

CAN A DRINFELD MODULE BE MODULAR?

DAVID GOSS

ABSTRACT. Let k be a global function field with field of constants \mathbb{F}_r , $r = p^m$, and let ∞ be a fixed place of k . In his habilitation thesis [Boc2], Gebhard Böckle attaches abelian Galois representations to characteristic p valued cusp eigenforms and double cusp eigenforms [Go1] such that Hecke eigenvalues correspond to the image of Frobenius elements. In the case where $k = \mathbb{F}_r(T)$ and ∞ corresponds to the pole of T , it then becomes reasonable to ask whether rank 1 Drinfeld modules over k are themselves “modular” in that their Galois representations arise from a cusp or double cusp form. This paper gives an introduction to [Boc2] with an emphasis on modularity and closes with some specific questions raised by Böckle’s work.

1. INTRODUCTION

Let k be a number field and let E be an “arithmetic object” over k such as an elliptic curve or abelian variety. Following Riemann, Artin, Weil, Hasse, and Grothendieck, one associates to E an L -series $L(E, s)$ via its associated Galois representations. Thus for each finite prime \mathfrak{p} of k , one obtains (or is conjectured to obtain) a canonical polynomial $f_{\mathfrak{p}}(u) \in \mathbb{Z}[u]$ and one sets

$$L(E, s) = \prod_{\mathfrak{p}} f_{\mathfrak{p}}(N\mathfrak{p}^{-s})^{-1}.$$

Using estimates, such as those arising from the Weil Conjectures, one sees that this Euler product converges on a non-trivial half-plane of the complex numbers \mathbb{C} to an analytic function.

Uncovering the properties of $L(E, s)$ is a major goal of number theory. The primary approach to this end, also initiated by Riemann, is to equate $L(E, s)$ with “known” or “standard” Dirichlet series via a reciprocity law. For instance, as recalled in Subsection 2.1, the Riemann zeta function completed with Γ -factors at the infinite primes, can also be obtained via an integral transform of a theta-function; the analytic properties of the zeta function are then consequences of those of the theta function. In general, for arbitrary E , one may, conjecturally(!) work the same way by replacing the theta-function with an “automorphic form.” In this fashion, one hopes to show that the function $L(E, s)$ has an analytic continuation and a functional equation under $s \mapsto k - s$ for some integer k .

The profundity of the task of attaching an appropriate automorphic form to $L(E, s)$ is attested to by noting that Fermat’s Last Theorem follows as a consequence when E is restricted to just the set of semi-stable elliptic curves over \mathbb{Q} [Wi1].

Now let k be a global function field over a finite field \mathbb{F}_r , $r = p^m$. Beginning with E. Artin’s thesis, number theorists learned how to attach L -series to arithmetic objects over k . Grothendieck [Gro1] presented a cohomological approach to these L -series which showed that they possess an analytic continuation (as a rational function in $u = r^{-s}$) and a functional equation of classical type. Moreover, the notion of automorphic form is supple enough to

work over k also. Thus a very natural, and important, problem was to investigate whether the class of L -series associated k would also be included in the class of standard L -series. With the recent work of L. Lafforgue ([Laf1], see also [Lau1]) this has now been established.

Lafforgue’s work builds on the ideas and constructions of V.G. Drinfeld and, in particular, his notion of an “elliptic module” [Dr1]. Elliptic modules (now called “Drinfeld modules”) are analogs of elliptic curves and abelian varieties. However, they are not projective objects; rather they are linear objects equipped with an exotic action by an affine sub-algebra of k . More precisely, following Drinfeld one picks a place of k , labels it “ ∞ ,” and then sets A to be those elements in k which have no poles away from ∞ . The ring A then becomes analogous to \mathbb{Z} and the field k analogous to \mathbb{Q} . What makes this analogy especially convincing is that Drinfeld modules arise over the algebraic closure of the completion k_∞ via “lattices” and “exponential functions” in a fashion rather analogous to what happens over the complex numbers with the classical exponential function and elliptic curves. Thus, for instance, the moduli spaces of Drinfeld modules of a given rank have *both* an algebraic and analytic description. Using the moduli curves of Drinfeld modules of rank 2, corresponding to rank 2 lattices, Drinfeld (ibid.) established his first general 2-dimensional reciprocity law. This then implies that elliptic curves with split-multiplicative reduction at ∞ are isogenous to Jacobian factors of these curves (see Section 3).

The analogy between Drinfeld modules and elliptic curves extends far beyond just the way these objects are constructed. Indeed, like an elliptic curve, one can associate to a Drinfeld module E its Tate modules with their canonical Galois representations and Frobenius actions. Thus, as with elliptic curves, it is completely natural to encode this information into a characteristic p valued L -function $L(E, s)$ where s lies in the space S_∞ (see Equation (52)). Moreover, as with elliptic curves, Drinfeld modules naturally have a theory of modular forms (defined in almost exactly the classical manner) associated to them [Go1]. In the case where the rank is 2, these modular forms naturally live on an “algebraic upper half-plane” which plays the role of the usual complex upper half-plane. Furthermore, these modular forms come equipped with an action of the “Hecke operators” which are again defined following classical theory. However, the relationship between the Hecke eigenvalues and the “ q -expansion” coefficients of a given eigenform was, and is, very mysterious unlike classical theory where it is quite transparent.

In classical theory the parameter q at ∞ satisfies $dq = *qdz$. In the characteristic p theory it satisfies $dq = *q^2dz$. As such, one is led to study both cusp forms (forms which vanish at all cusps) and the subspace of “double-cusp” forms (cusp forms which also have first order vanishing); both of these are readily seen to be Hecke-modules.

In this paper we report on the seminal work [Boc2] of Gebhard Böckle in which Galois representations are naturally associated to cusp and double cusp forms. Previous to this work, Böckle and R. Pink had developed a good cohomology theory associated to “ τ -sheaves” (which are a massive generalization of Drinfeld modules and form the correct category in which to discuss characteristic p valued L -series). Indeed, in [Boc1] Böckle used this cohomology to establish very generally good analytic properties for these L -functions of τ -sheaves. In [Boc2] the author applies the full power of this cohomological theory to cusp forms associated to rank 2 Drinfeld modules via the τ -sheaves naturally associated to the universal families lying over the moduli curves. By comparison with similar constructions in the étale topology, the associated Galois representations emerge. As the Hecke operators $T(I)$ in characteristic p satisfy $T(I^2) = T(I)^2$ for *all* I , one sees that the simple Galois factors are *abelian*. For

cuspidal forms which are *not* double cuspidal, these representations essentially arise from *finite* abelian extensions (split totally at ∞) but for most double cuspidal forms the representations have infinite image.

Classical theory immediately leads to an immense number of interesting questions about these representations and their associated L -series (which indeed have good analytic properties via the techniques in [Boc1]). As of now, one does not even have good guesses as to what the answers might be.

This paper is written in order to motivate interested number theorists to become involved in these basic issues. While [Boc2] is daunting in the great number of details that must be checked, this paper will be quite short on details. Rather we focus on the “big picture” of how the characteristic p theory compares with classical theory for both number fields and function fields. For ease of exposition we let $k = \mathbb{F}_r(T)$ and ∞ the place associated the pole of T as usual. Because the class number of k (in terms of divisors of degree 0) is 1, there exist many Drinfeld modules of rank 1 defined over k and it now makes sense to ask if any of them are “modular” in that their Galois representations arise from cusp forms.

In fact, while we now know that a Hecke eigenform f gives rise to a good L -series $L(f, s)$, we have no idea yet how to classify the functions which arise nor do we know any sort of “converse” theorems. Still, it makes sense to broaden the definition of “modularity” in order to allow one to capture the L -series of the rank 1 Drinfeld module *up to translation* (much as $\zeta(2s)$ is naturally associated to the classical theta function; see Subsection 2 just below). We then find that there are really *two* distinct notions of modularity depending on whether the cusp form is double cuspidal or not. We will see that the Carlitz module then becomes modular in both senses. Finally, in Subsection 4.10 we present a certain rank one Drinfeld module $C^{(-\theta)}$ (with $C_T^{(-\theta)}(x) := \theta x - \theta x^r$) whose associated Galois representations quite conceivably — with our current knowledge — might arise directly from a double cusp form. Classical theory certainly implies that the answer as to whether this Drinfeld module is truly modular or not should be very interesting.

It is my great pleasure to thank Gebhard Böckle for his immense patience in guiding me through his thesis. Without his careful answers to my many questions this paper would have been impossible. Indeed, it is my sincere hope that this work makes [Boc2] more accessible. Still, any mistakes in this paper are the fault of its author. This paper is an expanded version of a lecture presented at the Canadian Number Theory Association in May, 2002. It is also my pleasure to thank the Association for the opportunity to present these ideas. I also thank A. Greenspoon, D. Rohrlich and J.-P. Serre for their help with earlier versions of this work. Finally, I am very grateful to M. Ram Murty for suggesting that I write an exposition based on my presentation.

2. CLASSICAL MODULARITY OVER \mathbb{Q}

2.1. Theta functions and Dirichlet characters. The connection between modular forms and L -series is a central theme of modern number theory. We will summarize some of the relevant ideas in this section. An excellent source in this regard is [Kn1] which we follow rather closely.

The theory begins with Riemann’s original paper on the distribution of primes. Indeed, let

$$\theta(z) := \sum_{n=-\infty}^{\infty} e^{in^2\pi z} \quad (1)$$

be the classical theta function. One knows that $\theta(z)$ is analytic on the upper half-plane $\mathcal{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$, and one visibly sees that

$$\theta(z+2) = \theta(z). \quad (2)$$

Due to Jacobi and Poisson, one also has the much deeper formula

$$\theta(-1/z) = (z/i)^{1/2}\theta(z), \quad (3)$$

where, for the square root, one takes the principal value which is cut on the negative real axis. Transformation laws (2) and (3) are summarized by saying that $\theta(z)$ is a *modular form of weight* $1/2$ associated to the group Γ_θ of automorphisms of \mathcal{H} generated by $z \mapsto z+2$ and $z \mapsto -1/z$.

The well-known application of $\theta(z)$ to L -series, due to Riemann, then arises in the following fashion. Let

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}, \quad (4)$$

be the Riemann zeta function and let $\Gamma(s) = \int_0^{\infty} t^s e^{-t} \frac{dt}{t}$ be Euler's gamma function. One sets $\Lambda(s) := \zeta(s)\Gamma(s/2)\pi^{-s/2}$. Through the integral representation for $\Gamma(s)$ and a change of variables one finds

$$2\Lambda(s) = \int_0^{\infty} (\theta(it) - 1) t^{s/2} \frac{dt}{t}. \quad (5)$$

Equation (5) is, in turn, rewritten as

$$2\Lambda(s) = \int_0^1 \theta(it) t^{s/2} \frac{dt}{t} - \int_0^1 t^{s/2} \frac{dt}{t} + \int_1^{\infty} (\theta(it) - 1) t^{s/2} \frac{dt}{t}. \quad (6)$$

The second term on the right is readily computed to equal $2/s$. Via Equation (3), the first term on the right in Equation (6) is computed to be $\int_1^{\infty} (\theta(it) - 1) t^{1/2(1-s)} \frac{dt}{t} - \frac{2}{1-s}$. Thus, finally,

$$2\Lambda(s) = \int_1^{\infty} (\theta(it) - 1) t^{s/2} \frac{dt}{t} + \int_1^{\infty} (\theta(it) - 1) t^{1/2(1-s)} \frac{dt}{t} - \frac{2}{s(1-s)}. \quad (7)$$

The first two terms on the right can be shown to be entire in s . Moreover, from the invariance of the right hand side of (7) under $s \mapsto 1-s$, we deduce that

$$\Lambda(s) = \Lambda(1-s). \quad (8)$$

This is the famous *functional equation* for $\zeta(s)$, and surely one of the most sublime statements in mathematics.

Remark 1. The above argument actually gives *both* the analytic continuation of $\zeta(s)$ and the functional equation (8). In the following we will use “functional equation” to mean both an analytic continuation and invariance under $s \mapsto k-s$ for some integer k .

2.1.1. *L-series associated to modular forms.* The derivation of the functional equation (8) of $\zeta(s)$ from the properties (2), (3) of $\theta(z)$ is just the very tip of the iceberg as we shall see. We begin by recalling the general definition of a modular form.

Let $\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and let $z \in \mathcal{H}$. We set

$$\gamma z := \frac{az + b}{cz + d}. \quad (9)$$

The map $z \mapsto \gamma z$ is clearly an analytic automorphism of \mathcal{H} (the inverse automorphism arising from the inverse matrix). Note that γ and $-\gamma$ have the same action on \mathcal{H} . Let $\Gamma := SL_2(\mathbb{Z})$ and let $\tilde{\Gamma}$ be a subgroup of Γ . Finally, let k be a real number and assume that we have chosen a branch so that z^k is analytic on \mathcal{H} .

