ζ-PHENOMENOLOGY

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Abstract. It is well known that Euler experimentally discovered the functional equation of the Riemann zeta function. Indeed he detected the fundamental $s \mapsto 1-s$ invariance of $\zeta(s)$ by looking only at special values. In particular, via this functional equation, $\mathbb{Z}/(2)$ is realized as a group of symmetries of $\zeta(s)$. If one includes complex conjugation, one then has a group of symmetries of $\zeta(s)$ of order 4. In this paper, we use the theory of special-values of our characteristic p zeta functions to experimentally detect a natural symmetry group for these functions of cardinality $\mathfrak{c} = 2^{\aleph_0}$ (where $\mathfrak{c}$ is the cardinality of the continuum). This group appears as homeomorphisms of $\mathbb{Z}_p$ which stabilize both the positive and negative integers. The exact form that these symmetries will ultimately take is not known at this time.

1. Introduction

Euler's work on the zeta function has been an inspiration to us for many years. This work is briefly summarized in our Section 2, but we highly recommend [Ay1] to the reader. Euler was able to compute the the values of the Riemann zeta function at the positive even integers and at the negative integers. By very cleverly comparing them, he found the basic symmetry given by the famous functional equation of $\zeta(s)$. In particular, the lesson Euler taught us was that the special values are a window allowing one to glimpse very deep internal structure of the zeta function.

In the characteristic p theory, we have long had good results on special values in the basic case where the base ring A is $\mathbb{F}_q[T]$. At the positive integers, one had the brilliant analog of Euler's results due to L.Carlitz in the 1930's and 1940's (complete with an analogs of $2\pi i$, Bernoulli numbers, factorials, etc.). At the negative integers, one also had good formulas for the values of the characteristic p zeta function. However, all attempts to put the positive and negative integers together in an "$s \mapsto 1-s$" fashion failed.

The theory of these characteristic p functions works for any of Drinfeld's rings A (the affine ring of a complete smooth curve over $\mathbb{F}_q$ minus a fixed closed point ∞). It is however substantially harder to do explicit calculations for general A and so there are not that many specific examples yet to study.

In the 1990's Dinesh Thakur [Th1] looked at the "trivial-zeroes" of these zeta-functions for certain non-polynomial rings A. He found the intriguing phenomenon that such trivial-zeroes may have a higher order vanishing than naturally arises from the theory (current theory only gives a very classical looking lower bound on this vanishing in general, not the exact order!). More recently Javier Diaz-Vargas [DV1] extended Thakur's calculations. Both Thakur and Diaz-Vargas experimentally found that this general higher vanishing at $-j$ appears to be associated with j of a very curious type: the sum of the q-adic digits of j must be bounded.

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This paper is dedicated to my wife Rita.

We have been trying to come to grips with the implications of Thakur's and Diaz-Vargas's inspired calculations for a few years; see [Go2]. Quite recently we discovered a huge group of symmetries that seems to underlie their calculations. It is our purpose here to describe the evidence for this.

This paper is written to quickly explain these implications to the reader. We try as much as possible to stay away from general theory and keep the paper as self-contained as possible.

The symmetry group $S_{(q)}$ is introduced in Section 5. It is a group of homeomorphisms of $\mathbb{Z}_p$ obtained by simply rearranging the q-expansion coefficients. (In Remark 3, we will give one method of extending this action to all of the domain space $\mathbb{S}_\infty$ – see Definition 6 – of our characteristic p functions.) In particular, we readily see that $S_{(q)}$ stabilizes *both* the positive and negative integers; there is no mixing as in $s \mapsto 1-s$. Thus, perhaps, we have the "true" explanation for the failure to somehow put the positive and negative integers together as Euler did.

However all may not be lost here in terms of relating the positive and negative integers. Indeed, we *also* find that $S_{(q)}$ is realized as symmetries of Carlitz's "von Staudt" result where he calculated the denominator of his Bernoulli analogs. This is very exciting and highly mysterious to us.

The evidence linking $S_{(q)}$ to our zeta functions at both the positive and negative integers is presented in our last two subsections.

This paper grew out of my lecture at the workshop "Noncommutative geometry and geometry over the field with one element" at Vanderbilt University in May, 2008. It is my great pleasure to thank the organizers of this very interesting meeting for their kind hospitality and support.

2. Euler's creation of ζ-phenomenology

We recall here very briefly the fabulous first example of ζ-phenomenology: Euler's numerical discovery of the functional equation of the Riemann zeta function $\zeta(s)$. Our treatment here follows that of [Ay1]; we have also covered these ideas in [Go2].

