Complex Monge–Ampère equations and totally real submanifolds

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Abstract

We study the Dirichlet problem for complex Monge–Ampère equations in Hermitian manifolds with general (non-pseudoconvex) boundary. Our main result (Theorem 1.1) extends the classical theorem of Caffarelli, Kohn, Nirenberg and Spruck in \( \mathbb{C}^n \). We also consider the equation on compact manifolds without boundary, attempting to generalize Yau’s theorems in the Kähler case. As applications of the main result we study some connections between the homogeneous complex Monge–Ampère (HCMA) equation and totally real submanifolds, and a special Dirichlet problem for the HCMA equation related to Donaldson’s conjecture on geodesics in the space of Kähler metrics.

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1. Introduction

There are two primary purposes in this paper which are closely related. One is to study the Dirichlet problem for complex Monge–Ampère type equations in Hermitian manifolds. The other
is to characterize totally real submanifolds by solutions of the homogeneous Monge–Ampère equation using results from the first part. The latter is also one of the original motivations to our study of Monge–Ampère type equations on general Hermitian manifolds.

Let \((M^n, \omega)\) be a compact Hermitian manifold of (complex) dimension \(n \geq 2\) with smooth boundary \(\partial M\), and \(\tilde{M} = M \cup \partial M\). Let \(\chi\) be a smooth \((1, 1)\) form on \(M\) and \(\psi \in C^\infty(M \times \mathbb{R})\). We consider the Dirichlet problem for the complex Monge–Ampère equation

\[
\left(\chi + \frac{\sqrt{-1}}{2} \bar{\partial} \partial u\right)^n = \psi(z, u) \omega^n \quad \text{in} \quad \tilde{M}.
\]

(1.1)

Given \(\varphi \in C^\infty(\partial M)\), we seek solutions of Eq. (1.1) satisfying the boundary condition

\[
u = \varphi \quad \text{on} \quad \partial M.
\]

(1.2)

We require

\[
\chi u := \chi + \frac{\sqrt{-1}}{2} \bar{\partial} \partial u > 0 \quad (1.3)
\]

so that Eq. (1.1) is elliptic; we call such functions admissible. Accordingly we shall assume \(\psi > 0\). Eq. (1.1) becomes degenerate for \(\psi \geq 0\); when \(\psi \equiv 0\) it is usually referred as the homogeneous complex Monge–Ampère (HCMA) equation. Set

\[
\mathcal{H}_\chi = \{ \phi \in C^2(\tilde{M}) : \chi \phi > 0 \}, \quad \tilde{\mathcal{H}}_\chi = \{ \phi \in C^0(\tilde{M}) : \chi \phi \geq 0 \}.
\]

(1.4)

Two canonical cases that are very important in complex geometry and analysis correspond to \(\chi = \omega\) and \(\chi = 0\). For \(u \in \mathcal{H}_\omega\), as in the Kähler case, \(\omega_u\) is a Hermitian form on \(M\) and Eq. (1.1) describes one of its Ricci forms. For \(\chi = 0\), functions in \(\mathcal{H}_\chi\) are strictly plurisubharmonic, while those in \(\tilde{\mathcal{H}}_\chi\) plurisubharmonic.

The classical solvability of the Dirichlet problem was established by Cheng and Yau [18] for \(\psi = e^{2u}\) and \(\varphi = +\infty\), and by Caffarelli, Kohn, Nirenberg and Spruck [12] for smooth data, for strongly pseudoconvex domains in \(\mathbb{C}^n\). The main results of [12] were extended to strongly pseudoconvex Hermitian manifolds by Cherrier and Hanani [21,22] (for \(\chi = 0, \omega, -u\omega\) in (1.1)), and to general domains in \(\mathbb{C}^n\) by the first author [30] under the assumption of existence of a subsolution. This latter extension and its techniques have found useful applications in some important work; see, e.g., P.-F. Guan’s proof [35,36] of Chern–Levine–Nirenberg conjecture [19] and the papers of Chen [15], Blocki [10], and Phong and Sturm [56] on the Donaldson conjectures [23]. Our first purpose in this paper is to treat the Dirichlet problem in general (non-pseudoconvex) Hermitian manifolds.

**Theorem 1.1.** Suppose that \(\psi > 0\) and that there exists a subsolution \(u \in \mathcal{H}_\chi \cap C^4(\tilde{M})\) of (1.1)–(1.2):

\[
\begin{align*}
(\chi_u)^n \geq \psi(z, u) \omega^n & \quad \text{in} \quad \tilde{M}, \\
u = \varphi & \quad \text{on} \quad \partial M.
\end{align*}
\]

(1.5)

The Dirichlet problem (1.1)–(1.2) then admits a solution \(u \in \mathcal{H}_\chi \cap C^\infty(\tilde{M})\).
When \( \chi > 0 \), which is not assumed in Theorem 1.1, the conditions on \( u \) can be weakened: it is enough to assume \( u \in \bar{H}_{\chi} \); \( u \in C^2 \) and \( \chi_u > 0 \) in a neighborhood of \( \partial M \), and satisfies (1.5) in the viscosity sense. This shall be convenient in applications.

The Monge–Ampère equation is one of the most important partial differential equations in complex geometry and analysis. In the framework of Kähler geometry, it goes back at least to the Calabi conjecture [13] which asserts that any element in the first Chern class of a compact Kähler manifold is the Ricci form of a Kähler metric cohomologous to the underlying metric. In [71], Yau proved fundamental existence theorems for complex Monge–Ampère equations on compact Kähler manifolds (without boundary) and consequently solved the Calabi conjecture. Yau’s work also shows the existence of Kähler–Einstein metrics on Kähler manifolds with nonpositive first Chern class \( (c_1(M) \leq 0) \), proving another Calabi conjecture which was solved by Aubin [1] independently for \( c_1(M) < 0 \). In a series of work (e.g. [61–63]), Tian made important contributions to the Calabi conjecture when \( c_1(M) > 0 \); see also [2] and [64] for more references. More recently, Donaldson [23] made several conjectures concerning geodesics in the space of Kähler metrics which reduce to questions on special Dirichlet problems for the homogeneous complex Monge–Ampère (HCMA) equation; see also Mabuchi [51] and Semmes [57]. There has been interesting work in this direction, e.g. by Chen [15], Chen and Tian [16], Phong and Sturm [54–56], Blocki [10], and Berman and Demailly [7].

The HCMA equation \( (\psi \equiv 0 \text{ in } (1.1)) \), which is well defined on general complex manifolds, also arises in many other interesting geometric problems; see e.g. [19,5,59,48,11,70,35,36]. Because the HCMA equation is degenerate, the optimal regularity of its solution in general is only \( C^{1,1} \); see [3,28]. On the other hand, methods from complex analysis so far seem to have only been able to produce smooth or analytic solutions under special circumstances; in order to treat the equation using techniques of elliptic PDE theory one needs to introduce a metric on the manifold. In the full generality it seems most natural to consider Hermitian metrics as every complex manifold admits such a metric. Moreover, in many problems one needs to consider manifolds with non-pseudoconvex boundary. These are some of the major motivations to our study of the Dirichlet problem on general Hermitian manifolds. As an application of Theorem 1.1 we consider some connections between the HCMA equation and totally real submanifolds.

A submanifold \( X \) of a complex manifold \( M \) is totally real if for any \( p \in X \) the tangent space \( T_pX \) does not contain any complex line in \( T_pM \), i.e. \( J(T_pX) \cap T_pX = \{0\} \). In particular, \( \dim X \leq \frac{1}{2} \dim_R M \). The simplest example is the pair \( \mathbb{R}^n \subset \mathbb{C}^n \), and it is straightforward to verify that the function \( u(z) = |\Re z|^2 \) satisfies the HCMA equation \( (\bar{\partial} \partial u)^n = 0 \) in \( \mathbb{C}^n \setminus \mathbb{R}^n \). Another example is the affine hyperquadric

\[
\mathbb{Q}^n = \left\{ z_1^2 + z_2^2 + \cdots + z_{n+1}^2 = 1 \right\}
\]  

which was studied by Patrizio and Wong [53]. Note that \( |z|^2 = 1 + |\Re z|^2 \geq 1 \) in \( \mathbb{Q}^n \). Thus \( \mathbb{S}^n = \mathbb{Q}^n \cap \{|z| = 1\} \), the unit sphere in \( \mathbb{R}^{n+1} \), is a totally real submanifold of \( \mathbb{Q}^n \). It was proved in [53] that the function \( u = \cosh^{-1} |z|^2 \) is plurisubharmonic and satisfies \( (\bar{\partial} \partial u)^n = 0 \) in \( \mathbb{Q}^n \setminus \mathbb{S}^n \). In general any smooth Riemannian manifold is a totally real submanifold of its cotangent bundle under a canonical complex structure, and a theorem of Harvey and Wells [42] says that the minimum set of a \( C^2 \) plurisubharmonic function is totally real. In [38] Guillemin and Stenzel proved that if \( X \) is a compact real-analytic totally real submanifold of dimension \( n \) of a complex manifold \( M^n \), then there is a neighborhood \( M_1 \) of \( X \) and a nonnegative plurisubharmonic solution of \( (\bar{\partial} \partial u)^n = 0 \) in \( M_1 \setminus X \) such that \( X = u^{-1}(\{0\}) \) and \( u^2 \) is smooth and strictly plurisubharmonic in \( M_1 \). For compact symmetric spaces of rank one, Patrizio and Wong [53] found explicit
formulas for plurisubharmonic solutions of \((\partial \bar{\partial} u)^n = 0\) on corresponding Stein manifolds; see also [49] for related results. As our second goal in this paper we shall apply Theorem 1.1 to prove the following result.