Definition 1. Let $f(z)$ be an analytic function on \mathcal{H} . We say that $f(z)$ is an *unrestricted modular form of weight k associated to $\tilde{\Gamma}$* if and only if

$$f(\gamma z) = (cz + d)^k f(z) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Gamma}. \quad (10)$$

More generally, one allows certain constants, called “multiplier systems,” in front of $(cz + d)^k$ in Equation (10). For instance, multipliers are needed in order for $\theta(z)$ to be modular as in Equation (3).

Now, let χ be a primitive Dirichlet character modulo m . To χ one associates the L -series

$$L(\chi, s) := \sum_{n=1}^{\infty} \chi(n)n^{-s} = \prod_{p \text{ prime}} (1 - \chi(p)p^{-s})^{-1}, \quad (11)$$

where, by definition, $\chi(n) = 0$ if and only if $\gcd(n, m) > 1$. (So, if $\chi = \chi_0$ the trivial character, then $L(\chi, s) = \zeta(s)$.) To χ one also associates a theta function as follows. Let $z \in \mathcal{H}$ and set

$$\theta(\chi, z) := \begin{cases} \sum_{n=-\infty}^{\infty} \chi(n)e^{in^2\pi z/m} & \text{if } \chi(-1) = 1 \\ \sum_{n=-\infty}^{\infty} \chi(n)ne^{in^2\pi z/m} & \text{if } \chi(-1) = -1. \end{cases} \quad (12)$$

It is clear that $\theta(\chi_0, z) = \theta(z)$, and that

$$\theta(\chi, z + 2m) = \theta(\chi, z). \quad (13)$$

Moreover,

$$\theta(\chi, -1/z) := \begin{cases} w(\chi, m)(z/i)^{1/2}\theta(\bar{\chi}, z) & \text{if } \chi(-1) = 1 \\ -iw(\chi, m)(z/i)^{3/2}\theta(\bar{\chi}, z) & \text{if } \chi(-1) = -1, \end{cases} \quad (14)$$

where $|w(\chi, m)| = 1$ and $\bar{\chi}$ is the complex conjugate character. In particular, if $\chi = \bar{\chi}$, we obtain a modular form (of weights $1/2$ or $3/2$) for the group $\Gamma_{\theta, \chi}$ of automorphisms of \mathcal{H} generated by $z \mapsto -1/z$ and $z \mapsto z + 2m$.

One now sets

$$\Lambda(\chi, s) := \begin{cases} m^{s/2}\Gamma(s/2)\pi^{-s/2}L(\chi, s) & \text{if } \chi(-1) = 1 \\ m^{(s+1)/2}\Gamma\left(\frac{s+1}{2}\right)\pi^{-(s+1)/2}L(\chi, s) & \text{if } \chi(-1) = -1. \end{cases} \quad (15)$$

Using $\theta(\chi, z)$ and (14) one shows (Th. 7.19 of [Kn1])

$$\Lambda(\chi, s) = \begin{cases} w(\chi, m)\Lambda(\bar{\chi}, 1-s) & \text{if } \chi(-1) = 1 \\ -iw(\chi, m)\Lambda(\bar{\chi}, 1-s) & \text{if } \chi(-1) = -1. \end{cases} \quad (16)$$

When $\chi = \chi_0$, the functional equation for $\Lambda(\chi, s)$ is the one given above for the Riemann zeta function.

We can loosely characterize the results just presented by saying that Dirichlet characters are “modular” in that they arise from modular forms (albeit of fractional weight).

Starting with Yutaka Taniyama in the 1950’s, mathematicians began to suspect that the connection between L -series of abelian “arithmetic objects” defined over \mathbb{Q} , such as Dirichlet characters, and modular forms might also extend to “non-abelian objects” such as elliptic curves over \mathbb{Q} . That such a connection should exist at all is, at first glance, very surprising. Indeed, the space \mathcal{H} *already* has a deep connection with elliptic curves as every elliptic curve over \mathbb{C} is isomorphic, as a complex analytic space, to $E_z := \mathbb{C}/L_z$ where L_z is the lattice generated by $\{1, z\}$ for some $z \in \mathcal{H}$. This new relationship between \mathcal{H} and elliptic curves, via L -series and modular forms, is of a very different, and deeper, nature.

The modern rationale for the existence of this new connection is part of the general “Langlands philosophy.” To such an elliptic curve E one associates an L -series (the definition will be recalled below) $L(E, s) = L(E_{\mathbb{Q}}, s)$. This L -series, and its twists by Dirichlet characters, (also recalled below) are conjectured to satisfy certain functional equations; in turn these functional equations guarantee (due to Weil [We1]) that the L -series arises from a modular form of a specific type in essentially the same fashion as $\zeta(s)$ arises from $\theta(z)$. (The existence of such functional equations and modular forms is now, of course, well established, see below.)

As these ideas are crucial for us here, we will briefly recall them and refer the reader to [Kn1] (for instance) for more details. We begin by presenting more of the theory of modular forms. Let $\Gamma = SL_2(\mathbb{Z})$, as above, and let N be a positive integer. There is clearly a homomorphism from $SL_2(\mathbb{Z})$ to $SL_2(\mathbb{Z}/N)$ given by reducing the coefficients of the matrix modulo N . The kernel of this mapping is denoted by $\Gamma(N)$. Any subgroup of $SL_2(\mathbb{Z})$ which contains $\Gamma(N)$, for some $N \geq 1$, is called a “congruence subgroup;” we extend this notion to automorphisms of \mathcal{H} in the obvious sense. (For instance, the group Γ_θ is a congruence subgroup in this sense, see §I.5 of [Gu1]).

Let $\tilde{\Gamma}$ be a congruence subgroup of Γ . From now on we shall only consider modular forms for $\tilde{\Gamma}$ in the sense of Definition 1; that is, without multipliers and where the weight, k , is an integer. The quotient space $\tilde{\Gamma} \backslash \mathcal{H}$ is an open Riemann surface that may be compactified by adding a finite number of points called “cusps;” these cusps are in one to one correspondence with $\tilde{\Gamma} \backslash \mathbb{P}^1(\mathbb{Q})$. For instance, ∞ represents a cusp. The subgroup $\tilde{\Gamma}_\infty$ of $\tilde{\Gamma}$ which fixes ∞ is of the form $z \mapsto z + j$ where $j \in (\mathfrak{n}) \subseteq \mathbb{Z}$ is an ideal (and $\mathfrak{n} \geq 1$). If $f(z)$ is a modular form for $\tilde{\Gamma}$, it then automatically has a Fourier expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z / \mathfrak{n}} = \sum_{n=-\infty}^{\infty} a_n q_{\mathfrak{n}}^n, \quad (17)$$

where $q_{\mathfrak{n}} = e^{2\pi i z / \mathfrak{n}}$. Similar expansions are obtained at the other cusps by moving the cusp to ∞ via an element of $SL_2(\mathbb{Z})$. One says that the modular form $f(z)$ is *holomorphic* if, at every cusp, all terms associated to negative n in the associated expansion (17) vanish. One

says that a holomorphic form is a *cuspidal form* if all terms associated to $n = 0$ at the cusps also vanish.

One has holomorphic forms only if the weight k is non-negative. Of course, both the holomorphic and cusp forms of a given weight form vector spaces of \mathbb{C} which can be shown to be finite dimensional via standard results on algebraic curves.

Of primary importance in the theory are the congruence subgroups $\Gamma_0(N) \subset \Gamma$ defined by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv 0 \pmod{N} \right\}. \quad (18)$$

It is clear that for such subgroups one can choose \mathbf{n} in the expansion (17) at ∞ equal to 1, in which case we simply set $q = q_1 = e^{2\pi iz}$.

Now let $f(z) = \sum_{n=1}^{\infty} a_n q^n$ be a cusp form of weight k for $\Gamma_0(N)$. One sets

$$L(f, s) := \sum_{n=1}^{\infty} a_n n^{-s}. \quad (19)$$

In a fashion quite similar to that of Equation (5), one finds

$$(2\pi)^{-s} \Gamma(s) L(f, s) = \int_0^{\infty} f(it) t^s \frac{dt}{t}. \quad (20)$$

Recall that the functional equation for $L(\chi, s)$ arises from the action $z \mapsto -1/z$ on \mathcal{H} . Similarly the functional equation for $L(f, s)$ will arise from the action $z \mapsto -1/Nz$ on \mathcal{H} .

The matrix $\omega_N := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ is not in $\Gamma_0(N)$ but rather in its normalizer. As such there is an action of ω_N on cusp forms of a given weight for $\Gamma_0(N)$ and, to get a functional equation, one needs to further assume that $f(z)$ is an eigenfunction for this action. The eigenvalue ε will be ± 1 . With this added assumption, put

$$\Lambda(f, s) := N^{s/2} (2\pi)^{-s} \Gamma(s) L(f, s). \quad (21)$$

More generally let χ be a character (as above) which we now assume has conductor m which is prime to N . Set

$$L(f, \chi, s) := \sum_{n=1}^{\infty} \chi(n) a_n n^{-s}, \quad (22)$$

and

$$\Lambda(f, \chi, s) := (m^2 N)^{s/2} (2\pi)^{-s} \Gamma(s) L(f, \chi, s). \quad (23)$$

One then has the functional equations

$$\Lambda(f, s) = \varepsilon (-1)^{k/2} \Lambda(f, k - s), \quad (24)$$

and

$$\Lambda(f, \chi, s) = \varepsilon (-1)^{k/2} w(\chi, m) \chi(-N) \Lambda(f, \bar{\chi}, k - s) \quad (25)$$

with $|w(\chi, m)| = 1$.

We refer the reader to [Kn1] for a discussion of these functional equations which are due to Hecke (and which turn out, after all, to be very much in the spirit of Riemann's theory for $\zeta(s)$ as in Equations (6) and (7)).

Hecke also had a procedure for selecting those cusp forms $f(z)$ for which $L(f, s)$ has an infinite product expansion ("Euler product") similar to those given in Equations (4) and (11) *except* that the local factors will be degree 2 polynomials in p^{-s} for almost all primes p .

Hecke's idea can be very roughly sketched as follows: As above, every point $z \in \mathcal{H}$ gives rise to the elliptic curve $E_z := \mathbb{C}/L_z = \mathbb{C}/\{\mathbb{Z} + \mathbb{Z}z\}$; therefore, one can view modular forms as certain functions on elliptic curves (together with, in the case of $\Gamma_0(N)$, a cyclic subgroup C of order N ; for our purposes here we will simply ignore such subgroups altogether). Given an integer n , we can associate to E_z the set of all sub-lattices $\{L_z^{(i)}\}$ of L_z of index n as well as their associated elliptic curves $\{E_z^{(i)}\}$. This association depends only on L_z and thus E_z . If f is a function on elliptic curves, we can then define a new function $T(n)f$ on elliptic curves simply by summing up the values of f on the elliptic curves associated to the sub-lattices; i.e.,

$$T(n)f(E_z) := \sum_i f(E_z^{(i)}). \quad (26)$$

It turns out that the ‘‘Hecke operator,’’ $f \mapsto T(n)f$, gives rise to linear endomorphisms of both the space of modular forms and the space of cusp forms of a given weight. These operators form a commutative ring where $T(nm) = T(n)T(m)$ for coprime n and m but where, for p not dividing N , $T(p^2) \neq T(p)^2$. The important point is that those L -series, $L(f, s)$, which have Euler products are precisely those which are associated to *eigenvectors* (‘‘eigenfunctions’’) $f(z)$ for *all* Hecke operators $T(n)$. In order to establish this equivalence, one first shows that the Hecke eigenvalues are precisely the q -expansion coefficients of the (normalized) form $f(z)$.

2.2. L -functions of elliptic curves and elliptic modularity over \mathbb{Q} . Now we can turn to elliptic curves over \mathbb{Q} . Such a curve E is given by a Weierstrass equation of the form

$$y^2 = x^3 + ax + b \quad (27)$$

where $\{a, b\} \subset \mathbb{Q}$ and $\Delta := 4a^3 + 27b^2$ is non-zero. The association

$$z \in \mathcal{H} \mapsto \Delta(\mathbb{C}/L_z) \quad (28)$$

makes Δ a cusp form of weight 12 associated to the full modular group Γ .

In order to discuss to define the local L -factors of E we need to discuss its reduction at the finite primes p of \mathbb{Q} following §VII of [Si1]; for more information we refer the reader there. For simplicity we begin by assuming that $p \geq 5$ with associated additive valuation v_p . By the change of variables $(x, y) = (u^2\tilde{x}, u^3\tilde{y})$, $u \neq 0$, the Weierstrass equation (27) is changed into the Weierstrass equation

$$\tilde{y}^2 = \tilde{x}^3 + \tilde{a}\tilde{x} + \tilde{b}, \quad (29)$$

with $\tilde{a} = a/u^4$, $\tilde{b} = b/u^6$ and $\tilde{\Delta} = \Delta/u^{12}$. Thus, by the appropriate choice of u , one can find a Weierstrass equation for E where all the coefficients are integral at p ; in particular, of course, Δ then is integral at p also. Among all such equations, the ones where $v_p(\Delta)$ is a minimum are called ‘‘minimal Weierstrass equations’’ for E at p . Such an equation is not unique but it is easy to see that any two such equations will give rise to *isomorphic* curves upon reduction modulo p (obtained by reducing the coefficients of the Weierstrass equation modulo p).

