A RIEMANN HYPOTHESIS FOR CHARACTERISTIC p L-FUNCTIONS

DAVID GOSS

ABSTRACT. We propose analogs of the classical Generalized Riemann Hypothesis and the Generalized Simplicity Conjecture for the characteristic p L-series associated to function fields over a finite field. These analogs are based on the use of absolute values. Further we use absolute values to give similar reformulations of the classical conjectures (with, perhaps, finitely many exceptional zeroes). We show how both sets of conjectures behave in remarkably similar ways.

1. Introduction

The arithmetic of function fields attempts to create a model of classical arithmetic using Drinfeld modules and related constructions such as shtuka, $\mathbf{A}$-modules, τ-sheaves, etc. Let $\mathbf{k}$ be one such function field over a finite field $\mathbb{F}_r$ and let ∞ be a fixed place of $\mathbf{k}$ with completion $\mathbf{K} = \mathbf{k}_\infty$. It is well known that the algebraic closure of $\mathbf{K}$ is infinite dimensional over $\mathbf{K}$ and that, moreover, $\mathbf{K}$ may have infinitely many distinct extensions of a bounded degree. Thus function fields are inherently "looser" than number fields where the fact that $[\mathbb{C}\colon \mathbb{R}] = 2$ offers considerable restraint. As such, objects of classical number theory may have many different function field analogs.

Classifying the different aspects of function field arithmetic is a lengthy job. One finds for instance that there are two distinct analogs of classical L-series. One analog comes from the L-series of Drinfeld modules etc., and is the one of interest here. The other analog arises from the L-series of modular forms on the Drinfeld rigid spaces, (see, for instance, [Go2]). It is a very curious phenomenon that the first analog possesses no obvious functional equation whereas the second one indeed has a functional equation very similar to the classical versions. It is even more curious that the L-series of Drinfeld modules and the like seem to possess the correct analogs of the *Generalized Riemann Hypothesis* and the *Generalized Simplicity Conjecture* (see Conjecture 3 below). It is our purpose here to define these characteristic p conjectures and show just how close they are to their classical brethren.

That there might be a good Riemann Hypothesis in the characteristic p theory first arose from the ground-breaking work [W1] of Daqing Wan. In this paper, and in the simplest possible case, Wan computed the valuations of zeroes of an analog of the Riemann zeta function via the technique of Newton polygons. This immediately implied that these zeroes are all simple and lie on a "line." However, because of the great size of the function field arena (as mentioned above), it was not immediately clear how to then go on to state a Riemann Hypothesis in the function field case which worked for all places of $\mathbf{k}$ (as explained in this paper) and all functions arising from arithmetic.

Recently, the L-functions of function field arithmetic were analytically continued in total generality (as general as one could imagine from the analogy with classical motives). This is

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due to the forthcoming work of G. Boeckle and R. Pink [BP1] where an appropriate cohomology theory is created. This theory, combined with certain estimates provided by Boeckle, on the one hand, and Y. Amice [Am1], on the other, actually allows one to analytically continue the non-Archimedean measures associated to the L-series; the analytic continuation of the L-series themselves then arises as a corollary. In particular we deduce that *all* such L-functions, viewed at all places of $\mathbf{k}$, have remarkably similar analytic properties (for instance, their expansion coefficients all decay exponentially — see the discussion after Remarks 2).

Motivated by these results, we re-examined the work of Wan and those who came after him ([DV1], [Sh1]). In seeking to rephrase Wan's results in such a way as to avoid having to compute Newton polygons (which looks to be exceedingly complicated in general), we arrived at a statement involving only the use of *absolute values* of zeroes (as opposed to the absolute values of expansion coefficients which are used in Newton polygons). The use of absolute values in phrasing such a possible Riemann Hypothesis seems to be very fruitful. For instance, it offers a unification with local Riemann Hypotheses (which are always formulated in terms of absolute values of the zeroes). More strikingly, it also suggests a suitable reformulation of the classical GRH (with, perhaps, finitely many exceptional zeroes) as well as the simplicity conjectures (see Conjecture 6 and Proposition 7). Finally, as explained after Remarks 4, the conjectures presented here go a very long way towards explaining the lack of a classical style functional equation associated to the L-series of Drinfeld modules etc.

Upon examining these new "absolute value conjectures" in both theories, one finds that they behave remarkably alike. So much so that they seem to almost be two instances of one Platonic mold. This certainly adds to our sense that the function field statements may indeed be the correct ones. Moreover, because the algebraic closure of $\mathbf{K}$ is so vast and contains inseparable extensions, the function field theory offers insight into these statements not available in number fields. For instance, due to the existence of inseparable extensions, one needs both the function field analog of the GRH *and* the function field analog of the Simplicity Conjecture (Conjecture 7) to truly deduce that the zeroes (or almost all of them) lie on a line! Because $\mathbb{C}$ is obviously separable over $\mathbb{R}$ one only needs the GRH, (reformulated as Conjecture 6) classically.

It should be noted that we do not yet know the implications of our function field conjectures. However, it is our hope that such information will be found as a byproduct of the search for a proof of them. Moreover, because of the strong formal analogies between the number field and function field conjectures as presented here, we believe that insight obtained for one type of global field may lead to insight for the other.

2. Basic Statements

In this section we present the statements of the characteristic p Generalized Riemann Hypothesis as well as the characteristic p Generalized Simplicity Conjecture. We will begin by recalling the classical versions of these conjectures. We will work with classical abelian characters over a number field $\mathfrak{L}$. However, the reader will easily see the simple modifications necessary to handle other classical L-series. Moreover, of course, if χ is one such abelian character, then the analytic continuation of $L(\chi, s)$ has long been known. Let $\Lambda(\chi, s)$ be the completed L-series (so $\Lambda(\chi, s)$ contains the Γ-factors at the infinite primes). One knows that there is a functional equation relating $\Lambda(\chi, s)$ and $\Lambda(\bar{\chi}, 1-s)$.

Following Riemann's original paper, we define

$$\Xi(\chi, t) := \Lambda(\chi, 1/2 + it)\,.$$

It is very easy to see that the functional equation for Λ translates into one for Ξ relating $\Xi(\chi, t)$ and $\Xi(\bar{\chi}, -t)$.

Conjecture 1. The zeroes of $\Lambda(\chi, s)$ lie on the line $\{s = 1/2 + it \mid t \in \mathbb{R}\}$.

Conjecture 1 is obviously the *Generalized Riemann Hypothesis* for abelian L-series. It may clearly be reformulated (as Riemann did) in the following way.

Conjecture 2. The zeroes of $\Xi(\chi, t)$ are real.

All known zeroes of the Riemann zeta function, $\zeta(s) = L(\chi_0, s)$ where χ_0 is the trivial character and $\mathfrak{L} = \mathbb{Q}$, have been found to be simple. This is codified in the following conjecture which the author learned from J.-P. Serre. In it we *explicitly* assume that our number field $\mathfrak{L}$ is now the base field $\mathbb{Q}$.

Conjecture 3. 1. The zeroes of $\Lambda(\chi, s)$ should be simple.
2. $s = 1/2$ should be a zero of $\Lambda(\chi, s)$ only if χ is real and the functional equation of $\Lambda(\chi, s)$ has a minus sign.
3. If χ and χ' are distinct, then the zeroes of $\Lambda(\chi, s)$ not equal to $s = 1/2$ should be distinct from those of $\Lambda(\chi', s)$.

We shall call Conjecture 3 the *Generalized Simplicity Conjecture* ("GSC"). It is in fact expected to hold for general (not necessarily abelian) L-series (see e.g., Conjecture 8.24.1 of [Go1]).

We turn next to the function field versions of the above conjectures. As this theory is certainly not as well known as its classical counterparts, we begin by reviewing it. For more the reader can consult [Go1]; for a short and very readable introduction to Drinfeld modules the reader may consult [H1]. Let $\mathcal{X}$ be a smooth projective geometrically connected curve over the finite field $\mathbb{F}_r$, $r = p^m$, and let $\infty \in \mathcal{X}$ be a fixed closed point. Let $\mathbf{A}$ be the affine ring of $\mathcal{X} \backslash \infty$; so $\mathbf{A}$ is a Dedekind domain with finite class group and unit group equal to ${\mathbb{F}_r}^*$. We let $\mathbf{k}$ be the function field of $\mathcal{X}$ (= the quotient field of $\mathbf{A}$) and we let $\mathbf{K} = \mathbf{k}_\infty$. Finally we let $\mathbf{C}_\infty$ be the completion of a fixed algebraic closure of $\mathbf{K}$ equipped with its canonical topology.