**Theorem 1.2.** Let \(X^n\) be a \(C^3\) compact totally real submanifold of dimension \(n\) in a complex manifold \(M^n\), and \(\chi\) a \((1, 1)\)-form on \(M\). There exists a tubular neighborhood \(M_1\) of \(X\) and a (weak) solution \(u \in H^\chi(M_1) \cap C^{0,1}(\bar{M}_1)\) of the HCMA equation

\[
(\chi u)^n = 0 \quad \text{in } \bar{M}_1 \setminus X
\]

such that \(0 \leq u \leq 1\) on \(\bar{M}_1\), \(X = u^{-1}(0)\) and \(\partial M_1 = u^{-1}(1)\).

For the definition of weak solutions see [4]. The global Lipschitz regularity is the best possible as shown by the example \(u(z) = |\text{Im} z|\) in \(\mathbb{C}^n\), which is only \(C^{0,1}\) along \(\mathbb{R}^n\). It was known to Lempert and Szöke [49] that a plurisubharmonic solution to the homogeneous Monge–Ampère equation on Stein manifolds must have singularities along its minimum set. So the proof of Theorem 1.2 involves solving equation (1.7) with prescribed singularity, which is not always possible for general elliptic equations. As we shall see in Section 7, our proof of Theorem 1.2 makes use of Theorem 1.1 in an essential way that it cannot be replaced by the previous results of Cherrier and Hanani [21,22]. It is also different from the approach of Guillemin and Stenzel [38].

In her thesis [50] the second author proved the existence of a bounded plurisubharmonic solution to (1.7) for \(\chi = 0\). It would be interesting to prove \(u \in C^{1,1}(\bar{M}_1 \setminus X)\). This will be treated in [32].

Another interesting problem is to consider extensions of Yau’s theorems [71] to the Hermitian case, that is, to solve equation (1.1) on compact Hermitian manifolds without boundary. A difficult question seems to be how to derive \(C^0\) estimates, even for \(\chi = \omega\). Yau’s estimate in the Kähler case [71] makes use of Moser iteration based on his estimate for \(\Delta u\) and Sobolev inequality. His proof was subsequently simplified by Kazdan [44] for \(n = 2\), and by Aubin [1] and Bourguignon independently for arbitrary dimension (see e.g. [58] and [64]). Alternative proofs were given by Kolodziej [45] and Blocki [8] based on the pluripotential theory (see [6]) and the \(L^2\) stability of the complex Monge–Ampère operator (see [14]). All these proofs seem to heavily rely on the closedness or, equivalently, existence of local potentials of \(\omega\) and it is not clear to us whether any of them can be extended to the Hermitian case. In this paper we impose the following condition

\[
\partial \bar{\partial} \chi = 0, \quad \partial \bar{\partial} \chi^2 = 0
\]

which is equivalent to \(\partial \bar{\partial} \chi = 0\) and \(\partial \chi \wedge \bar{\partial} \chi = 0\). For \(\chi = \omega\), manifolds satisfying (1.8) were studied by Fino and Tomassini [26]. We have following extensions of theorems of Yau [71]. In Theorems 1.3 and 1.4 below, \((M, \omega)\) is a compact Hermitian manifold with \(\partial M = \emptyset\).

**Theorem 1.3.** Assume \(\psi_u \geq 0\), \(\chi\) satisfies (1.8), and that there exists a function \(\phi \in H^\chi \cap C^\infty(M)\) such that

\[
\int_M \psi(z, \phi(z))\omega^n = \int_M \chi^n.
\]
Then there exists a solution \( u \in H_\chi \cap C^\infty(M) \) of Eq. (1.1). Moreover the solution is unique, possibly up to a constant.

Consequently, if \( H_\chi \cap C^\infty(M) \neq \emptyset \) then for any \( \psi \in C^\infty(M) \) there is a unique constant \( c \) such that Eq. (1.1) has a solution in \( H_\chi \cap C^\infty(M) \) when \( \psi \) is replaced by \( c\psi \).

Cherrier [20] derived \( C^0 \) bounds without any condition for \( n = 2 \), and under some different conditions from (1.8) for \( n > 2 \).

Under stronger assumptions on \( \psi \) condition we can remove condition (1.8).

**Theorem 1.4.** Suppose \( H_\chi \cap C^\infty(M) \neq \emptyset \), \( \psi > 0 \) and

\[
\lim_{u \to -\infty} \psi(\cdot, u) = 0, \quad \lim_{u \to +\infty} \psi(\cdot, u) = \infty. \tag{1.10}
\]

Then Eq. (1.1) has a unique solution in \( H_\chi \cap C^\infty(M) \).

In [24], Donaldson proposed to generalize Yau’s theorems in a different direction; see [69,68,65,66] for some recent developments.

The degenerate complex Monge–Ampère equation is very important in geometry and analysis. There are many challenging open questions. Below we formulate some result for a special Dirichlet problem which, in the Kähler case, has been useful in the study of geodesics in the space of Kähler metrics; see, e.g. [10,15,56].

**Theorem 1.5.** Let \( M = N \times S \) where \( N \) is a compact Hermitian manifold without boundary, and \( S \) is a compact Riemann surface with smooth boundary \( \partial S \neq \emptyset \). Suppose \( \psi \geq 0 \), \( \psi^{\frac{1}{n}} \in C^2(\bar{M} \times R) \), and that there exists a subsolution \( u \in H_\chi \) satisfying (1.5). Then there exists a weak admissible solution \( u \in C^{1,\alpha}(\bar{M}) \), for all \( \alpha \in (0, 1) \) with \( \Delta u \in L^\infty(M) \) of the Dirichlet problem (1.1)–(1.2).

For applications in geometric problems it would be desirable to allow \( \chi \) to depend on \( u \) and its gradient \( \nabla u \); see for instance [27]. We shall prove Theorem 1.1 for \( \chi = \chi(\cdot, u) \) which is nondecreasing in \( u \), i.e. \( \chi(\cdot, u) - \chi(\cdot, v) \geq 0 \) for \( u \geq v \). For the general case we are able to deal with \( \chi = \chi(\cdot, u, \nabla u) \) of the form

\[
\chi(\cdot, u, \nabla u) = G(|\nabla u|^2)\omega + H(\cdot, u)\partial u \wedge \bar{\partial} u + \partial u \wedge \bar{\partial}(\cdot, u) + \alpha(\cdot, u) \wedge \bar{\partial} u + \chi^0(\cdot, u) \tag{1.11}
\]

where \( G, H, \alpha \) and \( \chi^0 \) are all smooth, under suitable assumptions on \( G \) and \( \chi^0 \). This will appear in [32].

The paper is organized as follows. In Section 2 we recall some basic facts and formulas for Hermitian manifolds, fixing notations along the way. We shall also construct some special local coordinates; see Lemma 2.1. These local coordinates will be used to derive estimates for \( \Delta u \) in Section 3 where we also derive gradient estimates, extending the arguments of Blocki [9] and P.-F. Guan [37] in the Kähler case. Section 4 concerns the boundary estimates for second derivatives. In Section 5 we come back to finish the global estimates for all (real) second derivatives which enable us to apply the Evans–Krylov Theorem for \( C^{2,\alpha} \) estimates; higher order estimates
and regularity then follow from the classical Schauder theory. In Section 6 we discuss the $C^0$ estimates and existence of solutions, completing the proof of Theorem 1.1 and Theorems 1.3–1.5. Section 7 contains the proof of Theorem 1.2. Finally, in Section 8 we discuss a Dirichlet problem for the HCM equation which is related to the Donaldson conjecture in the Kähler case.

