributions

$$E^f_1 = E^f_{(1,1)} \subset E^f_{(1,2)} \subset \ldots \subset E^f_{(1,l)} = E^{u,f},$$
$$E^f_l = E^f_{(l,l)} \subset E^f_{(l-1,l)} \subset \ldots \subset E^f_{(1,l)} = E^{u,f}.$$

Since both flags are uniquely integrable and the leaves of the corresponding foliations are at least C^{1+}, for any $1 \leq k \leq l$ the distribution $E^f_k = E^f_{(1,k)} \cap E^f_{(k,l)}$ also integrates uniquely to a Hölder foliation

$$V^f_k = W^f_{(1,k)} \cap W^f_{(k,l)}$$

with C^{1+} smooth leaves. Similarly, the distributions $E^f_{(i,j)} = E^f_{(1,j)} \cap E^f_{(i,l)}$, $1 \leq i \leq j \leq l$, integrate to Hölder foliations

$$W^f_{(i,j)} = W^f_{(1,j)} \cap W^f_{(i,l)}.$$

We use similar notation for the automorphism L: $E^L_{(i,j)} = E^L_i \oplus \ldots \oplus E^L_j$, and $W^L_{(i,j)}$ and V^L_i are the linear foliations tangent to $E^L_{(i,j)}$ and E^L_i respectively.

Since L is Anosov and f is C^1 close to L, there exists a bi-Hölder continuous homeomorphism $h : \mathbb{T}^d \to \mathbb{T}^d$ close to the identity in C^0 topology such that

$$h \circ L = f \circ h.$$

The conjugacy h takes the flag of weak foliations for L into the corresponding weak flag for f:

Lemma 2.1. *For any $1 \leq k \leq l$, $h(W^L_{(1,k)}) = W^f_{(1,k)}$.*

The proof is the same as that of Lemma 6.3 in [G08]. We give the argument for the reader's convenience.

Proof. Let $\tilde{h}$, $\tilde{f}$ and $\tilde{L}$ be the lifts of h, f and L to $\mathbb{R}^d$. Similarly we use the tilde sign to denote lifts of various foliations.

Since $\tilde{h}(\tilde{W}^{u,L}) = \tilde{W}^{u,f}$ we have that $\tilde{h}(\tilde{W}^L_{(1,k)}) \subset \tilde{W}^{u,f}$. Let $y \in \tilde{W}^{u,L}(x)$, then

$$y \in \tilde{W}^L_{(1,k)}(x) \text{ if and only if } \operatorname{dist}(\tilde{L}^n(x), \tilde{L}^n(y)) \le (\rho_k + \epsilon)^n \operatorname{dist}(x,y) \text{ for all } n > 0,$$

where dist is the standard metric on $\mathbb{R}^d$. Since $\tilde{h}$ is C^0 close to Id we further obtain that $y \in \tilde{W}^L_{(1,k)}(x)$ if and only if for all $n > 0$

$$\operatorname{dist}(\tilde{f}^n(\tilde{h}(x)), \tilde{f}^n(\tilde{h}(y))) = \operatorname{dist}(\tilde{h}(\tilde{L}^n(x)), \tilde{h}(\tilde{L}^n(y))) \le (\rho_k + \epsilon)^n \operatorname{dist}(x,y) + c.$$

The latter condition in turn is equivalent to $\tilde{h}(y) \in \tilde{W}^f_{(1,k)}(\tilde{h}(x))$. $\qquad \square$

We note that Lemma 2.1 holds for any sufficiently C^1-small perturbation of an Anosov automorphism of $\mathbb{T}^d$.

The coarse strategy of the proof of Theorem 1.1 is showing inductively that h is C^{1+} along $W^L_{(1,k)}$ for any k and thus along $W^L_{(1,l)} = W^u(L)$. By the same argument, h is C^{1+} along $W^s(L)$ and hence h is C^{1+} by Journé Lemma:

Lemma 2.2 (Journé [J88])**.** *Let $\mathcal{M}_j$ be a manifold and $\mathcal{F}^s_j$, $\mathcal{F}^u_j$ be continuous transverse foliations on $\mathcal{M}_j$ with uniformly smooth leaves, $j = 1, 2$. Suppose that $h : \mathcal{M}_1 \to \mathcal{M}_2$ is a homeomorphism that maps $\mathcal{F}^s_1$ into $\mathcal{F}^s_2$ and $\mathcal{F}^u_1$ into $\mathcal{F}^u_2$. Moreover, assume that the restrictions of h to the leaves of these foliations are uniformly $C^{r+\nu}$, $r \in \mathbb{N}$, $0 < \nu < 1$. Then h is $C^{r+\nu}$.*

The main steps of the proof of the Theorem are the following statements:

- $h(V^L_i) = V^f_i$;
- h is a C^{1+} diffeomorphism along V^L_i.

Their proofs are interdependent and organized into an inductive process given by Propositions 2.3 and 2.4.

Proposition 2.3. *If $h(V^L_i) = V^f_i$, then h is a C^{1+} diffeomorphism along V^L_i.*

The proof of this proposition is given in Section 2.2 below. Since $V^L_1 = W^L_{(1,1)}$, Lemma 2.1 implies that $h(V^L_1) = V^f_1$, and then Proposition 2.3 yields that h is C^{1+} along V^L_1. This provides the base of the induction. The inductive step is given by the following proposition.

Proposition 2.4. *Suppose that $h(V^L_i) = V^f_i$, $1 \le i \le k - 1$, and h is a C^{1+} diffeomorphism along $W^L_{(1,k-1)}$. Then $h(V^L_k) = V^f_k$ and h is a C^{1+} diffeomorphism along $W^L_{(1,k)}$.*

The proof of this proposition is given in Section 2.3 (and also uses an inductive argument). In the proof we only need to establish that $h(V^L_k) = V^f_k$. Then Proposition 2.3 implies the smoothness of h along V^L_k, and the smoothness along $W^L_{(1,k)}$ follows from the Journé Lemma 2.2.