Definition 1. The Bernoulli numbers, B_n, are defined by

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}\frac{B_n x^n}{n!}\,.$$

After many years of work, Euler computed the values $\zeta(2n)$, $n=1,2\cdots$ in terms of Bernoulli numbers and obtained the famous formula

$$B_{2n}=(-1)^{n+1}\frac{2(2n)!}{(2\pi)^{2n}}\zeta(2n)\,. \tag{1}$$

Euler then turned his attention to the values of $\zeta(s)$ at the negative integers where his work on special values becomes divinely inspired! Indeed, Euler did not have the notion of analytic continuation of complex valued functions to work with. Thus he relied on his instincts for beauty while working with divergent series; nevertheless, he obtained the right values.

Euler begins with the very well known expansion

$$\frac{1}{1-x}=1+x+x^2+x^3+\cdots+x^n+\cdots\,. \tag{2}$$

Clearly this expansion is only valid when $|x| < 1$, but that does not stop Euler. Upon putting $x = -1$, he deduces

$$1/2 = 1 - 1 + 1 - 1 + 1 \cdots , \tag{3}$$

where we simply ignore questions of convergence! He then applies the operator $x\frac{d}{dx}$ to Equation 2 and again evaluates at $x = -1$ obtaining

$$1/4 = 1 - 2 + 3 - 4 + 5 \cdots . \tag{4}$$

Applying the operator again, Euler finds the "trivial zero"

$$0 = 1 - 2^2 + 3^2 - \cdots , \tag{5}$$

and so on. Euler recognizes the sum on the right of these equations to be the values at the negative integers of the modified ζ-function

$$\zeta^*(s) := (1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} (-1)^{n-1}/n^s . \tag{6}$$

The wonderful point is, of course, that these values *are* the values rigorously obtained much later by Riemann. (N.B.: in [Ay1], our $\zeta^*(s)$ is denoted $\phi(s)$.)

Nine years later, Euler notices, at least for small $n \geq 2$, that his calculations imply

$$\frac{\zeta^*(1-n)}{\zeta^*(n)} = \begin{cases} \frac{(-1)^{(n/2)+1}(2^n-1)(n-1)!}{(2^{n-1}-1)\pi^n} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{7}$$

Upon rewriting Equation 7 using his gamma function $\Gamma(s)$ and the cosine, Euler then "hazards" to conjecture

$$\frac{\zeta^*(1-s)}{\zeta^*(s)} = \frac{-\Gamma(s)(2^s - 1)\cos(\pi s/2)}{(2^{s-1} - 1)\pi^s} , \tag{8}$$

which translates easily into the functional equation of $\zeta(s)$!

Remark 1. Note the important role played by the trivial zeroes in Equation 7 in that they render harmless our inability to calculate explicitly $\zeta^*(n)$, or $\zeta(n)$, at odd integers > 1.

Euler then calculates both sides of Equation 8 at $s = 1$ and obtains the same answer. To Euler, this is "strong justification" for his conjecture. Of course history has proved him to be spectacularly right!

From now on the symbol "$\zeta(s)$" will be reserved for characteristic p valued functions.

3. The factorial ideal

In order to later define Bernoulli elements in characteristic p, and so explain Carlitz's von Staudt result, we clearly need a good notion of "factorial".

We begin by reviewing the basic set-up of the characteristic p theory. We let $q = p^{n_0}$ where p is prime and n_0 is a positive integer. Let X be a smooth projective geometrically connected curve over the finite field $\mathbb{F}_q$ with q-elements. Choose ∞ to be a fixed closed point on X of degree d_∞ over $\mathbb{F}_q$. Thus $X - \infty$ is an affine curve and we let A be the ring of its global functions. Note that A is a Dedekind domain with finite class group and that $A^* = \mathbb{F}_q^*$. We let k denote the quotient field of A. The completion of k at ∞ is denoted k_∞ and the completion of a fixed algebraic closure of k_∞ (under the canonical topology) is denoted $\mathbb{C}_\infty$. We let $\mathbb{F}_\infty \subset k_\infty$ be the associated finite field. Set $q_\infty := q^{d_\infty}$ so that $\mathbb{F}_\infty \simeq \mathbb{F}_{q_\infty}$.

Of course the simplest example of such an A is $\mathbb{F}_q[T]$, $k = \mathbb{F}_q(T)$. In general though, A will not be Euclidean or factorial.

Let x be a transcendental element.

Definition 2. 1. For $i = 1, 2, \cdots$, we set $[i](x) := x^{q^i} - x$.
2. We define $L_0(x) \equiv 1$ and for $i = 1, 2, \cdots$, we set $L_i(x) := [i](x)[i-1](x)\cdots[1](x)$.
3. We define $D_0(x) \equiv 1$ and for $i = 1, 2, \cdots$, we set $D_i(x) := [i](x)[i-1](x)^q \cdots [1](x)^{q^{i-1}}$.

Elementary considerations of finite fields allow one to show the following proposition (see Prop. 3.1.6 [Go1]).