An earlier version (see [31]) of this article was posted on the arXiv in June 2009. We learned afterwards of the work of Cherrier and Hanani [20–22,39,40] where they already derived some of the estimates in this paper. We wish to thank Philippe Delanoë for bringing these beautiful papers to our attention. More recently, right before the current version was finished we received from Tosatti and Weinkove their paper [67] in which, among other very interesting results, they derived the $C^0$ estimates for balanced Hermitian manifolds. We thank them for sending us the preprint and for useful communications. Finally, the authors wish to express their gratefulness to Pengfei Guan, Fangyang Zheng, and the referees for their very insightful comments and suggestions.

2. Preliminaries

Let $M^n$ be a complex manifold of dimension $n$ and $g$ a Riemannian metric on $M$. Let $J$ be the induced almost complex structure on $M$ so $J$ is integrable and $J^2 = -\text{id}$. We assume that $J$ is compatible with $g$, i.e.

$$g(u, v) = g(Ju, Jv), \quad u, v \in TM;$$

(2.1)

such $g$ is called a Hermitian metric. Let $\omega$ be the Kähler form of $g$ defined by

$$\omega(u, v) = -g(u, Jv).$$

(2.2)

We recall that $g$ is Kähler if its Kähler form $\omega$ is closed, i.e. $d\omega = 0$.

The complex tangent bundle $T_C M = TM \times \mathbb{C}$ has a natural splitting

$$T_C M = T^{1,0} M + T^{0,1} M$$

(2.3)

where $T^{1,0} M$ and $T^{0,1} M$ are the $\pm \sqrt{-1}$-eigenspaces of $J$. The metric $g$ is obviously extended $\mathbb{C}$-linearly to $T_C M$, and

$$g(u, v) = 0 \quad \text{if } u, v \in T^{1,0} M, \text{ or } u, v \in T^{0,1} M.$$  

(2.4)

Let $\nabla$ be the Chern connection of $g$. It satisfies

$$\nabla_u (g(v, w)) = g(\nabla_u v, w) + g(v, \nabla_u w)$$

(2.5)

but may have nontrivial torsion. The torsion $T$ and curvature $R$ of $\nabla$ are defined by

$$\begin{cases} 
T(u, v) = \nabla_u v - \nabla_v u - [u, v], \\
R(u, v) w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u, v]} w, 
\end{cases}$$

(2.6)

respectively. Since $\nabla J = 0$ we have

$$g(R(u, v) J w, J x) = g(R(u, v) w, x) \equiv R(u, v, w, x).$$

(2.7)

Therefore $R(u, v, w, x) = 0$ unless $w$ and $x$ are of different type.
In local coordinates \((z_1, \ldots, z_n)\), we have

\[
J \frac{\partial}{\partial z_j} = \sqrt{-1} \frac{\partial}{\partial z_j}, \quad J \frac{\partial}{\partial \bar{z}_j} = -\sqrt{-1} \frac{\partial}{\partial \bar{z}_j}.
\]

(2.8)

Thus, by (2.4),

\[
g \left( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k} \right) = 0, \quad g \left( \frac{\partial}{\partial \bar{z}_j}, \frac{\partial}{\partial \bar{z}_k} \right) = 0.
\]

(2.9)

We write

\[
g_{j\bar{k}} = g \left( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k} \right), \quad \{g^{i\bar{j}}\} = \{g_{i\bar{j}}\}^{-1}.
\]

(2.10)

That is, \(g^{ij}g_{k\bar{j}} = \delta_{ik}\). The Kähler form \(\omega\) is given by

\[
\omega = \frac{\sqrt{-1}}{2} g_{j\bar{k}} dz_j \wedge d\bar{z}_k.
\]

(2.11)

For convenience we shall write

\[
\chi = \frac{\sqrt{-1}}{2} \chi_{j\bar{k}} dz_j \wedge d\bar{z}_k.
\]

The Christoffel symbols \(\Gamma^l_{jk}\) are defined by

\[
\nabla_{\frac{\partial}{\partial z_j}} \frac{\partial}{\partial z_k} = \Gamma^l_{jk} \frac{\partial}{\partial \bar{z}_l}.
\]

Recall that by (2.5) and (2.9),

\[
\begin{cases}
\nabla_{\frac{\partial}{\partial z_j}} \frac{\partial}{\partial \bar{z}_k} = \nabla_{\frac{\partial}{\partial \bar{z}_j}} \frac{\partial}{\partial z_k} = 0, \\
\nabla_{\frac{\partial}{\partial \bar{z}_j}} \frac{\partial}{\partial \bar{z}_k} = \Gamma^l_{jk} \frac{\partial}{\partial z_l} = \Gamma^l_{jk} \frac{\partial}{\partial \bar{z}_l}
\end{cases}
\]

(2.12)

and

\[
\Gamma^l_{jk} = g^{lm} \frac{\partial g_{k\bar{m}}}{\partial z_j}.
\]

(2.13)

The torsion is given by

\[
T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji} = g^{kl} \left( \frac{\partial g_{ji}}{\partial z_l} - \frac{\partial g_{il}}{\partial z_j} \right).
\]

(2.14)
while the curvature

\begin{equation}
R_{i}^{i} = R \left( \frac{\partial}{\partial z_{i}}, \frac{\partial}{\partial \bar{z}_{j}}, \frac{\partial}{\partial z_{k}}, \frac{\partial}{\partial \bar{z}_{l}} \right) = -g_{m\bar{l}} \frac{\partial^2 g_{k\bar{l}}}{\partial z_{i} \partial z_{j}} + g^{pq} g_{kq} \frac{\partial g_{pl}}{\partial z_{j}}. \tag{2.15}
\end{equation}

Note that from (2.6), (2.7) and (2.13) \( R_{i}^{i} = R_{i} = R_{i}^{i} = 0 \). By (2.15) and (2.14) we have

\begin{equation}
R_{i}^{j} - R_{k}^{j} = g_{m\bar{l}} \frac{\partial T_{k}^{m}}{\partial z_{j}} = g_{m\bar{l}} \nabla_{j} T_{k}^{m}, \tag{2.16}
\end{equation}

which also follows from the general Bianchi identity.

The traces of the curvature tensor

\begin{equation}
R_{k}^{l} = R_{i}^{l} R_{i}^{k}, \quad S_{i}^{j} = g^{k\bar{l}} R_{i}^{j k} = -\frac{\partial^2}{\partial z_{i} \partial \bar{z}_{j}} \log \det g_{k\bar{l}} \tag{2.17}
\end{equation}

are called the first and second Ricci tensors, respectively. Therefore one can consider extensions of Calabi–Yau theorem for \( S_{i}^{j} \); see [67].

The following special local coordinates, which will be used in our proof of \( a \ priori \) estimates for \(|\nabla u|\) and \( \Delta u \), seems of interest in themselves.

**Lemma 2.1.** Around a point \( p \in M \) there exist local coordinates such that, at \( p \),

\begin{equation}
g_{i\bar{j}} = \delta_{i\bar{j}}, \quad \frac{\partial g_{i\bar{j}}}{\partial z_{j}} = 0, \quad \forall i, j. \tag{2.18}
\end{equation}

**Proof.** Let \((z_{1}, z_{2}, \ldots, z_{n})\) be a local coordinate system around \( p \) such that \( z_{i}(p) = 0 \) for \( i = 1, \ldots, n \) and

\begin{equation}
g_{i\bar{j}}(p) := g \left( \frac{\partial}{\partial z_{i}}, \frac{\partial}{\partial \bar{z}_{j}} \right) = \delta_{i\bar{j}}.
\end{equation}

Define new coordinates \((w_{1}, w_{2}, \ldots, w_{n})\) by

\begin{equation}
w_{r} = z_{r} + \sum_{m \neq r} \frac{\partial g_{r\bar{r}}}{\partial z_{m}} (p) z_{m} z_{r} + \frac{1}{2} \frac{\partial g_{r\bar{r}}}{\partial z_{r}} (p) z_{r}^{2}, \quad 1 \leq r \leq n. \tag{2.19}
\end{equation}

We have

\begin{equation}
\tilde{g}_{i\bar{j}} := g \left( \frac{\partial}{\partial w_{i}}, \frac{\partial}{\partial \bar{w}_{j}} \right) = \sum_{r,s} g_{r\bar{s}} \frac{\partial z_{r}}{\partial w_{i}} \frac{\partial \bar{z}_{s}}{\partial \bar{w}_{j}}. \tag{2.20}
\end{equation}
It follows that
\[
\frac{\partial \tilde{g}_{ij}}{\partial w_k} = \sum_{r,s} grs \frac{\partial^2 z_r}{\partial w_i \partial w_k} \frac{\partial z_s}{\partial w_j} + \sum_{r,s,p} grp \frac{\partial z_r}{\partial w_p} \frac{\partial z_r}{\partial w_i} \frac{\partial z_s}{\partial w_j}. \tag{2.21}
\]

