January 31, 2000

Professor Richard Pink
Professor Gebhard Boeckle

Dear Richard and Gebhard:

I would like to show you a little result I found regarding the functional equations of classical $L$-series. I have found a simple reformulation of them just in terms of Galois theory. While this does not give any more information about the zeroes of the $L$-series it does bolster the connection between the classical and characteristic $p$ $L$-series. Indeed, as I will explain below, this reformulation allows one to deduce that the Taylor expansion of classical $L$-series (in $t$ where $s = 1/2 + it$) have a simple form which mirrors that predicted in characteristic $p$ by the “Generalized Riemann Hypothesis” and “Generalized Simplicity Conjecture.”

Let me first recall a little bit of the characteristic $p$ theory. For my purposes here, let $A := \mathbb{F}_p[T]$, $k := \mathbb{F}_p((T))$ and $K := \mathbb{F}_p((1/T))$. Let $C_\infty$ be the completion of a fixed algebraic closure of $K$ equipped with the canonical norm. Let $L/k$ be a finite Galois extension with group $G$ and let $\chi : G \to C_\infty^*$ be a character. (N.B. If, for instance, $G$ is a $p$-group then the only such character is the trivial one; however, for arbitrary $G$ there will obviously be many non-trivial $\chi$.)

As usual, put $S_\infty := C_\infty^* \times \mathbb{Z}_p = \text{the domain of definition (at } \infty \text{) of the characteristic } p \text{ } L\text{-series. In particular, (again as usual) for each } y \in \mathbb{Z}_p \text{ we obtain an entire power series } L(\chi, x, y) \text{ in } x^{-1} \text{ with various continuity requirements etc.}

As implied by my preprint on the “GRH” and “GSC,” in finite characteristic, if these conjectures hold for the characteristic $p$ $L$-functions then $L(\chi, x, y)$ may be written as a product $L(\chi, x, y) = p(\chi, x, y)\tilde{L}(\chi, x, y)$. Here $p(\chi, x, y) = \prod (1 - \alpha_y / x)$ is the product over the finitely many “anomalous roots” $\{\alpha_y\}$ expressly allowed by the conjectures, and is thus a polynomial in $x^{-1}$, and $\tilde{L}(\chi, x, y) = \prod (1 - \beta_y / x)$ is the product over the rest of the zeroes $\{\beta_y\}$ which are, by the conjectures, now in $K$ (the analog of $\mathbb{R}$). In particular $L(\chi, x, y)$ is then the product of a polynomial “fudge factor” and a power series with $K$-coefficients.

The “fudge factor” times “real power series” form of $L(\chi, s)$ is also found in classical theory, but where the fudge factor is now simply a constant. So putting finite characteristic aside, let me begin by establishing a simple result on complex power series.

**Definition 1.** Let $p(t) = \sum c_j t^j$ be a non-zero complex power series. We say that $p(t)$ is **almost real** if and only if $p(t) = \alpha h(t)$ where $\alpha \in \mathbb{C}$ and where $h(t)$ is a non-zero power series with real coefficients.

**Proposition 1.** A complex power series $p(t) = \sum c_j t^j$ is almost real if and only if the coefficients $c_j$ satisfy the “Galois functional equation”

$$c_j = wc_j$$

for a fixed complex number $w$ of absolute value 1.
Proof. Suppose that $p(t) = \alpha h(t)$ is almost real, where $\alpha$ is non-zero and $h(t) \in \mathbb{R}[t]$. Put $w := \overline{\alpha}/\alpha$; it is simple to check that with this $w$ the Galois functional equation holds. Conversely, assume the Galois functional equation and let $j_1$ and $j_2$ be two non-negative integers such that $c_{j_1} \neq 0$. Then

$$\frac{c_{j_2}}{c_{j_1}} = (wc_{j_2})/(wc_{j_1}) = c_{j_2}/c_{j_1};$$

thus $c_{j_2}/c_{j_1}$ is real. Now let $j_0$ be the smallest non-negative integer with $c_{j_0} \neq 0$. Then

$$p(t) = c_{j_0} \times t^{j_0}(1 + \sum_{i=1}^{\infty} b_i t^i)$$

with $b_i$ real, and the result is established.