Proposition 1. 1. *$[i](x)$ is the product of all monic irreducible polynomials in x whose degree divides i.*
2. *$L_i(x)$ is the least common multiple of all polynomials in x of degree i.*
3. *$D_i(x)$ is the product of all monic polynomials in x of degree i.*

As we will readily see later on (Proposition 3) the polynomials $D_i(x)$ and $L_i(x)$ are universal for the exponential and logarithm of general Drinfeld modules.

Our next goal is to use the functions $D_i(x)$ to define a factorial function á la Carlitz. Let j be an integer that we write q-adically as $j = \sum_{e=0}^{w} c_e q^e$ where $0 \leq c_e < q$ all e.

Definition 3. We set

$$\Pi_j(x) := \prod_{e=0}^{w} D_e(x)^{c_e} .$$

The function $\Pi_j(x)$ satisfies many of the same divisibility results as the classical $n!$.

Let A be an arbitrary affine ring as above. We now define the basic ideals of A of interest to us. Let $f(x) \in \mathbb{F}_q[x]$.

Definition 4. We set $\tilde{f} := (f(a))_{a \in A}$; i.e., $\tilde{f}$ is the ideal generated by the values of $f(x)$ on the elements of A.

In general one would expect these ideals to be trivial (i.e., equal to A itself) as the example $f(x) = x + 1$ shows. However, for the functions given in Definition 2, they are highly non-trivial.

Example 1. We show here that $\tilde{[i]} = \prod \mathfrak{P}$ where the product ranges over all primes of degree (over $\mathbb{F}_q$) dividing i. Let $\mathfrak{P}$ have degree dividing i; then modulo $\mathfrak{P}$ we have $a^{q^i} = a$ (or $a^{q^i} - a = 0$) for any a. Thus $\mathfrak{P}$ must divide $\tilde{[i]}$. Now let $a \in A$ be a uniformizer at $\mathfrak{P}$. Then clearly so is $[i](a)$. Therefore $\mathfrak{P}^2$ does not divide $\tilde{[i]}$. Finally, a moment's thought along these lines also shows that the only possible prime divisors of $\tilde{[i]}$ are those whose degree divides i.

Let $\mathfrak{P}$ be a prime of A with additive valuation $v_{\mathfrak{P}}$. Thakur observed that for a function $f(x)$, $v_{\mathfrak{P}}(\tilde{f}) = v_{\mathfrak{P}}(\tilde{f}_{\mathfrak{P}})$ where $\tilde{f}_{\mathfrak{P}}$ is the analog of $\tilde{f}$ constructed locally on the completion $A_{\mathfrak{P}}$ of A at $\mathfrak{P}$. As a consequence, we need only compute these valuations on $A = \mathbb{F}_q[T]$ where it is known that the ideals associated to the functions of Definition 2 and Definition 3 are generated by their values at $x = T$. Thus, using Theorem 9.1.1 of [Go1], we have the following basic factorization of $\tilde{\Pi}_j$.

Proposition 2. *Let $\mathfrak{P}$ be a prime of A of degree d. Then*

$$v_{\mathfrak{P}}(\tilde{\Pi}_j) = \sum_{e \geq 1} [j/q^{ed}] ,$$

where $[w]$ *is the greatest integer function,* $w \in \mathbb{Q}$.

In the fundamental case $A = \mathbb{F}_q[T]$, Proposition 2 was proved by W. Sinnott; it is clearly a direct analog of the calculation of the p-adic valuation of $n!$.

Finally, we explain the relationship with Drinfeld modules that the reader may skip as it is not needed for the remainder of the paper. As before, let k be the quotient field of A. Let L be a finite extension of k with O_L the ring of A-integers in L. Let ψ be a Drinfeld module of arbitrary rank over L with coefficients in O_L. Let $e(z) = z + \sum_{i\geq 1} e_i z^{q^i}$ and $l(z) = z + \sum_{i\geq 1} l_i z^{q^i}$ be the exponential and logarithm of the Drinfeld module (obtained say by embedding L into $\mathbb{C}_\infty$). Let $a \in A$.

Proposition 3. *The elements* $D_i(a)e_i$ *and* $L_i(a)l_i$ *lie in* O_L.

Proof. One has the basic recurrence relations

$$e(az) = \psi_a(e(z))$$

and

$$al(z) = l(\psi_a(z))\,.$$

The result now follows by induction and the definition of $D_i(x)$ and $L_i(x)$. □

4. Integral ζ-values

4.1. **Exponentiation of Ideals.** As mentioned in the introduction, we shall define these values here with a minimum of theory and refer the reader to Chapter 8 of [Go1] for the elided details. Our goal is to define an analog of n^j where n is a positive integer and j is an arbitrary integer. However, as general A is *not* factorial, we have to define "$\mathfrak{I}^j$" as an element of $\mathbb{C}_\infty^*$ for non-principal $\mathfrak{I}$. Here we immediately run into a notational issue in that the symbol "$\mathfrak{I}^j$" is universally reserved for taking the j-th power of the *ideal* $\mathfrak{I}$ in the Dedekind domain A. We do *not* change this; rather we will use "$\mathfrak{I}^{(j)}$" for the above element of $\mathbb{C}_\infty^*$ so that there will be no confusion.