For almost all primes $p \geq 5$ (the ‘‘good primes’’) the reduced minimal Weierstrass equation will also be an elliptic curve E_p over \mathbb{F}_p . Let n_p be the number of points on E_p over \mathbb{F}_p (which is obviously an isomorphism invariant) and put $a_p := p + 1 - n_p$. Finally we define the local L -factor $L_p(E, u)$ by

$$L_p(E, u) := \frac{1}{1 - a_p u + pu^2}. \quad (30)$$

A basic result, due to Hasse (Th. 10.5 of [Kn1]), establishes that $L_p^{-1}(E, u) = (1 - \alpha u)(1 - \beta u)$ where $|\alpha| = |\beta| = p^{1/2}$.

Suppose now E has “bad” reduction at a prime $p \geq 5$. Then from the Weierstrass equation one can see that the reduced curve E_p at p must have either a node or a cusp. If the reduced curve has a node with slopes in \mathbb{F}_p we say that E has “split multiplicative reduction at p ,” if the reduced curve has a node but where the slopes are not in \mathbb{F}_p then we say that E has “non-split multiplicative reduction.” If the reduced curve has a cusp, then we say that E has “additive reduction at p .” We can then describe the local factor at these finitely many bad primes as follows.

$$L_p(E, u) := \begin{cases} \frac{1}{1-u} & \text{if } E \text{ has split multiplicative reduction at } p \\ \frac{1}{1+u} & \text{if } E \text{ has non-split multiplicative reduction at } p \\ 1 & \text{if } E \text{ has additive reduction at } p. \end{cases} \quad (31)$$

For the primes $p = 2, 3$ one has an exactly similar story *but* where one has to use a more general form of the Weierstrass equation (27); again we refer the interested reader to [Si1] for the details.

Remark 2. We note for future use that if E does not have good reduction at p , but does acquire it over a finite extension, then E_p must have a cusp (since multiplicative reduction remains multiplicative reduction over any finite extension; see Prop. 5.4.b of [Si1]). In this case we see from Equation (31) that the local factor $L_p(u)$ is identically 1

Let B be the finite set of bad primes for E . The *conductor* of E/\mathbb{Q} , N_E , is defined by

$$N_E := \prod_{p \in B} p^{e_p} \quad (32)$$

where $e_p = 1$ if E has multiplicative reduction at p and $e_p \geq 2$ otherwise (in fact, equal to 2 if $p \geq 5$); see e.g., §A.16 of [Si1]. The conductor is a measure of how “twisted” the reduction of E at bad primes actually is.

The L -series of the elliptic curve E , $L(E, s)$, is then defined as

$$L(E, s) := \prod_{\text{all primes } p} L_p(E, p^{-s}). \quad (33)$$

Upon expanding the Euler product for $L(E, s)$ one obtains

$$L(E, s) = \sum_{n=1}^{\infty} c_n n^{-s}. \quad (34)$$

Let χ be a character of conductor m prime to $N := N_E$. We then define the twisted L -series $L(E, \chi, s)$ by

$$L(E, \chi, s) := \sum_{n=1}^{\infty} \chi(n) c_n n^{-s}. \quad (35)$$

One puts

$$\Lambda(E, s) := N^{s/2} (2\pi)^{-s} \Gamma(s) L(E, s), \quad (36)$$

and

$$\Lambda(E, \chi, s) := (m^2 N)^{s/2} (2\pi)^{-s} \Gamma(s) L(E, \chi, s). \quad (37)$$

It was conjectured (and is now a theorem) that $\Lambda(E, s)$ satisfies a functional equation of the form

$$\Lambda(E, 2 - s) = \pm \Lambda(E, s). \quad (38)$$

The sign ± 1 here can be expressed as a product over all places of \mathbb{Q} of “local signs.” Moreover the sign of E at ∞ is -1 and the sign at all good primes is $+1$. In particular, it is remarkable that the sign is then completely determined by the local signs at the *bad* primes; see e.g., [Roh1], [Ha1], [Ko1] and [Ri1]. Similarly, when the conductor of χ is prime to N , $\Lambda(E, \chi, s)$ was conjectured to satisfy (and is now known to satisfy)

$$\Lambda(E, \chi, s) = \pm w(m, \chi) \chi(-N) \Lambda(E, \bar{\chi}, 2 - s). \quad (39)$$

As one can see, the functional equations (38) and (39) are remarkably like the functional equations given above (in (24) and (25)) for $L(f, s)$ and $L(f, \chi, s)$ where f is a cusp form of weight 2. This ultimately led to the amazing expectation (the “Modularity Conjecture”) that for every E one could find a cusp form $f_E(z)$ of weight 2 for $\Gamma_0(N)$ such that $f_E(z)$ is an eigenform for all the Hecke operators and $L(E, \chi, s) = L(f_E, \chi, s)$ for all χ (of conductor prime to N). In particular, the conjectured analytic properties of $L(E, \chi, s)$ then follow immediately from those of $L(f_E, \chi, s)$. Indeed, the results of Weil [We1] characterize those Dirichlet series which arise from cusp forms (for $\Gamma_0(N)$) as *precisely* those satisfying functional equations (38) and (39).

The local L -factors of E can also be obtained from Galois representations associated to the elliptic curve as we will now explain. Let ℓ be a prime number. Then to each ℓ one attaches to E the “ ℓ -adic Tate module $T_\ell(E)$,” defined as the inverse limit of the ℓ^n -division points on E (§III.7 of [Si1]). One sees that the Tate module is a free \mathbb{Z}_ℓ -module of rank 2, and its cohomological dual is defined by

$$H^1(E, \mathbb{Q}_\ell) := \text{Hom}_{\mathbb{Z}_\ell}(T_\ell(E), \mathbb{Q}_\ell). \quad (40)$$

Both $T_\ell(E)$ and $H^1(E, \mathbb{Q}_\ell)$ are naturally modules for the Galois group $G := \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ (where $\bar{\mathbb{Q}}$ is a chosen algebraic closure), and one sees readily that this module is non-abelian (in that the Galois action factors through a non-abelian group).

The L -series $L(E, s)$ can also be expressed in terms of this “compatible system” of representations on $H^1(E, \mathbb{Q}_\ell)$ (for varying ℓ). At the good primes one obtains $L_p(u)^{-1}$ as the characteristic polynomial of the geometric Frobenius automorphism and at the bad primes one must first take the fixed subspace of the inertia elements (and then use a geometric Frobenius element etc.) see [Ta1]. This construction is the non-abelian version of the procedure used to define $L(\chi, s)$ for Dirichlet characters. It also justifies viewing the Modularity Conjecture as a non-abelian extension of the relationship between characters and theta-functions given in Subsection 2.1 as class field theory equates abelian characters of G with Dirichlet characters.

Once one knows that $L(E, s) = L(f_E, s)$ for a cusp form f_E of weight 2 (or, more technically correct, a “newform”) associated to $\Gamma_0(N)$ for some N , another remarkable sequence of results take over. Indeed, as mentioned above, the quotient space $\Gamma \backslash \mathcal{H}$ is compactified by attaching cusps and can be realized as a smooth projective curve $\bar{X}_0(N)$ over \mathbb{Q} . Using f_E one constructs (Th. 11.74 of [Kn1]) a certain elliptic curve E'/\mathbb{Q} which is a quotient of the Jacobian of $\bar{X}_0(N)$. By construction one also has $L(E', s) = L(f_E, s)$. The existence and properties of E' are due to Eichler and Shimura, as is the identification of $L(E', s)$ with $L(f_E, s)$ (via an “Eichler-Shimura relation” which connects the action of $T(p)$, p prime, with the Frobenius automorphism at p).

Clearly one should expect some relationship between E and E' . Indeed, recall that an “isogeny” between elliptic curves E_1 and E_2 is a surjective map (of elliptic curves) $E_1 \rightarrow E_2$ and once one has an isogeny $E_1 \rightarrow E_2$ it is easy to construct an isogeny $E_2 \rightarrow E_1$. If both E_1 and E_2 are defined over \mathbb{Q} , and if the map between them is *also* defined over \mathbb{Q} , then both elliptic curves will have the same local factors and L -series. Faltings [Fa1] tells us that the converse is also true; i.e., two elliptic curves over \mathbb{Q} with the same L -series (and thus the same local factors) are then automatically related by an isogeny (or are “isogenous”). In particular, we conclude that E and E' are isogenous. Consequently, using modular forms and curves, one obtains an amazing dictionary of the isogeny classes of elliptic curves over \mathbb{Q} !

As we have stated, the modularity conjecture for elliptic curves over \mathbb{Q} is now a theorem. The main work in establishing it was due to A. Wiles and Wiles and R. Taylor, [Wi1], [TaWi1]. The proof was then finished in [Di1], [CDT1], and [BCDT1].

A key part of the Wiles’ proof is the result of R. Langlands and J. Tunnell, [Tu1], [Lan1]. This establishes that certain non-abelian and complex-valued (and thus of finite image) representations of the absolute Galois group of \mathbb{Q} are modular in a similar sense as Dirichlet characters (in fact, one may view these representations as being non-abelian generalizations of Dirichlet characters). In other words the L -functions associated to these representations arise from certain cusp forms, which, in this case, are of weight 1. This gives yet another instance of the crucial role played by modular forms in classical arithmetic.

3. ELLIPTIC MODULARITY FOR $k = \mathbb{F}_r(T)$

In this section we will explain how the classification of isogeny classes of elliptic curves over \mathbb{Q} can be translated to the case of the global function field $k = \mathbb{F}_r(T)$ for a certain class of elliptic curves over k (e.g., those elliptic curves with split-multiplicative reduction at the place ∞ defined below). For a clear and thorough exposition of these ideas, we refer the reader to [GR1].

3.1. The L -series of an elliptic curve over k . Let \mathbb{F}_r be the finite field with $r = p^m$ elements and p prime. We let $k = \mathbb{F}_r(T)$ be the rational field in an indeterminate T . Let E be an elliptic curve over k . The L -series of E/k is defined in a completely analogous fashion to that of E/\mathbb{Q} as given above. More precisely, let w be a place of k with local ring \mathcal{O}_w and associated finite field \mathbb{F}_w . Put $Nw = |\mathbb{F}_w|$, which is a power of r . If w is a place of good reduction then one sets n_w to be the number of points on the reduction E_w and $a_w := Nw + 1 - n_w$. The local L -factor is

$$L_w(u) := \frac{1}{1 - a_w u + (Nw)u^2}. \quad (41)$$

If w is a place of bad reduction, then the local factor is defined as in (31). Finally, we put

$$L(E, s) = L(E/k, s) := \prod_{\text{all } w} L_w(Nw^{-s}). \quad (42)$$

3.2. Automorphic representations. In Subsections 2.1 and 2.1.1, we sketched some important aspects of the theory of classical elliptic modular forms. Using these forms, one can obtain the analytic properties of various L -series of arithmetic objects over \mathbb{Q} as described above. However, the general formalism and construction of L -series can be given in far

greater generality than just for objects over \mathbb{Q} . For instance, one can work with arbitrary global fields (such as k in the last subsection).

The correct generalization of cusp forms that works for arbitrary global fields is the notion of a “cuspidal automorphic representation” (see, e.g., [JL1], [BJ1], [Bor1], [Lan2]). For our purposes, it is sufficient to view such automorphic representations as being “generalized cuspidal (Hecke) eigenforms.” Indeed, a cuspidal automorphic representation π can be given an associated L -series $L(\pi, s)$ which arises from an Euler product and which has good properties (such as functional equations). Moreover, the L -series of an arithmetic object is *always* conjectured in the Langlands philosophy to equal the L -series of a certain associated cuspidal automorphic representation π_E (which completely generalizes the association $E \leftrightarrow f_E$ discussed in Subsection 2.2). In the function field case, this is now known to be a theorem due to the labors of Drinfeld, Lafforgue [Laf1] and others, (an excellent source in this regard is [Lau1] and its references).

In particular, due to the cohomological results of Grothendieck [Gro1] one knows that the L -series $L(E, s)$, and its twists by abelian characters (the generalization of $L(E, \chi, s)$, see (37)) have functional equations (in fact, one knows that, in this case, they are polynomials in r^{-s}). Thus the general theory of automorphic representations will associate to E/k an automorphic representation $\pi_E = \pi_{E/k}$ with $L(\pi_E, s) = L(E, s)$ [De1]. The reader should realize that this is very different than the case of elliptic curves E over \mathbb{Q} where one deduces the analytic properties of $L(E, s)$ at the same time as one finds the associated modular form f . In the case of E/k we know the analytic properties of the L -series directly, without having the associated $\pi_{E/k}$; in fact, one constructs $\pi_{E/k}$ from this knowledge.

What is lacking in the function field case is a concrete realization of the isogeny class of E , as was accomplished in the case of elliptic curves over \mathbb{Q} via the Jacobians of elliptic modular curves. It is precisely here that the work of V.G. Drinfeld comes in.