A particular instance is $\mathcal{X} = \mathbb{P}^1_{\mathbb{F}_r}$ and $\mathbf{A} = \mathbb{F}_r[T]$, etc. The reader may wish to first read this paper with only this instance in mind as the jump to general $\mathbf{A}$ is largely technical.

The "standard analogy" is that $\mathbf{A}$ plays the role classically played by $\mathbb{Z}$, $\mathbf{k}$ the role of $\mathbb{Q}$, $\mathbf{K} = \mathbf{k}_\infty$ the role of $\mathbb{R}$ and $\mathbf{C}_\infty$ the role of $\mathbb{C}$. We will have more to say on this below (see Remark 1).

As mentioned above, the basic "arithmetic" objects in the characteristic p theory are Drinfeld modules (originally presented in [Dr1]) and their various generalizations such as $\mathbf{A}$-modules. These arise in the following manner. Let L be some field over $\mathbb{F}_r$ and let $\mathbb{G}_a^n$ be the n-th Cartesian product of $\mathbb{G}_a$ over L. Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{G}_a^n$ and set

$$\tau(x) := \begin{pmatrix} x_1^r \\ \vdots \\ x_n^r \end{pmatrix};$$

one forms τ^i by composition for all non-negative integers i. Let $P\colon \mathbb{G}_a^n \to \mathbb{G}_a^n$ be a morphism of algebraic groups over L that is $\mathbb{F}_r$-linear on the geometric points. It is elementary to see that there is a "polynomial" $P(\tau) := \sum_{i=0}^m a_i\tau^i$, where the a_i are $n \times n$ matrices with coefficients in L, such that

$$P(x) = \sum_{i=0}^m a_i\tau^i(x) = P(\tau)(x)\,.$$

We denote the $\mathbb{F}_r$-algebra (under composition) of such maps by $\mathrm{End}_L^r(\mathbb{G}_a^n)$. The mapping which takes $P(\tau)$ to a_0 is readily checked to be a map of $\mathbb{F}_r$-algebras; we denote a_0 by P' or $P'(\tau)$.

Suppose now that L is also an "$\mathbf{A}$-field;" that is, there is an $\mathbb{F}_r$-algebra map $\imath\colon \mathbf{A} \to L$. Following Anderson [An1] an $\mathbf{A}$-*module*, ϕ, is then an $\mathbb{F}_r$-algebra map from $\mathbf{A}$ to $\mathrm{End}_L^r(\mathbb{G}_a^n)$, $a \in \mathbf{A} \mapsto \phi_a$, subject to the condition that $N_a := \phi'_a - \imath(a) \cdot I_n$ is nilpotent. The *dimension* of ϕ is n. Notice that if $n = 1$, then $N_a = 0$ for all $a \in \mathbf{A}$. A *Drinfeld module* is a 1-dimensional $\mathbf{A}$-module such that, for some $a \in \mathbf{A}$, $\phi_a(\tau) - \imath(a)\tau^0$ is non-trivial. (In other words, the "trivial" action $\psi_a(\tau) := \imath(a)\tau^0$ is not a Drinfeld module.)

Example 1. Let $\mathbf{A} = \mathbb{F}_r[T]$, $\mathbf{k} = \mathbb{F}_r(T)$, etc. The *Carlitz module* C (originally presented in 1935 in [C1]) is the Drinfeld module defined over $\mathbf{k}$ with

$$C_T(\tau) := T\tau^0 + \tau\,.$$

C is the simplest, and most basic, of all Drinfeld modules.

Let $M := \mathrm{Hom}_L^r(\mathbb{G}_a^n, \mathbb{G}_a)$ the $\mathbb{F}_r$-linear morphisms of affine algebraic groups defined over L. The $\mathbf{A}$-*motive* [An1] associated to an $\mathbf{A}$-module ϕ is $M_\phi = M$ viewed as an $L \otimes_{\mathbb{F}_r} \mathbf{A}$-module as follows: Let $l \in L$, $a \in \mathbf{A}$, and $m(x) \in M$. Then we set

$$(l \otimes a) \cdot m(x) = l \cdot m(\phi_a(\tau)(x)) \in \mathbb{G}_a\,.$$

Passing from the module to the motive is extremely useful in the theory, see Section 5 of [Go1]. The τ-*sheaves* are natural generalizations of $\mathbf{A}$-motives.

Remark 1. Suppose that L is a field over of $\mathbf{k}$ and let $T \in \mathbf{A}$ be non-constant. Then T plays two roles in the theory ("two T's"): In L the element T is a scalar whereas in $\mathbf{A}$ one knows that T is an operator (via some $\mathbf{A}$-module etc.). This is completely similar to the fact that an integer n plays two similar roles for elliptic curves over $\mathbb{Q}$. The "standard caveat" is that it is obviously impossible to separate the two distinct actions of an integer n via a module over "$\mathbb{Z} \otimes \mathbb{Z}$" as in the function field theory.

It is important to keep the two actions of an element T separate. To do so we follow [An1] and use a notational device: Let A, k, etc., be another copy of the basic algebras constructed above. There is an obvious isomorphism θ from "bold" to "non-bold" making the non-bold rings $\mathbf{A}$-algebras. When $\mathbf{A} = \mathbb{F}_r[T]$, it is customary to set $\theta := \theta(T) \in A$. The elements of the bold algebras will be the operators while the elements of the non-bold algebras will be the scalars. In this set-up, $\mathbf{A}$-modules etc., will always be defined over the non-bold scalars.

Example 2. Let C be the Carlitz module as in Example 1. Using the notation just introduced, the Carlitz module is the Drinfeld module defined over $\mathbb{F}_r(\theta)$ with

$$C_T(\tau) := \theta\tau^0 + \tau\,.$$

In particular, if $T \in \mathbf{A}$ and $x \in \mathbb{G}_a^n$, then the action $T \cdot x$ is now unambiguously given and one may suppress the use of ϕ etc. As a bonus, the "bold, non-bold" notation also provides an extremely useful way of classifying the constructions of function field arithmetic. For instance, the periods of Drinfeld modules or $\mathbf{A}$-modules are scalars; thus all constructions of Γ-functions take values in non-bold algebras, or are "scalar-valued." Similarly, the L-series of modular forms are scalar-valued. On the other hand, the L-series of Drinfeld modules, etc., are derived from Tate-modules exactly as in classical arithmetic. As the Tate modules are, by definition, modules over operator algebras, we see that these L-series are "operator-valued." We now describe these operator-valued L-series in some detail.

In all that follows, the reader should keep in mind the the L-series of an abelian variety over a number field or $L(\chi, s)$ mentioned above. The following definitions are given with the goal of making the characteristic p L-series as close as possible to these classical L-series.

We begin by explaining what is meant by a "Dirichlet series" in the characteristic p context. Let $S_\infty := \mathbf{C}_\infty \times \mathbb{Z}_p$; we view S_∞ as a topological abelian group whose operation is written additively. The space S_∞ will supply the "s" in "I^s." Let $\mathbb{F}_\infty \subset \mathbf{K}$ be the constant field at ∞. Let $\operatorname{sgn}\colon \mathbf{K}^* \to \mathbb{F}_\infty^*$ be a sign function; that is a morphism which is the identity on $\mathbb{F}_\infty^*$. An element $z \in \mathbf{K}^*$ is said to be *positive* (or *monic*) if and only $\operatorname{sgn}(z) = 1$; it is very elementary to see that the group $\mathcal{P}_+$ of principal and positively generated $\mathbf{A}$-fractional ideals of $\mathbf{k}$ is of finite index in the total group $\mathcal{I}$ of $\mathbf{A}$-fractional ideals. Let $\pi \in \mathbf{K}$ be a positive uniformizer. Let d_∞ be the degree of ∞ over $\mathbb{F}_r$ and set $d(z) := -d_\infty v_\infty(z)$ where $v_\infty(\pi) = 1$. If $a \in \mathbf{A}$ is positive, then $d(a)$ is the degree of the finite part of the divisor of a. We then put

$$\langle z\rangle = \langle z\rangle_\pi := \pi^{-v_\infty(z)} \cdot z\,.$$

Notice that $\langle z\rangle \in U_1$ where $U_1 \subset \mathbf{K}^*$ is the subgroup of 1-units. It is easy to see, via the binomial theorem, that U_1 is a $\mathbb{Z}_p$-module.