Differentiate (2.19) with respect to \( w_i \) and \( w_k \). We see that, at \( p \),
\[
\frac{\partial z_r}{\partial w_i} = \delta_{ri}, \quad \frac{\partial^2 z_r}{\partial w_i \partial w_k} = -\sum_{m\neq r} \frac{\partial g_m}{\partial z_m} \left( \frac{\partial z_m}{\partial w_i} \frac{\partial z_r}{\partial w_k} + \frac{\partial z_m}{\partial w_k} \frac{\partial z_r}{\partial w_i} \right) - \frac{\partial g_r}{\partial z_r} \frac{\partial z_r}{\partial w_i} \frac{\partial z_r}{\partial w_k}.
\]

Plugging these into (2.21), we obtain at \( p \),
\[
\begin{align*}
\frac{\partial \tilde{g}_{ii}}{\partial w_k} &= \frac{\partial g_{ii}}{\partial z_k} - \frac{\partial g_{ii}}{\partial z_k} = 0, \quad \forall i, k, \\
\frac{\partial \tilde{g}_{ij}}{\partial w_j} &= \frac{\partial g_{ij}}{\partial z_j} - \frac{\partial g_{ij}}{\partial z_j} = T^j_{ji}, \quad \forall i \neq j, \\
\frac{\partial \tilde{g}_{ij}}{\partial w_k} &= \frac{\partial g_{ij}}{\partial z_k}, \quad \text{otherwise.}
\end{align*}
\tag{2.22}
\]

Finally, switching \( w \) and \( z \) gives (2.18). □

**Remark 2.2.** If, in place of (2.19), we define
\[
w_r = z_r + \sum_{m\neq r} \frac{\partial g_m}{\partial z_r} (p) z_m z_r + \frac{1}{2} \frac{\partial g_r}{\partial z_r} (p) z_r^2, \quad 1 \leq r \leq n, \tag{2.23}
\]
then under the new coordinates \((w_1, w_2, \ldots, w_n)\),
\[
\begin{align*}
\frac{\partial \tilde{g}_{ii}}{\partial w_j} (p) &= 0, \quad \forall i, j, \\
\frac{\partial \tilde{g}_{ij}}{\partial w_k} (p) &= T^i_{ki}, \quad \forall i \neq k, \\
\frac{\partial \tilde{g}_{ij}}{\partial w_k} (p) &= \frac{\partial g_{ij}}{\partial z_k} (p), \quad \text{otherwise.}
\end{align*}
\tag{2.24}
\]

In [60] Streets and Tian constructed local coordinates satisfying
\[
g_{ij} = \delta_{ij}, \quad \frac{\partial g_{ij}}{\partial z_k} + \frac{\partial g_{kj}}{\partial z_i} = 0 \tag{2.25}
\]
and consequently, \( T^k_{ij} = 2 \frac{\partial g_{ij}}{\partial z_i} \) at a fixed point. In general it is impossible to find local coordinates satisfying both (2.18) and (2.25) simultaneously.

Let \( \Lambda^{p,q} \) denote differential forms of type \((p,q)\) on \( M \). The exterior differential \( d \) has a natural decomposition \( d = \partial + \bar{\partial} \) where
\[
\partial : \Lambda^{p,q} \to \Lambda^{p+1,q}, \quad \bar{\partial} : \Lambda^{p,q} \to \Lambda^{p,q+1}.
\]
Recall that \( \partial^2 = \bar{\partial}^2 = \partial \bar{\partial} + \bar{\partial} \partial = 0 \) and, by the Stokes theorem
\[
\int_M \partial \alpha = \int_{\partial M} \alpha, \quad \forall \alpha \in \Lambda^{n-1,n}.
\]
A similar formula holds for \( \bar{\partial} \).

For a function \( u \in C^2(M) \), \( \partial \bar{\partial} u \) is given in local coordinates by
\[
\partial \bar{\partial} u = \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j.
\]
Eq. (1.1) thus takes the form
\[
\det \left( \chi_{ij} + \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} \right) = \psi(z,u) \det g_{ij}.
\]
We use \( \nabla^2 u \) to denote the Hessian of \( u \):
\[
\nabla^2 u(X,Y) \equiv \nabla_Y \nabla_X u = Y(Xu) - (\nabla_Y X)u, \quad X,Y \in TM.
\]
By (2.12) we see that
\[
\nabla \partial \partial \nabla = \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}.
\]
Consequently, the Laplacian of \( u \) with respect to the Chern connection is
\[
\Delta u = g^{ij} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j},
\]
or equivalently,
\[
\Delta u \frac{\omega^n}{n} = \frac{\sqrt{-1}}{2} \partial \bar{\partial} u \wedge \omega^{n-1}.
\]
Integrating (2.31) (by parts), we obtain
\[
\frac{1}{n} \int_M \Delta u \omega^n = \frac{\sqrt{-1}}{2} \int_M \partial \bar{\partial} u \wedge \omega^{n-1}
\]
\[
= \frac{\sqrt{-1}}{2} \int_{\partial M} \bar{\partial} u \wedge \omega^{n-1} + \frac{\sqrt{-1}}{2} \int_M \bar{\partial} u \wedge \partial \omega^{n-1}
\]
\[
= \frac{\sqrt{-1}}{2} \int_{\partial M} \left( \bar{\partial} u \wedge \omega^{n-1} + u \partial \omega^{n-1} \right) + \frac{\sqrt{-1}}{2} \int_M u \partial \bar{\partial} \omega^{n-1}.
\]
It follows that the Chern Laplacian operator $\Delta$ is self-adjoint if $M$ is compact (without boundary) and $\omega$ is a Gauduchon metric, i.e., it satisfies $\partial \bar{\partial} \omega^{n-1} = 0$.

3. Global estimates for $|\nabla u|$ and $\Delta u$

Let $u \in H_x \cap C^4(\tilde{M})$ be a solution of (2.27). In this section we derive the following estimates

$$\max_M |\nabla u| \leq C_1 \left( 1 + \max_{\partial M} |\nabla u| \right),$$  \hfill (3.1)

$$\max_M \Delta u \leq C_2 \left( 1 + \max_{\partial M} \Delta u \right).$$  \hfill (3.2)

Here we emphasize that $C_1$ and $C_2$ depend only on geometric quantities (torsion and curvature) of $M$ and on $\chi$ as well as its covariant derivatives, but do not depend on $\inf \psi$ so the estimates (3.1) and (3.2) apply to the degenerate case ($\psi \geq 0$); see Propositions 3.1 and 3.3 for details.

For $\chi = \omega$ these estimates were derived by Cherrier and Hanani [39,40,21,22]. The estimate for $\Delta u$ is an extension of that of Yau [71]. The gradient estimate (3.1) was also independently recovered by Blocki [9] and P.-F. Guan [37] in the Kähler case, and by Xiangwen Zhang [72] for more general equations on compact Hermitian manifolds without boundary.

We shall first assume $\chi = \chi(\cdot, u)$ is positive definite. More precisely,

$$\chi = \chi(\cdot, u) \geq \epsilon \omega \quad (3.3)$$

where $\epsilon > 0$ may depend on $\sup_M |u|$. In this case we do not need the subsolution $u$ to derive (3.1) and (3.2). At the end of this section we shall remove assumption (3.3).

Throughout this section we use ordinary derivatives. For convenience we write in local coordinates,

$$u_i = \frac{\partial u}{\partial z_i}, \quad u_\bar{j} = \frac{\partial u}{\partial \bar{z}_j}, \quad u_{ij} = \frac{\partial^2 u}{\partial z_i \partial z_j}, \quad g_{ij} = \frac{\partial g_{ij}}{\partial z_k}, \quad g_{i\bar{j}j} = \frac{\partial^2 g_{i\bar{j}j}}{\partial z_k \partial \bar{z}_l},$$ etc.,

and $g_{i\bar{j}} = u_{i\bar{j}} + \chi_{i\bar{j}}, \quad \{g^{i\bar{j}}\} = \{g_{ij}\}^{-1}$.