Recall that the completion of k at ∞ is denoted k_∞ with $\mathbb{F}_\infty \subset k_\infty$ being the associated finite field; recall also that $q_\infty = q^{d_\infty}$. Fix an element $\pi \in k_\infty^*$ of order 1. Every element $x \in k_\infty^*$ has a unique decomposition:

$$x = \zeta_x \pi^{v_\infty(x)} u_x\,, \tag{9}$$

where $\zeta_x \in \mathbb{F}_\infty^*$, $v_\infty(x) \in \mathbb{Z}$, and $u_x \in k_\infty$ is a 1-unit (i.e., congruent to 1 modulo (π)) and depends on π). We say x is "positive" or "monic" if and only if $\zeta_x = 1$. Clearly the positive elements form a subgroup of finite index of k_∞^*.

Definition 5. We set $\langle x\rangle := u_x$ where u_x is defined in Equation 9.

As mentioned, the element $\langle x\rangle$ depends on π, but no confusion will result by not making this dependence explicit.

Note that $x \mapsto \langle x\rangle$ is a homomorphism from k_∞^* to its subgroup of 1-units.

As above, X is the smooth projective curve associated to k. For any fractional ideal I of A, we let $\deg_k(I)$ be the degree over $\mathbb{F}_q$ of the divisor associated to I on the affine curve $X - \infty$. For $\alpha \in k^*$, one sets $\deg_k(\alpha) = \deg_k((\alpha))$ where (α) is the associated fractional ideal; this clearly agrees with the degree of a polynomial in $\mathbb{F}_q[T]$.

Definition 6. Set $\mathbb{S}_\infty := \mathbb{C}_\infty^* \times \mathbb{Z}_p\,.$

The space $\mathbb{S}_\infty$ plays the role of the complex numbers in our theory in that it is the domain of "n^s." Indeed, let $s = (x, y) \in \mathbb{S}_\infty$ and let $\alpha \in k$ be positive. The element $v = \langle \alpha \rangle - 1$ has absolute value < 1; thus $\langle \alpha \rangle^y = (1 + v)^y$ is easily defined and computed via the binomial theorem.

Definition 7. We set

$$\alpha^s := x^{\deg_k(\alpha)} \langle \alpha \rangle^y \,. \tag{10}$$

Clearly $\mathbb{S}_\infty$ is a group whose operation is written additively. Suppose that $j \in \mathbb{Z}$ and α^j is defined in the usual sense of the canonical $\mathbb{Z}$-action on the multiplicative group. Let $\pi_* \in \mathbb{C}_\infty^*$ be a fixed d_∞-th root of π. Set $s_j := (\pi_*^{-j}, j) \in \mathbb{S}_\infty$. One checks easily that Definition 7 gives $\alpha^{s_j} = \alpha^j$. When there is no chance of confusion, we denote s_j simply by "j."

In the basic case $A = \mathbb{F}_q[T]$ one can now proceed to define zeta-values. However, in general A has non-principal and positively generated ideals. Fortunately there is a canonical and simple procedure to extend Definition 7 to them as follows. Let $\mathcal{I}$ be the group of fractional ideals of the Dedekind domain A and let $\mathcal{P} \subseteq \mathcal{I}$ be the subgroup of principal ideals. Let $\mathcal{P}^+ \subseteq \mathcal{P}$ be the subgroup of principal ideals which have positive generators. It is a standard fact that $\mathcal{I}/\mathcal{P}^+$ is a finite abelian group. The association

$$\mathfrak{h} \in \mathcal{P}^+ \mapsto \langle \mathfrak{h} \rangle := \langle \lambda \rangle \,, \tag{11}$$

where λ is the unique positive generator of $\mathfrak{h}$, is obviously a homomorphism from $\mathcal{P}^+$ to $U_1(K) \subset \mathbb{C}_\infty^*$.

Let $U_1(\mathbb{C}_\infty) \subset \mathbb{C}_\infty^*$ be the group of 1-units defined in the obvious fashion. The binomial theorem, again, shows that $U_1(\mathbb{C}_\infty)$ is a $\mathbb{Z}_p$-module. However, it is also closed under the unique operation of taking p-th roots; as such, $U_1(\mathbb{C}_\infty)$ is a $\mathbb{Q}_p$-vector space.