Now let I be any $\mathbf{A}$-fractional ideal and let e be the index of $\mathcal{P}_+$ in $\mathcal{I}$. Thus $I^e = (i)$ for positive $i \in \mathbf{A}$. Let $\mathbf{U}_1 \subset \mathbf{C}_\infty$ be the group of 1-units. Notice that $\mathbf{U}_1$ is a divisible group (in fact, a $\mathbb{Q}_p$-vector space); in particular, $\mathbf{U}_1$ is injective. Thus, if we set

$$\langle I\rangle := \langle i\rangle^{1/e}\,,$$

where we take the solitary root in $\mathbf{U}_1$, we obtain a unique morphism $\langle\ \rangle\colon \mathcal{I} \to \mathbf{U}_1$ extending the morphism defined on $\mathcal{P}_+$ with $\langle (a)\rangle = \langle a\rangle$ for a positive. Finally, if $s = (x, y) \in S_\infty$ then we set

$$I^s := x^{\deg I} \cdot \langle I\rangle^y\,.$$

If, for instance, $I = (i)$, where i is positive, then

$$(i)^s = x^{d(i)}\langle i\rangle^y\,.$$

Since U_1 is a $\mathbb{Z}_p$-module, the values $\langle I\rangle$, for I an $\mathbf{A}$-fractional ideal, are totally inseparable over $\mathbf{K}$. They generate a finite, totally inseparable, extension denoted by $\mathbf{K_V}$; we call $\mathbf{K_V}$ the *local value field*. Of course, $\mathbf{K_V}=\mathbf{K}$ when $\mathbf{A}$ has class number 1.

A Dirichlet series $L(s)$ is then a sum $L(s)=\sum_I c(I)I^{-s}$ where we sum over the ideals of $\mathbf{A}$ and where the elements $c(I)$ lie in a finite extension of $\mathbf{K}$ (in practice, actually a finite extension of $\mathbf{k}$). Notice that if we set $y=0$ and $u=x^{-1}$ in I^s, we obtain a characteristic p version of the classical power series arising from Artin-Weil L-series of function fields. Such characteristic p Dirichlet series may (and usually do) arise from Euler products over the finite primes in an obvious sense. Let $s=(x,y)$ be as above. By definition we have

$$L(s)=\sum_i c(I)x^{-\deg I}\langle I\rangle^{-y}\,.$$

Thus for each fixed $y\in\mathbb{Z}_p$ we obtain a formal power series $L(x,y)$ in x^{-1} with coefficients in a finite extension of $\mathbf{K_V}$.

As mentioned above, it is now known that all characteristic p L-series arising from arithmetic have analytic continuations to "essentially algebraic entire functions" on S_∞ (see Subsection 8.5 of [Go1]). That $L(s)$ is "entire" on S_∞ means, in practice, that for fixed $y\in\mathbb{Z}_p$ every power series $L(x,y)$ is entire (i.e., converges for all values of x^{-1}) *and* that the zeroes of this 1-parameter family of entire power series flow continuously.

The "essential algebraicity" rests on the following observation. Let π_* be a fixed d_∞-th root of π and let $a\in\mathbf{A}$ be positive with $d(a)=d$. Let j be a non-negative integer. Notice that, by definition,

$$(a)^{-(x\pi_*^j,-j)}=x^{-d}a^j\,;$$

that is, we have removed π from the definition. Clearly the function $x\mapsto(a)^{-(x\pi_*^j,-j)}$ is a polynomial in x^{-1} with algebraic coefficients. The essential algebraicity of $L(s)$ means that the same thing happens with $L(s)$; i.e., the functions

$$z_L(x,-j):=L(x\pi_*^j,-j)$$

are polynomials (called the "special polynomials") with algebraic coefficients.

Note that simple p-adic continuity implies that $L(x,y)$ is the interpolation of $L(x,-j)=z_L(x\pi_*^{-j},-j)$ to arbitrary $y\in\mathbb{Z}_p$.

It is now a straightforward exercise to use the algebraic elements $I^{-(\pi_*,-1)}$, for ideals I, to define v-adic analogs of the local value field $\mathbf{K_V}$ for finite primes v. We denote this v-adic local value field by $\mathbf{k}_{v,\mathbf{V}}$.

Using the above notational conventions, we let $\mathfrak{k}$ be a finite field extension of k contained in our complete, algebraically closed extension C_∞ of $K=k_\infty$. Let ϕ be a Drinfeld module (or an $\mathbf{A}$-module) that is defined over $\mathfrak{k}$. Let $\mathfrak{p}$ be a finite prime of k and let $\mathfrak{P}$ be a prime of $\mathfrak{k}$ lying over it with associated finite field $\mathbb{F}_\mathfrak{P}$. Almost all such primes $\mathfrak{P}$ are "good" for ϕ in that reducing the coefficients of ϕ modulo $\mathfrak{P}$ leads to a Drinfeld module $\phi^{(\mathfrak{P})}$ (or $\mathbf{A}$-module) over the finite field $\mathbb{F}_\mathfrak{P}$ of the same "rank" as ϕ (i.e., the $\mathbf{A}$-ranks of the various torsion modules prime to $\mathfrak{P}$ are the same). In complete analogy with abelian varieties, there is a "Frobenius endomorphism," $Fr_\mathfrak{P}$, of $\phi^{(\mathfrak{P})}$ and we set

$$f_\mathfrak{P}(t):=\det(1-tFr_\mathfrak{P}\mid T_v(\phi^{(\mathfrak{P})})),\,,$$

where $T_v(\phi^{(\mathfrak{P})})$ is the Tate module of $\phi^{(\mathfrak{P})}$ at a prime $v\neq\mathfrak{p}$ of $\mathbf{A}$. Again in analogy with abelian varieties, the polynomial $f_\mathfrak{P}(t)$ has coefficients in $\mathbf{A}$ which are independent of v

and has zeroes which satisfy the local Riemann hypothesis (established by Drinfeld); see Subsection 4.12 of [Go1]. Let $n\mathfrak{P} \subseteq \mathbf{A}$ be the ideal norm of $\mathfrak{P}$; thus. finally, one sets

$$L(\phi, s) := \prod_{\mathfrak{P}\ \text{good}} f_{\mathfrak{P}}(n\mathfrak{P}^{-s})^{-1}\,.$$

(See Part 3 of Remarks 1 for Euler factors at the finitely many "bad" primes.)

As mentioned above, due to very recent work of G. Boeckle, R. Pink, and the author, it is known that all $L(\phi, s)$ (and all associated partial L-series) have an analytic continuation at ∞ to an essentially algebraic entire function. This analytic continuation allows us to work in almost total generality in the function field theory in obvious contrast to our need for abelian L-series classically.

Remark 2. In the basic case of $\mathbf{A} = \mathbb{F}_r[T]$, the analytic continuation of these L-series was first obtained by Taguchi and Wan ([TW1], [TW2]). This was obtained by expressing the L-series as a Fredholm determinant in the manner of classical Dwork theory. It is not known at the present time whether one can use these results in the manner of [KS1]. Wan's original paper on the characteristic p Riemann hypothesis [W1], which uses elementary methods, arose originally as a response to a query from the present author as to whether the methods of [TW1], [TW2] could be used to obtain the exponential decay of certain L-series coefficients which can be computed in closed form (see Example 8.24.2 of [Go1]). In fact, as will be seen below, this exponential decay occurs in complete generality.