Suppose at a fixed point $p \in M$,

$$g_{ij} = \delta_{ij} \quad \text{and} \quad \{g_{i\bar{j}}\} \quad \text{is diagonal.} \quad (3.4)$$

Starting from

$$|\nabla u|^2 = g^{k\bar{i}} u_k u_{\bar{i}}, \quad \Delta u = g^{k\bar{i}} u_{k\bar{i}}, \quad (3.5)$$

by straightforward calculation, we see that

$$\left( |\nabla u|^2 \right)_i = u_k u_{i\bar{k}} + (u_{ki} - g_{k\bar{i}} u_{i\bar{k}}) u_{\bar{k}},$$

$$\left( \Delta u \right)_i = u_{k\bar{k}i} - g_{k\bar{k}i} u_{k\bar{i}}, \quad (3.6)$$
(\nabla u^2)_{ii} = u_{iik}u_{ki} + u_{ijk}u_{ki} + u_{iik}u_{ik} + (u_{ki} - g_{kli}u_i)^2 \\
- 2\Re\{g_{kli}u_{iij}u_{kj}\} + (g_{i\bar{p}i}g_{p\bar{k}i} - g_{ik\bar{i}})u_{k\bar{i}}. \quad (3.7)

(\Delta u)_{ii} = -u_{k\bar{k}i} - 2\Re\{u_{k\bar{k}i}\} + g_{k\bar{i}}u_{k\bar{i}} + (g_{ki\bar{i}} - g_{k\bar{i}i})(g_{p\bar{k}i}g_{p\bar{i}i} + R_{i\bar{k}\bar{i}}). \quad (3.8)

Next, differentiate Eq. (2.27) twice,
\begin{align*}
\mathfrak{g}^{ij}u_{ikk}u_{k\bar{k}} &= |\nabla u|^2 f_u + f_{zk}u_{k\bar{k}} + g_{i\bar{k}i}u_{k\bar{i}} - g^{ij}(\chi_{i\bar{j}})u_{k\bar{k}}. \quad (3.9) \\
\mathfrak{g}^{\bar{j}\bar{i}}u_{i\bar{k}k} &= \mathfrak{g}^{\bar{j}\bar{i}}\mathfrak{g}^{ij}u_{i\bar{j}k} + (\chi_{i\bar{j}})u_{k\bar{k}}^2 + (f)_{k\bar{k}} - R_{k\bar{i}i\bar{i}} - g^{ij}(\chi_{i\bar{j}})u_{k\bar{k}}. \quad (3.10)
\end{align*}

Note that
\[(\chi_{i\bar{j}}) = \nabla_k\chi_{i\bar{j}} + g_{i\bar{k}i}\chi_{i\bar{j}} + u_kD_u\chi_{i\bar{j}}. \quad (3.11)\]

Thus,
\[g_{k\bar{i}i}u_{i\bar{i}} + (\chi_{i\bar{i}}) = g_{k\bar{i}i}g_{i\bar{i}} + \nabla_k\chi_{i\bar{i}} + T_{i\bar{k}}u_{i\bar{k}} + u_kD_u\chi_{i\bar{i}}. \]

From (3.7) and (3.9) we see that
\[
\mathfrak{g}^{ij}(|\nabla u|^2)_{ii} = \mathfrak{g}^{ij}u_{i\bar{k}i}u_{ki} + \mathfrak{g}^{ij}u_{k\bar{i}i} - g_{k\bar{i}i}u_i^2 + g^{ij}R_{i\bar{k}i\bar{i}}u_{i\bar{k}} \\\n- 2|\nabla u|^2\mathfrak{g}^{ij}D_u\chi_{i\bar{i}} - 2g^{ij}\Re\{(\nabla_k\chi_{i\bar{i}} + T_{i\bar{k}}u_{i\bar{k}})u_{i\bar{i}}\} \\\n+ 2|\nabla u|^2 f_u + 2\Re\{(f_{zk} - g_{k\bar{i}i})u_{k\bar{i}}\}. \quad (3.12)
\]

**Proposition 3.1.** There exists $C_1 > 0$ depending on $\sup_M |u|$, $|\psi^{\frac{1}{n}}|_{C^1}$ and
\[
\sup_M |\chi|, \sup_M |\nabla\chi|, \inf\inf_M D_u\chi_{i\bar{i}}, \sup_M |T|, \inf\inf_M R_{j\bar{i}i\bar{i}} \quad (3.13)
\]
such that (3.1) holds.

**Proof.** Let $L = \inf_M u$ and $\phi = Ae^{L-u}$ where $A > 0$ is constant to be determined. Suppose that $e^{\phi}|\nabla u|^2$ attains its maximum at an interior point $p \in M$ where all calculations are done in this proof. By Lemma 2.1 we assume (3.4) and (2.18) hold at $p$ so that $g_{k\bar{i}i} = T_{i\bar{k}}$. Since $e^{\phi}|\nabla u|^2$ attains its maximum at $p$,
\[
\frac{|\nabla u|^2}{|\nabla u|^2} + \phi_i = 0, \quad \frac{|\nabla u|^2}{|\nabla u|^2} + \phi_{\bar{i}} = 0 \quad (3.14)
\]
and
\[
\frac{|\nabla u|^2}{|\nabla u|^2} - \frac{|(\nabla u)^2_{i\bar{i}}|^2}{|\nabla u|^4} + \phi_{i\bar{i}} \leq 0. \quad (3.15)
\]
By (3.6) and (3.14),

\[ \left| \left| \left( |\nabla u|^2 \right)_i \right| \right|^2 = \sum_k |u_k|^2 |u_{ki} - g_{kii}u_i|^2 - 2|\nabla u|^2 \Re \{g^{ij}u_{ij}\Phi_i\} - |u_i|^2 u_{ii}^2, \]  

(3.16)

Combining (3.15), (3.16) and (3.12), we obtain

\[ |\nabla u|^2 g^{ij} \Phi_i \Phi_j + 2 \Re \{g^{ij}u_{ij}\Phi_i\} \leq 2|\nabla f||\nabla u| + |\nabla u|^2 (C - 2f_u) + C|\nabla u|^2 \sum g^{ij} \]  

(3.17)

where \( C \) depends on the quantities in (3.13).

We have \( \Phi_i = -\Phi u_i, \Phi_{ij} = \Phi (u_i u_j - u_{ij}) \). Therefore,

\[ g^{ij} \Re \{u_{ij}\Phi_i\} = -\Phi |\nabla u|^2 + \Phi g^{ij} \chi_{ij} u_i u_j \geq -\Phi |\nabla u|^2 \]  

(3.18)

and, by assumption (3.3),

\[ g^{ij} \Phi_i \Phi_j = \Phi g^{ij} u_i u_j - \Phi + \Phi g^{ij} \chi_{ij} \geq -\Phi + \Phi g^{ij} u_i u_j + \epsilon \Phi \sum g^{ij}. \]  

(3.19)

Note that

\[ g^{ij} u_i u_j + \epsilon \sum g^{ij} \geq |\nabla u|^2 \min_i g^{ij} + \epsilon \sum g^{ij} \]

\[ \geq n\epsilon \frac{n-1}{n} |\nabla u|^2 \left( \det g^{ij} \right)^{\frac{1}{n}}. \]  

(3.20)

Thus

\[ n\epsilon \frac{n-1}{n} \psi^{-\frac{1}{n}} |\nabla u|^2 + g^{ij} u_i u_j + \epsilon \sum g^{ij} \]

\[ \leq 2(1 + \Phi^{-1} g^{ij} \Phi_i \Phi_j) \]

\[ \leq 4(|\nabla f||\nabla u|^{-1} - f_u)\Phi^{-1} + C_1\Phi^{-1} \sum g^{ij} + C_2. \]  

(3.21)

Choose \( A \) sufficiently large so that \( \epsilon \Phi \geq C_1 \). We see that

\[ |\nabla u|^2 \leq C_3 \left( \psi^{\frac{1}{n}} + D_u \psi^{\frac{1}{n}} + |\nabla \psi|^{\frac{1}{n}} \right). \]

This proves (3.1).

\[ \square \]

**Lemma 3.2.** Assume that (2.18) and (3.4) hold at \( p \in M \). Then at \( p \),

\[ g^{ij} (\Delta u)_{ij} \geq g^{ij} g^{jj} |u_{ij} + (\chi_{ij})_{j}|^2 + \Delta (f) - n^2 \inf_{j,k} R_{jkk} \]

\[ + c_1 (\Delta u + \tr \chi) \sum g^{ij} - (c_2 + c_3 |\nabla u|^2) \sum g^{ij} \]  

(3.22)

where \( c_1 = \inf R_{jkk} - \sup D_u \chi_{ij} \) and \( c_3 = 0 \) if \( \chi \) does not depend on \( u \).
Proof. Write
\[ u_k j i \bar{g} k i = \left[ u_{i j k} + (\chi_i j) k \right] \bar{g} k i - (\chi_i j) k \bar{g} j i. \] (3.23)

By Cauchy–Schwarz inequality,
\[ 2 \sum_{j \neq k} |\Re\left\{ \left[ u_{i j k} + (\chi_i j) k \right] \bar{g} k i \right\}| \leq \sum_{j \neq k} g^{ij} |u_{i j k} + (\chi_i j) k|^2 + \sum_{j \neq k} g_{j i} |\bar{g} j i|^2. \] (3.24)

