The impact of the infinite primes on the Riemann hypothesis for characteristic p valued L-series

David Goss

Department of Mathematics, Ohio-State University, Columbus, Ohio 43210, USA
goss@math.ohio-state.edu

Abstract. In [12] we proposed an analog of the classical Riemann hypothesis for characteristic p valued L-series based on known results for $\zeta_{\mathbb{F}_r[\theta]}(s)$ and two assumptions that have subsequently proved to be incorrect. The first assumption is that we can ignore the trivial zeroes of characteristic p L-series in formulating our conjectures. Instead, we show here how the trivial zeroes influence nearby zeroes and so lead to counter-examples of the original Riemann hypothesis analog. We then sketch an approach to handling such "near-trivial" zeroes via Hensel's and Krasner's Lemmas. Moreover, we show that $\zeta_{\mathbb{F}_r[\theta]}(s)$ is not representative of general L-series as, surprisingly, all its zeroes are near-trivial, much as the Artin-Weil zeta-function of $\mathbb{P}^1/\mathbb{F}_r$ is not representative of general complex L-functions of curves. The second assumption in [12] is that certain Taylor expansions associated to L-series of number fields would exhibit complicated behavior with respect to their zeroes. We present a simple argument that this is not so, and, at the same time, characterize functional equations.

Dedicated to Ram with great respect and affection on his 70-th birthday

1 Introduction

In the paper [12] (whose notations etc., we generally follow here) we defined a possible Riemann hypothesis for the zeroes of characteristic p L-functions. This work is based on the results of Wan [19], Diaz-Vargas and Thakur [7], Poonen (unpublished), and Sheats [15] for the zeta function $\zeta_{\mathbb{F}_r[\theta]}(s)$ ($s=(x,y)\in S_\infty$; see Section 2 for the definitions). In particular the zeroes of $\zeta_{\mathbb{F}_r[\theta]}(s)$ were found to be both simple and lie on the line $\mathbb{F}_r((1/T))$ just as the completed Riemann zeta function $\pi^{-s/2}\Gamma(s/2)\zeta(s)$, $s=1/2+it$, is conjectured to have only simple real zeroes in t. This finite characteristic Riemann hypothesis has two components to it. The first, Conjecture 4 of [12], is a basic finiteness statement about the extension fields generated by L-series zeroes (obviously no such conjecture need be made classically). The second, Conjecture 5 of [12], focuses on counting the number of zeroes of a given absolute value; based on the above examples, we postulated that, outside of finitely many exceptions, the absolute value determined the zero. (Both conjectures are recalled in Section 4 below; therefore we will drop the reference to [12] from now on.) We pointed out similarities between Conjecture 5 and the classical Riemann hypothesis for number fields.

Now in classical theory we work, of course, with the completed L-function, including the Gamma-factors arising from the infinite primes, as above for $\zeta(s)$. These Gamma-factors have poles at some of the negative integers (e.g., $\Gamma(s/2)$ has poles at the non-positive even integers) and so, by analytic continuation, they force the L-series to have associated "trivial zeroes." The functional equation then assures us that all other zeroes must lie in the critical strip. On the other hand, in the characteristic p case presented in [12] we ignored input from the infinite primes and their associated trivial zeroes. Indeed, our conjectures allow for finitely many exceptional cases for each y, and, for y a negative integer, there are only finitely many trivial zeroes; thus ignoring them appeared innocuous. However, since the publication of [12], we have realized that the infinite primes must also be taken into account in the characteristic p Riemann hypothesis (which, from the standpoint of classical algebraic number theory, is actually quite reasonable). It is our goal here to explain how the trivial zeroes lead to counter-examples for Conjecture 5 (though Conjecture 4 still appears to be valid). More precisely, the topology of our space S_∞, which is the natural domain of definition for the characteristic p L-series, permits us to inductively construct counter-examples using zeroes sufficiently close to trivial zeroes (we call such zeroes "near-trivial zeroes"). We also suggest some possible replacements for Conjecture 4 by sketching a procedure based on Hensel's Lemma to isolate the near-trivial zeroes. The original conjectures may then work for the remaining zeroes (which, following classical precedent, we call "critical zeroes").

However, there is now a great surprise. Upon searching for the critical zeroes of $\zeta_{\mathbb{F}_r[\theta]}(s)$, $s \in S_\infty$, we find that they do not exist! (More precisely, we prove this fact for $\mathbb{F}_p[\theta]$; the general $\mathbb{F}_r[\theta]$ case seems very likely to follow from Sheats' techniques — see also Corollary 3 of Section 6.) That is, the case of $\zeta_{\mathbb{F}_r[\theta]}(s)$ may not be representative of general characteristic p L-series, much as the Artin-Weil zeta-function of $\mathbb{P}^1$ over a finite field is not typical of the L-series of general curves over finite fields. Indeed, the Artin-Weil zeta-function of the projective line also has no critical zeroes. We will exhibit here a few examples of critical zeroes in the characteristic p theory and we hope to have more examples in the near future. However, as of this writing, they are a complete mystery. For instance where they lie, or even if there are infinitely many of them (for a given L-series and interpolation place) is not known. It is certainly possible that there may be more surprises ahead.

Let K be a complete, algebraically closed, non-Archimedean field and let $f(x) = \sum a_n x^n$ be an entire power series with coefficients in K. Let L be a complete subfield of K which contains the zeroes of $f(x)$. It is then a standard fact that there exists $\alpha \in K^*$ such that the Taylor coefficients of $\alpha f(x)$ at the origin lie in L; i.e., the coefficients of $f(x)$ lie on an L-line through the origin in K. As is well-known, in complex analysis the relationship between the zeroes and the Taylor coefficients (at the origin) of an arbitrary entire function is more complicated; e.g., $e^{2\pi ix} - 1$. (However, as Keith Conrad pointed out, a similar statement is true in complex analysis if we allow ourselves to multiply by the Taylor series of an invertible entire function; in non-Archimedean analysis such entire functions are non-zero constants.) During the writing of [12], we had assumed that Taylor expansions of classical L-series (of number fields) would exhibit similar complicated behavior. This assumption has also proved to be incorrect. Indeed, in our last section we give a simple argument which shows that the associated Taylor coefficients in t (where $s = 1/2 + it$) lie on a line through the origin in $\mathbb{C}$. This argument also characterizes the functional equation of the L-series via the description of complex conjugation given in Equation 9.

We now describe in more detail the contents of this paper. In Section 2 we review the definitions and interpolations of characteristic p L-series. We recall how the analogue of Artin L-series has trivial zeroes (essentially from the classical theory of Artin and Weil) at the negative integers. We also present a reasonable approach to trivial zeroes for the general L-series of a Drinfeld module; it is our belief that the work of Boeckle and Pink [4], [2] (esp., §4.5) will ultimately flesh out this construction.

In Section 3 we recall Krasner's Lemma in order to put it into a form more useful for calculations in characteristic p. In Section 4 we review the statements of Conjectures 4 and 5.

In Section 5 we construct counter-examples to Conjecture 5. Along the way we establish that the family of Newton polygons associated to an L-series possesses certain invariance properties, see Lemma 1. More precisely, the first n-segments of the Newton polygon will, generically, be an invariant of a natural group of translations. This leads us to believe that the family of Newton polygons itself will serve to distinguish between L-series (see Question 1 in this section). We also sketch the beginnings of an approach to remove these counter-examples by isolating the offending near-trivial zeroes via Hensel's Lemma. We give some examples of how this will work (as well as find a few critical zeroes) but total success here will need many more results on the structure of the zeroes.

In Section 6 we use the techniques of Diaz-Vargas and Sheats to study the effects of the trivial zeroes for $\zeta_{\mathbb{F}_r[\theta]}(s)$, $s \in S_\infty$, and we establish in Proposition 5 that all zeroes for $\zeta_{\mathbb{F}_p[\theta]}(s)$ are near-trivial. It is our belief that ultimately there should be some massive improvement of the techniques of Diaz-Vargas and Sheats to general characteristic p L-series enabling us, at least, to specify exactly where the near-trivial zeroes lie. Indeed, Boeckle has established the first of such results [2] by showing the logarithmic growth of the degrees of special polynomials in general. For $\mathbb{F}_r[T]$, this logarithmic growth is a first corollary of the techniques of [7] and [15]. As one sees by simple examples (such as in the discussion directly after Corollary 4), this logarithmic growth is analogous to the formula giving the growth of the number of zeroes of classical L-series in the critical strip.

It is natural to ask whether the characteristic p L-series satisfy some sort of "functional equation." From early work on such L-series, it was clear, from simple counting arguments, that such a functional equation could not be of the classical "$s \mapsto 1-s$" form. On the other hand, one consequence of classical functional equations of L-series is that they split the set of zeroes into a disjoint union of trivial zeroes and critical zeroes. Similarly, it is reasonable to expect that any result which specifies where the near-trivial zeroes lie in the characteristic p theory will play the role of the functional equation for the theory. In this regard, very recent work of Böckle [3] is highly encouraging as it establishes a profound connection between characteristic p valued L-series and characteristic p valued cusp forms. For instance, from [13] one sees that $\zeta_{\mathbb{F}_r[\theta]}(s+1-r)$ is associated to the cusp forms Δ (analog of the elliptic modular cusp form Δ) and $E^r_{(r-1)} \cdot \Delta$ where $E_{(r-1)}$ is an Eisenstein series.