3.3. A Quick introduction to Drinfeld modules. In 1973, V.G. Drinfeld introduced his “elliptic modules” [Dr1] which are now called “Drinfeld modules” in his honor. The analytic construction of Drinfeld modules is based on that of elliptic curves where the Archimedean place is singled out. Thus one begins by singling out a particular “infinite” place of $k = \mathbb{F}_r(T)$; the obvious one is “ ∞ ” where $v_\infty(1/T) = 1$ (so, naturally, ∞ corresponds to the usual point $\infty \in \mathbb{P}^1(\mathbb{F}_r)$). The ring $A := \mathbb{F}_r[T]$ consists of those rational functions which are regular away from ∞ . The field $K := k_\infty = \mathbb{F}_r((1/T))$ is a local field which contains A discretely and K/A is compact. The standard analogy is with $\mathbb{Z} \subset \mathbb{R}$ with \mathbb{R}/\mathbb{Z} being compact. The algebraic closure of K , denoted \bar{K} , is infinite dimensional over K and is not complete. However, v_∞ lifts to \bar{K} in a canonical way and every subextension $L \subset \bar{K}$ which is finite dimensional over K is, in fact, complete. Thus we may use analytic methods over L .

A \mathbb{Z} -lattice in \mathbb{C} is a discrete (in the standard topology on \mathbb{C}) \mathbb{Z} -submodule which may then be shown to have rank 1 or rank 2 (reflecting the fact that $[\mathbb{C} : \mathbb{R}] = 2$). The rank two lattices are precisely those that give rise to elliptic curves. An A -lattice M is a finitely generated, discrete (i.e., finitely many elements in any bounded ball with the metric generated by v_∞), A -submodule of \bar{K} . As M is finitely generated and obviously torsion-free, it is free of some rank $t = t_M$ and generates a finite extension of K . As $[\bar{K} : K] = \infty$, one can have lattices of

arbitrary rank. To M one attaches its exponential function

$$e_M(x) = x \prod_{0 \neq m \in M} (1 - x/m). \quad (43)$$

As M is discrete, it is easy to see that $e_M(x)$ converges for all $x \in \bar{K}$; that is, $e_M(x)$ is an entire non-Archimedean function. As M is finitely generated, the Taylor coefficients of $e_M(x)$ will lie in some finite extension of K . Consequently, if $x \in \bar{K}$ then $e_M(x)$ converges to an element of \bar{K} .

Non-Archimedean analysis is highly algebraic in nature. In particular, like polynomials, *all* entire non-Archimedean functions in 1-variable are surjective (as a function on \bar{K}) and are determined up to a constant by their divisors.

The main “miracle” of $e_M(x)$ is that the map $e_M(x): \bar{K} \rightarrow \bar{K}$ is actually \mathbb{F}_r -linear; thus one has $e_M(x+y) = e_M(x) + e_M(y)$. This is due to Drinfeld [Dr1] (but uses some combinatorial arguments on polynomials that have been known for ages). One then deduces the remarkable isomorphism of \mathbb{F}_r -vector spaces

$$e_M(x): \bar{K}/M \rightarrow \bar{K}. \quad (44)$$

The idea behind the analytic construction of Drinfeld modules is to carry over the natural quotient A -module structure on the left of (44) to \bar{K} via $e_M(x)$ (just as one carries over the \mathbb{Z} -module structure on $\mathbb{C}/\{\mathbb{Z} + \mathbb{Z}z\}$ to the associated elliptic curve).

More precisely, let $a \in A$ be a polynomial of degree d and let

$$E_a := \{e_M(\alpha) \mid \alpha \in a^{-1}M/M\}. \quad (45)$$

Thus $E_a \subset \bar{K}$ is a vector space over \mathbb{F}_r of dimension dt_M (and so $|E_a| = r^{dt_M}$). Put,

$$\psi_a(x) = ax \prod_{0 \neq \alpha \in E_a} (1 - x/\alpha). \quad (46)$$

As E_a is a finite set, $\psi_a(x)$ is a *polynomial*. The same combinatorics as used for $e_M(x)$ also establishes that $\psi_a(x)$ is an \mathbb{F}_r -linear function. As non-Archimedean entire functions are determined up to a constant by their divisors, a simple calculation then gives the basic identity

$$e_M(ax) = \psi_a(e_M(x)). \quad (47)$$

That is, the standard A -action on \bar{K} , $\{a, x\} \mapsto ax$ (on the left hand side of (44)) gets transferred over to the action $\{a, x\} \mapsto \psi_a(x)$ (on the right hand side of (44)). In particular, \bar{K} inherits a new A -module action called a “Drinfeld module.”

The mapping $a \in A \mapsto \psi_a(x)$ is readily seen to give an injection of A into the algebra of \mathbb{F}_r -linear polynomials (with composition of polynomials as multiplication). As it is an algebra map, it is uniquely determined by ψ_T which, as an \mathbb{F}_r -linear polynomial, is given by

$$\psi_T(x) = Tx + \sum_{i=1}^{t_M} a_i x^{r^i}, \quad (48)$$

where $a_{t_M} \neq 0$. The rank of the lattice M , t_M , is also the “rank” of the Drinfeld module.

As $\psi_a(x)$ is a polynomial in x for all $a \in A$, the notion of a Drinfeld module is really an algebraic one exactly as is the case with elliptic curves. Thus it makes sense over any field L containing A/\mathfrak{p} for any prime ideal \mathfrak{p} of A (including, obviously, $\mathfrak{p} = (0)$). Indeed, to get a rank t Drinfeld A -module over L , for a positive integer t , as done in (48) one just needs $\{a_i\}_{i=1}^t \subseteq L$ with $a_t \neq 0$ and “ T ” represents its image in L . It is common to denote this

image by “ θ ”; the use of θ allows us to distinguish when “ T ” is an operator via a Drinfeld module and “ $T = \theta \in L$ ” is a scalar.

Any Drinfeld module of rank t defined over \bar{K} can be shown to arise from a lattice M of the same rank via $e_M(x)$ as above; this is in exact agreement with the analytic theory of elliptic curves. For more on Drinfeld modules we refer the reader to [Hay1] or [Go4].

Example 1. Let C be the rank 1 Drinfeld module over $\mathbb{F}_r(T) = \mathbb{F}_r(\theta)$ defined by

$$C_T(x) := Tx + x^r = \theta x + x^r. \quad (49)$$

It is clear that C has rank 1 and is the simplest possible Drinfeld module. It is called the “Carlitz module” after the work of L. Carlitz [Ca1]. It is associated to a rank one lattice $M := A\xi$ where $0 \neq \xi \in \bar{K}$ and $\xi^{r-1} \in K$.

Remark 3. As mentioned in the introduction, the theory of Drinfeld modules actually exists in much greater generality where one replaces $\mathbb{F}_r(T)$ by an arbitrary global function field k of characteristic p and $\mathbb{F}_r[T]$ by the affine algebra A of functions regular away from a fixed place “ ∞ ” of k . The ring A is readily seen to be a Dedekind domain with finite class group and unit group equal to \mathbb{F}_r^* .

3.4. The rigid space Ω^2 . A Drinfeld module ψ of rank 2 over \bar{K} is given by

$$\psi_T(x) = Tx + g(\psi)x^r + \Delta(\psi)x^{r^2} = \theta x + g(\psi)x^r + \Delta(\psi)x^{r^2}, \quad (50)$$

where $\{g(\psi), \Delta(\psi)\} \subset \bar{K}$ and $\Delta(\psi) \neq 0$. From our last subsection, we know that ψ arises from a rank 2 A -lattice M of the form $Az_1 + Az_2$ where the discreteness of M is equivalent to $z_1/z_2 \in \bar{K} \setminus K$.

Definition 2. We set $\Omega^2 := \bar{K} \setminus K$.

The space Ω^2 was defined by Drinfeld in [Dr1] and, in fact, there is an Ω^i for all $i = 1, 2, \dots$. As we are only interested in Ω^2 here, from now on we shall simply denote it “ Ω .” The space Ω is clearly analogous to the $\mathbb{C} \setminus \mathbb{R}$, which, in turn, is precisely the upper and lower half-planes. Like $\mathbb{C} \setminus \mathbb{R}$, Ω has an analytic structure which allows one to use analytic continuation. This structure is called a “rigid analytic space.” Surprisingly, with this rigid structure Ω becomes a connected (but not simply connected) space unlike, of course, the classical upper half-plane \mathcal{H} . Rigid analysis allows one to handle non-Archimedean functions in a manner very similar to that of complex analytic functions.

The space Ω has an action of $\Gamma := GL_2(A)$ on it completely analogous to the classical action of $SL_2(\mathbb{Z})$ on \mathcal{H} . Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A)$ and $z \in \Omega$. Then, exactly as in (9), we set

$$\gamma z := \frac{az + b}{cz + d};$$

the map $z \mapsto \gamma z$ is a rigid analytic automorphism of Ω where the inverse transformation arises from the inverse matrix.

Let $0 \neq N \in A$. The definition of $\Gamma_0(N) \subseteq \Gamma$ is exactly the same as in the classical case (18). The rigid analytic space $X_0(N) := \Gamma_0(N) \backslash \Omega$ may then be realized as the underlying analytic space associated to an affine algebraic curve. As in the classical case, this space is compactified by adjoining a finite number of “cusps” and these cusps are given by $\Gamma_0(N) \backslash \mathbb{P}^1(k)$; we denote the compactified space by $\bar{X}_0(N)$. As in the number field case, $\bar{X}_0(N)$ may be realized canonically as a complete, smooth, geometrically connected curve

over k . Analogs of the constructions of Eichler and Shimura for elliptic curves can then be given on the Jacobian of $\bar{X}_0(N)$.

3.5. Modularity. Finally, Drinfeld [Dr1] established a reciprocity law which, in particular, identifies those automorphic representations given by elliptic curves over k which occur as quotients of the Jacobians of $\bar{X}_0(N)$.

Modularity for the class of elliptic curves over k with split-multiplicative reduction at ∞ is then accomplished via Drinfeld's reciprocity law coupled with the results of Grothendieck and Deligne mentioned above (which allow us to construct an associated automorphic representation), and the work of Y. Zarhin [Za1], [Za2] ($p \neq 2$, the case $p = 2$ is unpublished work of S. Mori; see Cor. XII.2.4 and Th. XII 2.5 of [M-B1]) establishing that the local factors of the elliptic curve over k determine its isogeny class (as Faltings was later to show for number fields).

4. MODULARITY FOR DRINFELD MODULES?

As we pointed out Subsection 2.1.1, classical elliptic modular curves play two (at least) distinct roles in number theory. The first role is their use in classifying elliptic curves (with various level structures) up to isomorphism. The second, very different, role lies in their use classifying elliptic curves over \mathbb{Q} up to isogeny.

Let $k = \mathbb{F}_r(T)$ as in the last section. Then we have seen how, for elliptic curves over k with split multiplicative reduction at ∞ , one classifies their isogeny classes via the moduli space of Drinfeld modules of rank 2. One is thus lead to ask whether Drinfeld modules themselves are “modular” in some reasonable sense. It is our goal here to explain finally how this may indeed be possible. In particular, just as the L -series of elliptic curves plays a crucial role in the modularity conjecture for elliptic curves over \mathbb{Q} , so too will the L -series of Drinfeld modules play an essential role here.

The basic idea is, roughly, that a Drinfeld module (or related object) will be called “modular” if its L -series can be obtained from the L -series of a rigid analytic cusp form under a simple translation of the argument.

The definition of such L -series proceeds very much like that the definition of L -functions of elliptic curves; one defines first the local Euler factor associated to a prime and then takes their product. We elaborate this construction first for the Carlitz module.

4.1. The L -series of the Carlitz module over k . Let $w = (f)$ be the prime ideal associated to a monic prime $f \in A$. Let $\mathbb{F}_w := A/w$ be the associated finite field and let $C(\mathbb{F}_w)$ be \mathbb{F}_w viewed as an A -module via the Carlitz action. As C has rank 1, it is easy to see that $C(\mathbb{F}_w)$ is isomorphic to $A/(g)$ for some monic $g \in A$. A simple calculation implies that $C_f(x) \equiv x^{r \deg f} \pmod{w}$; thus $f - 1$ annihilates $C(\mathbb{F}_w)$. Therefore $g \mid (f - 1)$ and counting points implies that, in fact, $g = f - 1$. Consequently, to w and C we define the local L -factor $L_w(C, u)$ by

$$L_w(C, u) = L_f(C, u) := \frac{1}{(1 - fu)}, \quad (51)$$

which the reader will see is a rank 1, characteristic p , version of (30).

We set

$$S_\infty := \bar{K}^* \times \mathbb{Z}_p, \quad (52)$$

which is a topological abelian group whose group operation will be written additively. For $s = (x, y) \in S_\infty$ and $a \in A$ monic we define

$$a^s := x^{\deg a} \cdot (a/T^{\deg a})^y, \quad (53)$$

where $(a/T^{\deg a})^y$ is defined using the binomial theorem (and converges in K as $a/T^{\deg a} = 1 + \{\text{higher terms in } 1/T\}$). In particular, note that if $s_i := (T^i, i)$, $i \in \mathbb{Z}$, then $a^{s_i} = a^i$; as such, we shall commonly write “ i ” for $s_i = (T^i, i) \in S_\infty$. One views S_∞ as a topological abelian group with the integers embedded (discretely) as a subgroup.