2.2. **Proof of Proposition 2.3.** In this subsection we write

$$\mathcal{V}^L \stackrel{\text{def}}{=} V_i^L, \quad \mathcal{V}^f \stackrel{\text{def}}{=} V_i^f, \quad \mathcal{E}^L \stackrel{\text{def}}{=} E_i^L = T\mathcal{V}^L, \quad \mathcal{E}^f \stackrel{\text{def}}{=} E_i^f = T\mathcal{V}^f.$$

The proof is an adaptation of arguments of de la Llave [dlL02]. First we show that h is Lipschitz along $\mathcal{V}^L$ as a limit of smooth maps with uniformly bounded derivatives. Then we prove that the measurable derivative of h along $\mathcal{V}^L$ is actually Hölder continuous. Both steps use Livšic Theorem for commutative and noncommutative cocycles and rely on conformality of L and f along $\mathcal{V}^L$ and $\mathcal{V}^f$ respectively. Conformality of f along $\mathcal{V}^f$ is crucial and to establish it we use a result from [KS10].

First we construct a map h_0 close to h and satisfying the following conditions:

(1) $h_0(\mathcal{V}^L) = \mathcal{V}^f$, moreover, $h_0(\mathcal{V}^L(x)) = \mathcal{V}^f(h(x))$ for all x in $\mathbb{T}^d$;
(2) $\sup_{x \in \mathbb{T}^d} d_{\mathcal{V}^f}(h_0(x), h(x)) < +\infty$, where $d_{\mathcal{V}^f}$ is the distance along the leaves;
(3) h_0 is C^{1+} diffeomorphism along the leaves of $\mathcal{V}^L$.

Let $\bar{\mathcal{V}}^L$ be the linear integral foliation of $E^{s,L} \oplus E_1^L \oplus \ldots \oplus E_{i-1}^L \oplus E_{i+1}^L \oplus \ldots \oplus E_l^L$. We define the map h_0 by intersecting local leaves:

$$h_0(x) = \mathcal{V}^{f,loc}(h(x)) \cap \bar{\mathcal{V}}^{L,loc}(x).$$

The map is well-defined and satisfies (2) since h is close to the identity. Condition (1) holds since $h(\mathcal{V}^L(x)) = \mathcal{V}^f(h(x))$ by the assumption, and (3) is satisfied since for any x the leaf $\mathcal{V}^f(h(x))$ is C^{1+} and C^1 close to $\mathcal{V}^L(x)$.

It follows easily as in [dlL02, Theorem 2.1] that

$$h = \lim_{n \to \infty} h_n, \quad \text{where} \quad h_n = f^{-n} \circ h_0 \circ L^n.$$

Indeed, let us endow the space of maps satisfying (1) and (2) with the metric $d(k_1, k_2) = \sup_x d_{\mathcal{V}^f}(k_1(x), k_2(x))$. Then, since f^{-1} contracts the leaves of $\mathcal{V}^f$, it follows that the map $k \mapsto f^{-1} \circ k \circ L$ is a contraction with the fixed point h.

Now we prove that h is Lipschitz along $\mathcal{V}^f$. For this it suffices to show that the derivatives of the maps h_n along $\mathcal{V}^L$ are uniformly bounded. We estimate

$$\|D_{\mathcal{V}^L(x)} h_n\| \leq \|Df^{-n}|_{\mathcal{E}^f(h_0(L^n x))}\| \cdot \|D_{\mathcal{V}^L(L^n x)} h_0\| \cdot \|L^n|_{\mathcal{E}^L(x)}\|$$

$$= \| \left(Df^n|_{\mathcal{E}^f(f^{-n}(h_0(L^n x)))} \right)^{-1} \| \cdot \|D_{\mathcal{V}^L(L^n x)} h_0\| \cdot \|L^n|_{\mathcal{E}^L}\|$$

$$\leq \| \left(Df^n|_{\mathcal{E}^f(h_n(x))} \right)^{-1} \| \cdot \|L^n|_{\mathcal{E}^L}\| \cdot \sup_z \|D_{\mathcal{V}^L(z)} h_0\|.$$

Since $D_{\mathcal{V}^L} h_0$ is continuous on $\mathbb{T}^d$, the supremum on the right is finite. Now we show that the product $\|(Df^n|_{\mathcal{E}^f(y)})^{-1}\| \cdot \|L^n|_{\mathcal{E}^L}\|$ is uniformly bounded in y and n.

We concentrate on the case when $\mathcal{V}^f$ is two-dimensional. The one-dimensional case is similar except for conformality of L along $\mathcal{V}^L$ and of f along $\mathcal{V}^f$ is trivial. Since L is irreducible it is diagonalizable over $\mathbb{C}$. Therefore, as the eigenvalues of $L|_{\mathcal{E}^L}$ have the same modulus, $L|_{\mathcal{E}^L}$ is conformal with respect to some norm on $\mathcal{E}^L$. We can assume that our background norm $\|\cdot\|$ is chosen so that $L|_{\mathcal{E}^L}$ is conformal.

By the assumption of the theorem, $D_p f^n$ is conjugate to L^n whenever $f^n p = p$. It follows that $D_p f^n|_{\mathcal{E}^f(p)}$ is also diagonalizable over $\mathbb{C}$ and has eigenvalues of the same modulus. To obtain conformality of $Df|_{\mathcal{E}^f}$, we apply the following result to vector bundle $\mathcal{E} = \mathcal{E}^f$ and cocycle $F = Df|_{\mathcal{E}^f}$.