Finally, let $L(\chi, s)$, $s = 1/2+it \in \mathbb{C}$, be the L-series associated to a number field and abelian character χ. Let $\Lambda(\chi, s)$ be the completed L-series which includes the Gamma-factors. In our last section, Section 7, we present an elementary argument that the Taylor coefficients of $\Lambda(\chi, s)$ about $t = 0$ are, up to multiplication by a non-zero constant, real numbers. This is contrary to what was expected during the writing of [12].

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2 Trivial zeroes

As mentioned in the introduction, we will follow the notation of [12]. We begin by briefly reviewing the definition of L-functions of Galois characters, Drinfeld modules, etc. Let $\mathcal{X}$ be a smooth, projective, geometrically connected curve over the finite field $\mathbb{F}_r$ where $r = p^m$ and p is prime. Let $\infty \in \mathcal{X}$ be a fixed closed point of degree d_∞ over $\mathbb{F}_r$. Let $\mathbf{k}$ be the function field of $\mathcal{X}$ and let $\mathbf{A}$ be the subring of those functions which are regular outside of ∞. In the theory, one views $\mathbf{k}$ as the analogue of $\mathbb{Q}$ and $\mathbf{A}$ as the analogue of $\mathbb{Z}$; indeed, $\mathbf{A}$ is a Dedekind domain with finite class group and group of units ${\mathbb{F}_r}^*$.

Let w be an arbitrary place of $\mathbf{k}$. We let $|x|_w$ be the normalized absolute value (= multiplicative valuation) at w with associated additive valuation $\nu_w(x)$; by definition $\nu_w(t) = 1$ if $t \in \mathbf{k}$ has a simple zero at w. We let $\mathbf{k}_w$ be the completion of $\mathbf{k}$ with respect to w and we denote its finite field of constants by $\mathbb{F}_w$. We let $d_w := [\mathbb{F}_w \colon \mathbb{F}_r]$ be the degree of w over $\mathbb{F}_r$. We let $\mathbf{C}_w$ be the completion of a fixed algebraic closure of $\mathbf{k}_w$ equipped with the canonical extension of $|x|_w$. In particular we put $\mathbf{K} := \mathbf{k}_\infty$. Thus $\mathbf{K}, \mathbf{C}_\infty$ are the analogues of $\mathbb{R}, \mathbb{C}$, respectively, whereas $\mathbf{k}_v$, for a finite v, is the analogue of $\mathbb{Q}_p$ etc.

We now let A, k, K etc., be another copy of these rings. There is an obvious isomorphism θ from the "bold" to the "non-bold" which makes k, K, C_∞ into $\mathbf{A}$-fields. As in [12], we view the non-bold fields as being the "scalars" over which we can define Drinfeld modules etc., equipped with an action by the "operators" in $\mathbf{A}$. The basic example is the Carlitz module C defined for $\mathbf{A} = \mathbb{F}_r[T]$. Here we put $\theta = \theta(T) \in k$ and

$$C_T := \theta\tau^0 + \tau,$$

where $\tau\colon C_\infty \to C_\infty$ is the r-th power mapping of the field C_∞; thus the Carlitz module is obviously defined over $k = \mathbb{F}_r(\theta)$. In fact, it is easy to see that C gives rise to an obvious family of Drinfeld modules over $\mathrm{Spec}(A)$. (It will always be clear to the reader when "C" is being used to denote a field or the Carlitz module.)

More generally, let $L \subset C_\infty$ be a finite extension of k and let ω be a place of L. We say ω is a "finite place" (or finite prime) if it lies over a prime of A; otherwise it is an "infinite place" (or infinite prime). This notion carries over to $\mathbf{k}$ etc., in the obvious fashion.

Let ψ be a Drinfeld $\mathbf{A}$-module of rank d defined over L. Let $\mathfrak{P}$ be a finite prime of L lying over the prime $\mathfrak{p}$ of A with finite residue class fields $\mathbb{F}_\mathfrak{P}$ and $\mathbb{F}_\mathfrak{p}$ respectively. As usual, we define the norm $n\mathfrak{P}$ of $\mathfrak{P}$ to be $\mathfrak{p}^f$ where f is the residue field degree. All but finitely many such primes $\mathfrak{P}$ are good for ψ in that one can reduce ψ modulo $\mathfrak{P}$ to obtain a Drinfeld module $\psi^\mathfrak{P}$ over $\mathbb{F}_\mathfrak{P}$ with the same rank as ψ. Associated to $\mathbb{F}_\mathfrak{P}$ there is the Frobenius endomorphism $\mathrm{Fr}_\mathfrak{P}$ of $\psi^\mathfrak{P}$, and one sets

$$P_\mathfrak{P}(u) := \det(1 - u\mathrm{Fr}_\mathfrak{P} \mid T_v(\psi^\mathfrak{P}))\,;$$

here $v \subset \mathbf{A}$ is a non-trivial prime such that $\theta(v) \neq \mathfrak{p}$ and T_v is the v-adic *Tate module* of $\psi^\mathfrak{P}$. It is easy to see that the $\mathbf{A}_v$-module $T_v(\psi)$ is free of rank d. In complete accordance with classical theory, $P_\mathfrak{P}(u)$ has $\mathbf{A}$-coefficients which are independent of the choice of v. Moreover, as Drinfeld has shown, its roots inside $\mathbf{C}_\infty$ satisfy the local Riemann hypothesis in terms of their absolute values, see, e.g., §4.12 of [11].

Now assume that we have chosen a notion of "sign" on $\mathbf{K}$; that is, a homomorphism $\mathrm{sgn}\colon \mathbf{K}^* \to \mathbb{F}_\infty^*$ which is the identity on $\mathbb{F}_\infty^* \subset \mathbf{K}^*$. Elements x with $\mathrm{sgn}(x) = 1$ are called "positive." Let $\pi \in \mathbf{K}^*$ be a fixed positive uniformizer and let $a \in \mathbf{K}$ be a positive element with a pole of order e at ∞. We set

$$\langle a \rangle = \langle a \rangle_\pi := \pi^e a\,.$$

It is clear that $\langle a \rangle \in U_1$, where U_1 is the multiplicative group of 1-units in $\mathbf{K}^*$, and that $\langle ab \rangle = \langle a \rangle \langle b \rangle$ for positive a and b. Note that the binomial theorem makes U_1 into a $\mathbb{Z}_p$-module.

Let a be a positive element of $\mathbf{A}$. We set $\langle (a) \rangle := \langle a \rangle$ thereby giving a homomorphism from the group of principal and positively generated $\mathbf{A}$-fractional ideals to U_1. Let $\hat{U}_1 \supset U_1$ be the group of all 1-units in $\mathbf{C}_\infty$. As

$\hat{U}_1$ may be seen to be a $\mathbb{Q}_p$-vector space (p-th roots may be uniquely taken), a simple argument implies that $\langle ? \rangle$ extends uniquely to a homomorphism from the group of all fractional ideals to $\hat{U}_1$. Let $\mathbf{K_V} \subset \mathbf{C}_\infty$ be the subfield obtained by adjoining $\langle I \rangle$ to $\mathbf{K}$ where I ranges over all fractional ideals of $\mathbf{A}$. As U_1 is a $\mathbb{Z}_p$-module and the class number of $\mathbf{A}$ is finite, we conclude that $\mathbf{K_V}$ is a finite, totally inseparable, extension of $\mathbf{K}$.

Let $S_\infty := \mathbf{C}_\infty^* \times \mathbb{Z}_p$ and let $s = (x, y) \in S_\infty$. For a non-zero ideal $I \subseteq \mathbf{A}$ we put

$$I^s := x^{\deg I} \langle I \rangle^y \,.$$

Let $\pi_* \in \mathbf{C}_\infty$ be a fixed d_∞-th root of π and let j be an integer. It is then easy to see that

$$(i)^{s_j} = i^j$$

for positive $i \in \mathbf{A}$ and $s_j := (\pi_*^{-j}, j) \in S_\infty$. We frequently write "$j$" for "$s_j$."

By abuse of language, we also write $n\mathfrak{P}$ for $\theta^{-1}(n\mathfrak{P})$ whenever no confusion will arise. Thus, finally, the L-series $L(\psi, s)$, $s \in S_\infty$, of ψ over the field L is defined by

$$L_\infty(\psi, s) = L(\psi, s) := \prod_{\mathfrak{P} \text{ good}} P_\mathfrak{P}(\theta^{-1}(n\mathfrak{P})^{-s})^{-1} = \prod_{\mathfrak{P} \text{ good}} P_\mathfrak{P}(n\mathfrak{P}^{-s})^{-1} \,.$$

The local Riemann hypothesis assures us that these L-series converge on the "half-plane" of S_∞ defined by $\{(x, y) \mid |x|_\infty > t\}$, for some positive real t, to a $\mathbf{C}_\infty$-valued function. In a similar fashion one can construct L-series of pure T-modules, τ-sheaves, etc.