We now define the L -function $L(C, s)$, $s \in S_\infty$, of the Carlitz module by

$$L(C, s) = L(C, x, y) := \prod_{f \text{ monic prime}} L_f(C, f^{-s}) = \prod_f (1 - f f^{-s})^{-1} = \prod_f (1 - f^{1-s})^{-1}. \quad (54)$$

Upon expanding (54), we find

$$L(C, s) = \sum_{n \text{ monic}} n^{1-s} = \sum_{e=0}^{\infty} x^{-e} \left(\sum_{\substack{n \text{ monic} \\ \deg n=e}} n \langle n \rangle^{-y} \right). \quad (55)$$

In this case, elementary estimates (§8.8 of [Go4]) allow us to establish that $L(C, x, y)$ is an entire power series for all $y \in \mathbb{Z}_p$. Moreover, the resulting function on S_∞ is also continuous and its “zeroes flow continuously.” (The best technical definition of this concept is via non-Archimedean Frechet spaces as in [Boc1].)

Now let $y = -i$ for i a non-negative integer. The same elementary estimates also allow us to show that $L(C, x, -i)$ is a polynomial in x^{-1} ; one then immediately deduces that $L(C, x/T^i, -i) \in A[x^{-1}]$. As the set of non-positive integers is dense in \mathbb{Z}_p , we see that the set of *special polynomials* $\{L(C, x/T^i, -i)\}$ determines $L(C, s)$ as a function on S_∞ . In Subsection 4.7 we will see that such polynomials are *cohomological* in nature which will be the key towards handling the L -series of an arbitrary Drinfeld module.

Remark 4. Implicit in the definition of $L(C, s)$ is the “zeta-function of A ” defined by

$$\zeta_A(s) = \prod_{f \text{ monic prime}} (1 - f^{-s})^{-1}; \quad (56)$$

so $L(C, s) = \zeta_A(s-1) = \zeta_A(s-s_1)$ (s_1 as above). Clearly the analytic properties of $\zeta_A(s)$ follow from those of $L(C, s)$. Let now $i = s_i$ be a positive integer which is divisible by $(r-1)$ and let ξ be the period of the Carlitz module (as in Example 49). It is then easy to see that $0 \neq \zeta_A(i)/\xi^i \in \mathbb{F}_r(T)$ which is a version of the classical result of Euler on zeta-values at positive even integers.

Remark 5. It is natural to wonder if there is some obstruction to interpolating the set $\{L(C, x/T^i, -i)\}$ at a finite prime v of k . In fact, there is none (see §8 of [Go4] or [Boc1]). Just as one obtains functions on $\bar{K}^* \times \mathbb{Z}_p$, so too does one obtain functions on $\bar{k}_v^* \times S_v$ where \bar{k}_v is the algebraic closure of the completion k_v and $S_v := \varprojlim_j \mathbb{Z}/(p^j(r^{\deg v} - 1))$. These functions have remarkably similar properties to $L(C, s)$, $s \in S_\infty$. While we do not emphasize such v -adic functions here for space considerations, their existence is an important and natural part of the theory.

4.2. A quick introduction to T -modules and τ -sheaves. In order to understand the general L -series of Drinfeld modules, and their possible “modular” interpretation we need to expand the category of objects under study.

We will begin first with T -modules. This is an idea due to Greg Anderson [An1] (see also §5 of [Go4]), based on Drinfeld’s notion of “shtuka” or “elliptic sheaf.” The idea behind it is to replace the use of polynomials in 1 variable in the definition of a Drinfeld module in Subsection 3.3 with polynomials in many variables.

Thus let L be any extension of \mathbb{F}_r and consider the algebraic group $E := \mathbb{G}_a^e$ over L , where

\mathbb{G}_a is the additive group. Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_e \end{pmatrix} \in E$. There are two obvious types of \mathbb{F}_r -linear

endomorphisms of E as an algebraic group. The first is the r^i -th power mapping defined by

$x^{r^i} := \begin{pmatrix} x_1^{r^i} \\ \vdots \\ x_e^{r^i} \end{pmatrix}$. The second is $x \mapsto Mx$ where $M \in M_e(L) = \{e \times e \text{ matrices with coefficients}$

in $L\}$. It is then easy to see that any \mathbb{F}_r -linear endomorphism of E is just a combination of these; i.e., it can be written $\sum_{j=0}^t M_j x^{r^j}$, for $M_j \in M_e(L)$. We let $\text{End}_{\mathbb{F}_r}(E)$ be the set of all \mathbb{F}_r -linear endomorphisms viewed as an \mathbb{F}_r -algebra under composition.

As is standard, we let $I_e \in M_e(L)$ be the identity matrix.

Now let L be a field equipped with an \mathbb{F}_r -algebra map $\iota: A = \mathbb{F}_r[T] \rightarrow L$. We set $\theta := \iota(T)$ as before. A T -module over L is then a pair $E = (E_{\text{gp}}, \psi_E)$ where E_{gp} is an algebraic group isomorphic to \mathbb{G}_a^e , for some e , and $\psi = \psi_E: A \rightarrow \text{End}_{\mathbb{F}_r}(E_{\text{gp}})$ is an injection of \mathbb{F}_r -algebras. This injection is uniquely determined by ψ_T which is further required to satisfy

$$\psi_T(x) = (\theta I_e + N)x + \sum_{i=1}^t M_i x^{r^i}, \quad (57)$$

for some (possibly vanishing) $M_i \in M_e(L)$ and *nilpotent* $N \in M_e(L)$.

Remark 6. One can restate (57) as

$$\psi_T(x) = \Theta x + \sum_{i=1}^t M_i x^{r^i}, \quad (58)$$

where θ is the only eigenvalue for $\Theta \in M_e(L)$ (i.e., the characteristic polynomial of Θ is $(\lambda - \theta)^e$).

The reader may well wonder why one allows the existence of the nilpotent matrix N in (57). The reason is that its existence allows us to introduce a *tensor product* into the theory, [An1].

Example 2. Let e be arbitrary and set $\psi_T(x) = \theta I_e x = \theta x$. This is indeed a T -module under the above definition albeit a not very interesting one. Furthermore, note that when $e = 1$, we do *not* get a Drinfeld module. Indeed, this “trivial T -module” is precisely the case ruled out in the definition of Drinfeld modules.

Example 3. (See [AT1].) Let $L = k = \mathbb{F}_r(T)$ and let ι be the identity mapping. Let n be a positive integer and $C_{\text{gp}}^{\otimes n} := \mathbb{G}_a^n$. Let N_n be the $n \times n$ matrix

$$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix},$$

and V_n the $n \times n$ matrix

$$\begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 1 & \dots & 0 \end{pmatrix}.$$

We then set $C^{\otimes n}$ to be the injection of A into $\text{End}_{\mathbb{F}_r}(C_{\text{gp}}^{\otimes n})$ given by

$$C_T^{\otimes n}(x) := (\theta I_n + N_n)x + V_n x^r. \quad (59)$$

We then have a T -module $\mathfrak{C}^{\otimes n} := (C_{\text{gp}}^{\otimes n}, C^{\otimes n})$. We call $\mathfrak{C}^{\otimes n}$ the “ n -th tensor power of the Carlitz module.”

One commonly uses $C^{\otimes n}$ interchangeably with $\mathfrak{C}^{\otimes n}$. Clearly $C^{\otimes 1}$ coincides with the Carlitz module C as defined in Example 1.

In order to explain how the tensor product appears in the theory, we begin with a dual construction originally due to Drinfeld. Let ψ be a Drinfeld module of rank d over a field L . Let

$$M := \text{Hom}_L^{(r)}(\mathbb{G}_a, \mathbb{G}_a) \quad (60)$$

be the vector space of \mathbb{F}_r -linear homomorphisms of the additive group to itself as an algebraic group over L . We make the group M into a left module over $L \otimes_{\mathbb{F}_r} \mathbb{F}_r[T] \simeq L[T]$ via ψ as follows: Let $f(x) \in M$, $a \in \mathbb{F}_r[T]$ and $l \in L$. Then we put

$$l \otimes a \cdot f(x) := lf(\psi_a(x)). \quad (61)$$

It is easy to see (using a right division algorithm) that M is free over $L[T]$ of rank d . The module M is called the “motive” of ψ .

More generally, let $E = (E_{\text{gp}}, \psi)$ be an arbitrary T -module. We define its motive $M = M(E)$ as the group of \mathbb{F}_r -linear morphisms of E_{gp} to \mathbb{G}_a over L , exactly as (60). The action of $L[T]$ on M is defined as in (61). The T -module E is said to be *abelian* if and only if its motive M is finitely generated over $L[T]$. In this case, M is then free over $L[T]$ of finite rank which is also the *rank of E* .

For instance, Drinfeld modules are exactly the 1-dimensional abelian T -modules. As an exercise, the reader may check that $C^{\otimes n}$ of Example 3 is abelian (of rank 1) whereas the trivial T -module of Example 2 is not.

The motive M of a T -module also comes equipped with a canonical endomorphism τ defined by

$$\tau f(x) := f^r(x). \quad (62)$$

Notice that $\tau(lf) = l^r \tau(f)$ for $l \in L \subset L[T]$ whereas $\tau(a \cdot f) = a \cdot \tau(f)$ for $a \in A \subset L[T]$; we call such a mapping “partially Frobenius-linear.” In Anderson’s theory [An1], it is the interplay between the T -action and the partially Frobenius-linear τ -action that allows one to pass back and forth between a T -module and its motive.

A “ τ -sheaf” is then just a globalization of M viewed as an $L[T]$ -module equipped with the action of τ . More precisely, let X be a scheme over \mathbb{F}_r .

Definition 3. (See [BP1] or [Boc1]) A *coherent τ -sheaf* on X is a pair $\underline{\mathcal{F}} := (\mathcal{F}, \tau)$ consisting of a coherent sheaf \mathcal{F} on $X \times_{\mathbb{F}_r} \mathbb{A}^1$ and a partially Frobenius-linear mapping $\tau = \tau_{\mathcal{F}}: \mathcal{F} \rightarrow \mathcal{F}$. A *morphism* of τ -sheaves is a morphism of the underlying coherent sheaves which commutes with the τ -actions.

Therefore M , with the standard action of τ (62), canonically gives a τ -sheaf on $\text{Spec}(L) \times \mathbb{A}^1$. We call a τ -sheaf $\underline{\mathcal{F}}$ *locally-free* if \mathcal{F} is locally-free on $X \times \mathbb{A}^1$. We call $\underline{\mathcal{F}}$ a *strict τ -sheaf* if it is locally-free and τ is injective. (Our strict τ -sheaves are the “ τ -sheaves” of [Gal].) The τ -sheaves arising from T -modules, for instance, are strict in this definition.

The *rank* of a strict τ -sheaf is just the rank of the underlying vector bundle.

Example 4. We will describe here the τ -sheaf $\underline{\mathcal{C}} = (\mathcal{C}, \tau)$ on $\text{Spec}(\mathbb{F}_r(\theta))$ associated to the Carlitz module C . The underlying space for the vector bundle is $\text{Spec}(\mathbb{F}_r(\theta)) \times \mathbb{A}^1 \simeq \text{Spec}(\mathbb{F}_r(\theta)[T])$; for the moment let us call this product Y . Over Y the coherent module \mathcal{C} given by $M = M(C)$ is isomorphic to the structure sheaf \mathcal{O}_Y . The action of τ is then easily checked to be given by

$$\tau\left(\sum h_i(\theta)T^i\right) := (T - \theta) \sum h_i^r(\theta)T^i. \quad (63)$$

Remark 7. Example 4 suggests the following general construction of rank 1 strict τ -sheaves. Let $Y = \text{Spec}(\mathbb{F}_r(\theta)[T])$ as in the example and let $g(\theta, T)$ be an arbitrary non-trivial function in $\mathbb{F}_r(\theta)[T]$. We then define the τ -sheaf $\underline{\mathcal{F}}_g$ to have underlying sheaf \mathcal{O}_Y and $\tau = \tau_g$ -action given by

$$\tau_g\left(\sum h_i(\theta)T^i\right) := g(\theta, T) \sum h_i^r(\theta)T^i. \quad (64)$$

As an example, let $0 \neq \beta \in \mathbb{F}_r(\theta)$. One then has the general rank 1 Drinfeld module $C^{(\beta)}$ defined over $\mathbb{F}_r(\theta)$ by

$$C_T^{(\beta)}(x) := \theta x + \beta x^r, \quad (65)$$

(so $C^{(1)}$ is just the Carlitz module). The associated τ -sheaf is then $\underline{\mathcal{F}}_g$ for $g(\theta, T) = \frac{1}{\beta}(T - \theta)$. Thus one sees how small a subset of all rank 1 τ -sheaves is occupied by the rank 1 Drinfeld modules.