In classical theory, one constructs p-adic L-series out of the special values of complex L-series. We will briefly describe here how a very analogous construction works for the special polynomials mentioned above; the reader may choose to ignore this construction during a first reading. In any case, let v be a finite prime of $\mathbf{A}$ and set $S_v := \mathbb{Z}/(r^{\deg v} - 1) \times \mathbb{Z}_p$. Let $(x, y) \in S_\infty$. Since the Euler factors in $L(\phi, s)$ have coefficients in $\mathbf{A}$, it makes sense to also study them v-adically. Thus, one defines the v-adic L-series associated to ϕ etc., by using an Euler product over the finite primes of $\mathfrak{k}$ **not** lying over v (in the obvious sense using the map θ). One obtains in this fashion an essentially algebraic entire function $L_v(x_v, s_v)$ on the space $\mathbf{C}_v^* \times S_v$, where $\mathbf{C}_v$ is the completion of a fixed algebraic closure of $\mathbf{k}_v$ equipped with its canonical topology (see Subsection 8.3 of [Go1]). However, it is also easy to see that this function is just the v-adic interpolation of the special polynomials z_L mentioned above. Thus we see how very close the theory is at all the places of $\mathbf{k}$ and the central role played by the special polynomials.

Example 3. Let $\mathbf{A} = \mathbb{F}_r[T]$, $k = \mathbb{F}_r(\theta)$. Note that ideals are in one to one correspondence with monic polynomials. Thus we set

$$\zeta_A(s) := \sum_{n \in \mathbf{A}\ \text{monic}} n^{-s} = \prod_{f\ \text{monic prime}} (1 - f^{-s})^{-1}\,,$$

If we expand out n^{-s}, we find

$$\zeta_A(x, y) = \sum_{n\ \text{monic}} x^{-\deg n} \langle n \rangle^{-y} = \sum_{j=0}^{\infty} x^{-j} \left(\sum_{\deg n = j} \langle n \rangle^{-y} \right)\,.$$

The function $\zeta_A(s)$ is obviously **the** analog of the Riemann zeta function for $\mathbf{A}$. It interpolates v-adically to

$$\zeta_{A,v}(x_v, s_v) = \prod_{f \neq v \text{ monic prime}} (1 - x_v^{-\deg f} f^{-s_v})^{-1} = \sum_{j=0}^{\infty} x_v^{-j} \left(\sum_{\substack{n \text{ prime to } v \\ \deg n = j}} n^{-s_v} \right).$$

$\zeta_A(s)$ has special values and "trivial zeroes" in line with what is known for the Riemann zeta function (see Subsections 8.12, 8.13 and 8.18 of [Go1]).

Example 4. Let $\mathbf{A} = \mathbb{F}_r[T]$ and let C be the Carlitz module over k as in Example 2. The Tate modules of the Carlitz module have rank 1 and are very similar to the Tate modules of roots of unity. Thus a simple calculation gives

$$L(C, s) = \zeta_A(s-1)\,.$$

There is an obvious v-adic version of this result.

Let $k^{\rm sep} \subset C_\infty$ be the separable closure of k and let $G := \mathrm{Gal}(k^{\rm sep}/k)$. Let $\rho\colon G \to GL_m(\mathbf{C}_\infty)$ be a representation of Galois type (i.e., factoring through a finite extension of k). A completely similar theory holds for the L-series $L(\rho, s)$ formed in the obvious fashion.

Remarks 1. 1. In practice, the essential algebraicity at ∞ of an L-series $L(s)$ given above actually arises as a consequence of its being entire. Indeed, by construction $z_L(x-j)$ is certainly an entire power series in x^{-1}. Now in most cases, such as the L-series of a Drinfeld module, one concludes that $z_L(x, -j)$ also has $\mathbf{A}$-coefficients. The only way this can happen is that almost all the coefficients are 0; i.e., $z_L(x, -j)$ is a polynomial. The other cases usually factor a function of this sort and so their essential algebraicity can also be deduced.
2. The special polynomials $z_L(x, -j)$ lie at the heart of the analytic continuation given by the author, G. Boeckle and R. Pink. Indeed, Boeckle and Pink [BP1] give a cohomological expression for these polynomials which leads to an estimate of the degree (in x^{-1}) of $z_L(x, -j)$. Moreover, Boeckle has used this estimate to establish that this degree grows *logarithmically* with j. It is then relatively easy to translate this into a logarithmic growth statement for the measures associated to $L(s)$ (at all places of $\mathbf{k}$). As explained in [Ya1], this is enough to establish that the integrals for the functions $L(s)$ (again, at all places of $\mathbf{k}$) converge everywhere.
3. It is obviously desirable to have Euler factors at all the finite primes (as opposed to just the "good" primes). It seems likely that the work of Boeckle and Pink will be able to provide these in line with that is known for abelian varieties classically. In any case, it is easy to see that there are many examples of Drinfeld modules with no bad primes, even when they are defined over k (unlike the situation with abelian varieties).

The proof of the analytic continuation of the L-series $L(s)$ at the various places of $\mathbf{k}$, mentioned in Part 2 of Remarks 1, works quite similarly whether the place is ∞ or a finite prime. This reinforces previous experience that the theories at ∞ and at a finite prime v are substantially the same. For instance, as mentioned above they are all interpolations of the special polynomials $z_L(x, -j)$.

Therefore it is somewhat reasonable to expect that an analog of Conjecture 1 should work for *all* the interpolations of $L(s)$ at all the places of $\mathbf{k}$. As we have mentioned when $r = p$, Daqing Wan calculated in [W1] the Newton polygons at ∞ of the power-series $\zeta_A(x, y)$ (as in Example 3) for all $y \in \mathbb{Z}_p$. Wan found that these polygons were always simple implying that the zeroes (in x^{-1}) of $\zeta_A(x,y)$ are always in $\mathbf{K} = \mathbf{k}_\infty$ and are themselves simple. As $\mathbb{R} = \mathbb{Q}_\infty$, this is obviously in keeping with the classical Conjectures 2 and 3.

Wan's proof was simplified by D. Thakur and J. Diaz-Vargas in [DV1]. Finally, based on some work of B. Poonen for $r = 4$, the general case (all r) was established by J. Sheats in [Sh1]. Still, it was not clear how to proceed to a "good" version of the Generalized Riemann Hypothesis, etc., in the function field case. For instance, there are examples of zeroes of $\zeta_A(s)$ (in the obvious definition) which do not belong to $\mathbf{K_V}$ for some non-polynomial $\mathbf{A}$. Moreover, Daqing Wan had mentioned to the author that the calculation at ∞ for $\zeta_{\mathbb{F}_r[\theta]}(s)$, could easily be modified to establish the *same* result v-adically when $\deg v = 1$ (or at least for those s_v in an open subset of S_v for such v; see Proposition 9).

It is well-known (and important for what follows in the next section) that non-Archimedean analysis is quite algebraic when compared to complex analysis. This is brought out by the result that *all* entire functions are determined up to a constant by their zeroes (with multiplicity of course) and that these zeroes are algebraic over any complete field containing the coefficients. In particular it makes sense to analyze such functions by studying the extension fields obtained by adjoining the zeroes. Let $L(s)$, $s = (x,y) \in S_\infty$, be one of the characteristic p L-series arising from arithmetic (via a Drinfeld module or $\mathbf{A}$-module) and let $\mathbf{K}_L$ be the finite extension of $\mathbf{K_V}$ obtained by adjoining the coefficients of $L(s)$. (In practice, $\mathbf{K}_L$ will usually be $\mathbf{K_V}$ itself, some finite constant field extension of $\mathbf{K_V}$ obtained by adjoining the values of certain characters, or some finite extension obtained by adjoining "complex multiplications" etc.) Let $\mathbf{K}_L(y)$ be the extension of $\mathbf{K}_L$ obtained by adjoining the roots of $L(x,y)$ for each y. Thus, a-priori, $\mathbf{K}_L(y)$ is merely some algebraic extension of $\mathbf{K}_L$.

Now in any extension of local, or global, function fields, the most important part is the maximal separable subfield. Indeed it is well known that the total extension is uniquely determined by its degree over the maximal separable subfield (see, eg., 8.2.12 of [Go1]). Thus a first function field version of Conjecture 1 (or Conjecture 2) is the following.