[KS10, Theorem 1.3] *Let $\mathcal{E}$ be a Hölder continuous linear bundle with two-dimensional fibers over a compact Riemannian manifold $\mathcal{M}$. Let $F : \mathcal{E} \to \mathcal{E}$ be a Hölder continuous linear cocycle over a transitive Anosov diffeomorphism $f : \mathcal{M} \to \mathcal{M}$. If for each periodic point $p \in \mathcal{M}$, the return map $F_p^n : \mathcal{E}_p \to \mathcal{E}_p$ is diagonalizable over $\mathbb{C}$ and its eigenvalues are equal in modulus, then F is conformal with respect to a Hölder continuous Riemannian metric on $\mathcal{E}$.*

We denote by $\|\cdot\|_x^f$ the norm induced by the metric on $\mathcal{E}^f(x)$ given by the theorem. The conformality of $Df|_{\mathcal{E}^f}$ with respect to this norm means that

$$\|Df(v)\|_{f(x)}^f = c(x) \cdot \|v\|_x^f \quad \text{for any } x \in \mathbb{T}^d \text{ and } v \in \mathcal{E}^f(x).$$

Clearly, $c(x) = \|Df|_{\mathcal{E}^f(x)}\|^f$, the norm of $Df : (\mathcal{E}^f(x), \|\cdot\|_x^f) \to (\mathcal{E}^f(f(x)), \|\cdot\|_{f(x)}^f)$.

We set

$$a(x) = \|L|_{\mathcal{E}^L(x)}\| = \|L|_{\mathcal{E}^L}\| \quad \text{and} \quad b(x) = c(h(x)) = \|Df|_{\mathcal{E}^f(h(x))}\|^f.$$

The function $a(x)$ is constant in our context, however we will keep the variable for consistency with $b(x)$. Since L is conformal on $\mathcal{E}^L$, $a(x)$ satisfies

$$a_n(x) \overset{\text{def}}{=} a(x)a(Lx)\cdots a(L^{n-1}x) = \|L^n|_{\mathcal{E}^L}\|.$$

The function $b(x)$ is Hölder continuous, and using the relation $f^m \circ h = h \circ L^m$ and the conformality of $Df|_{\mathcal{E}^f}$ we obtain

$$b_n(x) \overset{\text{def}}{=} b(x)b(Lx)\cdots b(L^{n-1}x) =$$
$$= \|Df|_{\mathcal{E}^f(h(x))}\|^f \cdot \|Df|_{\mathcal{E}^f(h(Lx))}\|^f \cdots \|Df|_{\mathcal{E}^f(h(L^{n-1}x))}\|^f = \|Df^n|_{\mathcal{E}^f(h(x))}\|^f.$$

We claim that the functions a and b are cohomologous, i.e. the exists a continuous function $\phi : \mathbb{T}^d \to \mathbb{R}_+$ such that

$$a(x)/b(x) = \phi(Lx)/\phi(x).$$

This follows from the Livšic Theorem [Liv71], [KH95, Theorem 19.2.1] once we show that $a_n(p) = b_n(p)$ for any periodic point $p = L^n p$. We note that $b_n(p) = \|Df^n|_{\mathcal{E}^f(h(p))}\|^f$ is the modulus of the eigenvalues of $Df^n|_{\mathcal{E}^f(h(p))}$ since this linear map is conformal with respect to norm $\|\cdot\|_{h(p)}^f$. A similar statement holds for $a_n(p)$ and $L^n|_{\mathcal{E}^L}$. The coincidence of the periodic data for f and L implies that indeed $a_n(p) = b_n(p)$ and hence the functions a and b are cohomologous. Using conformality we obtain that

$$\|L^n|_{\mathcal{E}^L}\| \cdot \|(Df^n|_{\mathcal{E}^f(h(x))})^{-1}\|^f = \|L^n|_{\mathcal{E}^L}\| \cdot \left(\|Df^n|_{\mathcal{E}^f(h(x))}\|^f\right)^{-1} =$$
$$= a_n(x)/b_n(x) = \phi(L^n x)/\phi(x)$$

is uniformly bounded since ϕ is continuous on $\mathbb{T}^d$. Since the norm $\|\cdot\|^f$ is equivalent to $\|\cdot\|$ we obtain that $\|(Df^n|_{\mathcal{E}^f(y)})^{-1}\| \cdot \|L^n|_{\mathcal{E}^L}\|$ is uniformly bounded in y and n. We conclude that $\|D_{\mathcal{V}^L(x)} h_n\|$ is uniformly bounded in x and n, and hence h is Lipschitz along $\mathcal{V}^f$.

A similar argument shows that $\|(D_{\mathcal{V}^L(x)} h_n)^{-1}\|$ is uniformly bounded and hence h is bi-Lipschitz along $\mathcal{V}^f$. In particular, $D_{\mathcal{V}^L} h$ exists and is invertible almost everywhere.

Differentiating $f \circ h = h \circ L$ along $\mathcal{V}^L$ on a set of full Lebesgue measure we obtain

$$Df|_{\mathcal{E}^f(h(x))} \circ D_{\mathcal{V}^L(x)} h = D_{\mathcal{V}^L(Lx)} h \circ L|_{\mathcal{E}^L(x)},$$

i.e., the cocycles $Df|_{\mathcal{E}^f(h(x))}$ and $L|_{\mathcal{E}^L(x)}$ are cohomologous with transfer function $D_{\mathcal{V}^L(x)} h$. The bundle $\mathcal{E}^f$ is trivial since it is close to the trivial bundle $\mathcal{E}^L$. Therefore, $Df|_{\mathcal{E}^f(h(x))}$ and $L|_{\mathcal{E}^L(x)}$ can be viewed as Hölder continuous $GL(2,\mathbb{R})$-valued cocycles over the automorphism L. Moreover, the existence of conformal metrics implies that they are cohomologous to cocycles with values in the conformal subgroup. We remark that in general measurable transfer functions are not necessarily continuous [PW01, Section 9]. However, for conformal cocycles the measurable transfer function coincides almost everywhere with a Hölder continuous one. This follows from [Sch99, Theorem 6.1] or from [PP97, Theorem 1] after reducing cocycles to orthogonal ones by factoring out the norms. See also [S10] for stronger results on $GL(2,\mathbb{R})$-valued cocycles. We conclude that $D_{\mathcal{V}^L(x)} h$ is Hölder continuous, and hence h is a C^{1+} diffeomorphism along $\mathcal{V}^L$. $\qquad\square$

2.3. **Proof of Proposition 2.4.** The proof is based on the following proposition.