Now let $\chi$ be a non-trivial finite abelian character associated to a Galois extension of number fields $L/k$. Let $\Lambda(\chi, s)$ be the classical (complex) $L$-series and let $\Lambda(\chi, s)$ be the completed $L$-function with the Euler factors at the infinite primes. As is standard $\Lambda(\chi, s)$ is entire and there is a functional equation connecting $\Lambda(\chi, s)$ and $\Lambda(\chi, 1 - s)$. In particular,

$$\Lambda(\chi, 1 - s) = w(\chi)\Lambda(\chi, s),$$

where $w(\chi)$ has absolute value 1.

I follow Riemann and put $\Xi(\chi, t) := \Lambda(\chi, 1/2 + it)$; from the characteristic $p$ point of view this is very useful as now the zeroes in $t$ are expected to be in $\mathbb{R} = \mathbb{Q}_\infty$ (just as in characteristic $p$ most zeroes are “expected” to lie in $K = k_\infty$). The function $\Xi(\chi, t)$ is still entire and we write

$$\Xi(\chi, t) := \sum_{n=0}^{\infty} a_n t^n.$$

Proposition 2. We have

$$\Lambda(\chi, 1 - s) = w(\chi)\Lambda(\chi, s)$$

if and only if the elements $\{a_n\}$ satisfy

$$\bar{\alpha}_n = w(\chi)a_n,$$

for all $n$.

Proof. We know that $\overline{\Lambda(\chi, s)} = \Lambda(\overline{\chi}, \overline{s})$. Thus we see

$$\overline{\Xi(\chi, t)} = \overline{\Lambda(\chi, 1/2 + it)} = \Lambda(\chi, 1/2 - it) = \Xi(\chi, -t).$$

On the other hand, from the Taylor expansion of $\Xi(\chi, t)$ we find clearly

$$\Xi(\chi, t) = \sum_{n=0}^{\infty} \bar{\alpha}_n(t)^n.$$

Substituting $\overline{t}$ for $t$ we deduce

$$\Xi(\chi, -t) = \Lambda(\chi, 1/2 - it) = \sum_{n=0}^{\infty} \bar{\alpha}_n t^n.$$
But the functional equation immediately gives us
\[ \Xi(\chi, -t) = \Lambda(\chi, 1/2 - it) \]
\[ = \Lambda(\chi, 1 - (1/2 + it)) \]
\[ = w(\chi) \Lambda(\chi, 1/2 + it) \]
\[ = w(\chi) \Xi(\chi, t) = w(\chi) \sum_{n=0}^{\infty} a_n t^n. \]

The only if part follows immediately. The if part follows since these calculations are reversible.

**Corollary 1.** The existence of a classical style functional equation for \( \Lambda(\chi, s) \) is equivalent to having the Taylor expansion of \( \Lambda(\chi, 1/2 + it) \) at the origin \( t = 0 \) being almost real.

**Proof.** This follows directly from Proposition 1.

The corollary is the result mentioned at the beginning of this letter. Notice also that if \( \chi = \chi \) is a real character then one can establish some vanishing results for \( \{a_n\} \). I looked in some standard texts and did not find it anywhere (I thought e.g., it might be a problem in Whittaker and Watson...).

I originally started to think about these things because the analogy with the characteristic \( p \) theory also leads one to imagine that \( \Lambda(\chi, 1/2 + it) \) factors over its zeroes. (I know that this is essentially true for the Riemann zeta function but do not know how general it actually holds.) Such a factorization morally gives
\[ \Lambda(\chi, 1/2 + it) = ct^m \prod_{\rho} (1 - t/\rho) \]
where \( \rho \) runs over the non-zero zeroes. If the GRH is true then \( \{\rho\} \subset \mathbb{R} \); in any case, the functional equation implies that both \( \rho \) and \( \overline{\rho} \) are zeroes. Thus one would suspect Corollary 1. Still I was surprised that it did work out so easily without worrying about factorization.

With best wishes,

David