Let $\underline{\mathcal{F}}$ and $\underline{\mathcal{G}}$ be two coherent τ -sheaves. We define the *tensor product τ -sheaf* $\underline{\mathcal{F}} \otimes \underline{\mathcal{G}}$ to have underlying coherent sheaf $\mathcal{F} \otimes_{\mathcal{O}_{X \times \mathbb{A}^1}} \mathcal{G}$ with $\tau_{\underline{\mathcal{F}} \otimes \underline{\mathcal{G}}} := \tau_{\underline{\mathcal{F}}} \otimes \tau_{\underline{\mathcal{G}}}$. One can check, for instance, that the tensor product of strict τ -sheaves is again a strict τ -sheaf.

Example 5. Let $\underline{\mathcal{C}}$ be the τ -sheaf associated to the Carlitz module, as in Example 4. One can now easily form the n -th tensor power τ -sheaf $\underline{\mathcal{C}}^{\otimes n}$. As in [AT1], this τ -sheaf is isomorphic to the canonical τ -sheaf associated to $C^{\otimes n}$ which also justifies the latter’s name.

We will also identify $C^{\otimes n}$ with its associated τ -sheaf in later applications.

Remark 8. Anderson [An1] (also §5.5 of [Go4]) gives a very important “purity” condition that insures in general that the tensor product of the τ -sheaves associated to two T -modules also arises from a T -module.

Remark 9. As the reader will hopefully have come to see, T -modules and τ -sheaves are two sides to the same coin, so to speak. Indeed, with T -modules one focuses on the realization of

A as certain algebraic endomorphisms of \mathbb{G}_a^e for some e . On the other hand, with τ -sheaves, one emphasizes, and generalizes, the associated motives of the T -modules. As we shall see in Subsection 4.7, it is the τ -sheaf formalism that is essential in establishing the basic analytic properties of L -series in the characteristic p theory. However, Drinfeld modules over $\mathbb{F}_r((1/T))$ arise from lattices and such lattices are needed, at least, for properties of special-values of L -functions such as given in Example 4. It is therefore natural to ask about the relationship of general T -modules to lattices. In our next subsection we will discuss what is known in this regard; it turns out that the answer is essential for Böckle's theory.

4.3. Uniformization of general T -modules. As before, let $k = \mathbb{F}_r(T)$, $\iota: A \rightarrow k$ the identity map and $\theta = \iota(T)$. Let $K := \mathbb{F}_r((1/\theta))$ with fixed algebraic closure \bar{K} . So we are back in the analytic set-up of Subsection 3.3. Let $E = (E_{gp}, \psi)$ be a t -module of dimension e defined over a finite extension $L \subset \bar{K}$ of K . Without loss of generality we can, and will, suppose that $E_{gp} \simeq \mathbb{G}_a^e$.

By definition (Equation 57) one knows that $\psi_T = (\theta I_e + N)x + \{\text{higher terms}\}$ with N nilpotent. Clearly the action $\psi_{T,*}$ of T on the Lie algebra of E is then given by $\theta I_e + N$. One now formally looks for an exponential function \exp_E associated to E of the form

$$\exp_E = \sum_{i=0}^{\infty} Q_i x^{r^i}$$

where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_e \end{pmatrix} \in \text{Lie}(E)$, x^{r^i} is defined in the obvious fashion, $Q_0 = I_e$ and the Q_i are

$e \times e$ matrices with coefficients in L . As in the Drinfeld module case (Equation 47), \exp_E is further required to satisfy

$$\exp_E(\psi_{T,*}x) = \psi_T(\exp_E(x)). \quad (66)$$

Using (66), one readily, and uniquely, finds the coefficient matrices Q_i and that $\exp_E(x)$ is *entire* (i.e., converges for all x).

However, as Anderson discovered, as soon as $e > 1$ a fundamental problem arises in that there exist abelian T -modules E where $\exp_E(x)$ is *not* surjective on geometric points (that is, over \bar{K}). Anderson [An1] gives some necessary and sufficient conditions for the geometric surjectivity of $\exp_E(x)$. We will focus here on the criterion Anderson calls “rigid analytic triviality.”

Let M be the T -motive of E . Let $L\{T\}$ be the Tate algebra of all power series $\sum_{j=0}^{\infty} c_j T^j$

where $c_j \in L$ all j and $c_j \rightarrow 0$ as $j \rightarrow \infty$.

Definition 4. 1. We set $M\{T\} := M \otimes_{L[T]} L\{T\}$ with its obvious $L\{T\}$ -module structure. We equip $M\{T\}$ with a τ -action by setting

$$\tau(m \otimes \sum c_j T^j) := \tau m \otimes \sum c_j^r T^j.$$

2. We let $M\{T\}^\tau \subset M\{T\}$ be the A -module of τ -invariants.

3. The module M is said to be *rigid analytically trivial over L* if the natural map $M\{T\}^\tau \otimes_A L\{T\} \rightarrow M\{T\}$ is an isomorphism.

It is important to note that Definition 4 makes sense for general arbitrary τ -modules over L .

Anderson then proves that $\exp_E(x)$ is surjective on geometric points if and only if there is a finite extension L' of L such that the motive M of E over L' is rigid analytically trivial (over L'). This condition is preserved under tensor products.

If $\exp_E(x)$ is geometrically surjective, then its kernel (as a homomorphism of groups) \mathcal{L} is called the “lattice of E .” One can show that \mathcal{L} is an A -module of the same rank as M .

Definition 5. We say that E , and M , is *uniformizable over a field L* if and only if it is rigid analytically trivial over L . We say that E , and M , are *uniformizable* if and only if there is a finite extension L' of L over which they are uniformizable.

As before, this notion can be extended to arbitrary τ -modules M (with no obvious exponential function attached!).

As an example, $C^{\otimes n}$ is uniformizable over $\mathbb{F}_r((1/\theta_1))$, $\theta_1 := (-\theta)^{1/(r-1)}$, all $n \geq 1$, while $C^{\otimes m(r-1)}$ is uniformizable over $\mathbb{F}_r((1/\theta))$ for all $m \geq 1$.

If E is uniformizable over L then the τ -invariants $M\{T\}^\tau$ form a free A -module of rank equal to the rank of M . The converse is also true (and is an unpublished result of Urs Hartl): If the τ -invariants over L form a free module of rank equal to that of M , then M is uniformizable over L .

Remark 10. Implicit in the above statement is the assertion that if $M\{T\}^\tau$ has rank equal to that of M , then one obtains *no further* invariants by passing to any finite extension L' . This is indeed true and can be seen directly. Indeed, the invariants over any finite extension will have the same rank. Thus, if m is one such invariant, there is an $f \in A$ such that fm is an invariant over L . One then sees that this forces m to be defined over L also.

Let E be a uniformizable T -module which is defined over L . Let $L' \subset \bar{K}$ be the finite extension generated by the lattice \mathcal{L} of E . One then sees that L' is the smallest extension of L over which E is uniformizable.

Example 6. We will present here the rank 1 Drinfeld module $C^{(-\theta)}$ defined over $\mathbb{F}_r(\theta)$ by

$$C_T^{(-\theta)}(x) = \theta x - \theta x^r. \quad (67)$$

Using the explicit knowledge of the period ξ of the Carlitz module (see Example 1), one sees readily that the lattice of $C^{(-\theta)}$ lies in $\mathbb{F}_r((1/\theta))$; thus $C^{(-\theta)}$ is uniformizable over $\mathbb{F}_r((1/\theta))$.

4.4. Tate modules of Drinfeld modules and T -modules over $\mathbb{F}_r(\theta)$. One approach to constructing the L -series of an elliptic curve over \mathbb{Q} mentioned in Subsection 2.2 is through the use of its Tate-modules. We will use the same approach here to define the L -series of general Drinfeld modules and T -modules and, in the next subsection, we will present the construction for τ -sheaves.

Thus let $E = (E_{\text{gp}}, \psi_E)$ be an abelian T -module defined over $k = \mathbb{F}_r(\theta)$. Let $v = (g)$ be the prime associated to a monic irreducible $g \in A$. We define the v^i torsion points of E to be the kernel of the map $x \mapsto \psi_{E, g^i}(x)$ where $x \in E_{\text{gp}}(\bar{k})$ and \bar{k} is a fixed algebraic closure of k ; we denote this kernel by “ $E[v^i]$.” Clearly, $E[v^i]$ inherits an A -structure and it can then be shown that $E[v^i] \simeq (A/v^i)^t$ where t is the rank of E . The v -adic Tate module of E is then defined by

$$T_v(E) := \varprojlim_i E[v^i]. \quad (68)$$

Thus $T_v(E)$ is a free A_v -module of rank t . Finally, we set

$$H_v^1(E) = \text{Hom}_{A_v}(T_v(E), k_v). \quad (69)$$

The various A_v -modules, $\{H_v^1(E)\}$, form a compatible system of Galois representations as with elliptic curves. Using geometric Frobenius elements and invariants of inertia, again as in the case of elliptic curves, one obtains local L -factors $L_f(E, u)$ for monic primes $f \in A$ with $L_f(E, u)^{-1} \in A[u]$. One then defines the L -function of E , $L(E, s)$ for $s \in S_\infty$, by

$$L(E, s) := \prod_f L_f(E, f^{-s}). \quad (70)$$

One sees easily that $L(C, s)$, with the above definition, agrees with $L(C, s)$ as given in (54).

Two T -modules are said to be isogenous if there is a finite surjective map between them (i.e., a surjective map of the underlying algebraic groups which commutes with the A -actions). It is known that the isogeny class for Drinfeld modules and many T -modules ([Tag1], [Tag2], [Tam1]) is determined by the associated L -series (as one can read off the local L -factors from the L -series).

In [Ga2] Gardeyn shows that an abelian T -module is uniformizable if and only if the Tate action of the decomposition group at ∞ has finite image.

4.5. The L -series of a τ -sheaf over k . As is discussed in [Ga1], general T -modules are not the proper setting in which to analyze the local factors of their associated L -series. It is relatively easy to define the appropriate notion of “good” prime for a T -module E (one just wants to insure that one can reduce the T -action of E to obtain a T -module of the *same* rank over the quotient field.) However, outside of the case of Drinfeld modules, one then loses the connection between good primes for E and good (=unramified) primes for the compatible system $\{H_v^1(E)\}$. Moreover, even in the case of Drinfeld modules, τ -sheaves are needed in order to describe the factors at the bad primes, see Example 7.

The techniques for defining the L -function of a τ -sheaf goes back to work of Anderson [An1] on T -modules. Let E be an abelian T -module with associated motive $M = M(E)$ as in Subsection 4.2 and let \bar{M} be constructed in the same fashion as M but over the algebraic closure \bar{k} of k . Let $v = (g)$ be a prime of A . Then Anderson shows that the Galois module $H_v^1(E)$ is isomorphic to the Galois module $H_v^1(M)$ defined by

$$H_v^1(M) := \varprojlim_i (\bar{M}/g^i \bar{M})^\tau, \quad (71)$$

(where $N^\tau := \{\lambda \in N \mid \tau\lambda = \lambda\}$ for any τ -module N).

The above definitions immediately carry over to the case of τ -sheaves $\underline{\mathcal{F}} = (\mathcal{F}, \tau)$ over k ; one obtains local factors $L_f(\underline{\mathcal{F}}, u)$ again using inertial invariants and characteristic polynomials of geometric Frobenius elements. The idea of G. Böckle, R. Pink and F. Gardeyn (again following work of Anderson), is to show that $L_f(\underline{\mathcal{F}}, u)$ be expressed in terms of the τ -action itself. Indeed, at a bad prime f (where there are non-trivial invariants of inertia) Gardeyn [Ga3] constructs a “maximal model” $\underline{\mathcal{F}}^M = (\mathcal{F}^M, \tau^M)$ of $\underline{\mathcal{F}}$ (which may be viewed as a “Néron model” for $\underline{\mathcal{F}}$). The point is that the special fiber $\underline{\mathcal{F}}_{\text{sp}} = (\mathcal{F}_{\text{sp}}, \tau_{\text{sp}})$ of $\underline{\mathcal{F}}^M$ is a τ -sheaf on $\text{Spec}(\mathbb{F}_f)$ (where \mathbb{F}_f is the residue field at f). One then sees that

$$L_f(\underline{\mathcal{F}}, u)^{-1} = \det_A(1 - u\tau \mid H^0(\mathcal{F}_{\text{sp}})), \quad (72)$$

which establishes, for instance, that $L_f(\underline{\mathcal{F}}, u)^{-1} \in A[u]$.

Example 7. Let ψ be a Drinfeld module over k . In [Ga1], Gardeyn presents the local factors $L_f(\psi, u)$ at the bad primes f . It is shown that if ψ has bad reduction at f but ψ obtains good reduction over a finite extension L of k (and a prime of L above f) then $L_f(\psi, u) = 1$.