Proposition 2.5. *Assume that $h(W_{(i,k)}^L) = W_{(i,k)}^f$, $h(V_i^L) = V_i^f$ and h is a C^{1+} diffeomorphism along V_i^L. Then $h(W_{(i+1,k)}^L) = W_{(i+1,k)}^L$.*

We apply Proposition 2.5 inductively with $i = 1, \ldots, k-1$. At every step the assumption of the proposition is fulfilled due to the assumptions in the Proposition 2.4 and the conclusion of Proposition 2.5 at the previous step. We obtain the conclusion of Proposition 2.4 at the final step when $W_{(i+1,k)}^L = W_{(k,k)}^L = V_k^L$.

It remains to prove Proposition 2.5. We will use the following simplified notation:

$$(\mathcal{W}^L, \mathcal{V}^L, \mathcal{U}^L) = (W_{(i,k)}^L, V_i^L, W_{(i+1,k)}^L),$$
$$(\mathcal{W}^f, \mathcal{V}^f, \mathcal{U}^f) = (W_{(i,k)}^f, V_i^f, W_{(i+1,k)}^f).$$

We note that $\mathcal{V}^L$ and $\mathcal{U}^L$ are slow and fast sub-foliations of $\mathcal{W}^L$ respectively. Similarly, $\mathcal{V}^f$ and $\mathcal{U}^f$ are slow and fast sub-foliations in $\mathcal{W}^f$. We also note that $\mathcal{U}^f = \mathcal{W}^f \cap W_{(i+1,l)}^f$. The foliation $W_{(i+1,l)}^f$ is a fast part of the unstable foliation and hence is C^∞ inside the unstable leaves, see for example [KS07, Proposition 3.9]. Therefore, the foliation $\mathcal{U}^f$ is C^{1+} inside the leaves of $\mathcal{W}^f$ and the holonomies between the leaves of $\mathcal{V}^f$ along $\mathcal{U}^f$ are uniformly C^{1+}.

Let $\mathcal{F} = h^{-1}(\mathcal{U}^f)$. Then $\mathcal{F}$ is a continuous foliation with continuous leaves that subfoliates $\mathcal{W}^L$. We need to show that $\mathcal{F} = \mathcal{U}^L$. Since $\mathcal{V}^L = h^{-1}(\mathcal{V}^f)$, $\mathcal{F}$ is topologically transverse to $\mathcal{V}^L$, that is, any leaf of $\mathcal{F}$ and and any leaf of $\mathcal{V}^L$ in the same leaf of $\mathcal{W}^L$ intersect at exactly one point. First we prove an auxiliary statement that gives some insight into relative structure of $\mathcal{F}$ and $\mathcal{V}^L$.

For any point $a \in \mathbb{T}^d$ and any $b \in \mathcal{F}(a)$ we denote by $H_{a,b} : \mathcal{V}^L(a) \to \mathcal{V}^L(b)$ the holonomy along the foliation $\mathcal{F}$, i.e. for every $x \in \mathcal{V}^L(a)$ we define $H_{a,b}(x)$ to be the unique point of intersection $\mathcal{F}(x) \cap \mathcal{V}^L(b)$.

Lemma 2.6. *For any point $a \in \mathbb{T}^d$ and any $b \in \mathcal{F}(a)$ the holonomy map $H_{a,b}$ is a restriction to $\mathcal{V}^L(a)$ of a parallel translation inside $\mathcal{W}^L$.*

Proof. For any point $c \in \mathbb{T}^d$ and any $d \in \mathcal{U}^f(c)$ we denote by $\tilde{H}_{c,d} : \mathcal{V}^f(c) \to \mathcal{V}^f(d)$ the holonomy along the foliation $\mathcal{U}^f$, which is C^{1+} as we noted above. Since $\mathcal{F} = h^{-1}(\mathcal{U}^f)$ and $h(\mathcal{V}^L) = \mathcal{V}^f$ we have

$$H_{a,b} = h^{-1} \circ \tilde{H}_{h(a),h(b)} \circ h.$$

Since h is a C^{1+} diffeomorphism along $\mathcal{V}^L$ we conclude that $H_{a,b}$ is C^{1+}.
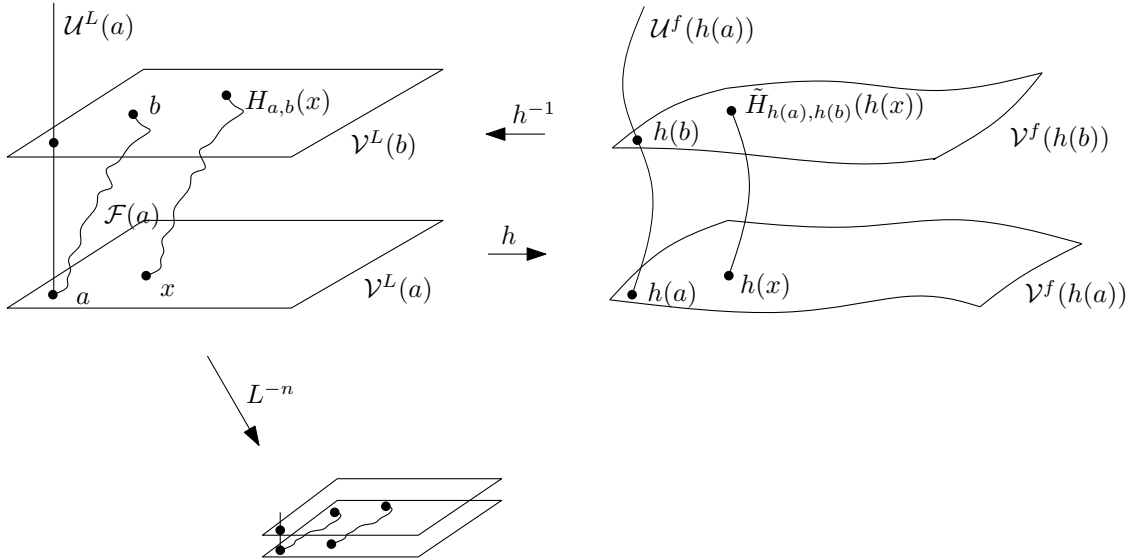