Ultimately, as with classical theory, Euler factors at the finitely many bad primes ought to be added into the definition, see Remark 2. Such factors are defined in the classical fashion using invariants of inertia etc. Recent work of F. Gardeyn [9] establishes that these local factors satisfy exactly what one would expect and are amazingly analogous to those defined for elliptic curves at the bad primes. In any case, unlike classical theory, there are also many examples where all finite primes are good.

Now let $L^{\text{sep}} \subset C_\infty$ be the separable closure of L. Let $\overline{\mathbb{Q}}_p$ be a fixed algebraic closure of $\mathbb{Q}_p$ and let V be a finite dimensional $\overline{\mathbb{Q}}_p$-vector space. Let $\rho\colon \operatorname{Gal}(L^{\text{sep}}/L) \to \operatorname{Aut}_{\overline{\mathbb{Q}}_p}(V)$ be a representation of Galois type (i.e., which factors through the Galois group of a finite Galois extension of L). As explained in §8 of [11], the classical definition of Artin L-series is easily modified to define a $\mathbf{C}_\infty$-valued L-series $L(\rho, s)$, $s \in S_\infty$, whose Euler product converges on the half-plane $\{(x, y) \mid |x|_\infty > 1\}$.

We will refer to $L(\psi, s)$, for ψ a Drinfeld module, as an L-series of "Drinfeld type," and $L(\rho, s)$, for a Galois representation ρ, as an L-series of "Galois type." Both types of L-series will be referred to as "L-series of arithmetic type" from now on.

Let $L(s)$, $s = (x, y) \in S_\infty$, be an L-series of arithmetic type. For fixed $y \in \mathbb{Z}_p$, $L(x, y)$ is an power series in x^{-1} with coefficients in a finite extension

$\mathbf{K}_L(y)$ of $\mathbf{K}$. (To avoid any possibility of confusion we will always use a subscripted "L" to refer to a particular L-series and not a field L.) For instance, if $L(s) = L(\psi, s)$ is an L-series of Drinfeld type, then $\mathbf{K}_L(y)$ is a subfield of $\mathbf{K}_{\mathbf{V}}$. In fact, let u be the order of the strict class group of $\mathbf{A}$ (which is the quotient of the group of all $\mathbf{A}$-fractional ideals modulo the subgroup of principal and positively generated ideals). Then if $y \in u\mathbb{Z}_p$, one sees easily that $\mathbf{K}_L(y) = \mathbf{K}$. If $L(s)$ is an L-series of Galois type, then $\mathbf{K}_L(y)/\mathbf{K}$ may contain a finite extension of constant fields.

Let $\pi_* \in \mathbf{C}_\infty$ be a fixed d_∞-th root of our fixed parameter π, and let j be a non-negative integer. As is standard, we put

$$z_L(x, -j) := L(x\pi_*^j, -j)\,.$$

The finiteness of the class number of $\mathbf{A}$ implies that $z_L(x, -j)$ is a power series whose coefficients lie in a finite extension of $\mathbf{k}$ and are integral over $\mathbf{A}$. For $L(\psi, s)$ of Drinfeld type, recent work [4], [2] expresses these power series in terms of the cohomology of certain "crystals" and thus establishes that they are actually polynomials in x^{-1} (in [3] such crystals are also shown to arise from characteristic p valued cusp forms on the Drinfeld upper half-plane). A similar statement in the case of an L-series of Galois type had previously been shown using elementary estimates and the classical theory of Weil.

Definition 1. The polynomials $z_L(x, -j)$ are the *special polynomials* of $L(s)$.

The cohomological description of the special polynomials is critical in the analytic continuation of an L-series $L(\psi, s)$ of Drinfeld type to an essentially algebraic entire function on all of S_∞ (as defined in §8.5 of [11]). Indeed, as in [2], one may use the cohomological description to give a *logarithmic* bound on the growth of the degrees (in x^{-1}) of $z_L(x, -j)$ as a function of j from which the analytic continuation is readily deduced. For the Carlitz module C, whose L-series is easily seen to be $\zeta_{\mathbb{F}_r[\theta]}(s-1)$, this bound was originally shown by H. Lee [14] using elementary methods; see Th. VIII of [16] for a statement of related results. This bound also follows from the work of Diaz-Vargas and Sheats as in Section 6. In fact, the logarithmic bound on the degrees of the special polynomials for any L-series of Galois type may be established by elementary, non-cohomological means.

Let $L(\rho, s)$ be an L-series of Galois type. In Sections 8.12 and 8.17 of [11] there is a "double congruence" relating $z_L(x, -j)$ to the incomplete characteristic 0-valued L-function $\hat{L}(\rho \otimes \omega^{-j}, t)$ of ρ twisted by powers of the Teichmüller character; this incomplete L-series is defined by the usual Euler product but taken only over the finite primes. By Weil's Theorem (= the Artin Conjecture in this context), this incomplete L-series is a polynomial in $t = r^{-s}$ which is divisible by the finite Euler product taken only over the infinite primes. By using such double congruences infinitely often, one deduces that $z_L(x, -j)$, and thus $L(\rho, (x, -j))$, have a number of canonical zeroes.

Definition 2. The zeroes just described are the called the trivial zeroes of $L(\rho, s)$, and $z_L(x-j)$, at $y=-j$.

We will use the expression "trivial zeroes" to refer to the union of the trivial zeroes at $-j$ for all j. It is very easy to see that the trivial zeroes belong to a finite extension of $\mathbf{K}$.

Note that as $L(\rho,(x,-j))$ is a polynomial in x^{-1}, there are obviously only finitely many trivial zeroes for each j. As ρ is of Galois type, the trivial zeroes for $z_L(x,-j)$ are easily seen to be in the algebraic closure of $\mathbb{F}_r \subset \mathbf{C}_\infty$ and so have ∞-adic absolute value 1.

Example 1. In Example 3 of [12] we discussed the basic example $\zeta_A(s)$, $s \in S_\infty$, of the zeta-function of $A = \mathbb{F}_r[\theta]$. Note obviously that $d_\infty = 1$ for such an A and a "positive" polynomial is just a monic polynomial. Let $s=(x,y)$; one finds easily that

$$\zeta_A(s) = \sum_{e=0}^{\infty} x^{-e} \left(\sum_{\substack{n \text{ monic} \\ \deg(n)=e}} \langle n \rangle^{-y} \right) . \tag{1}$$

For a non-negative integer j one then has

$$z_{\zeta_A}(x,-j) = \sum_{e=0}^{\infty} x^{-e} \left(\sum_{\substack{n \text{ monic} \\ \deg(n)=e}} n^j \right) . \tag{2}$$

For $e \gg 0$, the sum in parentheses vanishes (in fact, Lee (loc. cit.) shows that one can choose $e > l_r(j)/(r-1)$ where $l_r(j)$ is the sum of the r-adic digits of j). When j is positive and divisible by $r-1$, then $z_{\zeta_A}(x,-j)$ has a simple zero at $x=1$. Thus $\zeta_A(s)$ has a simple trivial zero at $s_{-j} = (\pi^j, -j)$, just as the Riemann zeta function has a simple trivial zero at negative even integers. A very similar story happens for general $L(\rho, s)$ of Galois type.

For r and small positive j not divisible by $r-1$, computer calculations have shown that $z_{\zeta_A}(x,-j)$ is a polynomial in x^{-1} which is irreducible over $\mathbb{F}_r(T)$ and has associated Galois group equal to the full symmetric group. If $j \equiv 0 \pmod{r-1}$, a similar statement is true computationally once $1-x^{-1}$ is factored out.

For an L-series $L(\psi, s)$ of Drinfeld type, as well as other more general L-functions, one should find trivial zeroes in a very similar fashion. For simplicity let $\mathbf{A} = \mathbb{F}_r[T]$. Then, for a Drinfeld module (and, more generally, for pure, uniformizable T-modules), the factors at the infinite primes (whose zeroes are then the trivial zeroes) should conjecturally arise in a fashion completely analogous to the one used above for $L(\rho, s)$. Indeed, as mentioned before, the theory of Böckle and Pink [4] computes

$$Z_L(x, -j) = \prod_{\mathfrak{P} \text{ good}} P_{\mathfrak{P}}(n\mathfrak{P}^j x^{-\deg_{\mathbb{F}_r} \mathfrak{P}})^{-1}$$

via the cohomology of the crystal associated to the $\mathbf{A}$-module $\psi \otimes C^{\otimes j}$ (where C is the Carlitz module) through an associated trace formula. Each local factor $P_{\mathfrak{P}}(n\mathfrak{P}^j u)$ at a good prime $\mathfrak{P}$ may be computed via the canonical Galois action on the Tate module $T_v(\psi \otimes C^{\otimes j})$.