Conjecture 4. The maximal separable (over $\mathbf{K}_L$) sub-extension of $\mathbf{K}_L(y)$ is *finite* over $\mathbf{K}$ for all $y \in \mathbb{Z}_p$. Similarly, the maximal separable subfields of the extensions obtained by adjoining the v-adic zeroes should also be finite for each $s_v \in S_v$.

Remarks 2. 1. Notice that the conjecture is vacuously true for $y = -j$ since, obviously, the special polynomials have only finitely many zeroes to begin with.
2. At ∞ one can show that the maximal separable (over $\mathbf{K}$) subfield of $\mathbf{K}_L(y)$ is independent of the choice of sign function or uniformizer. Thus it depends only on y and the underlying "motive" used to construct the L-series. (The formal module of such a motive may always be extended to separable elements. We believe that these extended modules may be of importance in studying the zeroes of characteristic p L-series.)

Conjecture 4 is clearly in line with the few known results about such extensions (e.g., the results of Wan, Diaz-Vargas etc.). However it suffers from the drawback of not being a statement directly about the zeroes themselves. On the other hand, the analytic continuation of these functions using integral calculus (mentioned above) is based on the a-priori estimates of Y. Amice in non-Archimedean functional analysis; these estimates are nicely reviewed in [Ya1]. In particular, the logarithmic growth of the degrees of the special polynomials translates into the exponential decay of the coefficients of the L-series when expressed as power-series as above. (In the particular case where $\mathbf{A} = \mathbb{F}_r[T]$ such exponential decay for L-series of Drinfeld modules, etc., was originally mentioned to the author by D. Wan.)

Remark 3. The exponential decay of the coefficients has a number of consequences. For instance, let $a \in \mathbf{A}$ be non-constant. One can construct the "Carlitz polynomials" (as in [Ca1] or Subsection 8.22 of [Go1]) for the ring $\mathbb{F}_r[a] \simeq \mathbb{F}_r[T] \subseteq \mathbf{A}$ by simple substitution of a for T. Then the above mentioned exponential decay and the main result of [Ca1] imply that all L-series arising from arithmetic have convergent expansions in these Carlitz polynomials at ∞. Such expansions are somewhat similar to Fourier expansions classically.

The exponential decay also suggests refining Conjecture 4 in the following fashion. Let us write

$$L(x,y) = \prod_i \left(1 - \beta_i^{(y)}/x\right) \,,$$

where the elements $\{\beta_i^{(y)}\}$ are algebraic over $\mathbf{K}_L$ and are the zeroes of $L(x,y)$. Clearly we are only interested in non-zero $\beta_i^{(y)}$ and we call $\{\lambda_i^{(y)} := 1/\beta_i^{(y)}\}$ the "reciprocal zeroes" of $L(s)$; standard non-Archimedean function theory tells us that the reciprocal zeroes are discrete in the sense that their absolute values tend to infinity. There is an obvious v-adic version of this product. We are thus lead to the following refinement of Conjecture 4 which focuses directly on the zeroes.

Conjecture 5. There exists a positive real number $b = b(y)$ such that if $\delta \geq b$, then there is at most **one** reciprocal zero (taken without multiplicities) of $L(x,y)$ of absolute value δ. (This conjecture, again, is meant to apply to all the interpolations of $L(s)$ at all the places of $\mathbf{k}$.)

In other words, for fixed $y \in \mathbb{Z}_p$ almost all zeroes are uniquely determined by their absolute values etc.

Remarks 3. 1. Conjecture 5 is also vacuously true for $y = -j$. Perhaps a more refined version might be non-trivial in this case also.
2. Conjecture 5 appears to us to be the correct version of the Generalized Riemann Hypothesis in the characteristic p setting. However, it should probably still be viewed as a working version since we do not yet even know what the implications of this conjecture are (unlike, obviously, classical theory).
3. Let $f(t)$ be an entire power series with coefficients in a finite extension of $\mathbb{Q}_p$. As there are only finitely many extensions of $\mathbb{Q}_p$ of bounded degree, *any* finite bound on the number of zeroes of $f(t)$ of fixed degree will suffice to establish that the zeroes of $f(t)$ generate a finite extension of $\mathbb{Q}_p$. This argument does not work in finite characteristic however. As the reader will see, Conjecture 5 is the best one can hope for in finite characteristic; see Example 6 of our next section.

Our next result establishes that Conjecture 5 implies Conjecture 4. We show it at ∞ with the v-adic result being completely analogous.

Proposition 1. *Assume* Conjecture 5 *and let $b(y)$ be as in its statement. Let $\beta_i^{(y)}$ be a root such that $|\lambda_i^{(y)}| = 1/|\beta_i^{(y)}| > b(y)$. Then $\beta_i^{(y)}$ is totally inseparable over $\mathbf{K}_L$.*

Proof. Let $\overline{\mathbf{K}} \subset \mathbf{C}_\infty$ be the algebraic closure. Let σ be an automorphism of $\overline{\mathbf{K}}$ which fixes $\mathbf{K}_L$. Then $\sigma(\beta_i^{(y)})$ is also a zero of $L(x,y)$ of the *same* absolute value. Thus Conjecture

5 implies that $\sigma(\beta_i^{(y)})$ actually equals $\beta_i^{(y)}$. This is equivalent to the total inseparability of $\beta_i^{(y)}$. □

Thus we only have finitely many zeroes that might contribute separable elements and hence the maximal separable subfield is obviously finite. This is Conjecture 4. Ultimately, one would like an arithmetic characterization of this maximal separable subfield.

It is explicitly allowed in Conjecture 5 that one can throw out finitely many zeroes. This is necessary for the Conjecture to apply at all the places of $\mathbf{k}$. Indeed, when one interpolates v-adically, one removes the Euler-factors lying over v. This process in fact adds finitely many zeroes v-adically which may be arbitrary in behavior, as our next example attests.

Example 5. Let $\zeta_A(s)$ be as in Example 3 which we write as

$$\zeta_A(s) = \sum_{n \text{ monic}} n^{-s}\,.$$

So, for $j \geq 0$ we have

$$z_\zeta(x, -j) = \sum_{j=0}^{\infty} x^{-j} \Big(\sum_{\deg n = j} n^j \Big)\,.$$

As mentioned, these power-series are actually polynomials in x^{-1}. Let v be a finite prime associated to an irreducible monic f of degree d. Let $s_v \in S_v$. To interpolate v-adically, one takes a sequence e_t of natural numbers with the property that e_t goes to ∞ in the Archimedean absolute value but to s_v in the p-adic absolute value. In particular, this process will eliminate all monic n which are divisible by f. Thus $z_\zeta(x, -j)$ is transformed v-adically into

$$(1 - x_v^{-d} f^j) z_\zeta(x_v, -j)\,.$$

Therefore we have added the zeroes of $1 - x_v^{-d} f^j$ to the zeroes of $z_\zeta(x_v, -j)$. Note that there will be many zeroes of $1 - x_v^{-d} f^j$ of the same size if $d > 1$ is prime to p.

Example 5 also makes clear the importance of having Euler factors at *all* finite primes in the definition of L-series and not just the good primes; indeed, we add (finitely many) zeroes for each Euler factor left out. A refined version of these conjectures should ultimately take this into account. In any case, throwing out finitely many zeroes also allows Conjecture 5 to have some surprising consequences that make the characteristic p and classical cases seem even closer; see Section 3.

Conjecture 5 has a direct analog in classical theory.

Conjecture 6. *Let $e \geq 0$. Then there are at most* **two** *zeroes (taken without multiplicity) of $\Xi(\chi, t)$ of absolute value e.*

Proposition 2. *Conjecture 6 implies the Generalized Riemann Hypothesis up to possibly finitely many exceptional zeroes (which are then the non-critical real zeroes of $\Lambda(\chi, s)$).*

Proof. We begin by assuming that χ is a real valued character. Let z be a zero of $\Xi(\chi, t)$. Then complex conjugation, and the functional equation, imply that z, $-z$, $\bar{z}$ and $-\bar{z}$ are also zeroes of $\Xi(\chi, t)$ with of course the same absolute value. If z is non-real and non-purely imaginary, then these elements are distinct which is a violation of the conjecture. The purely imaginary zeros of $\Xi(\chi, t)$ are the real zeroes of $\Lambda(\chi, s)$ of which there are only finitely many by the functional equation.