Combining (3.8), (3.10), (3.23), and (3.24), we derive
\[ g_i j (\Delta u) i = g_i j g_j k \bar{g} k i + (\chi_i j) k \bar{g} j i - (\chi_i j) k g_j k \bar{g} j i \] (3.25)
where
\[ G_i j = g_j i g_k j i \chi_k j - 2 \Re\left\{ (\chi_i j) k \bar{g} j i \right\} + (\chi_i j) k \bar{g} j i \]
Next,
\[ (\chi_i j) k k = \chi_i j k k + 2 \Re\left\{ u_k \left( D_k \nabla_k \chi_i j + g_i k \bar{g} j k \chi j k \right) \right\} + u_k |u|^2 D_k u \chi_i j + u_k D_k u \chi_i j \]
where
\[ \chi_i j k k = \nabla_k \nabla_k \chi_i j - R_{k l i j} \chi_k + g_i k g_k j \chi j k + 2 \Re\left\{ g_i j k \nabla_k \chi j i \right\}. \]
It follows from (3.11) and the identity
\[ |g_{i j k}|^2 + |g_{k j i}|^2 - 2 \Re\left\{ g_{i j k} g_{j k i} \right\} = |g_{i j k} - g_{j k i}|^2 = |T_{i j k}|^2 \]
that
\[ G_i j = \nabla_k \nabla_k \chi_i j - R_{k l i j} \chi_k + T_{i j k} \| T_{i j k} \| \chi j k - 2 \Re\left\{ T_{i j k} \nabla_k \chi j i \right\} + 2 \Re\left\{ u_k \nabla_k D_k \chi_i j - T_{i j k} u_k D_k \chi j i \right\} + |\nabla u|^2 D_k u \chi_i j + \Delta u D_k \chi_i j. \] (3.26)
Plugging this into (3.25) we prove (3.22).

Proposition 3.3. There exists constant \( C_2 > 0 \) depending on
\[ |u|_{C^1(\bar{M})}, \quad |\psi|_{C^2(\bar{M} \times \mathbb{R})}, \quad |\chi|_{C^2(\bar{M} \times \mathbb{R})} \]
and the geometric quantities (curvature and torsion) of \( M \), such that (3.2) holds. If both \( \chi \) and \( \psi \) are independent of \( u \), then \( C_2 \) does not depend on \( |\nabla u|_{C^0(\bar{M})} \).
Proof. Let \( a = \sup \text{tr} \chi \) and consider the function \( \Phi = e^{\phi}(a + \Delta u) \) where \( \phi = Ae^{L-u} \) and \( L = \inf_M u \) as in the proof of Proposition 3.1. Suppose \( \Phi \) achieves its maximum at an interior point \( p \in M \) where we assume (2.18) and (3.4) hold. We have (all calculations are done at \( p \) below)

\[
\frac{(\Delta u)_i}{a + \Delta u} + \phi_i = 0, \\
\frac{(\Delta u)_{ii}}{a + \Delta u} - \frac{|(\Delta u)_i|^2}{(a + \Delta u)^2} + \phi_{ii} \leq 0.
\]  

(3.27)

(3.28)

Write

\[
(\Delta u)_i = u_{i\bar{j}j} + (\chi_{i\bar{j}j}) + \lambda_i
\]

where, by (3.6), (3.4), (2.18) and (3.11),

\[
\lambda_i = -(\chi_{i\bar{j}j}) - g_{l\bar{k}i}u_{l\bar{k}i} = -\nabla_j \chi_{i\bar{j}} + T_{ij} \chi_{i\bar{j}} - u_j Du \chi_{i\bar{j}}.
\]

We have by (3.27),

\[
|(a + \Delta u)_i|^2 = |u_{i\bar{j}j} + (\chi_{i\bar{j}j})|^2 + 2\Re\{(\Delta u)_i\bar{\lambda}_i\} - |\lambda_i|^2
\]

\[
= |u_{i\bar{j}j} + (\chi_{i\bar{j}j})|^2 - 2(a + \Delta u) \Re\{\phi_i \bar{\lambda}_i\} - |\lambda_i|^2.
\]  

(3.29)

By Cauchy–Schwarz inequality,

\[
g_{i\bar{i}} |u_{i\bar{j}j} + (\chi_{i\bar{j}j})|^2 = \sum_i g_{i\bar{i}} \left| \sum_j g_{j\bar{j}}^{1/2} g_{jj}^{-1/2} (u_{i\bar{j}j} + (\chi_{i\bar{j}j})) \right|^2
\]

\[
\leq \sum_i g_{i\bar{i}} \sum_j g_{j\bar{j}} |u_{i\bar{j}j} + (\chi_{i\bar{j}j})|^2 \sum_k g_{kk}
\]

\[
= (\text{tr} \chi + \Delta u) \sum_{i,j} g_{i\bar{i}} g_{j\bar{j}} |u_{i\bar{j}j} + (\chi_{i\bar{j}j})|^2.
\]  

(3.30)

From (3.28), (3.22), (3.29) and (3.30) we derive

\[
(a + \Delta u) g_{i\bar{i}} \phi_i + 2g_{i\bar{i}} \Re\{\phi_i \bar{\lambda}_i\} \leq -c_1(\text{tr} \chi + \Delta u) \sum_i g_{i\bar{i}} - \Delta(f)
\]

\[
+ n^2 \inf_{j,k} R_{j\bar{j}k\bar{k}} + (c_2 + c_3|\nabla u|^2) \sum_i g_{i\bar{i}}.
\]  

(3.31)

By (3.19) and using the following inequality as in Yau [71]

\[
\left( \sum g_{i\bar{i}} \right)^{n-1} \geq \frac{\sum g_{i\bar{i}}}{g_{1\bar{1}} \cdots g_{n\bar{n}}} = \frac{\text{tr} \chi + \Delta u}{\text{det}(\chi_{i\bar{j}} + u_{i\bar{j}})} = \frac{\text{tr} \chi + \Delta u}{\psi},
\]

(3.32)
we see that
\[
\phi^{-1}g^i{}_{\bar j} \phi_i \geq \frac{\epsilon}{2\psi \frac{1}{\psi^r}} (\text{tr } \chi + \Delta u)^{\frac{1}{\psi^r}} - 1 + \frac{\epsilon}{2} \sum g^i{}_{\bar j} + g_i{}^j u_i{}_{\bar j}. \tag{3.33}
\]

Note also that
\[
2g^i{}_{\bar j} \Re \{\phi_i \lambda_i\} = 2\phi g^i{}_{\bar j} \Re \{u_i \lambda_i\} \geq -\phi g^i{}_{\bar j} (u_i{}_{\bar j} + |\lambda_i|^2).
\]

Consequently, when \( A \) is chosen so that \( \epsilon \phi \geq 2(1 - c_1) \) we have from (3.31), (3.33) and (3.32),
\[
(\text{tr } \chi + \Delta u)^{\frac{1}{\psi^r}} \leq C (1 + |\psi^{\frac{1}{\psi^r}}|) \]
or
\[
a + \Delta u \leq A|\lambda|^2 + c_2 + c_3|\nabla u|^2.
\]

The proof is complete. \( \square \)

Finally, to remove assumption (3.3) we need assume \( \chi(\cdot, u) \) to be nondecreasing in \( u \). In this case \( \chi(\cdot, u) - \chi(\cdot, \bar u) \geq 0 \). We may therefore replace \( u \) by \( v = u - \bar u \) and \( \chi(\cdot, u) \) by
\[
\chi'(\cdot, v) \equiv \chi(\cdot, v + \bar u) + \frac{\sqrt{-1}}{2} \partial \bar \partial u.
\]

Note that \( \chi'(\cdot, v) \geq \chi_d(\cdot, v) > 0 \) and \( \chi'_u(\cdot, v) = \chi_u(\cdot, u) \). Propositions 3.1 and 3.3 thus apply to \( v \).

4. Boundary estimates for second derivatives

In this section we derive \( \text{a priori} \) estimates for second derivatives (the real Hessian) on the boundary
\[
\max_{\partial M} |\nabla^2 u| \leq C. \tag{4.1}
\]

We shall follow techniques developed in [34,29,30] using subsolutions.

We begin with a brief review of formulas for covariant derivatives which we shall use in this and following sections.