FIGURE 1

To show that $H_{a,b}$ is the restriction of a parallel translation, we prove that the differential $DH_{a,b} = \mathrm{Id}$. We apply L^{-n} and denote $a_n = L^{-n}(a)$, $b_n = L^{-n}(b)$. Since $\mathcal{F} = h^{-1}(\mathcal{U}^f)$ and f preserves the foliation $\mathcal{U}^f$, L preserves $\mathcal{F}$ and we can write

$$H_{a,b} = L^n \circ H_{a_n,b_n} \circ L^{-n}.$$

Differentiating and denoting $D_{a_n} H_{a_n, b_n} = \mathrm{Id} + \Delta_n$ we obtain

$$D_a H_{a,b} = L^n|_{\mathcal{V}^L} \circ D_{a_n} H_{a_n, b_n} \circ L^{-n}|_{\mathcal{V}^L} = \mathrm{Id} + L^n|_{\mathcal{V}^L} \circ \Delta_n \circ L^{-n}|_{\mathcal{V}^L}.$$

Since L is conformal on $\mathcal{V}^L$ with respect to some inner product,

$$\|L^n|_{\mathcal{V}^L}\| \cdot \|L^{-n}|_{\mathcal{V}^L}\| \leq C \quad \text{and hence} \quad \|L^n|_{\mathcal{V}^L} \circ \Delta_n \circ L^{-n}|_{\mathcal{V}^L}\| \leq C\|\Delta_n\| \quad \text{for all } n.$$

It remains to show that $\|\Delta_n\| \to 0$. This follows easily by differentiating the equation

$$H_{a_n, b_n} = h^{-1} \circ \tilde{H}_{h(a_n), h(b_n)} \circ h.$$

Indeed, we obtain

$$D_{a_n} H_{a_n, b_n} = (D_{b_n} h)^{-1} \circ D_{h(a_n)} \tilde{H}_{h(a_n), h(b_n)} \circ D_{a_n} h.$$

Since $\mathrm{dist}(a_n, b_n) \to 0$ as $n \to \infty$ we obtain that $D_{h(a_n)} \tilde{H}_{h(a_n), h(b_n)} \to \mathrm{Id}$. Also, by choosing a subsequence if necessary, we can assume that $\lim a_n = \lim b_n = c \in \mathbb{T}^d$. Thus $D_{a_n} h \to D_c h$, $D_{b_n} h \to D_c h$ and hence $\Delta_n = D_{a_n} H_{a_n, b_n} - \mathrm{Id} \to 0$ as $n \to \infty$. $\quad\square$

Now let a be a fixed point of L and let B be the unit ball in $\mathcal{U}^L(a)$ centered at a. If $B \subset \mathcal{F}(a)$, then $\mathcal{U}^L(a) = \mathcal{F}(a)$ by invariance under L. Since L is irreducible and $\mathcal{U}^L$ is invariant, the leaf $\mathcal{U}^L(a)$ is dense in $\mathbb{T}^d$. It follows that the set of points x such that $\mathcal{U}^L(x) = \mathcal{F}(x)$ is dense in $\mathbb{T}^d$ and hence $\mathcal{U}^L = \mathcal{F}$. Therefore, it suffices to show that $B \subset \mathcal{F}(a)$.

We argue by contradiction. Assume that there is $z_1 \in B$ such that $z_1 \notin \mathcal{F}(a)$. Let

$$x_1 = \mathcal{V}^L(z_1) \cap \mathcal{F}(a).$$

Since $\mathcal{V}^L$ has dense leaves we can choose a sequence $\{b_n, n \geq 1\} \subset \mathcal{V}^L(a)$ so that $b_n \to x_1$ as $n \to \infty$. Let

$$y_n = H_{a, x_1}(b_n),$$

where H_{a, x_1} is the holonomy map along $\mathcal{F}$ from $\mathcal{V}^L(a)$ to $\mathcal{V}^L(x_1)$. Continuity of $\mathcal{F}$ implies that the sequence y_n converges to a point $x_2 \in \mathcal{F}(a)$. Moreover, Lemma 2.6 implies that $\{x_1, x_2\}$ is a parallel translation of $\{a, x_1\}$.

We continue this procedure inductively to construct the sequence $\{x_n, n \geq 1\} \subset \mathcal{F}(a)$. Let

$$z_n = \mathcal{V}^L(x_n) \cap \mathcal{U}^L(a).$$

Then according to the construction

$$(2.1) \qquad d_{\mathcal{U}^L}(z_n, a) = n \cdot d_{\mathcal{U}^L}(z_1, a) \quad \text{and} \quad d_{\mathcal{V}^L}(x_n, z_n) = n \cdot d_{\mathcal{V}^L}(x_1, z_1).$$

For every n we take $N(n)$ to be the smallest integer such that $L^{-N(n)}(z_n) \in B$. Since L^{-1} contracts $\mathcal{U}^L$ exponentially faster than $\mathcal{V}^L$, equation (2.1) implies that

$$d_{\mathcal{V}^L}(L^{-N(n)}(x_n), L^{-N(n)}(z_n)) \to \infty \quad \text{as} \ n \to \infty.$$

This contradicts an obvious bound due to compactness of B:

$$\max_{z \in B} d_{\mathcal{V}^L}(z, \mathcal{V}^L(z) \cap \mathcal{F}(a)) < \infty.$$
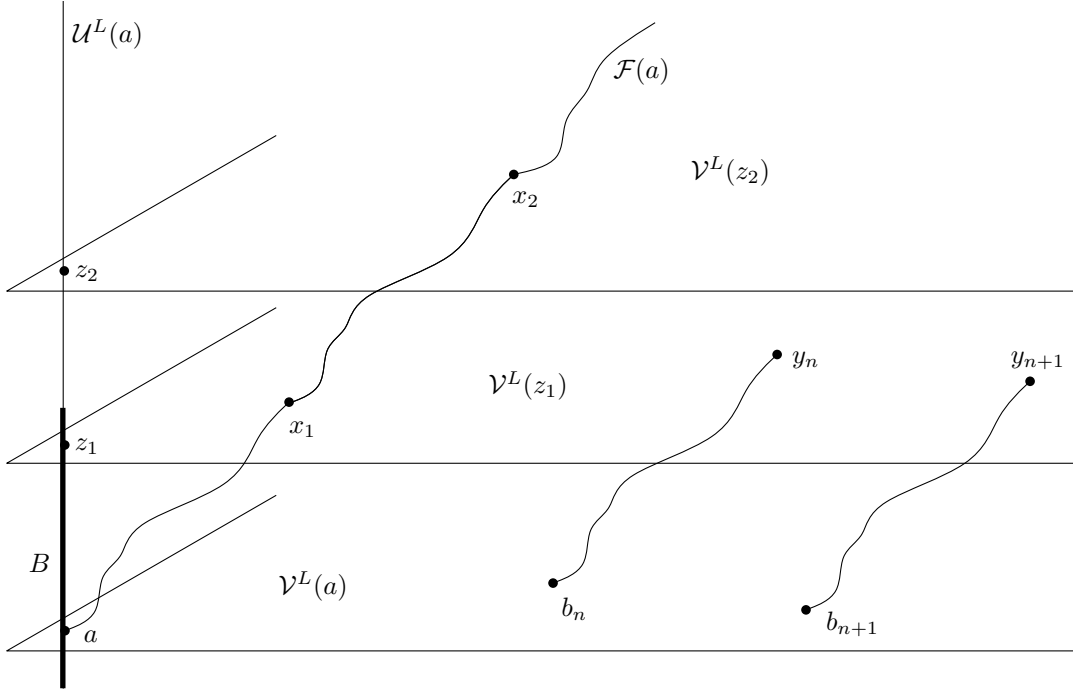
FIGURE 2

Thus we conclude that $\mathcal{F} = \mathcal{U}^L$. □

3. GENERICITY

In this section we show that toral automorphisms satisfying the assumptions of Theorem 1.1 are generic in $SL(d, \mathbb{Z})$. We would like to thank A. Gorodnik, P. Sarnak, and D. Speyer for helpful discussions on this topic.

We consider $SL(d, \mathbb{R})$, $d \geq 2$, as a subset of the Euclidean space of $d \times d$ matrices equipped with the norm $\|A\| = (\text{Tr}(A^*A))^{1/2}$. We denote

$$B_{\mathbb{R}}(T) = \{A \in SL(d, \mathbb{R}) : \|A\| \leq T\} \quad \text{and} \quad B_{\mathbb{Z}}(T) = \{A \in SL(d, \mathbb{Z}) : \|A\| \leq T\}.$$

It is known [DRS93, Theorem 3.1] that number of matrices in $B_{\mathbb{Z}}(T)$ grows as the Haar volume of $B_{\mathbb{R}}(T)$. More precisely

$$\#B_{\mathbb{Z}}(T) \sim vol(B_{\mathbb{R}}(T)) \sim c\,T^{d^2-d}.$$

Let $E(T)$ be the subset of $B_{\mathbb{Z}}(T)$ that consists of matrices that do *not* satisfy the assumptions of Theorem 1.1, i.e. are either reducible (over $\mathbb{Q}$), or non-Anosov, or have at least three eigenvalues of the same modulus.

Proposition 3.1. *There exists $\delta > 0$ such that $\#E(T) \ll T^{d^2-d-\delta}$.*

To prove the above proposition we first show the following.

Lemma 3.2. *The set $E = \cup_{T>0} E(T)$ lies in a finite union of algebraic hypersurfaces in $SL(d, \mathbb{R})$.*

Proof. We consider $A \in SL(d, \mathbb{Z})$ and denote its eigenvalues by $r_1, r_2, ..., r_d$. We will describe explicit relations on the entries of matrices in E as symmetric polynomials in $r_1, r_2, ..., r_d$. Since the eigenvalues are the roots of the characteristic polynomial χ_A, such a polynomial can be expressed as a polynomial in the coefficients of χ_A, and hence as one in the entries of A.

Suppose that A is irreducible and χ_A has three roots of the same modulus, in particular, $d \geq 3$. Then χ_A must have either two pairs of complex conjugate eigenvalues of the same modulus or a pair of complex conjugate eigenvalues of the same modulus as a real eigenvalue. In the first case, the eigenvalues satisfy

$$P = \prod_{i,j,k,l} (r_i r_j - r_k r_l) = 0, \quad \text{where } 1 \leq i, j, k, l \leq d \text{ are distinct,}$$

and in the second case they satisfy

$$P = \prod_{i,j,k} (r_i r_j - r_k^2) = 0, \quad \text{where } 1 \leq i, j, k \leq d \text{ are distinct.}$$

Suppose that A is not Anosov, i.e. it has an eigenvalue of modulus 1. If A has a complex pair r_i, r_j on the unit circle, then $r_i r_j = 1$ and the same holds for the product of all other eigenvalues since $\det A = 1$. Thus we obtain a symmetric polynomial relation

$$P = \prod_{i \neq j} (r_i r_j - \prod_{k \neq i,j} r_k) = 0.$$

Similarly, if $r_i = 1$ or $r_i = -1$ for some i, we have

$$P = \prod_i (r_i - \prod_{k \neq i} r_k) = 0.$$

These relations are non-trivial if $d \geq 3$. For $d = 2$, A is not Anosov if and only if $|TrA| \leq 2$. Such matrices lie in affine hyperplanes $TrA = k$, $k = 0, \pm 1, \pm 2$.

Finally, suppose that A is reducible, i.e. its characteristic polynomial χ_A is reducible over $\mathbb{Q}$. Since A is in $SL(d, \mathbb{Z})$, χ_A is reducible over $\mathbb{Z}$ and the factors have constant terms equal to 1 or -1. It easy to see that having such a factor of a given degree imposes a nontrivial constrain on the coefficients of χ_A. Alternatively, one can give relations on the roots as before. For example, if χ_A has a factor of degree k with constant term 1, then some product of k eigenvalues is 1 and hence the eigenvalues satisfy the relation

$$P = \prod_{i_1, ..., i_k} (r_{i_1} \cdots r_{i_k} - 1) = 0, \quad \text{where } 1 \leq i_1, ..., i_k \leq d \text{ are distinct.}$$

$\square$

Now we deduce Proposition 3.1 from the following result by A. Nevo and P. Sarnak, which is a particular case of Lemma 4.2 in [NS08].