Suppose that ψ is defined over a finite extension L of k. Let $\sigma\colon L \to C_\infty$ be a k-embedding and let L_σ be the completion of L under the induced absolute value. Let $\psi^\sigma \otimes C^{\otimes j}$ be the T-module defined over L_σ obtained by applying σ to the coefficients of $\psi \otimes C^{\otimes j}$ (note that σ acts as the identity on the coefficients of C and its tensor powers). Via a fundamental result of Anderson [1], the module $\psi^\sigma \otimes C^{\otimes j}$ is *uniformizable* and arises from a lattice $M_{\sigma,j}$. The action of the decomposition group at the infinite place defined by σ can then be computed via the Galois action on this lattice. Therefore it must factor through a finite Galois extension precisely because the lattice generates a finite extension of L_σ. Consequently the associated characteristic polynomial of Frobenius, which should conjecturally impart trivial zeroes to $L(\psi, s)$, will have constant (i.e., in the algebraic closure of $\mathbb{F}_r \subset \mathbf{C}_\infty$) coefficients just as it does for $L(\rho, s)$. The product of such characteristic polynomials over all infinite primes should then give all the trivial zeroes at $y = -j$. A very similar description is expected for general $\mathbf{A}$. Finally, recent work of Gardeyn [9] shows that we can expect a somewhat similar treatment of trivial zeroes in much greater generality (when, e.g., our t-module is not uniformizable); one ends up with good factors at the infinite primes but where these factors may not have constant coefficients.

Example 2. As mentioned above, the L-series $L(C, s)$ of the Carlitz module C over $\mathbb{F}_r(\theta)$ is $\zeta_{\mathbb{F}_r[\theta]}(s-1)$. From Example 1 we see then that $L(C, s)$ has a trivial zero at $-j+1$ where j runs over the positive integers divisible by $r-1$. This is what is predicted by the above prescription.

Let v be a closed point in $\mathrm{Spec}(\mathbf{A})$ and let $L(s)$ be an L-series of arithmetic type. Then the logarithmic growth of the degrees of the special polynomials also allows one to establish the v-adic interpolation $L_v(x, y)$ of the L-functions given above. The function $L_v(x, y)$ is naturally defined on the space $\mathbf{C}_v^* \times S_v$, where $\mathbf{C}_v$ is the completion of an algebraic closure of the local field $\mathbf{k}_v$ and S_v is the completion of $\mathbb{Z}$ with respect to a certain topology (one sees easily that S_v is isomorphic to the product of a finite cyclic group H_v with $\mathbb{Z}_p$; see §8.3 of [11]). We will continue to use x for the first variable (now in $\mathbf{C}_v^*$) and y for the second variable (now in S_v); thus, in this case, our notation here differs slightly from that of [12]. These functions have analytic properties completely similar to those possessed by the original functions on S_∞; so we again have a

1-parameter family of entire power series in x^{-1} with very strong continuity properties in the variable y.

Notice that the zeroes of all these entire functions are algebraic over the base completion of $\mathbf{k}$ (i.e., $\mathbf{K}$ or $\mathbf{k}_v$) by standard non-Archimedean function theory.

Example 3. Let $\mathbf{A} = \mathbb{F}_r[T]$ and let v be a prime of degree d. The v-adic interpolation of $\zeta_A(s)$ will be denoted $\zeta_{A,v}(x,y)$. One has

$$\zeta_{A,v}(x,y) = \sum_{e=0}^{\infty} x^{-e} \left(\sum_{\substack{n \text{ monic} \\ \deg(n)=e \\ (n,v)=1}} n^y \right) , \tag{3}$$

where $x \in \mathbf{C}_v^*$ and $y \in S_v$.

Remark 1. Let $\rho : \mathrm{Gal}(L^{\mathrm{sep}}/L) \to \mathrm{Aut}_{\overline{\mathbb{Q}}_p}(V)$ be a representation of Galois type, as above, with L-series $L(\rho, s)$. Let j be a non-negative integer. It is important to note that as the trivial zeroes of $z_L(x, -j)$ are constants, they also have v-adic absolute value 1; thus their effect v-adically is very limited. Under the above conjectures on the Galois modules associated to Drinfeld modules, etc., a similar remark should ultimately hold in complete generality.

Notice that the very act of interpolating $L(\rho, s)$ v-adically also removes the Euler factors at the primes above v in $z_L(x, -j)$. In other words, let

$$z_L(v; x, -j) := z_L(x, -j)\hat{z}_L(v; x, -j)$$

where

$$\hat{z}_L(v; x, -j) := \prod_{\substack{\mathfrak{P} | v \\ \mathfrak{P} \text{ good}}} P_{\mathfrak{P}}(n\mathfrak{P}^{-(x\pi_*^j, -j)}) .$$

Then $z_L(v; x, -j) = L_v(x, -j)$ where $-j \in S_v$ (and is obviously also a polynomial in x^{-1}).

Definition 3. The zeroes of $\hat{z}_L(v; x, -j)$ are the *v-adic trivial zeroes* of $L(\rho, s)$ at $-j$.

The impact of Remark 1 is precisely that we can ignore v-adically the ∞-adic trivial zeroes (i.e., the trivial zeroes of $z_L(x, -j)$) and work as above. As usual, the union over all j of these zeroes is the set of all v-adic trivial zeroes. They lie in a finite extension of $\mathbf{k}_v$.

For an L-series $L(\psi, s)$ of Drinfeld type, the v-adic trivial zeroes are given in exactly the same way. The main difference is that the existence of the ∞-adic trivial-zeroes and their v-adic influence is conjectural for such L-series at this moment.

Example 4. We continue examining the basic case of Example 1. Let v be a finite prime of degree d in $\mathbb{F}_r[T]$ associated to a monic irreducible $f(T)$ and let j be a non-negative integer. Then the v-adic trivial zeroes of $\zeta_A(s)$ are the elements $x \in \mathbf{C}_v$ with $0 = 1 - f^j x^{-d}$; these are considered with the obvious multiplicities when d is divisible by p.

Note the remarkable similarity between the ∞-adic and v-adic trivial zeroes. For instance, let $\zeta_A(s)$ be as in Examples 1 and 4, and let $v = (f)$ be a prime of degree 1. Then the ∞-adic trivial zeroes occur at $(\pi^j, -j) \in S_\infty$ for $j > 0$ and divisible by $(r-1)$, while the v-adic trivial zeroes occur at $(f^j, -j) \in \mathbf{C}_v^* \times S_v$ for non-negative j. Obviously both π and f are parameters in their respective local fields.

We finish this section by using the above ideas to factor the special polynomials. We begin at ∞ and let $L(s)$, $s \in S_\infty$, be an L-series of arithmetic type. Let j be a non-negative integer. Then, as we have seen, $L(x, -j)$ is a polynomial in x^{-1} and, conjecturally (in the case $L = L(\psi, s)$), there is a polynomial factorization

$$L(x, -j) = L_{\text{triv}}(x, -j) L_{\text{nontriv}}(x, -j)\,, \tag{4}$$

where $L_{\text{triv}}(x, -j)$ is the product of the factors arising from the infinite primes and whose zeroes are the trivial zeroes at $-j$. It is important to note that these polynomials may be trivial (i.e., the constant polynomial 1). The zeroes of $L_{\text{nontriv}}(x, -j)$ are referred to as the "non-trivial zeroes at $-j$." Now let v be a finite prime and view $-j$ as lying in S_v. Let

$$L_{v,\text{triv}}(x, -j) = \prod_{\substack{\mathfrak{P} | v \\ \mathfrak{P} \text{ good}}} P_{\mathfrak{P}}(n\mathfrak{P}^j x^{-\deg n\mathfrak{P}})\,,$$

and "rename" $z_L(x, -j)$ as $L_{v,\text{nontriv}}(x, -j)$. Then by the v-adic construction we have the factorization

$$L_v(x, -j) = L_{v,\text{triv}}(x, -j) L_{v,\text{nontriv}}(x, -j)\,. \tag{5}$$

The zeroes of $L_{v,\text{nontriv}}(x - j)$ are then called the "v-adic non-trivial zeroes at $-j$," etc. Again, it is possible that these polynomials will be identically 1. Both the factorization at ∞ and at finite primes can be put in exactly the same form by setting $L_\infty(x, -j) := L(x, -j)$, $L_{\infty,\text{triv}}(x, -j) := L_{\text{triv}}(x, -j)$, etc.

In Section 5 we will see that there is, conjecturally, a further decomposition of these polynomials.

Remark 2. The reader may well wonder why, besides the obvious classical analogies, one would want to have Euler factors at the finitely many bad primes in the definition of an arithmetic L-series $L(s)$. However, we have seen how removing Euler factors adds zeroes to an L-series. These zeroes might then unnecessarily enlarge the splitting field associated to $L(s)$ and y (i.e., the algebraic extension $\mathbf{K}_L^{\rm tot}(y)$ of $\mathbf{K}_L(y)$ obtained by adjoining the zeroes of $L(x,y)$). So, from the viewpoint of splitting fields at least, having such local factors is quite desirable.