We postpone the proof when χ is non-real to the next section. □

Remarks 4. 1. The reason that Conjecture 6 allows two zeroes where Conjecture 5 allows only one lies with functional equations. The functional equation of $L(\chi, s)$ precisely allows us to move the critical line over to the real axis, and so make the connection to the characteristic p theory. On the other hand, it also introduces a new symmetry for $L(\chi, s)$ and so adds to the count of zeroes.
2. The use of absolute values in these conjectures offers a unification between the global and local Riemann Hypotheses (by "local" we mean a Riemann Hypothesis for a variety or Drinfeld module etc., over a finite field). Indeed, all local Riemann Hypotheses are stated in terms of absolute values of the zeroes.
3. Let χ be a real-valued character associated to a number field. It is easy to see that, modulo possible real non-critical zeroes of $L(\chi, s)$, Conjecture 6 is equivalent to Conjecture 2. However, in the function field case, it is also easy to see that Conjecture 5 is much stronger than merely assuming that almost all zeroes are totally inseparable over $\mathbf{K}_L$ etc.
4. It is known that there are no real zeroes for the Riemann zeta function. In any case, one can certainly use the sign of the real part of a zero to pose a refinement of Conjecture 6 which will work for all zeroes. Only time will tell if there is any utility in such statements.
5. As in the proof of Proposition 2, Conjecture 6 may be restated as saying that a zero of $\Xi(\chi, t)$ is uniquely determined up to multiplication by ± 1 by its absolute value.

Crucial to the proof of Proposition 2 is the fact that functional equation of $\Xi(\chi, t)$ takes t to $-t$ and so is *absolute value preserving*. This suggests very strongly that any sort of functional equation for our characteristic p L-series should also be given by invariance under mappings of this sort. However, any non-trivial such automorphism would automatically provide more than 1 zero of a given absolute value. Perversely, therefore, by Part 3 of Remarks 3 we see that a classical style functional equation in characteristic p would lead to a Riemann hypothesis with *no* punch!

It remains to present the analog of the Generalized Simplicity Conjecture in the characteristic p case. We first make some definitions. Let $\bar{k}$ be a fixed algebraic closure of k and let k^{sep} be the separable closure. Let $G := \text{Gal}(k^{\text{sep}}/k)$ and let $\rho\colon G \to \mathbf{C}_\infty^*$ be a character which factors through a finite abelian extension. (We restrict ourselves to abelian characters for comparison with the classical case where we are only using abelian characters; however see below). Let G_1 be the Galois group of the maximal constant field extension of $\mathbf{K}$ over $\mathbf{K}$ and let $\sigma \in G_1$. Obviously ρ^σ is also a character of G of the same type. We call the orbit of ρ under these automorphisms the *Galois packet* generated by ρ.

The Galois packet of a classical character χ is just $\{\chi, \bar{\chi}\}$.

The characteristic p version of the Simplicity Conjecture can now be given. We state it at ∞ with the obvious v-adic version being left to the reader.

Conjecture 7. 1. Let $L(s) = L(\rho, s)$, $s = (x, y)$, be the characteristic p L-series of ρ. Then for fixed y, almost all zeroes of $L(x, y)$ are simple.
2. Let ρ and ρ' be two characters as above but from distinct Galois packets. Let $y \in \mathbb{Z}_p$. Then there is a number $c = c(y)$ such that the absolute values of the zeroes of $L(\rho, x, y)$ and $L(\rho', x, y)$ which are $> c$ are distinct.

Remark 4. The abelian representation ρ may be viewed as a "simple motive over k;" that is, $L(\rho, s)$ has no "obvious" factors. (Indeed, as the degree of ρ is 1, every Euler factor is linear and so obviously does not factor.) Similarly, due to the work of Richard Pink [P1] on the Serre Conjecture for Drinfeld modules, a Drinfeld $\mathbb{F}_r[T]$-module over $\mathbb{F}_r(\theta)$ is also a simple motive.

Now let $\overline{\mathbb{Q}_p}$ be an algebraic closure of $\mathbb{Q}_p$ and let V be a finite dimensional $\overline{\mathbb{Q}_p}$-vector space. Let $G = \text{Gal}(k^{\text{sep}}/k)$ and let ρ now be an irreducible representation of G into $\text{Aut}_{\overline{\mathbb{Q}_p}}(V)$ of Galois type (i.e., which factors through a finite Galois extension of k). As explained in Subsection 8.10 of [Go1], one works with such characteristic 0 representations in order to get the correct (=classical) local factors at the ramified primes. Let R_p be the integers of $\overline{\mathbb{Q}_p}$ and let M_p be the maximal ideal. Since ρ factors through a Galois representation, the coefficients of the classical Euler factors at any prime obviously belong to R_p. The corresponding *characteristic* p L-series is then defined by reducing the classical Euler factors modulo M_p and viewing the reduced polynomial as having $\mathbf{C}_\infty$-coefficients; one then forms the obvious Euler product. In a similar way we can reduce ρ itself modulo M_p. If this *reduced* representation is still irreducible, then ρ gives rise to a simple (characteristic p!) motive and the Simplicity Conjecture should also be true for such ρ.

We shall discuss Conjecture 7 in more detail at the end of our next section (after Corollary 1). However, note that even classically one throws out finitely many zeroes in the Generalized Simplicity Conjecture (e.g., $s = 1/2$)!

3. Missing Zeroes and the Generalized Simplicity Conjecture

Let $p(z)$ be a polynomial with coefficients in some field and with $p(0) = 1$. Basic high school algebra tells us that every field containing the roots of $p(z)$ must also contain its coefficients. Now let $p(z)$ be an entire non-Archimedean function with $p(0) = 1$ (all L-series in the characteristic p theory are of this form). As $p(z)$ factors over its zeroes, we see analogously that every *complete* field containing the zeroes also contains the coefficients.

As is universally known the above discussion is totally false for complex entire functions. Indeed, complex entire functions need not have zeroes at all (e.g., e^z) or the zeroes can lie in strictly smaller complete fields than the coefficients (e.g. $1 - e^{2\pi i z}$). From the point of view of non-Archimedean analysis such entire functions have "missing zeroes."

In this section we will show how Conjecture 5 implies an analog of "missing zeroes" for the characteristic p L-functions. That is, we will show how it implies that almost all zeroes lie in strictly smaller subfields than one would a-priori believe they should – this makes the function field theory and the classical theory look even closer than one would at first believe. In the process we will re-examine the Simplicity Conjectures for both function fields and number fields. We will do this by examining similar abelian L-series in tandem for both cases. The reader will then see how to handle other L-functions in both theories.

As above, let $G = \text{Gal}(k^{\text{sep}}/k)$ and Let $\rho\colon G \to \mathbf{C}_\infty^*$ be a character arising from a finite abelian extension $\mathfrak{k} = \mathfrak{k}_\rho$ of k. We explicitly assume that the values of ρ are **not** contained in $\mathbf{K}$. Thus ρ is analogous to a non-real classical abelian character χ. We fix one such complex character χ and let $\mathfrak{L}/\mathfrak{L}_1$ be a finite abelian extension of number fields such that χ is defined on $\text{Gal}(\mathfrak{L}/\mathfrak{L}_1)$. We do not assume that the base $\mathfrak{L}_1$ of this extension is $\mathbb{Q}$ (while, for reasons which will be clear later, we **do** make the analogous assumption in the function field case). We let $L(\rho, s)$, $s \in S_\infty$, be the L-series of ρ at ∞ (which will be enough for our purposes) and let $L(\chi, s)$, s a complex number, be the classical L-series of χ.

The symbol "χ" will always be reserved for complex abelian characters of number fields and "ρ" will be reserved for characteristic p valued characters of function fields.

We begin by applying Conjecture 5 directly to $L(\rho, s)$ where $s = (x, y) \in S_\infty$. Notice first that $\mathbf{K}_L$ clearly equals $\mathbf{K}_\mathbf{V}(\rho) :=$ the constant field extension of $\mathbf{K}_\mathbf{V}$ obtained by adjoining the values of ρ. As in Proposition 1, we deduce immediately the following result.