In local coordinates \( z = (z_1, \ldots, z_n) \), \( z_j = x_j + \sqrt{-1} y_j \), we use notations such as
\[
v_i = \nabla_{\partial z_i} v, \quad v_{ij} = \nabla_{\partial z_j} \nabla_{\partial z_i} v, \quad v_{xi} = \nabla_{\partial x_i} v, \quad \text{etc.}
\]

Recall that
\[
v_{ij} - v_{ji} = 0, \quad v_{ij} - v_{ji} = T^l_{ij} v_l. \tag{4.2}
\]
By straightforward calculations,

\[
\begin{aligned}
vi \bar{j}k - vi \bar{k}j &= \overline{T_{jk}^l v_i l}, \\
vi \bar{j}k - vi \bar{k}j &= -g^{lm} R_{kji \bar{m}} v_l, \\
vijk - vikj &= T_{ik}^l v_l.
\end{aligned}
\]  

\[\text{(4.3)}\]

Therefore,

\[
vi \bar{j}k - vki \bar{j} = (vi \bar{j}k - vik \bar{j}) + (vik \bar{j} - vki \bar{j})
\]

\[= -g^{lm} R_{kji \bar{m}} v_l + T_{ik}^l v_l + \nabla_j T_{ik}^l v_l \]

\[= -g^{lm} R_{i \bar{k}jm} v_l + T_{ik}^l v_l \]

\[\text{(4.4)}\]

by (2.16), and

\[
vijk - viks = (vijk - vijk) + (vijk - viks)
\]

\[= T_{ik}^l v_l + T_{ik}^l v_l + \nabla_j T_{ik}^l v_l. \]

\[\text{(4.5)}\]

Since

\[
\frac{\partial}{\partial x_k} = \frac{\partial}{\partial z_k} + \frac{\partial}{\partial \bar{z}_k}, \quad \frac{\partial}{\partial y_k} = \sqrt{-1} \left( \frac{\partial}{\partial z_k} - \frac{\partial}{\partial \bar{z}_k} \right),
\]

we see that

\[
\begin{aligned}
vzixj - vzjixl &= vij - vji = T_{ij}^l v_l, \\
vziyj - vyjzil &= \sqrt{-1} (vij - vji) = \sqrt{-1} T_{ij}^l v_l, \\
vzi \bar{j}xk - vxi\bar{j}zk &= (vi \bar{j}k + vi \bar{k}j) - (vik \bar{j} + vki \bar{j})
\]

\[= (vijk - vikj) + (vijk - vikj)
\]

\[= -g^{lm} R_{i \bar{j}km} v_l + T_{ik}^l v_l + \overline{T_{jk}^l v_l}. \]

\[\text{(4.6)}\]

Similarly,

\[
vzizjyk - vykzi\bar{z}j = \sqrt{-1} \left( (vij \bar{k} - vi \bar{j}k) - (vki \bar{j} - vki \bar{j}) \right)
\]

\[= \sqrt{-1} \left( (vij \bar{k} - vi \bar{j}k) - (vki \bar{j} - vki \bar{j}) \right)
\]

\[= \sqrt{-1} ( -g^{lm} R_{i \bar{j}km} v_l + T_{ik}^l v_l - \overline{T_{jk}^l v_l}). \]

\[\text{(4.7)}\]

For convenience we set

\[
t_{2k-1} = x_k, \quad t_{2k} = y_k, \quad 1 \leq k \leq n - 1; \quad t_{2n-1} = y_n, \quad t_{2n} = x_n.
\]
By (4.7), (4.8) and the identity
\[ g_{i\bar{j}} T_{k\bar{i}}^l = T_{k\bar{i}}^l - g_{i\bar{j}} T_{l\bar{k}}^i \chi_{\bar{i}j} \] (4.9)
we obtain for all \( 1 \leq \alpha \leq 2n \),
\[ |g_{i\bar{j}} u_{\alpha i\bar{j}}| \leq |(f)_\alpha| + |g_{i\bar{j}} (\chi_{\alpha i\bar{j}})_{\alpha u}| + |g_{i\bar{j}} (u_{\alpha i\bar{j}} - u_{\alpha j\bar{i}})| \leq C (1 + g_{i\bar{j}} g_{\bar{j}i}). \] (4.10)
We also record here the following identity which we shall use later: for a function \( \eta \),
\[ g_{i\bar{j}} \eta_i u_{\alpha i\bar{j}} = g_{i\bar{j}} \eta_i (2 u_{\alpha j\bar{i}} + \sqrt{-1} u_{\alpha y\bar{j}}) = 2 \eta_n - 2 g_{i\bar{j}} \eta_i \chi_{\alpha i\bar{j}} + \sqrt{-1} g_{i\bar{j}} \eta_i u_{\alpha y\bar{j}}. \] (4.11)
We now start to derive (4.1). We assume
\[ |u| + |\nabla u| \leq K \quad \text{in } \widetilde{M}. \] (4.12)
Set
\[ \psi \equiv \min_{|u| \leq K, z \in \bar{M}} \psi(z,u) > 0, \quad \bar{\psi} \equiv \max_{|u| \leq K, z \in \bar{M}} \psi(z,u). \] (4.13)
Let \( \sigma \) be the distance function to \( \partial M \). Note that \(|\nabla \sigma| = \frac{1}{2}\) on \( \partial M \). There exists \( \delta_0 > 0 \) such that \( \sigma \) is smooth and \( \nabla \sigma \neq 0 \) in
\[ M_{\delta_0} := \{ z \in M : \sigma(z) < \delta_0 \}, \]
which we call the \( \delta_0 \)-neighborhood of \( \partial M \). We can therefore write
\[ u - u = h \sigma, \quad \text{in } M_{\delta_0} \] (4.14)
where \( h \) is a smooth function.
Consider a boundary point \( p \in \partial M \). We choose local coordinates \( z = (z_1, \ldots, z_n), z_j = x_j + \sqrt{-1} y_j \), around \( p \) in a neighborhood which we assume to be contained in \( M_{\delta_0} \) such \( \frac{\partial}{\partial x_n} \) is the interior normal direction to \( \partial M \) at \( p \) where we also assume \( g_{i\bar{j}} = \delta_{ij} \). (Here and in what follows we identify \( p \) with \( z = 0 \).) For later reference we call such local coordinates regular coordinate charts.
By (4.14) we have
\[ (u - u)_{x_n} = h_{x_n} \sigma + h \sigma_{x_n} \]
and
\[ (u - u)_{j\bar{k}} = h_{j\bar{k}} \sigma + h \sigma_{j\bar{k}} + h_j \sigma_{\bar{k}} + h_{\bar{k}} \sigma_j. \]
Since \( \sigma = 0 \) on \( \partial M \) and \( \sigma_{x_n}(0) = 2|\nabla \sigma| = 1 \), we see that
\[(u - u)_{\kappa_n}(0) = h(0)\]

and
\[(u - u)_{\kappa^j}(0) = (u - u)_{\kappa_n}(0)\sigma_{\kappa^j}(0), \quad j, k < n. \tag{4.15}\]

Similarly,
\[(u - u)_{\kappa^\alpha\kappa^\beta}(0) = -(u - u)_{\kappa_n}(0)\sigma_{\kappa^\alpha\kappa^\beta}, \quad \alpha, \beta < 2n. \tag{4.16}\]

It follows that
\[\left|u_{\kappa^\alpha}(0)\right| \leq C, \quad \alpha, \beta < 2n, \tag{4.17}\]

where \(C\) depends on \(|u|_{C^1(M)}\), \(|u|_{C^1(\bar{M})}\), and the principal curvatures of \(\partial M\).

To estimate \(u_{\kappa^\alpha\kappa_n}(0)\) for \(\alpha \leq 2n\), we will follow [30] and employ a barrier function of the form
\[v = (u - u) + t\sigma - N\sigma^2, \tag{4.18}\]

where \(t, N\) are positive constants to be determined. Recall that \(u \in C^2\) and \(\chi_u > 0\) in a neighborhood of \(\partial M\). We may assume that there exists \(\epsilon > 0\) such that \(\chi_u \geq \epsilon\omega\) in \(M_\delta\). Locally, this gives
\[\{u_{\kappa^j} + \chi \kappa^j(\cdot, u)\} \geq \epsilon\{g_{k\kappa^j}\}. \tag{4.19}\]

The following is the key ingredient in our argument.

**Lemma 4.1.** For \(N\) sufficiently large and \(t, \delta\) sufficiently small,
\[g^{i\bar{j}}v_{i\bar{j}} \leq -\frac{\epsilon}{4}\left(1 + \sum g^{i\bar{j}}g_{i\bar{j}}\right) \quad \text{in } \Omega_\delta, \]
\[v \geq 0 \quad \text{on } \partial \Omega_\delta\]

where \(\Omega_\delta = M \cap B_\delta\) and \(B_\delta\) is the (geodesic) ball of radius \(\delta\) centered at \(p\).