3 Krasner's Lemma

In this section we recall Krasner's Lemma and put it in a form which is particularly useful in characteristic p.

Let $\mathcal{K}$ be an arbitrary field which is complete under a general (not necessarily discrete) non-trivial non-Archimedean absolute value $|?|$. The characteristic of $\mathcal{K}$ may also be completely arbitrary. Let $\overline{\mathcal{K}}$ be a fixed algebraic closure of $\mathcal{K}$ equipped with the canonical extension of $|?|$. Let $\mathcal{F}$ be a subfield of $\overline{\mathcal{K}}$ with maximal separable (over $\mathcal{K}$) subfield $\mathcal{F}_s$. In particular, $\overline{\mathcal{K}}_s = \mathcal{K}^{\rm sep}=$ the separable closure of $\mathcal{K}$ in $\overline{\mathcal{K}}$.

Let $\alpha \in \overline{\mathcal{K}}$.

Definition 4. If α is not totally inseparable over $\mathcal{K}$ then we set

$$\delta(\alpha) = \delta_{\mathcal{K}}(\alpha) := \min_{\sigma \neq id}\{|\sigma(\alpha) - \alpha|\}\,,$$

where σ runs over the non-identity $\mathcal{K}$-injections of $\mathcal{K}(\alpha)$ into $\overline{\mathcal{K}}$. If α is purely inseparable over $\mathcal{K}$, then we set $\delta(\alpha) = 0$.

Notice that if $\mathrm{char}(\mathcal{K}) = p > 0$ then

$$\delta(\alpha^{p^i}) = \delta(\alpha)^{p^i}$$

for $i \geq 0$.

Now let β be another element in $\overline{\mathcal{K}}$. Krasner's Lemma is then stated as follows.

Proposition 1. *Suppose that α is separable over $\mathcal{K}(\beta)$ and that $|\beta - \alpha| < \delta(\alpha)$. Then $\mathcal{K}(\alpha) \subseteq \mathcal{K}(\beta)$.*

Proof. By the separability assumption, the result follows if one knows that there are no non-trivial embeddings of $\mathcal{K}(\alpha,\beta)$ over $\mathcal{K}(\beta)$. But if τ is any such injection then one has

$$|\tau(\alpha) - \alpha| = |(\tau(\alpha) - \beta) + (\beta - \alpha)| \leq |\beta - \alpha| < \delta(\alpha)$$

as $|\tau(\alpha) - \beta| = |\tau(\alpha - \beta)| = |\beta - \alpha|$. Thus $\tau = id$.

Corollary 1. *Let α be any element in $\overline{\mathcal{K}}$ and suppose that $|\beta - \alpha| < \delta(\alpha)$. Then $\mathcal{K}(\alpha)_s \subseteq \mathcal{K}(\beta)_s \subseteq \mathcal{K}(\beta)$.*

Proof. Suppose that $\mathcal{K}$ has characteristic $p > 0$. Now, for some $i \geq 0$ one knows that α^{p^i} is separable over $\mathcal{K}$. As

$$|\beta^{p^i} - \alpha^{p^i}| = |\beta - \alpha|^{p^i} < \delta(\alpha)^{p^i} = \delta(\alpha^{p^i})\,,$$

the result follows from the proposition.

Corollary 2. *Suppose that $|\beta - \alpha| < \delta(\alpha)$. Then $\delta(\beta) \leq \delta(\alpha)$ with equality if and only if $\mathcal{K}(\beta)_s = \mathcal{K}(\alpha)_s$.*

Proof. Let σ be an injection of $\mathcal{K}(\beta)$ into $\overline{\mathcal{K}}$ over $\mathcal{K}$. Then

$$\beta - \sigma(\beta) = (\beta - \alpha) + (\alpha - \sigma(\alpha)) + (\sigma(\alpha) - \sigma(\beta))\,.$$

The first and third terms on the right have the same absolute value. Moreover, by assumption, if the second term is non-zero then its absolute value is the greatest of the three; thus it is also the absolute value of $\beta - \sigma(\beta)$. The result now follows.

Let $\mathcal{K}$ have characteristic 3 and let $\lambda \in \mathcal{K}$ with $|\lambda| > 1$. Using $\alpha := \lambda^{1/2}$ and $\beta := \alpha + \lambda^{1/3^i}$, for some $i > 0$, one sees that the above corollary cannot be strengthened to an equality between $\mathcal{K}(\alpha)$ and $\mathcal{K}(\beta)$ in general.

Finally, the reader may trivially establish an Archimedean analogue of Krasner's Lemma upon defining $\delta(\alpha) := |\alpha - \bar{\alpha}|/2$ for a complex number α.

4 Review of some conjectures from [12]

Since they are used so often in this paper, we recall Conjectures 4 and 5. Let $L(s)$, $s = (x, y) \in S_\infty$, be an L-function of arithmetic type. We write

$$L(x, y) = \sum_{e=0}^{\infty} a_e(y) x^{-e}\,.$$

For each $y \in \mathbb{Z}_p$, this power series has coefficients in the finite extension $\mathbf{K}_L(y)$ of $\mathbf{K}$. As in Remark 2, we let $\mathbf{K}_L^{\text{tot}}(y)$ be the extension of $\mathbf{K}_L(y)$ obtained by adjoining the zeroes of $L(x, y)$; we let $\mathbf{K}_{L,s}^{\text{tot}}(y)$ be its maximal separable (over $\mathbf{K}_L(y)$) subfield.

The essential part of an algebraic extension of function fields in 1-variable over a finite field, whether local or global, is the maximal separable subfield. Indeed, well-known arguments show that totally-inseparable extensions are defined uniquely by their degree (see Corollary 8.2.13 of [11]).

Conjecture 4 of [12]. The field $\mathbf{K}_{L,s}^{\text{tot}}(y)$ is a finite extension of $\mathbf{K}$.

The obvious v-adic analogue of the above conjecture is also postulated in [12]

Viewed as power series in x^{-1} for fixed y, $L(x, y)$ has an associated Newton polygon in $\mathbb{R}^2$. (To distinguish between the characteristic p theory, we use X and Y for the coordinates of $\mathbb{R}^2$.)

In $\mathbf{C}_\infty$ we may write

$$L(x,y) = \prod_i (1 - \beta_i^{(y)}/x)\,.$$

Obviously only non-zero $\beta_i^{(y)}$ are of interest, in which case we set $\lambda_i^{(y)} := 1/\beta_i^{(y)}$. The valuation (using ν_∞) of $\lambda_i^{(y)}$, and so $\beta_i^{(y)}$, is computed by the Newton polygon of $L(x,y)$. Standard theory shows that the $\beta_i^{(y)}$ tend to 0 as i tends to ∞, whereas the $\lambda_i^{(y)}$ also tend to ∞; in fact, with a little thought one sees that this can be made uniform with respect to y. We call the $\beta_i^{(y)}$ (resp. $\lambda_i^{(y)}$) the "zeroes in x" of $L(s)$ (resp. "zeroes in x^{-1}"). An advantage of using $\beta_i^{(y)}$ as opposed to $\lambda_i^{(y)}$ is that the slope of a side of the Newton polygon equals the valuation of the corresponding element $\beta_i^{(y)}$; for $\lambda_i^{(y)}$ one needs to multiply by -1.

Conjecture 5 of [12]. There exists a positive real number $b = b(y)$ such that if $\delta \geq b$, then there exists at most one zero in x^{-1} of $L(x,y)$ of absolute value δ.

In other words, outside of finitely many anomalous cases, zeroes are uniquely determined by their absolute values. The conjecture is also formulated v-adically.

Conjecture 5 is based on the examples of Wan, Sheats, etc., and appears to play a role similar to the classical Generalized Riemann Hypothesis. Indeed in [12] we showed how it leads to a variant of the classical Generalized Riemann Hypothesis for number fields. It implies Conjecture 4 simply because one can then easily show that almost all zeroes of $L(s)$ are totally inseparable over $\mathbf{K}_L(y)$. To show that $\mathbf{K}_L^{\rm tot}(y)$ is itself a finite extension of $\mathbf{K}_L(y)$ (and so of $\mathbf{K}$), one factors $L(x,y)$ into the L-series of "simple motives" and then applies the Generalized Simplicity Conjecture (Conjecture 7 of [12]).

Our next section explains how to use the trivial zeroes to find counter-examples to Conjecture 5. We also suggest a reasonable modification of Conjecture 5.

5 The counter-examples

Let $L(s)$, $s = (x,y) \in S_\infty$, be an arithmetic L-series which we continue to write as $\sum_{e=0}^\infty a_e(y)x^{-e}$. Let n be a positive integer and let $y_0 \in \mathbb{Z}_p$. Suppose that the first n slopes of the Newton polygon of $L(x,y_0)$ (as a function of x^{-1}) are finite.