Proposition 3. *Let $y \in \mathbb{Z}_p$ be fixed. Then* Conjecture 5 *implies that almost all zeroes of $L(\rho, x, y)$ are totally inseparable over $\mathbf{K}_L$.*

Now we examine $L(\chi, s)$ or rather $\Xi(\chi, t)$. Note firstly that the analog of $\mathbf{K}_\mathbf{V}$ in this case is obviously $\mathbb{R}$ and the analog of $\mathbf{K}_L$ is obviously $\mathbb{C}$ (which equals $\mathbb{R}$ adjoined with the values of χ). The functional equation relates $\Xi(\chi, t)$ and $\Xi(\bar{\chi}, -t)$. Suppose that z is a zero of $\Xi(\chi, t)$. Playing off the functional equation for $\Xi(\chi, t)$ and complex conjugation, we deduce that $\bar{z}$ is also a zero for $\Xi(\chi, t)$. This is all the information that one may deduce directly. Thus applying Conjecture 6 directly to $\Xi(\chi, t)$ implies that the zeroes of $\Xi(\chi, t)$ belong to $\mathbb{C}$; i.e., nothing interesting!

Now let us return to the characteristic p theory. We introduce another ingredient into the mix by simply noting that $L(\rho, s)$ divides the zeta function of the ring of integers $\mathcal{O}_\mathfrak{k}$ of $\mathfrak{k}$. This zeta function is also an essentially algebraic entire function. Note that $\mathbf{K}_\zeta = \mathbf{K}_\mathbf{V}$ which is strictly smaller than $\mathbf{K}_L$. By applying Conjecture 5 to $\zeta_{\mathcal{O}_\mathfrak{k}}(s)$ we deduce immediately the next result which improves dramatically Proposition 3.

Proposition 4. *Let $y \in \mathbb{Z}_p$ be fixed. Then* Conjecture 5 *implies that almost all zeroes of $L(\rho, x, y)$ are totally inseparable over $\mathbf{K}_\mathbf{V}$.*

Proof. By Proposition 3, Conjecture 5 implies that almost all zeroes of $\zeta_{\mathcal{O}_\mathfrak{k}}(x, y)$ are totally inseparable over $\mathbf{K}_\mathbf{V} = \mathbf{K}_\zeta$. This is then obviously true for $L(\rho, x, y)$. □

Remarks 5. 1. In fact, we shall do even better once we bring the first part of the Generalized Simplicity Conjecture into the mix; see Proposition 8.
2. Notice that, as $\mathbf{K}_\mathbf{V}$ is totally inseparable over $\mathbf{K}$, Proposition 4 immediately implies the total inseparability of almost every zero all the way down to $\mathbf{K}$ itself.
3. Again we remind the reader that there are v-adic versions to the above results.

Next, let us return to the classical case. We have seen that for $\Xi(\chi, t)$, Conjecture 6 says nothing interesting directly. However, we shall now play the same game as we did in characteristic p and we shall obtain an analogous improvement. We know that our L-series divides the zeta function $\zeta_\mathfrak{L}(s)$ of $\mathfrak{L}$. We shall denote the "Ξ-version" of this zeta function by $\Xi(\chi_0, t)$ where χ_0 is the appropriate trivial character.

Proposition 5. Conjecture 6 *implies that almost all zeroes of $\Xi(\chi, t)$ are real.*

Proof. As above we deduce that Conjecture 6 implies that almost all zeroes of $\Xi(\chi_0, t)$ are real. We therefore immediately deduce the same for $\Xi(\chi, t)$. □

From Part 3 of Remarks 4 we obtain the following corollary.

Corollary 1. *Let χ be an abelian character of a number field. Then, modulo possible real non-critical zeroes of $L(\chi, s)$ (of which there can be only finitely many),* Conjecture 6 *is equivalent to the GRH for $L(\chi, s)$.*

There is even more that can be gleaned from this line of thought. Let ρ, ρ' be two characteristic p abelian characters as above.

Proposition 6. *Assume that ρ and ρ' are in the same Galois packet and fix $y \in \mathbb{Z}_p$. Then* Conjecture 5 *implies that almost all zeroes of $L(\rho, x, y)$ and $L(\rho', x, y)$ are equal.*

Proof. Let σ be the automorphism taking ρ to ρ'. Extend σ to the full algebraic closure. Let β be a zero of $L(\rho, x, y)$ where $1/|\beta|$ is sufficiently large so that Conjecture 5 applies for both $L(\rho, x, y)$ and $L(\rho', x, y)$. Then $\sigma(\beta) = \beta$ by Proposition 4. On the other hand, $\sigma(\beta)$ is a zero of $L(\rho', x, y)$. Thus by Conjecture 5 it must be the unique zero of its absolute value. □

The obvious v-adic version of Proposition 6 also holds.

Proposition 6 explains why Part 2 of the function field Generalized Simplicity Conjecture (Conjecture 7) is formulated as it is.

Remarks 6. 1. Proposition 6 rules out a general elementary proof by Newton Polygons of Conjecture 5.
2. Proposition 6 therefore gives some credence to viewing the Galois action as being the characteristic p analog of *both* the classical action of complex conjugation and the functional equation.

Our next result shows how close the classical and function field Generalized Simplicity Conjectures are.

Proposition 7. *Let χ and χ' be two complex characters from different Galois packets (i.e., χ and χ' are not complex conjugates). Then the Generalized Simplicity Conjecture (*Conjecture 3*) and the Generalized Riemann Hypothesis imply that all zeroes, except possibly $t = 0$, of $\Xi(\chi, t)$ and $\Xi(\chi', t)$ have distinct absolute values.*

Proof. Assume GRH. Then, using complex conjugation and the functional equation as before, we see that the $\Xi(\chi, t)$ and $\Xi(\bar{\chi}, t)$ fill out all zeroes of a fixed absolute value. Thus the result follows immediately from GSC. □

Thus the difference between the classical and function field Generalized Simplicity Conjectures lies in comparing L-series associated to characters in the same Galois packet. In characteristic zero we expect infinitely many different zeroes, whereas by Proposition 6, in characteristic p we expect only finitely many different zeroes.

The function field theory appears to offer insight into the arithmetic meaning of simplicity of zeroes. Indeed, let us return to the case of $L(\rho, s)$ as above. Recall that we have supposed that $\rho\colon \mathrm{Gal}(k^{\mathrm{sep}}/k) \to \mathbf{C}_\infty^*$. Applying the function field Generalized Simplicity Conjecture now gives the next result.

Proposition 8. *The first part of* Conjecture 7 *(assumed along with* Conjecture 5*) implies that, for fixed $y \in \mathbb{Z}_p$ almost all the zeroes of $L(\rho, x, y)$ belong to $\mathbf{K}_\mathbf{V}$.*

Proof. The first part of Conjecture 7 (applied to $L(\rho, s)$) implies that almost all the zeroes are simple. Combining this with Conjecture 5, and a simple argument using Newton Polygons, we find that almost all zeroes of $L(\rho, x, y)$ are actually in $\mathbf{K}_L$. On the other hand these zeroes are also totally inseparable over the subfield $\mathbf{K}_\mathbf{V}$ of $\mathbf{K}_L$ by Proposition 4. Note that $\mathbf{K}_L = \mathbf{K}_\mathbf{V}(\rho)$ is separable over $\mathbf{K}_\mathbf{V}$. Thus, the only way that this can happen is that the zeroes belong to $\mathbf{K}_\mathbf{V}$. □

The obvious v-adic version of the proposition is also true. As $\mathbf{K}_\mathbf{V}$ is strictly smaller than $\mathbf{K}_L$ we have deduced an analog of the missing zeroes phenomenon for function fields! Other examples can be easily worked out. While it is true that in non-Archimedean analysis we cannot have *all* zeroes in too small a field, there is in fact no contradiction. Indeed, we are allowed in our conjectures to throw out finitely many zeroes which avoids any difficulties (as simple examples attest).