Lemma 1. *The mapping $\mathcal{P}^+ \to U_1(\mathbb{C}_\infty)$ given by $\mathfrak{h} \mapsto \langle \mathfrak{h} \rangle$ has a unique extension to $\mathfrak{I}$ (which we also denote by $\langle ? \rangle$).*

Proof. As $U_1(\mathbb{C}_\infty)$ is a $\mathbb{Q}_p$-vector space, it is a divisible group; thus the extension follows by general theory. The uniqueness then follows by the finitude of $\mathcal{I}/\mathcal{P}^+$. □

If $s \in \mathbb{S}_\infty$ and I as above, we now set

$$I^s := x^{\deg_k(I)} \langle I \rangle^y \,. \tag{12}$$

Thus if $\alpha \in k$ is positive one sees that $(\alpha)^s$ agrees with α^s as in Equation 10.

For a fractional ideal $\mathfrak{I}$ and integer j, as promised, we now put $\mathfrak{I}^{(j)} := \mathfrak{I}^{s_j}$. Thus if $a \in k$ is positive then $(a)^{(j)} = a^j$ by definition.

The values $\mathfrak{I}^{(j)}$ are obviously determined multiplicatively by $\mathfrak{I}^{(1)}$. Furthermore, suppose $\mathfrak{I}^t = (i)$ where i is positive where t is a positive integer (which always exists as the ideal class group is finite) and put $\mathfrak{i} = \mathfrak{I}^{(1)}$. Then we have the basic formula

$$\mathfrak{i}^t = i \,. \tag{13}$$

From this it is very easy to see that the values $\mathfrak{I}^{(1)}$ generate a finite extension V of k in $\mathbb{C}_\infty$. It is also easy to see that $\mathfrak{I}$ becomes principal in this field and is generated by $\mathfrak{I}^{(1)}$.

4.2. **The ζ-values.** Let j be an arbitrary integer.

Definition 8. We formally put

$$\zeta(j) := \sum_{\mathfrak{I}} \mathfrak{I}^{(-j)} = \sum_{\mathfrak{I}} \mathfrak{I}^{s_{-j}} \,;$$

where $\mathfrak{I}$ ranges over the ideals of A and $s_{-j} \in \mathbb{S}_\infty$ was defined after Definition 7 .

Because the analysis is non-Archimedean, the sum $\zeta(j)$ clearly converges to an element of $\mathbb{C}_\infty$ for $j > 0$. At the non-positive integers we must regroup the sum. More precisely, for $j \geq 0$ we write

$$\zeta(-j) = \sum_{e=0}^{\infty} \left(\sum_{\deg_k(\mathfrak{I}) = e} \mathfrak{I}^{(j)} \right) \,, \tag{14}$$

where, as above, $\deg_k(\mathfrak{I})$ is the degree over $\mathbb{F}_q$ of $\mathfrak{I}$. As is now well-known (see, e.g., Chapter 8 of [Go1]) for sufficiently large e the sum in parentheses vanishes. Thus the value is an algebraic integer over A.

We note that the values given above are special values of a function $\zeta(s)$ defined for all $s \in \mathbb{S}_\infty$.

5. The Group $S_{(q)}$

In this section we will introduce the automorphism groups of interest to us. These will be subgroups of the group of homeomorphisms of $\mathbb{Z}_p$ and and they will stabilize – and so permute – both the positive and negative integers.

Let q continue to be a power of p and let $x \in \mathbb{Z}_p$. Write x q-adically as

$$x = \sum_{i=0}^{\infty} c_i q^i \tag{15}$$

where $0 \leq c_i < q$ for all i. If x is a non-negative integer (so that the sum in Equation 15 is obviously finite), then we set

$$\ell_q(x) = \sum_i c_i \,. \tag{16}$$

Let ρ be a permutation of the set $\{0, 1, 2, \ldots\}$.

Definition 9. We define $\rho_*(x)$, $x \in \mathbb{Z}_p$, by

$$\rho_*(x) := \sum_{i=0}^{\infty} c_i q^{\rho(i)} \,. \tag{17}$$

Clearly $x \mapsto \rho_*(x)$ gives a representation of ρ as a set permutation (in fact, as we will see in Proposition 4, a homeomorphism) of $\mathbb{Z}_p$.

Definition 10. We let $S_{(q)}$ be the group of permutations of $\mathbb{Z}_p$ obtained as ρ varies over all permutations of $\{0, 1, 2, \ldots\}$.

Remark 2. We use the notation "$S_{(q)}$" to avoid confusion with the symmetric group S_q on q-elements.

Note that if q_0 and q_1 are powers of p, and $q_0 \mid q_1$, then $S_{(q_1)}$ is naturally realized as a subgroup of $S_{(q_0)}$.

The next proposition gives the basic properties of the mapping $\rho_*(x)$.