Lemma 1. *There is an non-trivial open neighborhood $U(y_0,n)$ of y_0 such that if $y \in U(y_0,n)$, then the first n segments of the Newton polygon in x^{-1} of $L(x,y)$ are the same as those for $L(x,y_0)$.*

Proof. The functions $a_e(y)$ are continuous. Moreover, let ν_∞ be the additive valuation associated to ∞. Then, from [2], one also has exponential lower bounds on $\nu_\infty(a_e(y))$ which are independent of y. The result follows directly.

A completely analogous v-adic result follows in the same way.

Remark 3. a. We can use the Newton polygons of $L(x, y)$ to define equivalence relations on $\mathbb{Z}_p$ (or its v-adic analogue $S_v = \mathbb{Z}_p \times H_v$ where H_v is a finite abelian group etc.) in the following fashion. Let n be a fixed positive integer. Let $y_i \in \mathbb{Z}_p$, $i = 1, 2$, be such that the Newton polygon of $L(x, y_i)$ has n finite slopes for each i. We then say that $y_1 \sim_n y_2$ if and only if the Newton polygons of both $L(x, y_1)$ and $L(x, y_2)$ have the same first n segments. If $y \in \mathbb{Z}_p$ does not have n finite slopes, then, by definition, y will only be equivalent to itself. It is clear that $\sim_n$ is an equivalence relation which only depends on $L(s)$ and n.
b. The impact of Lemma 1 is precisely that an equivalence class of $\sim_n$ consisting of more than one element is then open in $\mathbb{Z}_p$.
c. Let $y \in \mathbb{Z}_p$ belong to an open equivalence class E_y under $\sim_n$ and let m be the least non-negative integer such that $U := y + p^m\mathbb{Z}_p \subseteq E_y$. Thus, on U, the first n-segments of the Newton polygon are an invariant of the maps $z \mapsto z + \beta$ where $\beta \in p^m\mathbb{Z}_p$. We believe that such statements may be viewed as possible "micro-functional-equations" for (the Newton polygon of) $L(x, y)$. See Section 6 for an example worked out in detail.

It seems reasonable that the family of Newton polygons associated to L-series actually determine the L-series itself. We state this more succinctly in the following question.

Question 1. Let $A = \mathbb{F}_r[T]$ and let ϕ_1 and ϕ_2 be two non-isogenous Drinfeld modules over $\mathbb{F}_r(\theta)$ of the same degree. Does the family of Newton polygons serve to distinguish between $L(\phi_i, s)$ ($s \in S_\infty$) for $i = 1, 2$?

Obviously, there are many variants of Question 1 that may also be formulated.
We can now construct the counter-examples.

Example 5. Let $\mathbf{A}$ be arbitrary but where $d_\infty > 1$. If j is a positive integer divisible by $r^{d_\infty} - 1$ then $\zeta_A(s)$ has trivial zeroes at $(\zeta\pi_*^j, -j)$, where ζ runs over the d_∞-th roots of 1 with multiplicity. Thus there is a segment of the Newton polygon of $\zeta_A(x, -j)$ (in x^{-1}) which has slope j/d_∞ and whose projection to the X-axis has length $\geq d_\infty$. Lemma 1 now assures us that all y sufficiently close to $-j$ will possess this property. We now construct a counterexample to Conjecture 5 inductively. Let $y_0 = r^{d_\infty} - 1$. Let $y_1 = y_0 + p^{t_1}(r^{d_\infty} - 1)$ where t_1 is a non-negative integer chosen (in accordance with Lemma 1) so that the first n segments of the Newton polygons at $-y_0$ and $-y_1$ are the same and where these segments include the one associated to the trivial zeroes at $-y_0$. Now construct y_2 in the same fashion but where we choose t_2 to also be greater than t_1 etc. The sequence $\{y_i\}$ clearly converges to a p-adic integer $\hat{y}$.

The Newton polygon in x^{-1} of $\zeta_A(x,-\hat{y})$ will have infinitely many segments whose projection to the X-axis will have lengths $\geq d_\infty$. There are then two cases to discuss:
1. d_∞ is a pure power of p. In this case we cannot directly conclude that there are infinitely many zeroes of $\zeta_A(x,-\hat{y})$ which are not uniquely determined by their absolute value simply because we do not know a-priori that the zeroes of $\zeta_A(x,-\hat{y})$ are not totally inseparable. However, if one also assumes the Generalized Simplicity Conjecture (Conjecture 7 of [12]), then almost all such zeroes cannot be totally inseparable and so Conjecture 5 must now be false.
2. d_∞ is not a pure p-th power. In this case, there are at least two distinct d_∞-roots of unity. One can then choose the t_i sufficiently large so that the distinct trivial zeroes separate the nearby zeroes. In this case, one obtains a counter-example unconditionally.

One can often use Krasner's Lemma to obtain similar constructions as in the following example.

Example 6. Let $\mathbf{A} = \mathbb{F}_r[T]$. Let f be a prime of degree $d > 1$ with associated place v and assume that d is not a pure p-th power. Then the v-adic trivial zeroes of $\zeta_A(s)$ at $-j$ are the roots of $1 - f^j x^{-d}$. Let $j \not\equiv 0 \pmod{d}$ and let α be one such root. Then $\alpha \notin \mathbf{k}_v$. Moreover, it is easy to see that, in the notation of Section 3, we have

$$\delta_{\mathbf{k}_v}(\alpha) = |\alpha|_v\,.$$

Let $y \in S_v$ be sufficiently close to $-j$ so that $\zeta_{A,v}(x,y)$ has a zero β with $|\beta - \alpha|_v < |\alpha|_v$. By Corollary 1, the separable degree of $\mathbf{k}_v(\beta)$ is greater than 1. As such this β possesses a non-trivial Galois conjugate β' which is also a zero of $\zeta_{A,v}(x,y)$ of the same absolute value. One can now proceed as in Example 5 to obtain a counter-example to the v-adic version of Conjecture 5.

In the above example, it is easy to see that all v-adic trivial zeroes belong to a finite extension of $\mathbf{k}_v$. Thus, Krasner's Lemma does not allow us to deduce a counter-example to Conjecture 4.

Conjecture 5 may still remain valid in its original form in the much more limited case where there exists only one (including multiplicity!) trivial zero of a given absolute value. Indeed, the techniques used in the above counter-examples do not work in his case.

Ideally, one would like to negate the effects of the trivial zeroes which permit the above counter-examples. Classically one removes the effects of the trivial zeroes through the use of the Gamma-factors (as in the introduction), which are the Euler factors arising from the infinite primes, and the functional equation. Indeed, the functional equation assures us that the trivial zeroes are quite far from the critical zeroes (= all "non-trivial" zeroes).

In the characteristic p case that we are studying, it has been known for a long time that the Gamma-functions do not seem to be related to the trivial zeroes of L-series. A philosophical explanation for this phenomenon comes from the "two T's" approach; indeed, L-series have values in the field $\mathbf{C}_\infty$ whereas Gamma-functions, as they are related to exponential functions and their periods, must take values in C_∞.

In any case, we need to find other methods in the characteristic p theory. We now sketch an approach to removing the "harmful" effects of trivial zeroes based on Hensel's Lemma. The idea is simply to isolate those zeroes which are influenced by the trivial zeroes so that they can be removed from the conjectures and handled separately. Whether the definition of the L-series should be altered, as in the classical case, to physically remove these zeroes is unknown.

In order to isolate those zeroes which are sufficiently close to trivial zeroes, an affirmative answer to the following question about trivial zeroes would give the nicest situation. This question seems reasonable in view of examples and ramification considerations. So let w be a place of $\mathbf{k}$ (either ∞ or a finite place) and consider the w-adic interpolation of an L-series $L(s)$ of arithmetic type. Recall that, from Equations 4 and 5, we have a factorization

$$L_w(x,-j) = L_{w,\mathrm{triv}}(x-j)L_{w,\mathrm{nontriv}}(x,-j)\,.$$

Let $e>0$ be a real number. Then, standard non-Archimedean analysis leads to a rational factorization

$$L_w(e;x,-j) = L_{w,\mathrm{triv}}(e;x,-j)L_{w,\mathrm{nontriv}}(e;x,-j) \tag{6}$$

where $L_w(e;x,-j)$ is the product of $1-\beta/x$ where β runs through all zeroes in x of $L_w(x,-j)$ with $\nu_w(\beta)=e$; etc. (Recall that the zeroes in x of $L_w(x,y)$ uniformly tend to 0 so that their valuations tend to ∞.)

Question 2. Let j be a non-negative integer. Does there exists a constant $C>0$ (depending only on $L(s)$ and w) so that for $e>C$ the polynomials $L_{w,\mathrm{triv}}(e;x-j)$ and $L_{w,\mathrm{nontriv}}(e;x,-j)$ are relatively prime polynomials in x^{-1}?