Lemma 3.3. *Let G be a subgroup of $GL(m, \mathbb{R})$ isomorphic to $SL(d, \mathbb{R})$. Let $v \in \mathbb{Z}^m$ and $V = Gv$ be the orbit of v. Assume that there is a polynomial $P \in \mathbb{Q}[x_1, ..., x_m]$ that does not vanish identically on V. Then there exists $\delta = \delta(P) > 0$ such that*

$$\#\{A \in G \cap GL(m, \mathbb{Z}) : \|A\| \leq T, \ P(Av) = 0\} \ll T^{d^2 - d - \delta}.$$

We apply this proposition with $m = d^2$ and identify $\mathbb{R}^m$ with $\mathrm{Mat}_{d \times d}(\mathbb{R})$ as follows, first d coordinates are identified with the first column, next d coordinates with the second column, etc. We embed $SL(d, \mathbb{R})$ into $GL(m, \mathbb{R})$ diagonally $g \mapsto g \times g \times \ldots \times g$ (d times). It is easy to see that under these identifications, the action of $SL(d, \mathbb{R})$ on $\mathbb{R}^m$ is the same as the matrix multiplication on the left in $\mathrm{Mat}_{d \times d}(\mathbb{R})$.

We take $v \in \mathbb{Z}^m = \mathrm{Mat}_{d \times d}(\mathbb{Z})$ to be the identity matrix. Then V is identified with $SL(d, \mathbb{R})$ and applying Lemma 3.3 to the polynomials given in Lemma 3.2 yields Proposition 3.1. We note that the norm in Lemma 3.3 comes from $GL(m, \mathbb{R})$ and is different from the norm we have defined on $SL(d, \mathbb{R})$. However this does not make a difference for the asymptotics since the norms are equivalent.

Appendix A. Some examples

by Rafael de la Llave

We consider matrix cocycles over an Anosov diffeomorphism g of a manifold $\mathcal{M}$. Such a cocycle is given by a function $A : \mathcal{M} \to GL(d, \mathbb{R})$. Our goal is to construct examples of (a) cocycles which are conformal at periodic points but are not uniformly quasi-conformal and (b) Anosov diffeomorphisms such that the restriction of the derivative to an invariant distribution gives a cocycle as in (a).

For a matrix $A \in GL(d, \mathbb{R})$ we denote by $K(A) = \|A\| \cdot \|A^{-1}\|$ its *quasi-conformal distortion* with respect to a norm $\|.\|$ on $\mathbb{R}^d$. A is called *conformal* with respect to a given norm if $K(A) = 1$. For example, if A is diagonalizable over $\mathbb{C}$ and all its eigenvalues are of the same modulus, then A is conformal with respect to a norm given by a diagonalization of A. We say that a cocycle $A : \mathcal{M} \to GL(d, \mathbb{R})$ is *uniformly quasi-conformal* if $K(A^n(x))$ is uniformly bounded in x and n, where

$$A^n(x) = A(g^{n-1}x) \cdots A(gx)A(x).$$

Unlike conformality, uniform quasi-conformality does not depend on the choice of a norm.

Examples of (a) were already constructed in [KS10] but did not give rise to examples of (b). We also note that our examples are contained in infinite dimensional families which include linear automorphisms of the tori, and so they show that parametric rigidity is also impossible.

Example A.1. *Let g be an Anosov diffeomorphism of a manifold $\mathcal{M}$. There exists a family of $SL(3, \mathbb{R})$-valued cocycles A_ϵ, $|\epsilon| < 1$, over g such that:*

- A_0 is a constant conformal matrix;
- $A_\epsilon(x)$ is jointly analytic in ϵ and x;
- For any ϵ and any periodic point $p = g^n p$, $A_\epsilon^n(p)$ is conformal in some norm;
- For any $\epsilon \neq 0$, the cocycle A_ϵ is not uniformly quasi-conformal.

Note that in Example A.1 we can take g to be any Anosov diffeomorphism and we do not even require transitivity.

Remark A.2. *One can construct similar examples taking values in $SL(d, \mathbb{R})$ for $d \geq 3$. It was shown in [KS10, Theorem 1.3] that there are no such examples when $d = 2$, and the result is trivial when $d = 1$.*

The second example shows that this phenomenon is also possible in derivative cocycles.

Example A.3. *There exists d (e.g., $d = 9$) and an analytic family of analytic maps $f_\epsilon : \mathbb{T}^d \to \mathbb{T}^d$ such that:*

- *f_0 is an Anosov linear automorphism of $\mathbb{T}^d$;*
- *For any ϵ, Df_ϵ preserves a three dimensional bundle E;*
- *For any ϵ and any periodic point $p = f^n p$, $Df_\epsilon^n|_{E(p)}$ is conformal in some norm;*
- *For any $\epsilon \neq 0$, $Df_\epsilon|_E$ is not uniformly quasi-conformal.*

A.1. **Construction of Example A.1.** We pass to a finite power $f \equiv g^N$ of g for which there exist two fixed points x_1, x_2 and two balls B_1, B_2 around them such that for every sequence $\sigma \in \{1, 2\}^{\mathbb{N}}$ there exists a point x^* such that

$$(A.1) \qquad f^j(x^*) \in B_{\sigma_j}.$$

This can be easily arranged using Markov partitions. Of course, the point x^* is far from being unique, as any point on its local stable manifold would have the same itinerary.