Proposition 4. *Let $\rho_*(x)$ be defined as above.*
1. *The mapping $x \mapsto \rho_*(x)$ is continuous on $\mathbb{Z}_p$.*
2. *("Semi-additivity") Let x, y, z be three p-adic integers with $z = x + y$ and where there is no carry over of q-adic digits. Then $\rho_*(z) = \rho_*(x) + \rho_*(y)$.*
3. *The mapping $x \mapsto \rho_*(x)$ stabilizes the non-negative integers.*
4. *The mapping $x \mapsto \rho_*(x)$ stabilizes the negative integers.*
5. *Let n be a non-negative integer. Then $\ell_q(n) = \ell_q(\rho_*(n))$.*
6. *Let n be an integer. Then $n \equiv \rho_*(n) \pmod{q-1}$.*

Proof. To see Part 1, let j be a positive integer. We want to show that the first q^j expansion coefficients of $\rho_*(x)$ and $\rho_*(y)$ are the same if $x \equiv y \pmod{q^t}$ for some positive integer t. Let ϕ be the inverse permutation to ρ (as functions on the non-negative integers). Choose t so that q^t is greater than $\phi(e)$ for $e = 0, ..., q^j - 1$. Parts 2 and 3 are obvious. To see Part 4, let n be a negative integer and let j be a positive integer chosen so that $q^j + n$ is non-negative. Then q-adically we have

$$n = (q^j + n) - q^j = (q^j + n) + (q-1)q^j + (q-1)q^{j+1} + \cdots\,, \tag{18}$$

as $-1 = q - 1 + (q-1)q + (q-1)q^2 \cdots$. On the other hand, $\rho_*(n)$ will now clearly also have almost all of its q-adic coefficients equal to $q-1$ and the result is clear. Part 5 is clear and implies Part 6 for non-negative n as then we have $n \equiv \ell_q(n) \pmod{q-1}$. Thus suppose n is negative. As in Equation 18 write $n = (q^j + n) - q^j$ with $q^j + n$ non-negative. By Part 2, we have

$$\rho_*(n) = \rho_*(q^j + n) + \rho_*(-q^j)\,. \tag{19}$$

Clearly $\rho_*(-q^j)$ has almost all coefficients equal to $q-1$ with the rest equaling 0; thus we can write $\rho_*(-q^j) = m - q^t$ for some t where m is positive and divisible by $q-1$. Part 6 for non-negative integers now implies that modulo $q-1$ we have

$$\begin{aligned} \rho_*(n) &\equiv \rho_*(q^j + n) + m - q^t \\ &\equiv (q^j + n) - q^t \\ &\equiv 1 + n - 1 \\ &\equiv n\,. \end{aligned}$$

□

Thus, by Parts 3 and 4 of Proposition 4, ρ_* permutes both the nonpositive and nonnegative integers.

Notice further that the injection $x \mapsto p^e x$ (e a positive integer) is not in $S_{(p)}$ as it is not surjective. However, let n be a positive integer. Then clearly $p^e n = \rho_*(n)$ for infinitely many $\rho_* \in S_{(p)}$ (which may vary with n). Note, however, that multiplication by p will change the set of q-adic digits of an integer if $q > p$, etc. Thus in this case pn will not equal $\rho_*(n)$ for any $\rho_* \in S_{(q)}$.

The reader can readily see that the cardinality of $S_{(p)}$ is $\mathfrak{c}$ (where $\mathfrak{c}$ is the cardinality of the continuum).

Remark 3. We finish this section by by presenting one method of extending ρ_* to a homeomorphism of all of $\mathbb{S}_\infty$ where ρ is a permutation of $\{0,1,2,\cdots\}$ as before; perhaps there are others. The reader may skip this on a first reading. The discussion in Subsection 6.1 leads us to demand the following property of any such extension:

$$\rho_*(s_{-t}) = s_{-\rho_*(t)}\,, \tag{20}$$

for integers t where $\rho_*(t)$ is given in Definition 9. Moreover, it is clear that to give an extension of ρ_* to $\mathbb{S}_\infty$, we need only give an action of ρ on $\mathbb{C}_\infty^*$. Let $\mathbb{O} \subset \mathbb{C}_\infty$ be the ring of integers and let Λ be a chosen set of representatives of $\mathbb{O}/(\pi_*)$ where π_* is defined after Definition 7. We assume that $\mathbb{F}_\infty \subset \Lambda$, where we recall that $\mathbb{F}_\infty$ is the finite field inside k_∞. A little thought then assures us that we can write every element $x \in \mathbb{C}_\infty$ uniquely as