Moreover, if there does not exist a finite extension L of k over which ψ obtains good reduction, then $L_f(\psi, u)^{-1} \in \mathbb{F}_r[u] \subset A[u]$. This is remarkably similar to the case of elliptic curves (31). It would be interesting to establish exactly which polynomials in $\mathbb{F}_r[u]$ actually occur for a given Drinfeld module ψ . Note also that all rank 1 Drinfeld modules have potentially good reduction (since they are all geometrically isomorphic to the Carlitz module). As such, the local factors at the bad primes in the rank 1 case are all identically 1 as one expects.

Remark 11. In [Boc1], the local L -factors of a τ -sheaf $\underline{\mathcal{F}}$ are defined directly as in Equation (72) without using Galois representations. However, to any τ -sheaf one can attach a constructible étale sheaf of A_v -modules which is a natural Galois module. One can use this Galois module as we have used $H_v^1(E)$ for a T -module to define the L -factor (and, indeed, in the T -module case the Galois module is isomorphic to $H^1(E)$). Therefore one can always use the classical Galois formalism to define local L -factors in general.

We will finish this subsection by describing briefly the Galois representations associated to τ -sheaves $\underline{\mathcal{F}}_g = (\mathcal{F}_g, \tau_g)$ where $\mathcal{F}_g = \text{Spec}(\mathbb{F}_r(\theta)[T])$, $0 \neq g(\theta, T) \in \mathbb{F}_r(\theta)[T]$, and τ_g is given by (64). As these sheaves have rank 1, we obtain 1-dimensional v -adic representations which we denote by $\rho_{g,v}$. Let $f(\theta) \in \mathbb{F}_r[\theta]$ be a monic irreducible polynomial of degree d with roots $\{\bar{\theta}, \bar{\theta}^r, \dots, \bar{\theta}^{r^{d-1}}\}$. Set

$$g^f(T) := \prod_{i=0}^{d-1} g(T, \bar{\theta}^{r^i}) \in \mathbb{F}_r[T]. \quad (73)$$

For instance if $g(T, \theta) = \prod_i (h_i(T) - \theta)$, where $h_i(T)$ does not involve θ , then $g^f(T) = \prod_i f(h_i(T))$. Suppose now that $g^f(T) \in A_v^*$ and, finally, let Frob_f be the geometric Frobenius at $(f(\theta))$. Then one has

$$\rho_{g,v}(\text{Frob}_f) = g^f(T). \quad (74)$$

(I am indebted to Böckle for pointing out this simple and elegant formula.)

Now let C be the Carlitz module. One knows that C corresponds to the function $g(\theta, T) = T - \theta$. Let v be as above and denote $\rho_{g,v}$ by $\rho_{C,v}$. Let $f(\theta)$ be a monic prime with v relatively prime to $f(T)$. Then one has

$$\rho_{C,v}(\text{Frob}_f) = f(T) \in A_v^* \quad (75)$$

which agrees with (51) and where we recall that we use the dual action to define L -series.

More generally let $C^{(\beta)}$ be the general rank 1 Drinfeld module over $\mathbb{F}_r(\theta)$ as in Remark 7 with associated function $g(\theta, T) = \frac{1}{\beta}(T - \theta)$. Let $\rho_{C^{(\beta)},v}$ be the associated v -adic representation. Then, as β is *constant* in T (so that $\beta^f(T)$ is also constant in T), one finds

$$\rho_{C^{(\beta)},v} = \chi_\beta \rho_{C,v} \quad (76)$$

where χ_β is an \mathbb{F}_r^* -valued Galois character which is independent of v .

4.6. Special polynomials and Carlitz tensor powers. Recall that in the case of the L -series $L(C, s)$ of the Carlitz module the functions $L(C, x/T^i, -i)$, i a non-negative integer, actually belong to $A[x^{-1}]$. Let $\underline{\mathcal{F}}$ now be a τ -sheaf with L -series $L(\underline{\mathcal{F}}, s)$. The case of the Carlitz module suggests looking at the power series $L(\underline{\mathcal{F}}, x/T^i, -i)$ for i as above. In our next subsection we will establish that these *special power series* are in fact rational functions with A -coefficients (and, naturally, called the *special functions of $L(\underline{\mathcal{F}}, s)$*). Essential to the proof is the equality

$$L(\underline{\mathcal{F}}, x/T^i, -i) = L(\underline{\mathcal{F}} \otimes C^{\otimes i}, x, 0). \quad (77)$$

Equation (77) follows directly from looking at the associated Galois representations. In particular, one concludes for non-negative integers i that

$$L(\underline{\mathcal{F}}, s - i) = L(\underline{\mathcal{F}}, s - s_i) = L(\underline{\mathcal{F}} \otimes C^{\otimes i}, s). \quad (78)$$

4.7. Crystals and their cohomology. In this subsection we review briefly the theory of “crystals” associated to τ -sheaves developed by R. Pink and G. Böckle [BP1] (see also [Boc1] and [Boc2]). Let $\underline{\mathcal{F}} = (\mathcal{F}, \tau)$ be a τ -sheaf on a scheme X where τ acts nilpotently, that is, $\tau^m = 0$ for some $m > 0$. From Equation (72) we see that the L -factors of $\underline{\mathcal{F}}$ will be trivial (identically 1) at every prime; thus the associated global L -series will also be trivial. Therefore, from the L -series point of view, such τ -sheaves are negligible. The idea of Böckle and Pink is to make this precise by passing to a certain quotient category. More precisely, the category of “crystals over X ” is the quotient category of τ -sheaves modulo the subcategory of nilpotent τ -sheaves.

The category of crystals has the advantage that a cohomology theory may be developed for it. This cohomology theory is very closely related to coherent sheaf cohomology but which possesses only the first three of the canonical six functors $\{Rf_!, f^*, \otimes, f_*, f^!, \text{Hom}\}$. However, the cohomology of crystals does possess a Lefschetz trace formula. As such, by using (77), Böckle and Pink establish that the special power series associated to $L(\underline{\mathcal{F}}, s)$ are rational functions. If, for instance, $\underline{\mathcal{F}}$ is locally free, then one obtains an entire function whose special rational functions are polynomials (Th. 4.15 of [Boc1]). Furthermore, Böckle establishes that the degrees of these special polynomials $L(\underline{\mathcal{F}}, x/T^i, -i)$ grow *logarithmically* in i . This is then enough to establish that general $L(\underline{\mathcal{F}}, s)$ have meromorphic interpolations at all the places of k .

4.8. Modular forms in characteristic p . We now have all the techniques necessary to begin studying modular forms in characteristic p which we present in this subsection. Let Ω be the Drinfeld upper-half plane as given in Definition 2. Based on the discussion given in Subsection 2.1.1, the notion of a “congruence subgroup” $\tilde{\Gamma}$ of $\Gamma := GL_2(A)$ is obvious as is the notion of an unrestricted modular form of weight j (where j is now an integer) for $\tilde{\Gamma}$ (upon replacing “analytic” with “rigid analytic” in Definition 1).

Thus, following classical theory, we clearly need to describe what happens at the cusps $\tilde{\Gamma} \backslash \mathbb{P}^1(k)$ and to do this one needs only treat the special case of the cusp ∞ . As before let $\tilde{\Gamma}_\infty$ be the subgroup of $\tilde{\Gamma}$ that fixes ∞ . One sees that $\tilde{\Gamma}$ consists of mappings of the form $z \mapsto \alpha z + b$ where α belongs to a subgroup H of \mathbb{F}_r^* and $b \in I$ where $I = (i)$ is an ideal of A . We set $e_\infty(z) := e_C(\xi z/i)$ where $e_C(z)$ is the exponential of the Carlitz module and ξ is its period. Finally we set $q := e_\infty(z)^{-e}$ where e is the order of H . In [Go1], it is shown that q is a parameter at the cusp ∞ .

With the above choice of parameter q , the definitions of *holomorphic form* and *cuspidal form* are exactly the same as their complex counterparts. One can show (ibid.) that holomorphic forms are sections of line bundles on the associated compactified moduli curves; therefore for fixed subgroup and weight, they form finite dimensional \bar{K} -vector spaces.

There are now 2 distinct cases of subgroups $\tilde{\Gamma}$ of interest to us. The first case is $\tilde{\Gamma} = \Gamma$ and the second case is the full congruence subgroup

$$\tilde{\Gamma} = \Gamma(N) = \left\{ \gamma \in GL_2(A) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

for some polynomial $N \in A$. In the first case, the parameter q at ∞ is $e_C(\xi z)^{1-r}$ and in the second case it is $q = e_C(\xi z/N)^{-1}$.

Remark 12. In the classical elliptic modular theory one has $dq = cq \cdot d\tau$ for some non-zero constant c . Thus one sees that cusp forms of weight 2 correspond to holomorphic differential forms on the associated complete moduli curve. For $\Gamma(N)$, with $N \in A$, one computes readily that $dq = cq^2 \cdot dz$ with $c \neq 0$. Thus cusp forms with zeroes of order 2 at every cusps correspond to holomorphic differential forms on the associated complete modular curve. Such cusp forms are called “double cusp forms.”

We denote the space of cusp forms of weight j associated to N by $S(N, j)$ and the subspace of double cusp forms by $S^2(N, j)$. A simple calculation implies that a cusp form f for $GL_2(A)$ *automatically* becomes a double cusp form for $\Gamma(N)$ for any non-constant N .

Remark 13. Recall that after Definition 1 we mentioned “multiplier systems” that allow one to obtain a (slightly) generalized notion of modular forms. One such multiplier is $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-t} = (ad - bc)^{-t}$ where t is an integer; one then says that the modular form has *type* t (see, e.g., [Ge1] or Definition 5.1 of [Boc2]).

Example 8. As in (50), a rank two Drinfeld A -module ψ is uniquely determined by $\psi_T(x) = Tx + g(\psi)x^r + \Delta(\psi)x^{r^2}$ where $\Delta(\psi) \neq 0$. Let $z \in \Omega$ and let $L_z := A + Az$ be the associated rank 2 A -lattice and $\psi^{(z)}$ the associated rank 2 Drinfeld module. As in the classical case, the maps $g: z \mapsto g(\psi^{(z)})$ and $\Delta: z \mapsto \Delta(\psi^{(z)})$ define rigid analytic modular forms for the group $GL_2(A)$ of weights $r - 1$ and $r^2 - 1$ respectively (and type 0). Moreover, Δ is easily seen to be a cusp form as it is classically.

Remark 14. When working with modular forms associated to congruence subgroups there is a major difference between classical theory and the theory developed in [Boc2]. Indeed, Böckle needs to work with the *full* moduli spaces attached to congruence subgroups and in particular the moduli space of Drinfeld modules of rank 2 with level I structure (i.e., a basis for the i -division points). This moduli space contains *many* different geometric components (as does its classical counterpart). As in Drinfeld’s original paper [Dr1], these components are best handled through the use of the adèles. In particular, in [Boc2], §5.5, Böckle develops a theory of types for adelic modular forms which generalizes that given in Remark 13 above.

The definition of the Hecke operators $T(I)$ for ideals I of A is then modeled on classical theory. In particular Böckle [Boc2] presents naturally defined Hecke-operators in the adelic setting (and for general base rings A) which depend on the type, weight and level involved. Moreover, these Hecke operators do *not* fix the components of the underlying moduli spaces. In particular they therefore differ from the ones defined in [Go1], [Go2] and [Ge1]; the latter Hecke operators fix the components of the moduli spaces but cannot be defined for general base rings A . When recalling the Eichler-Shimura isomorphism given in [Boc2], this is an important consideration; a comparison between the two viewpoints is given in Example 6.13 of [Boc2].

One sees naturally that the cusp forms of a given weight are stable under the Hecke operators, but also, when $\tilde{\Gamma} = \Gamma(N)$, so are the double cusp forms (which is highly remarkable from the classical viewpoint!). Moreover, as in the classical case, the Hecke operators form a commutative ring of endomorphisms of these spaces.

Thus there are really three Hecke stable spaces of interest: $S(N, j)$, $S^2(N, j)$ and the quotient space $S(N, j)/S^2(N, j)$.

Remark 15. Classical Hecke operators, as covered in Subsection 2.1.1, have the property that $T(p^2) \neq T(p)^2$ for p prime. Remarkably, in the characteristic p case, one finds that $T(I^2) = T(I)^2$ for *any* ideal I (including, precisely, the case of I prime); thus the Hecke operators are *strongly multiplicative*. Indeed, the classical computation of $T(\wp^2)$, \wp prime, works and one sees that the terms different from $T(\wp)^2$ are weighted with integer factors divisible by $p = 0 \in \mathbb{F}_r$. This commutativity is essentially the reason that the Hecke operators give rise to *abelian* representations as in our next subsection.

4.9. Galois representations associated to cusp forms. In this subsection we summarize very briefly the results of [Boc2] on Galois representations associated to cusp forms in characteristic p . Fix $N \in A$ and view $S(N, j)$ as a Hecke-module. As the ring of Hecke operators is commutative we can decompose $S(N, j)$ into generalized eigenspaces. Let $\{M_1, \dots, M_\lambda\}$ denote the simple Hecke subfactors of the *true* eigenspaces. Every simple Hecke subfactor of $S(N, j)$ is then isomorphic to one of the M_i .