We construct the family of cocycles with required properties over f to illustrate the idea and then indicate how to carry out similar construction over g. We take

$$A_\epsilon(x) = \begin{pmatrix} R_\beta & \epsilon\varphi(x) \\ 0 & 1 \end{pmatrix}, \quad \text{where} \quad R_\beta = \begin{pmatrix} \cos 2\pi\beta & \sin 2\pi\beta \\ -\sin 2\pi\beta & \cos 2\pi\beta \end{pmatrix},$$

β an irrational number, and $\varphi : \mathcal{M} \to \mathbb{R}^2$ an analytic function satisfying some properties to be specified later. We observe that

$$A_\epsilon^n(x) = \begin{pmatrix} R_{n\beta} & \epsilon\widetilde{\varphi}_n(x) \\ 0 & 1 \end{pmatrix}, \quad \text{where}$$

$$\widetilde{\varphi}_n(x) = \sum_{j=1}^n R_{(n-j)\beta}\, \varphi(f^j(x)) = R_{n\beta} \sum_{j=1}^n R_{-j\beta}\, \varphi(f^j(x)).$$

Clearly, the eigenvalues of $A_\epsilon^n(x)$ are $e^{2\pi in\beta}$, $e^{-2\pi in\beta}$, 1. Since β is irrational, all of them are different, and $A_\epsilon^n(x)$ is diagonalizable. Therefore $A_\epsilon^n(p)$ is conformal in some norm whenever $f^n(p) = p$. We construct a function φ and a point x^* such that

$$\Big\| \sum_{j=1}^n R_{-j\beta}\,\varphi(f^j(x^*)) \Big\| \to \infty \quad \text{as } n \to \infty,$$

which implies that A_ϵ is not uniformly quasiconformal for every $\epsilon \neq 0$.

Since the C^∞ case is easier we will discuss it first. We choose an increasing sequence of integers $\mathcal{J} \equiv \{j_k\}_{k=1}^\infty$ such that

$$\{j_k\beta \mod 1\} \to 0 \quad \text{and hence} \quad \|R_{-j_k\beta} - \mathrm{Id}\| \to 0 \text{ as } k \to \infty.$$

We take a sequence σ such that

(A.2)
$$\sigma_\ell = \begin{cases} 1 & \text{if } \ell \in \{j_k\}_{k\in\mathbb{N}} \\ 2 & \text{if } \ell \notin \{j_k\}_{k\in\mathbb{N}} \end{cases}$$

and consider x^* that satisfies (A.1) for the sequence (A.2). Now, we choose a function φ so that

$$\varphi(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ if } x \in B_1 \quad \text{and} \quad \varphi(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ if } x \in B_2.$$

Then we have

$$\Big\| \sum_{j=1}^n R_{-j\beta}\varphi(f^j x^*) \Big\| = \Big\| \Big(\sum_{j_k \leq n} R_{-j_k\beta} \Big) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Big\| \to \infty \quad \text{as } n \to \infty.$$

The analytic case is slightly more complicated since we cannot use functions with compact support. We define a sequence $\mathcal{J}$ in a different way

$$j \in \mathcal{J} \quad \text{if and only if} \quad \measuredangle\Big(R_{-j\beta}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Big) < \pi/3.$$

Consider a corresponding point x^* and an analytic function φ satisfying

$$\Big| \varphi(x) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Big| \leq 10^{-8} \text{ if } x \in B_1 \quad \text{and} \quad \Big| \varphi(x) - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Big| \leq 10^{-8} \text{ if } x \in B_2.$$

Such functions can easily be obtained by modifying the corresponding C^∞ examples. Since our new sequence has asymptotic density $1/3$ in $\mathbb{N}$ a straightforward estimate implies divergence of corresponding sum.

The point of Example A.1 is that the matrices that diagonalize $A_\epsilon^n(p)$ are not bounded uniformly in p, so while $A_\epsilon^n(p)$ is conformal in some norm, $K(A_\epsilon^n(p))$ are not uniformly bounded with respect to the standard norm.

Similar constructions can be carried out for the initial diffeomorphism g. Instead of balls B_1 and B_2 one needs to work with neigborhoods of $\{x_1, g(x_1), ..., g^{N-1}(x_1)\}$ and $\{x_2, g(x_2), ..., g^{N-1}(x_2)\}$ and redefine φ accordingly.

A.2. **Construction of Example A.3.** Let B and C be hyperbolic integer matrices with determinant 1 such that $r > 1$ is an eigenvalue of C, and $re^{\pm 2\pi i\beta}$ are eigenvalues of B for some irrational β. Then

$$f_0(x, y) = (Bx, Cy),$$

is an Anosov toral automorphism. As in Example A.1 we pass to a finite power if necessary, this only changes the value of r. To embed Example A.1 into a diffeomorphism we consider a perturbation of the form

$$f_\epsilon(x, y) = (Bx + \epsilon\psi(y), Cy),$$

where ψ takes values in the two dimensional subspace U corresponding to the eigenvalues $re^{\pm 2\pi i\beta}$. Let v be an eigenvector of C corresponding to r. Then the three dimensional space $W = U \oplus \mathbb{R}v$ is invariant under f_ϵ. The restriction of the differential Df_ϵ to W is of the form rA_ϵ, where

$$A_\epsilon(y) = \begin{pmatrix} R_\beta & \frac{\epsilon}{r}\frac{\partial\psi}{\partial v}(y) \\ 0 & 1 \end{pmatrix}.$$

We consider this restriction as a cocycle over Anosov automorphism $g(y) = Cy$. Thus Example A.3 is reduced to Example A.1 provided we can solve the cohomological equation

$$\frac{1}{r}\frac{\partial\psi}{\partial v} = \varphi,$$

where φ as in Example A.1. If the vector v is Diophantine then by a theorem of Kolmogorov this cohomological equation has a solution ψ if and only if $\int \varphi = 0$ (see e.g. [R75]). The vector v is algebraic and hence Diophantine as an eigenvector of integral matrix. Additional condition $\int \varphi = 0$ can be accommodated since we have a lot of freedom outside the balls B_1 and B_2.

The existence of the matrices B and C satisfying the required properties can be easily seen if one finds an integer coefficients monic polynomials with corresponding properties. We communicated the above question to Professor F. Voloch who kindly formulated it and posted it at `http://mathoverflow.org`. In less than a day we obtained several responses [V] from Buzzard and other participants. For example, one can take polynomials

$$x^3 + 3x^2 + 2x - 1 \quad \text{and} \quad x^6 - 2x^4 - 3x^2 - 1,$$

as characteristic polynomials of B and C, which gives a 9 dimensional example.

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