$$x = \sum_{j \gg -\infty} c_j \pi_*^j \tag{21}$$

where $\{c_j\} \subseteq \Lambda$. We rewrite this, a bit perversely, as

$$x = \sum_{j \gg -\infty} c_j \pi_*^{-(-j)}\,. \tag{22}$$

Thus we define

$$\rho_*(x) := \sum_{j \gg -\infty} c_j \pi_*^{-\rho_*(-j)}\,, \tag{23}$$

where, again, $\rho_*(-j)$ is given in Definition 9. This definition makes sense precisely because ρ_* gives rise to permutations of both the positive and negative integers and so there is no mixing. The same argument as in the first Part of Proposition 4 tells us that this action is again a homeomorphism of $\mathbb{C}_\infty$ and $\mathbb{C}_\infty^*$. It is then easy to see that this action has the desired properties on $\mathbb{S}_\infty$. Of course the action of ρ depends on the choice of Λ but it is substantially more canonical on $\mathbb{F}_\infty((\pi_*))^* \subset \mathbb{C}_\infty^*$. Additionally, it appears almost certain that the mapping $x \mapsto \rho_*(x)$ is not locally analytic (or even differentiable) on $\mathbb{C}_\infty$. Perhaps, however, there will ultimately be ways to compensate for this.

6. $S_{(q)}$ as symmetries of $\zeta(s)$

In this last section, we present the evidence showing how $S_{(q)}$ and its subgroups arise as symmetries of the ζ-values of Section 4.2. The evidence we have is Eulerian by its very nature as we *only* use the special values. However, unlike Euler, we cannot now guess at the mechanism that exhibits these groups as automorphisms of the full two-variable zeta function.

As we saw in Proposition 4, the group $S_{(q)}$ permutes both the positive and negative integers. Each set of integers separately gives evidence that $S_{(q)}$ acts as symmetries of $\zeta(s)$. However, it may ultimately turn out that both types of evidence are really manifestations of the same underlying symmetries.

We begin with the evidence from the negative integers.

6.1. **Evidence from ζ-values at negative integers.** In this section we present the evidence that $S_{(q)}$ acts as symmetries of $\zeta(s)$ arising from the negative integers. We believe that this evidence has greater impact than the evidence given in the next subsection (at the positive integers) because it represents actual symmetries associated to the zeroes of $\zeta(s)$.

We shall see, experimentally at least, that the orders of vanishing of $\zeta(s)$ at negative integers appear to be invariants of the action of $S_{(q)}$.

Here is what is known about such vanishing in general. Recall that $q_\infty := q^{d_\infty}$ where d_∞ is the degree of ∞ over $\mathbb{F}_r$. Let j be a positive integer which is divisible by $q_\infty - 1$. Then it is known that $\zeta(-j) = 0$ (see, e.g. Section 8.13 of [Go1]). The theory that exists at this moment then naturally gives very classical looking *lower bounds* on the order of vanishing of these "trivial zeroes." In the case $d_\infty = 1$ this bound is 1; when $d_\infty > 1$, it may be greater than 1. As our examples here all have $d_\infty = 1$, we refer the interested reader to [Go1] for the general case.

The first example is $A = \mathbb{F}_q[T]$. Here it is known [Sh1] that *all* zeroes are simple and that $\zeta(-j) \neq 0$ for $j \not\equiv 0 \pmod{q-1}$. By Part 6 of Proposition 4, we have $j \equiv \rho_*(j) \pmod{q-1}$. Thus the next proposition follows immediately.

Proposition 5. *Let $A = \mathbb{F}_q[T]$. Then the order of vanishing of $\zeta(s)$ at $-j$, j positive, is an invariant of the action of $S_{(q)}$ on the positive integers.*

By itself, Proposition 5 is certainly not overwhelming evidence for realizing $S_{(q)}$ as a symmetry of the full zeta function. However, in the case of some non-polynomial A with ∞ rational, Dinesh Thakur [Th1], Theorem 5.4.9 of [Th2], and Javier Diaz-Vargas [DV1] have produced some fundamentally important calculations on zeroes at negative integers. They found examples of trivial zeroes where the natural lower bound was *not* the exact order of vanishing; of course, this is something that never happens in classical theory. Trivial zeroes where the order of vanishing is greater than the predicted lower bound are thus called "non-classical." (In general a zeta value at the negative integers is called "non-classical" whenever the vanishing is higher than predicted [N.B.: the prediction could be that the value is nonzero!].)

The calculations of Thakur and Diaz-Vargas seem to imply that the non-classical trivial zeroes occur at $-j$ where $\ell_q(j)$ is *bounded*. Moreover, their calculations *also* continue to exhibit $S_{(q)}$ invariance of the orders of vanishing of even these non-classical trivial zeroes.

As such we expect Proposition 5 to remain true for general A where q will need to be replaced by q_∞.

As analytic objects on $\mathbb{S}_\infty$, our zeta functions are naturally 1-parameter families of entire power series where the parameter is $y \in \mathbb{Z}_p$. Having such a huge group acting on $\mathbb{Z}_p$ may ultimately give us good control of the family.

6.2. **Evidence from ζ-values at positive integers.** We now finish by discussing the evidence arising from Carlitz's von Staudt result.