To each M_i corresponds to a true Hecke eigenform $f_i \in S(N, j)$. Let \mathfrak{P} be a prime of A not dividing N and suppose that $T(\mathfrak{P})f_i = \alpha_{i, \mathfrak{P}}f_i$ (where we use the adelic Hecke operators of [Boc2]). Via the general cohomological formalism of crystals, Böckle attaches to each f_i a rank 1 τ -sheaf $\underline{\mathcal{M}}_i$; this is done in a non-canonical fashion.

Let v be a prime of A . The general theory of τ -sheaves, as in Remark 11, then gives us a continuous 1-dimensional v -adic Galois representation $\rho_i = \rho_{f_i}$ for each i (which *is* indeed canonical!). We call the compatible system of representations obtained this way the *Böckle system (of Galois representations) attached to f_i* . The Eichler-Shimura relation established in [Boc2] in this context then implies that

$$\rho_i(\text{Frob}_{\mathfrak{P}}) = \alpha_{i, \mathfrak{P}}$$

for \mathfrak{P} prime to N and v . In particular, we conclude that $\alpha_{i, \mathfrak{P}} \neq 0$.

Remark 16. In [Boc2], Theorem 13.2, the above result is only established for cusp forms of weight n , type $n - 1$ and level $I \neq A$. There is also a general “yoga” which allows one to change types, after increasing the level; by using compatibilities of the Galois representations attached to modular forms, the above result can be extended to arbitrary types independent of the weight, cf. [Boc2] Lemma 5.32 and Remark 6.12. This is important to us since we want to consider cusp forms of level A (attached the full modular group) and type 0. A more conceptual proof which avoids this yoga can be given by proving an Eichler-Shimura isomorphism for fixed level and arbitrary weight n for all types $l \geq n - 1$, where however the τ -sheaf $\underline{\mathcal{M}}_i$ mentioned above have to be twisted suitably. “Untwisting” by powers of $C^{\otimes(r-1)}$ on the Galois side, one then obtains the result.

While the above process attaches Galois representations to cusp forms for any type l , for $l < n - 1$ there is *no* τ -sheaf associated to the representation. This is similar to the classical situation where the inverse of the Tate motive is not represented by a geometric object but the corresponding cyclotomic character obviously has an inverse.

Remark 17. If the cusp form is *not* double-cuspidal, then the associated family of Galois representations arises essentially from a finite character. To be more precise, if the weight is n and the type is $n - 1$, then there is indeed a finite character. For other types and the same weight, the Galois representations get twisted by some natural 1-dimensional characters

associated to Drinfeld modular varieties of rank 1 Drinfeld modules. Moreover, Böckle establishes that the class of finite characters which arise are *all* finite characteristic p valued characters allowed by the explicit class field theory of rank 1 Drinfeld modules. That is one obtains those finite dimensional characters of abelian extensions of $k = \mathbb{F}_r(\theta)$ which are *totally-split* at ∞ . Moreover, one obtains such representations for arbitrary weights > 2 (for 2 there are some modifications involving the trivial character). It is reasonable to view the associated cusp forms as being rather analogous to the theta-series assigned to finite characters classically as in Equation (12). Explicitly constructing such cusp forms in characteristic p is certainly now a very interesting problem.

Remark 18. Although the choice of τ -sheaf $\underline{\mathcal{M}}_i$ associated to f_i is not canonical, the associated Galois representations are and depend only on the Hecke eigenvalues. As such, one can define $L(f_i, s) := L(\underline{\mathcal{M}}_i, s)$ in an unambiguous fashion. The results of Böckle in Subsection 4.7 then imply the analytic continuation of $L(f_i, s)$ (at all places of k).

Question 1. Classically one can twist modular forms by characters simply by multiplication of the Fourier coefficients. Can one define such twists for the Böckle systems in characteristic p ?

Remark 19. It is very important for us that Böckle’s theory *does* establish at least one (so far!) constraint on the τ -sheaves that may arise from modular forms. Indeed, Böckle shows that such τ -sheaves arise from decomposing a τ -sheaf (via “complex multiplications”) which is defined and *uniformizable* (in the sense of Definition 5) over $\mathbb{F}_r((1/\theta))$.

Remark 20. The reader may well be asking why one works with $\Gamma(N)$ as opposed to $\Gamma_0(N)$. One does not use $\Gamma_0(N)$ because one needs a fine moduli space (i.e., a representable functor) for Böckle’s constructions and $\Gamma_0(N)$ is not associated with a representable functor. Indeed, Böckle begins with the τ -sheaf $\underline{\mathcal{F}}_N$ associated to the universal family of Drinfeld modules associated to $\Gamma(N)$. The representations arise by relating the Böckle-Pink cohomology of the symmetric powers of $\underline{\mathcal{F}}_N$ with étale cohomology.

4.10. Modularity for rank 1 Drinfeld modules. Let $\underline{\mathcal{F}}$ be any τ -sheaf defined over k and let $L(\underline{\mathcal{F}}, s)$, $s \in S_\infty$, be its L -series. From Equation (78) we see that the L -series of $\underline{\mathcal{F}}$ and $\underline{\mathcal{F}} \otimes C^{\otimes n}$ are simple integral *translates* of each other. Thus, from the point of view of L -series, the τ -sheaves $\underline{\mathcal{F}}$ and $\underline{\mathcal{F}} \otimes C^{\otimes n}$ are equivalent.

The above observation will guide our definition of “modularity.” In fact, there are really two notions of “modularity” implicit in the theory. Let $k = \mathbb{F}_r(\theta)$ as before and let k^{sep} be a fixed separable closure.

Definition 6. We say that a Drinfeld module ψ defined over $k = \mathbb{F}_r(\theta)$ is *modular of class I* if and only if its L -series is an integral translate of the L -series of a finite \mathbb{F}_r^* -valued character of $\text{Gal}(k^{\text{sep}}/k)$ which has trivial component at ∞ (see Remark 17). We say ψ is *modular of class II* if and only if its L -series is an integral translate of $L(f, s)$ where f is a double cusp form of some weight and level.

Remark 21. In general for a Drinfeld module ϕ to possibly have its Galois representations (possibly twisted by those of $C^{j(r-1)}$) arise from one of the τ -sheaves \mathcal{M}_i , it must have *abelian* Galois representations on its Tate modules. Thus it is either of rank 1 or has “complex multiplication” when the rank $d > 1$. Since the Galois-image is abelian, the latter means that the ring A' of endomorphism of ϕ is commutative and a finite extension of A . Furthermore

by Prop. 4.7.17 of [Go4] $k' := A' \otimes_A k$ as well as $K' := k' \otimes_k K$ (where we recall $K = k_\infty = \mathbb{F}_r((1/T))$) are fields. Since ϕ must be uniformizable over K , it therefore must also satisfy the weaker condition that it is uniformizable over K' .

Example 9. Let $C^{(\beta)}$ be the general rank 1 Drinfeld module given in Equation (65) with $\beta \in \mathbb{F}_r(\theta)$. Suppose that $\beta = \alpha^{r-1}$ with $\alpha \in \mathbb{F}_r((1/\theta))$; thus over $\mathbb{F}_r((1/\theta))$ one has $C_a^{(\beta)}(x) = \alpha^{-1}x \circ C_a(x) \circ \alpha x$. In particular, C and C^β are isomorphic over $\mathbb{F}_r((1/\theta))$. Thus the character χ_β of Equation (76) has trivial component at ∞ and $C^{(\beta)}$ is modular of class I. As an example, one can take $\beta := \frac{\theta+1}{\theta} = 1 + \frac{1}{\theta}$ and then find α via the binomial theorem applied to $1/(r-1)$.

In particular, the Carlitz module is obviously then modular of class I. We now show how it is also modular of class II.

Example 10. Let Δ be as in Example 8. It is shown in [Go2] that if $\mathfrak{P} = (\mathfrak{p})$ then

$$T(\mathfrak{P})\Delta = \mathfrak{p}^{r-1}\Delta, \quad (79)$$

where we have used the Hecke operators from [Go2]. If instead we had used the Hecke operators as in [Boc2], we would obtain

$$T(\mathfrak{P})\Delta = \mathfrak{p}^{(r-r^2)}\Delta, \quad (80)$$

cf. Example 6.13 of [Boc2]. Thus the L -function of the Böckle system of Galois representations associated to Δ equals $L(C, s + r^2 - r + 1)$. In particular C is therefore modular of class II.

It should be pointed out that, in line with the results mentioned in Subsection 4.3, $C^{\otimes(r-1)}$ is actually uniformizable over $\mathbb{F}_r((1/\theta))$.

Note that in particular, $\zeta_A(s + 1 - r) = L(\Delta, s)$. This should be compared with the classical formula of Subsection 2.1 where the theta function $\theta(\tau)$ naturally gives $\zeta(2s)$ (i.e., one needs the factor $s/2$ in the integral (5)).

In fact, in [Go2] the exact *same* result (79) is also established for the cusp form $g^r \Delta$ of weight $(2r + 1)(r - 1)$ (where g is also defined in Example 8). From the classical viewpoint this is *highly* surprising!

Recall that in Example 6 we discussed the rank one Drinfeld module $C^{(-\theta)}$ which is uniformizable over $\mathbb{F}_r((1/\theta))$. There is *no* known obstruction for the v -adic representations associated to $C^{(-\theta)}$ (or any Drinfeld module defined over $\mathbb{F}_r(\theta)$ which is uniformizable over $\mathbb{F}_r((1/\theta))$) to be the Böckle system arising from some double cusp form. We are thus led to the following question.

Question 2. Does there exist a double cusp form of some weight and level whose Böckle system of Galois representations is the same as the system arising from $C^{(-\theta)} \otimes C^{\otimes j}$ for some $j \geq 0$ with $j \equiv 0 \pmod{r-1}$?

In other words, is $C^{(-\theta)}$ modular of class II with the integer giving the translation being divisible by $r - 1$?

Remark 22. Recall that we defined the finite Galois character χ_β associated to $C^{(\beta)}$ in Equation (76). When $\beta = -\theta$ standard calculations involved in the $(r - 1)$ -st power reciprocity law for $\mathbb{F}_r[T]$ tell us that the finite part of the conductor of $\chi_{-\theta}$ (in the usual sense of class field theory) is (T) (see, e.g., the Exercises to §12 of [Ros1]). Thus the level in Question 2 should almost certainly be (T) . Predicting the weight is much more difficult as the functor Rf_* on crystals does not preserve purity; thus there is as yet no obvious guess for the weight.

The answers to Question 2 and its refinement (Remark 22), as well as the analogous questions for arbitrary rank 1 Drinfeld modules over $\mathbb{F}_r(\theta)$ uniformizable over $\mathbb{F}_r((1/\theta))$, will be very interesting. Classical theory leads us to expect “good” reasons for the answer whether affirmative or negative.

4.11. Final remarks. There are any number of interesting problems and comments that virtually leap at one from Böckle’s constructions. We mention just a few here.

The first obvious problem is to characterize the “Dirichlet series” that arise from cusp or double-cusp forms; i.e., what special properties does $L(f, s)$ possess besides an analytic continuation (which, after all, exists for *all* L -series of τ -sheaves)? Classically, such information is contained in the functional equations satisfied by the Dirichlet series. Moreover there are some questions [Go6] about the zeroes of the characteristic p functions that seem to be quite natural. Furthermore, the analogy with classical theory would suggest that the answer to these questions would involve some sort of “functional equation” in the characteristic p theory. However, at present, one does not know even how to guess at the formulation of such a functional equation.

Secondly, the example of Δ and $g^r \Delta$ as well as the results of Böckle mentioned in Remark 17 shows that the relationship between the q -expansion of an eigenform and its Hecke eigenvalues is *very* different from that known classically. In fact, one does not yet have formulae which allow one to characterize the q -expansion coefficients from the Hecke eigenvalues. An obvious problem is to find additional structure that allows one to distinguish the different cusp forms which have the same L -function (or, even, the same up to translation). As of now there is no obvious guess here also.

One would also like an explicit basis of eigenforms for the complement of the space of double cusp forms in the space of cusp forms. There are examples in [Boc2] where such forms are given by Poincaré series, but no general construction is now known.

The theory of Drinfeld modules exists in the very general set-up where A can be the ring of functions in any global field k of finite characteristic regular away from a fixed place ∞ . Virtually all of the theory discussed above goes over directly in this general set-up. However, when A is not factorial, the reader should keep in mind that there are NO Drinfeld modules defined over k itself; rather one must work over some Hilbert class field.

Finally the theory of rigid modular forms exists for Drinfeld modules of *all* ranks. In the case $A = \mathbb{F}_r[T]$ there is a compactification for these general moduli schemes of arbitrary rank due to M. Kapranov [Ka1]. In [Go3], it is shown that Kapranov’s compactification, and coherent cohomology, allow one to conclude the finite dimensionality of spaces of modular forms in general. It is very reasonable to expect that Böckle’s techniques will also work here too, thus producing another huge class of rank 1 τ -sheaves which will also need to be understood and somehow classified.

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DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY, 231 W. 18TH AVE., COLUMBUS, OHIO 43210

E-mail address: `goss@math.ohio-state.edu`