Let us assume that the above question may be answered in the affirmative and let e be as in its statement. As $L_{w,\mathrm{triv}}(e;x,-j)$ and $L_{w,\mathrm{nontriv}}(e;x,-j)$ are relatively prime, Hensel's Lemma now applies to polynomials which are close to $L_w(e;x,-j)$ (see, eg., Thm. 4.1 of [8]).

Now let y be chosen sufficiently close to (but *not* equal to) $-j$ so that the first m segments of the Newton polygons are the same, where m is large enough so that the segment associated to e is among the first m chosen. It is reasonable to assume that $L_w(e;x,y)$, with the obvious definition, is also then close enough to $L_w(e;x,-j)$ for Hensel's Lemma to apply. Thus, under these assumptions, $L_w(e;x,y)$ will inherit a rational factorization

$$L_w(e;x,y)=L_{w,\mathrm{triv}}(e;x,y)L_{w,\mathrm{nontriv}}(e;x,y)\,. \tag{7}$$

In other words, outside of finitely many exceptional e, we would then be able to isolate those zeroes of $L_w(x,y)$ which are influenced by the trivial zeroes.

Remark 4. Even if Question 2 is answered in the negative, one can still proceed as follows. By using the Euclidean algorithm, perhaps repeatedly, we can find a factor $d_w(e;x,-j)=1+\cdots$ such that one has a relatively prime factorization

$$L_w(e;x,-j)=L_{w,\mathrm{triv}}(e;x,-j)d_w(e;x,-j)\times L_{w,\mathrm{nontriv}}(e;x,-j)/d_w(e;x,-j)\,,$$

and such that the trivial zeroes are precisely the zeroes (discounting multiplicity) of the factor on the left. One can now use Hensel's Lemma as above to this factorization.

The zeroes of $L_{w,\mathrm{triv}}(e;x,y)$ are called the "near-trivial zeroes associated to e and y," etc. They are precisely the zeroes which are influenced by the original trivial zeroes. (N.B.: If y is actually a negative integer itself, there is nothing a-priori to rule out having a near-trivial zero also being an actual trivial zero for y.) The rest of the zeroes are called the "critical zeroes" (in analogy with classical theory) and these are the ones Conjecture 5 may indeed apply to.

It remains to deal with Conjecture 4. Assuming that Conjecture 5 is established somehow for critical zeroes, the only issue that remains is to somehow establish that the field generated by all the near-trivial zeroes for a given y is also finite over $\mathbf{k}_w$. However, the degree of $L_{w,\mathrm{triv}}(e;x,y)$ is bounded (the example of $L(\rho,s)$ will suffice to convince the reader that this is so). Thus it would suffice to bound the discriminants of the maximal separable subfield of the splitting field of $L_{w,\mathrm{triv}}(e;x,y)$ (as there are only finitely many separable extensions of a local function field of bounded degree and discriminant, see Prop. 8.23.2 of [11]). Needless to say, such a problem never comes up in classical theory. However, the following examples give some evidence in favor of such bounds in the characteristic p theory.

Example 7. Let $\mathbf{A}=\mathbb{F}_3[T]$ and let v correspond to a monic prime f of degree 2. Let $z_{\zeta_A}(x,-j)$ be as in Equation 2; one computes easily that $z_{\zeta_A}(x,-5)=1+(T-T^3)x^{-1}$. Thus

$$z_{\zeta_A}(v;x,-5)=\zeta_{A,v}(x,-5)=(1-f^5x^{-2})(1+(T-T^3)x^{-1})\,.$$

The first factor gives the trivial zeroes and the second gives the non-trivial zeroes. Clearly these two factors are relatively prime. Thus for $y\in S_v$ sufficiently close to -5, Hensel's Lemma may be used. Note also that $f^{5/2}$ is obviously a separably algebraic element. Thus, if y is also close enough to -5 so that Krasner's Lemma applies, then we find that the near-trivial zeroes at y associated to $5/2$ generate $\mathbf{k}_v(\sqrt{f})$. Indeed, if β is a near-trivial zero associated to $f^{5/2}$ then β will also satisfy a quadratic equation over $\mathbf{k}_v$ (and so we deduce equality of fields as opposed to merely inclusion as in Lemma 1). Thus Corollary 2 implies that $\delta_{\mathbf{k}_v}(\beta) = \delta_{\mathbf{k}_v}(f^{5/2}) = |f^{5/2}|_v$.

Example 8. We continue with the set-up of Example 7. One has $z_{\zeta_A}(x, -4) = (1 - x^{-1})$. Thus

$$z_{\zeta_A}(v; x, -4) = \zeta_{A,v}(x, -4) = (1 - f^4 x^{-2})(1 - x^{-1})\,.$$

In this case, Hensel's Lemma implies that near-trivial zeroes associated to $\pm f^2$ are in $\mathbf{k}_v$.

Simple considerations of Newton polygons imply that both $(T^3 - T, -5)$ and $(1, -4)$ are critical zeroes for $\zeta_{A,v}(x, y)$.

6 The analytic behavior of $\zeta_{\mathbb{F}_p[\theta]}(s)$, $s \in S_\infty$

We will use the techniques and results of Diaz-Vargas [7] (see also §8.24 of [11]) and Sheats [15] to describe the influence of the trivial zeroes for $\zeta_A(s)$, $A = \mathbb{F}_r[\theta]$ and $s \in S_\infty$. We will see that, contrary to what we first expected, all zeroes of $\zeta_{\mathbb{F}_p[\theta]}(s)$ are near-trivial. In fact, examples lead us to expect this to hold for all r; the proof will take a detailed analysis of Sheats' method which we hope to return to in later works.

Our first result along these lines concerns the valuation at ∞ of the zeroes of $\zeta_A(s)$. Let $s = (x, y) \in S_\infty$ and write

$$\zeta_A(s) = \sum_{i=0}^{\infty} a_i(y) x^{-i}\,. \tag{8}$$

As before, let ν_∞ be the normalized valuation at ∞ with $\nu_\infty(1/T) = 1$.

Proposition 2. *We have $\nu_\infty(a_i(y)) \equiv 0 \pmod{r-1}$ for all i and y.*

Proof. Let j be a non-negative integer and (in the notation of [15])

$$S'_k(j) = \sum_{\substack{n \in \mathbb{F}_r[T] \\ n \text{ monic} \\ \deg(n) = k}} n^j\,.$$

The main result in [15] is to establish a formula originated by Carlitz for $\deg(S'_k(j))$ (this formula is then used to compute $\nu_\infty(a_i(y))$ and the Newton polygon of $\zeta_A(x, y)$). The formula expresses $\deg(S'_k(j))$ in terms of a certain

$k+1$-tuple, called the "greedy element," $(x_1, \ldots, x_{k+1})$ of non-negative integers such that $\sum x_t = j$ in such a way that there is no carry-over of p-adic digits and such that the first k elements are both positive and divisible by $r-1$. From this formula one obtains a formula for $\nu_\infty(a_i(y))$ by choosing j sufficiently close to $-y$ (see Equation 2.2 and Lemma 2.1 of [15]). The result follows simply by noting that this formula is linear and involves only the first k-terms of the given $k+1$-tuple.

Corollary 3. *Let $\alpha \in \mathbf{K}$ be a zero of $\zeta_A(x,y)$ for $y \in \mathbb{Z}_p$. Then $\nu_\infty(\alpha)$ is positive and divisible by $r-1$.*

Proof. The fact that $\nu_\infty(\alpha)$ is positive follows easily from general theory. To see that it is divisible by $r-1$, note that Sheats' work shows that the Newton polygon of $\zeta_A(x,y)$ only has segments of vertical length 1 (i.e., their projection to the X-axis has unit length). Thus the divisibility follows immediately from the proposition.

We now set $r = p$ in order to use Diaz-Vargas' simple techniques to compute the greedy element and to avoid problems involving carry-over of p-adic digits (this carry-over is what makes the general $\mathbb{F}_r$-case so subtle). Let n be a positive integer and let $\sim_n$ be as in Part a of Remark 3. Our first goal is to describe explicitly the equivalence classes of $\sim_n$ in $\mathbb{Z}_p$. Let $y \in \mathbb{Z}_p$ be a non-negative integer which we write p-adically as $\sum_{t=0}^{w} c_t p^t$ where $0 \le c_t < p$ for all t. We set $l(y) = l_p(y) := \sum_t c_t$ as usual. If $y \in \mathbb{Z}_p$ is not a non-negative integer then we set $l(y) = \infty$.

Proposition 3. *a. Let j be a non-negative integer. Then the degree in x^{-1} of $\zeta_A(x,-j)$ is $[l(j)/(p-1)]$ (where $[?]$ is the standard greatest integer function).*
b. Let $y \in \mathbb{Z}_p$. Then $\zeta_A(x,y)$ has at least n distinct slopes if and only if $[l(-y)/(p-1)] \ge n$.

Proof. The first part follows immediately from Diaz-Vargas' construction of the greedy element (see e.g., the proof of Lemma 8.24.11 of [11]). The second part follows from the first part and the fact that all segments of the Newton polygon of $\zeta_A(x,y)$ are known to have projections to the X-axis of unit length.