To finish, we will present an example which involves complex multiplication by totally inseparable elements. This leads to certain L-series where the v-adic versions of Conjectures 5 and 7 are true, but where the obvious analog of the v-adic version of Proposition 8 is not valid. That is, where one can find infinitely many non-trivial (over the v-adic analog of $\mathbf{K_V}$) totally inseparable roots for a fixed $s_v \in S_v$. This highlights the crucial role that separability plays in Proposition 8. In order to do so, we first present Wan's cogent observation that the ∞-adic techniques presented in [W1], [DV1], and [Sh1] for $\mathbb{F}_r[T]$ also work v-adically when $\deg v = 1$.

Proposition 9. *Let v be prime of degree 1 in $\mathbf{A} = \mathbb{F}_r[T]$. Let j be an element of the ideal $(r-1)S_v$. Then the zeroes (in x_v) of $\zeta_{A,v}(x_v, j)$ are simple, in $\mathbf{k}_v$ and uniquely determined by their absolute value.*

Proof. As $\deg v = 1$, it is clear that $\mathbf{K}$ and $\mathbf{k}_v$ are isomorphic. Moreover, without loss of generality, we can set $v = (T)$. As the Newton polygon does not depend on the choice of uniformizer, we choose our positive uniformizer to be $\pi = 1/T$ and we begin by letting j be a positive integer divisible by $r-1$. Now the coefficient of x^{-d} in $\zeta_A(x, -j)$ is precisely the sum of $\langle n \rangle^j$ where $\deg n = j$ and n is monic. On the other hand, the coefficient of x_v^{-d} in $\zeta_{A,v}(x_v, -j)$ is the sum of n^j such that n is monic of degree d and $n \not\equiv 0 \bmod v$. This last condition is the same as saying that n has non-vanishing constant term.

The set $\{\langle n \rangle\}$, where n is monic, ranges over all polynomials $f(1/T)$ in $1/T$ with constant term 1 and degree (in $1/T$) $< d$. Moreover, as j is divisible by $r-1$, the set $\{\langle n \rangle^j\}$ is the same as the set $f(1/T)^j$ where $f(u)$ is a monic polynomial of degree $< d$ and has non-vanishing constant term.

Let us denote by $\zeta_{A,v}(x, -j)$ the function obtained by replacing x_v by x in $\zeta_{A,v}(x_v, -j)$ and applying the isomorphism $\mathbf{k}_v \to \mathbf{K}$ given by $T \mapsto 1/T$. The above now implies that

$$(1 - x^{-1})^{-1} \zeta_{A,v}(x, -j) = \zeta_A(x, -j)\,.$$

The result for positive j divisible by $r-1$ follows immediately. The general result then follows by passing to the limit. ∎

For the example, we let $\mathbf{A} = \mathbb{F}_r[T]$ with $r = 2$. Let $\mathbf{A}' := \mathbb{F}_r[\sqrt{T}]$, $\mathbf{k}' := \mathbb{F}_r(\sqrt{T})$ and $\mathbf{K}' := \mathbb{F}_r((1/\sqrt{T}))$. Form A', k' etc., in the obvious fashion. Note that in this case $\mathbf{K_V} = \mathbf{K}$. For each $a \in \mathbf{A}$ let $a' \in \mathbf{A}'$ be its unique square-root. Let π be a positive uniformizer for $\mathbf{K}$ and let $\pi' = \sqrt{\pi}$ be the uniformizer in $\mathbf{K}'$. Finally let $v = (g)$ be a prime of $\mathbf{A}$ of degree 1 with $v' = (g')$ the unique prime of $\mathbf{A}'$ above it.

Example 6. Let ψ be the Drinfeld $\mathbf{A}$-module defined over k' given by

$$\psi_T(\tau) := \theta\tau^0 + (\theta + \sqrt{\theta})\tau + \tau^2\,.$$

It is simple to check that $\mathbf{A}'$ acts as complex multiplications of ψ; indeed, ψ is just the Carlitz module C' for $\mathbf{A}'$ ($C'_{\sqrt{T}}(\tau) = \sqrt{\theta}\tau^0 + \tau$) as one readily checks. Let $L(\psi, s)$, $s \in S_\infty$ be the L-series of ψ over k'. As ψ has complex multiplication, $L(\psi, s)$ factors into the product of L-series associated to Hecke characters. In this case, it is simple to work out what happens directly. For each monic prime $f \in \mathbf{A}$, let f' be its unique square-root in $\mathbf{A}'$. Let

$$L(s) := \prod_{f \text{ monic prime}} (1 - f' f^{-s})^{-1}\,.$$

It is easy to see that $L(s)$ is the L-series of the Hecke character Θ with $\Theta(f) = f'$. One checks readily that

$$L(\psi, s) = L(s)^2 .$$

Thus we need only focus on $L(s)$.

It clear that $\mathbf{K}_L$, for our L-series $L(s)$, equals $\mathbf{K}'$ (=$\mathbf{K}(\Theta)$ defined in the obvious fashion) and so is totally inseparable over $\mathbf{K} = \mathbf{K}_{\mathbf{V}}$. Upon expanding $L(s)$ we find

$$L(s) = \sum_{n \text{ monic}} n' n^{-s} .$$

Thus,

$$L(s) = \sum_n x^{-\deg n} n' \langle n \rangle^{-y} .$$

Let $\langle z \rangle_{\pi'}$ be the 1-unit part of an element of $z \in \mathbf{K}'$ defined with respect to π'. Thus

$$L(s) = \sum_{n' \text{ monic}} x^{-\deg n} n' \langle n' \rangle_{\pi'}^{-2y} .$$

But notice that the degree in T of n is clearly the degree in $\sqrt{T}$ of n'. Thus, finally,

$$L(s) = \sum_{n' \text{ monic}} x^{-\deg n'} n' \langle n' \rangle_{\pi'}^{-2y} .$$

If we form $\zeta_{A'}(s)$ in the obvious fashion, then we have shown that

$$L(s) = \zeta_{A'}(x\pi', 2y - 1) .$$

Thus the results of Wan, Thakur, and Diaz-Vargas tell us that the zeroes of $L(s)$ are simple, in $\mathbf{K}' = \mathbf{K}(\Theta)$ and uniquely determined by their absolute value. As such they are indeed totally inseparable over $\mathbf{K}$. Thus both Conjecture 5 and Conjecture 7 are true for $L(s)$.

We now form the v-adic functions $L_v(\psi, x_v, s_v)$, $L_v(x_v, s_v)$, etc. Note that, by definition,

$$L_v(x_v, s_v) = \prod_{f \neq g} (1 - f' x_v^{-\deg f} f^{-s_v})^{-1} .$$

Note further that the v-adic version, $\mathbf{k}_{v,\mathbf{V}}$, of $\mathbf{K}_{\mathbf{V}}$ obviously equals $\mathbf{k}_v$ since $\mathbf{A}$ has class number 1. As we are assuming $\deg v = 1$ and $r = 2$, we see that $S_v = \mathbb{Z}_p$. As above we find $L_v(\psi, x_v, s_v) = L_v(x_v, s_v)^2$ and

$$L_v(x_v, s_v) = \zeta_{A',v'}(x_v, 2s_v - 1) .$$

Proposition 9, and the results of [W1], [DV1] now tell us that the zeroes of $L_v(x_v, s_v)$ are simple, uniquely determined by their absolute values, and in

$$\mathbf{k}'_{v'} = \mathbb{F}_2((\sqrt{T})) = \mathbf{k}_v(\Theta) = \mathbf{k}_v(f')$$

for any prime $f \neq v$. So both Conjecture 5 and Conjecture 7 are true for $L_v(x_v, s_v)$. Moreover, if λ is one such zero, one may easily compute $v_{g'}(\lambda)$ which is seen to be odd. Since the elements of $\mathbb{F}_2((T))$ are precisely the squares in $\mathbb{F}_2((\sqrt{T}))$ (and so have even valuation), we deduce immediately that $\lambda \notin \mathbf{k}_v = \mathbf{k}_{v,\mathbf{V}} = \mathbb{F}_2((T))$.

The same calculation performed at ∞ will give an even valuation and so fails to show that infinitely many zeroes are not in $\mathbf{K} = \mathbf{K}_{\mathbf{V}}$ (though this is indeed likely).

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Department of Mathematics, The Ohio State University, 231 W. 18th Ave., Columbus, Ohio 43210

E-mail address: goss@math.ohio-state.edu