The work of David Hayes on "sign-normalized" rank one Drinfeld modules (see [Hay1] or Chapter 7 of [Go1]) shows the existence of a special Drinfeld module ψ with the following properties: It is defined over the ring of integers in a certain Hilbert Class Field of k (ramified at ∞) lying in $\mathbb{C}_\infty$ which we denote by H^+. Let $\mathfrak{I}$ be an ideal of A. Then the product of all $\mathfrak{I}$-division values of ψ lies in H^+ and is an explicit generator of $\mathcal{O}^+\mathfrak{I}$ where $\mathcal{O}^+$ is the ring of A-integers. The lattice L associated to ψ may be written $A\xi$ for a transcendental element $\xi \in \mathbb{C}_\infty$.

Let $T := V \cdot H^+$ be the compositum of V and H^+ where V is defined after Equation 13. The following result is shown in [Go1] (Theorem 8.18.3).

Theorem 1. *Let j be a positive integer divisible by $q_\infty - 1$ and let $\zeta(j)$ be defined as in* Definition 8. *Then*

$$0 \neq \zeta(j)/\xi^j \in T\,. \tag{24}$$

Theorem 1 was established in the basic $\mathbb{F}_q[T]$-case by L. Carlitz in the 1930's; in this case, both V and H^+ equal k.

Let $\mathcal{O}_T$ be the A-integers of T.

Definition 11. Let j be divisible by $q_\infty - 1$. We define $\widetilde{BC}_j$ to be the $\mathcal{O}_T$ fractional ideal generated by $\tilde{\Pi}_j \zeta(j)/\xi^j$.

We call the fractional ideal $\widetilde{BC}_j$ the "j-th Bernoulli-Carlitz fractional ideal" as, again, these were originally defined (as elements in $\mathbb{F}_q(T)$) by Carlitz in the 1930's.

Definition 12. Let $\mathfrak{d}_j := \{a \in A \mid a\widetilde{BC}_j \subseteq \mathcal{O}_T\}$.

We call $\mathfrak{d}_j$ the "A-denominator of $\widetilde{BC}_j$;" it is obviously an ideal of A.

For $A = \mathbb{F}_q[T]$, Carlitz ([Ca1], [Ca2], [Ca3]) gives an explicit calculation of $\mathfrak{d}_j$ which we now recall. Let $q = p^{n_0} > 2$ for the moment.

Theorem 2. (Carlitz) *There are two conditions on j:*
1. *$h := \ell_p(j)/(p-1)n_0$ is integral.*
2. *$q^h - 1$ divides j.*

If j satisfies both conditions, then $\mathfrak{d}_j$ is the product of all prime ideals of degree h. If j does not satisfy both conditions, then $\mathfrak{d}_j = (1)$.

Carlitz's result gives us the first (historically) indication of an action of $S_{(q)}$ on ζ-values. This is given in the following corollary.

Corollary 1. *Let $\mathfrak{P}$ be a prime ideal of A of degree h. Then* $\operatorname{ord}_{\mathfrak{P}} \mathfrak{d}_j$ *is an invariant of the action of $S_{(q^h)}$ on the positive integers divisible by $q-1$,*

What about $q = 2$. Here ([Ca3]) the same result holds if $h \neq 2$. More precisely Carlitz established the following result.

Theorem 3. (Carlitz) *Let $q = 2$ and consider the system given in* Theorem 2. *If this system is consistent for $h = \ell_2(j) \neq 2$, then $\mathfrak{d}_j$ is the product of all prime ideals of degree h. If it is consistent for $h = 2$, then for j even we have*

$$\mathfrak{d}_j = (T^2 + T + 1)\,, \tag{25}$$

while, for j odd, we have

$$\mathfrak{d}_j = (T^2 + T \cdot T^2 + T + 1)\,. \tag{26}$$

If the system is inconsistent and j is of the form $2^\alpha + 1$ (so $\ell_2(j) = 2$), then

$$\mathfrak{d}_j = (T^2 + T)\,. \tag{27}$$

If it is inconsistent and j cannot be written as $2^\alpha + 1$, then $\mathfrak{d}_j = (1)$.

Corollary 2. *($q = 2$) Let $\mathfrak{P}$ be a prime of A of degree h and suppose that first that $h \neq 1$. Then* $\operatorname{ord}_{\mathfrak{P}} \mathfrak{d}_j$ *is an invariant of the action of $S_{(2^h)}$ on the positive integers. If $h = 1$ then* $\operatorname{ord}_{\mathfrak{P}} \mathfrak{d}_j$ *is an invariant of the subgroup $\tilde{S}_{(4)}$ of $S_{(4)}$ arising from permutations of* $\{0, 1, 2, \ldots\}$ *fixing* 0.

It is reasonable to expect these symmetries to persist when Carlitz's results are generalized to arbitrary A where, again, one will need to replace q with q_∞.

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