Note that Part a of the proposition allows one to compute explicitly the zeroes in y of $a_i(y)$ (as defined in Equation 8).

Now let $y \in \mathbb{Z}_p$ be chosen so that $[l(-y)/(p-1)] \ge n$ and expand $-y$ p-adically as $\sum_{t=0}^{\infty} c_i p^i$ (where it may happen that all but finitely many of the c_i vanish). Set

$$y_n = \sum_{i=0}^{e} c_i p^i$$

where $\sum_{i=0}^{e} c_i = n(p-1)$ and $c_e \neq 0$. Clearly $y_n \equiv 0 \pmod{p-1}$.

Proposition 4. *a. We have* $-y_n \sim_n y$.
b. y_n *is the smallest element in the set of positive integers* i *with* $-i \sim_n y$.
c. Let y *and* z *be in* $\mathbb{Z}_p$. *Then* $y \sim_n z$ *if and only if* $y_n = z_n$.

Proof. This again follows from Diaz-Vargas' construction of the greedy element.

Thus the open equivalence classes under $\sim_n$ are in one to one correspondence with negative integers $-j$ with $l(j) = n(p-1)$ (and, in particular, j is divisible by $p-1$). Let E be the equivalence class of one such $-j$ and write j p-adically as $\sum_{t=0}^{u} c_t p^t$ where $c_u \neq 0$. Let $\beta \in p^{u+1}\mathbb{Z}_p$. It is then clear that E is stable under the mapping $z \mapsto z + \beta$.

We finish this section by reworking the above results in a way which makes more transparent the close connection all zeta zeroes (at ∞!) have with trivial zeroes. Thus let $y \in \mathbb{Z}_p$ be arbitrary and let α be a zero (in x) of $\zeta_A(x, y)$. From Corollary 3 we know that $\nu_\infty(\alpha)$ is both positive and divisible by $p - 1 = r - 1$. Set $j := \nu_\infty(\alpha)$.

Proposition 5. *Let* $n(j, y)$ *be the number of zeroes* β *of* $\zeta_A(x, y)$ *with* $\nu_\infty(\beta) \leq j$.
a. We have $n(j, y) = l(j)/(p-1)$.
b. We have $-j \sim_n y$.
c. The zero of $\zeta_A(x, -j)$ *corresponding to* α *is precisely the trivial zero of* $\zeta_A(x, -j)$.

Proof. Clearly the trivial zero of $\zeta_A(x, -j)$ has valuation j and it is easy to see that this is the unique zero of $\zeta_A(x, -y)$ of highest valuation. The result now follows as before.

Corollary 4. *We have* $n(j, y) = O(\log(j))$.

Let $y = -1$. The i-th slope of the Newton polygon of $\zeta_A(x, -1)$ is $p^{i+1} - 1$ and it is easy to see that $n(p^{i+1} - 1, -1) = i$ is asymptotic to $\log_p(p^{i+1} - 1)$. Thus the number of zeroes of $\zeta_A(x, -1)$ of valuation $\leq x$, for a positive real x, is asymptotic to $\log_p(x)$. Of course many other such examples may be worked out.

In any case, one sees that all zeroes of $\zeta_{\mathbb{F}_p[\theta]}(s)$ are near-trivial. For general $\mathbb{F}_r[T]$, calculations indicate that Parts b and c of Proposition 5 should remain valid. If so, then Part c of Proposition 5 may ultimately afford an explanation why the results of Wan, Diaz-Vargas, Thakur, Poonen and Sheats were obtainable by elementary means. Moreover, it also shows that, as of this writing, we have had precious little experience with critical zeroes.

It is also reasonable to expect that Corollary 4 will be true for all arithmetic L-series at all primes. A much more interesting question is whether some version of Part a of Proposition 5 will be true. That is, is the analogue of $n(j, y)$ independent of y?

7 Taylor expansions of classical L-series

Let $\mathbf{A} = \mathbb{F}_r[T]$ and consider $\zeta_A(s)$, $s = (x, y) \in S_\infty$, as in Example 1. It is clear from Equation 1 that for all $y \in \mathbb{Z}_p$, $\zeta_A(x, y)$ is a power series in x^{-1} with coefficients in $\mathbf{K} = \mathbf{k}_\infty$. In this section we will establish in great generality a very similar result for the complex analytic functions $\Xi(\chi, t)$ of [12] (the definition of $\Xi(\chi, t)$ will be recalled below). Consequently, as mentioned in the introduction, these Taylor expansions reflect the (conjectured!) rationality of their zeroes in a simpler fashion than one finds for arbitrary entire complex functions.

Definition 5. Let $p(t) = \sum c_j t^j$ be a non-zero complex power series. We say that $p(t)$ is *almost real* if and only if

$$p(t) = \alpha h(t)$$

where $\alpha \in \mathbb{C}^*$ and where $h(t)$ is a non-zero power series with real coefficients.

Proposition 6. *A complex power series $p(t) = \sum c_j t^j$ is almost real if and only if the coefficients c_j satisfy the "Galois functional equation"*

$$\overline{c_j} = w c_j$$

for a fixed complex number w of absolute value 1*.*

Proof. Suppose that $p(t) = \alpha h(t)$ is almost real, where α is non-zero and $h(t) \in \mathbb{R}[[t]]$. Put $w := \overline{\alpha}/\alpha$; it is simple to check that with this w the Galois functional equation holds. Conversely, assume the Galois functional equation and let j_1 and j_2 be two non-negative integers such that $c_{j_1} \neq 0$. Then

$$\overline{c_{j_2}/c_{j_1}} = \overline{c_{j_2}}/\overline{c_{j_1}} = (wc_{j_2})/(wc_{j_1}) = c_{j_2}/c_{j_1}\,;$$

thus c_{j_2}/c_{j_1} is real. Now let j_0 be the smallest non-negative integer with $c_{j_0} \neq 0$. Then

$$p(t) = c_{j_0} \times t^{j_0}(1 + \sum_{i=1}^{\infty} b_i t^i)$$

with b_i real, and the result is established.

Now let χ be a non-trivial finite abelian character associated to a Galois extension of number fields L/k. Let $L(\chi, s)$ be the classical (complex) L-series and let $\Lambda(\chi, s)$ be the completed L-function with the Euler factors at the infinite primes. As is standard $\Lambda(\chi, s)$ is entire and there is a functional equation connecting $\Lambda(\chi, s)$ and $\Lambda(\overline{\chi}, 1-s)$. In particular,

$$\Lambda(\overline{\chi}, 1-s) = w(\chi)\Lambda(\chi, s)\,,$$

where $w(\chi)$ has absolute value 1. We then set $\Xi(\chi,t) := \Lambda(\chi, 1/2+it)$ following Riemann. Let

$$\Xi(\chi,t) = \sum_{n=0}^{\infty} a_n t^n \,,$$

be the Taylor expansion of $\Xi(\chi,t)$ about the origin.

Proposition 7. *We have*

$$\Lambda(\overline{\chi}, 1-s) = w(\chi)\Lambda(\chi, s)$$

if and only if the coefficients $\{a_n\}$ *satisfy*

$$\overline{a_n} = w(\chi) a_n \,,$$

for all n.

Proof. We know that $\overline{\Lambda(\chi,s)} = \Lambda(\overline{\chi}, \overline{s})$. Thus we see

$$\overline{\Xi(\chi,t)} = \overline{\Lambda(\chi, 1/2+it)} = \Lambda(\overline{\chi}, 1/2 - i\overline{t}) = \Xi(\overline{\chi}, -\overline{t})\,;$$

consequently, $\overline{\Xi(\chi,\overline{t})} = \Xi(\overline{\chi}, -t)$. On the other hand, the functional equation immediately gives us

$$\begin{aligned} \Xi(\overline{\chi}, -t) &= \Lambda(\overline{\chi}, 1/2 - it) \\ &= \Lambda(\overline{\chi}, 1 - (1/2+it)) \\ &= w(\chi)\Lambda(\chi, 1/2+it) = w(\chi)\Xi(\chi,t)\,. \end{aligned}$$

Consequently we deduce that

$$\overline{\Xi(\chi,\overline{t})} = w(\chi)\Xi(\chi,t)\,. \tag{9}$$

The only if part now follows upon substituting in the power series for $\Xi(\chi,t)$. The if part follows since these calculations are reversible.

Theorem 1. *The existence of a classical style functional equation for* $\Lambda(\chi,s)$ *is equivalent to the Taylor expansion at the origin* $t=0$ *of* $\Lambda(\chi, 1/2+it)$ *being an almost real power series.*

Proof. This follows directly from Propositions 7 and 6.

For Dedekind zeta functions a completely similar result may easily be established along with some vanishing of the Taylor coefficients. In many instances it is known that classical L-series may be factored as infinite products over their zeroes. Such a factorization gives another approach to showing that the Taylor expansion of $\Xi(\chi,t)$ is almost real.

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