**Proof.** This lemma was first proved in [30] for domains in \(C^n\). For completeness we include the proof here with minor modifications. By (4.19) we have
\[g^{i\bar{j}}(u_{i\bar{j}} - u_{i\bar{j}}) \leq g^{i\bar{j}}(u_{i\bar{j}} + \chi_{i\bar{j}}(\cdot, u) - u_{i\bar{j}} - \chi_{i\bar{j}}(\cdot, u)) \leq n - \epsilon g^{i\bar{j}}g_{i\bar{j}}. \tag{4.20}\]

Obviously,
\[g^{i\bar{j}}\sigma_{i\bar{j}} \leq C_1 g^{i\bar{j}}g_{i\bar{j}}\]

for some constant \(C_1 > 0\) under control. Thus
\[g^{i\bar{j}}v_{i\bar{j}} \leq n + \{C_1(t + N\sigma) - \epsilon\}g^{i\bar{j}}g_{i\bar{j}} - 2N g^{i\bar{j}}\sigma_{i\bar{j}} \quad \text{in } \Omega_\delta.\]
Let $\lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of $\{u_{ij} + \chi_{ij}\}$ (with respect to $\{g_{ij}\}$). We have $g^{ij}g_{ij} = \sum \lambda_k^{-1}$ and

$$g^{ij}\sigma_i\sigma_j \geq \frac{1}{2\lambda_n} \quad (4.21)$$

since $|\nabla \sigma| \equiv \frac{1}{2}$ where $\sigma$ is smooth. By the arithmetic–geometric mean-value inequality,

$$\epsilon g^{ij}g_{ij} \geq -\frac{\epsilon}{4}(1 + g^{ij}g_{ij}) \quad \text{in } \Omega_{\delta}$$

if we require $\delta$ to satisfy $C_1 N \delta \leq \frac{\epsilon}{4}$ in $\Omega_{\delta}$.

On $\partial M \cap B_{\delta}$ we have $v = 0$. On $M \cap \partial B_{\delta}$,

$$v \geq t\sigma - N\sigma^2 \geq (t - N\delta)\sigma \geq 0$$

if we require, in addition, $N\delta \leq t$. \qed

**Remark 4.2.** For the real Monge–Ampère equations, Lemma 4.1 was proved in [29] both for domains in $\mathbb{R}^n$ and in general Riemannian manifolds, improving earlier results in [43,34,33].

**Lemma 4.3.** Let $w \in C^2(\overline{\Omega}_{\delta})$. Suppose that $w$ satisfies

$$g^{ij}w_{ij} \geq -C_1(1 + g^{ij}g_{ij}) \quad \text{in } \Omega_{\delta}$$

and

$$w \leq C_0 \rho^2 \quad \text{on } B_{\delta} \cap \partial M, \quad w(0) = 0$$

where $\rho$ is the distance function to the point $p$ (where $z = 0$) on $\partial M$. Then $w_n(0) \leq C$, where $v$ is the interior unit normal to $\partial M$, and $C$ depends on $\epsilon^{-1}$, $C_0$, $C_1$, $|w|_{C^0(\overline{\Omega}_{\delta})}$, $|u|_{C^1(\overline{M})}$ and the constants $N$, $t$ and $\delta$ determined in Lemma 4.1.

**Proof.** By Lemma 4.1, $Av + B\rho^2 - w \geq 0$ on $\partial \Omega_{\delta}$ and

$$g^{ij}(Av + B\rho^2 - w)_{ij} \leq 0 \quad \text{in } \Omega_{\delta}$$

when $A \gg B$ and both are sufficiently large. By the maximum principle,

$$Av + B\rho^2 - w \geq 0 \quad \text{in } \overline{\Omega}_{\delta}.$$
Consequently,
\[ Av_v(0) - w_v(0) = D_v(Av + B\rho^2 - w)(0) \geq 0 \]
since \( Av + B\rho^2 - w = 0 \) at the origin. □

We next apply Lemma 4.3 to estimate \( u_{t\alpha x_n}(0) \) for \( \alpha < 2n \), following [12]. For fixed \( \alpha < 2n \), we write \( \eta = \sigma_{tn}/\sigma_{xn} \) and define
\[ \mathcal{T} = \nabla \frac{\partial}{\partial t} - \eta \nabla \frac{\partial}{\partial x_n}. \]
We wish to apply Lemma 4.3 to
\[ w = (u_{yn} - \varphi_{yn})^2 + \mathcal{T}(u - \varphi). \]

By (4.12),
\[ |\mathcal{T}(u - \varphi)| + (u_{yn} - \varphi_{yn})^2 \leq C \text{ in } \Omega_\delta. \]

On \( \partial M \) since \( u - \varphi = 0 \) and \( \mathcal{T} \) is a tangential differential operator, we have
\[ \mathcal{T}(u - \varphi) = 0 \text{ on } \partial M \cap B_\delta \]
and, similarly,
\[ (u_{yn} - \varphi_{yn})^2 \leq C\rho^2 \text{ on } \partial M \cap B_\delta. \]

We compute next
\[ g^{ij}(T u)_{ij} = g^{ij}(u_{ta,ij} + \eta u_{x_n,ij}) + g^{ij} \eta_{ij} u_{xn} + 2g^{ij} Re(\eta_iu_{x_n,j}). \]

By (4.10) and (4.11),
\[ |g^{ij}(u_{ta,ij} + \eta u_{x_n,ij})| \leq |T(f)| + C_1(1 + g^{ij}g_{ij}) \]
and
\[ 2|g^{ij} Re(\eta_iu_{x_n,j})| \leq g^{ij}u_{yn,i}u_{yn,j} + C_2(1 + g^{ij}g_{ij}). \]

Applying (4.10) again, we derive
\[ g^{ij}(u_{yn} - \varphi_{yn})_2)_{ij} = 2g^{ij}(u_{yn} - \varphi_{yn})_i(u_{yn} - \varphi_{yn})_j + 2(u_{yn} - \varphi_{yn})g^{ij}(u_{yn} - \varphi_{yn})_i \]
\[ \geq g^{ij}u_{yn,i}u_{yn,j} - 2g^{ij}\varphi_{yn,i}\varphi_{yn,j} + 2(u_{yn} - \varphi_{yn})g^{ij}(u_{yn,i} - \varphi_{yn,i}) \]
\[ \geq g^{ij}u_{yn,i}u_{yn,j} - |f|_n - C_3(1 + g^{ij}g_{ij}). \]
Finally, combining (4.23)–(4.26) we obtain
\[ \tilde{g}^{ij} \left[ (u_{yn} - \varphi_{yn})^2 \pm T(u - \varphi) \right]_{ij} \geq -C_4 (1 + \tilde{g}^{ij} \tilde{g}_{ij}) \quad \text{in } \Omega_\delta. \] (4.27)

Consequently, we may apply Lemma 4.3 to \( w = (u_{yn} - \varphi_{yn})^2 \pm T(u - \varphi) \) to obtain
\[ |u_{ta, x_n}(0)| \leq C, \quad \alpha < 2n. \] (4.28)

By (4.6) we also have
\[ |u_{x_n t_\alpha}(0)| \leq C, \quad \alpha < 2n. \] (4.29)

It remains to establish the estimate
\[ |u_{x_n x_n}(0)| \leq C. \] (4.30)

Since we have already derived
\[ |u_{ta, t_\alpha}(0)|, |u_{ta, x_n}(0)|, |u_{x_n t_\alpha}(0)| \leq C, \quad \alpha, \beta < 2n, \] (4.31)

it suffices to prove
\[ 0 \leq \chi_{n\tilde{n}}(0) + u_{n\tilde{n}}(0) = \chi_{n\tilde{n}}(0) + u_{x_n x_n}(0) + u_{y_n y_n}(0) \leq C. \] (4.32)

Expanding \( \det(u_{ij} + \chi_{ij}) \), we have
\[ \det(u_{ij}(0) + \chi_{ij}(0)) = a(u_{n\tilde{n}}(0) + \chi_{n\tilde{n}}(0)) + b \] (4.33)

where
\[ a = \det(u_{\alpha\beta}(0) + \chi_{\alpha\beta}(0)) \bigg|_{1 \leq \alpha, \beta \leq n-1} \]

and \( b \) is bounded in view of (4.31). Since \( \det(u_{ij} + \chi_{ij}) \) is bounded, we only have to derive an \textit{a priori} positive lower bound for \( a \), which is equivalent to
\[ \sum_{\alpha, \beta \leq n} (u_{\alpha\beta}(0) + \chi_{\alpha\beta}(0)) \xi_{\alpha} \xi_{\beta} \geq c_0 |\xi|^2, \quad \forall \xi \in \mathbb{C}^{n-1}, \] (4.34)

for a uniform constant \( c_0 > 0 \).

**Proposition 4.4.** There exists \( c_0 = c_0(\psi^{-1}, \varphi, u) > 0 \) such that (4.34) holds.
Proof. Let $T_C \partial M \subset T_C M$ be the complex tangent bundle of $\partial M$ and

$$T^{1,0} \partial M = T^{1,0} M \cap T_C \partial M = \{ \xi \in T^{1,0} M : d\sigma(\xi) = 0 \}.$$ 

In local coordinates,

$$T^{1,0} \partial M = \{ \xi = \xi_i \frac{\partial}{\partial \bar{z}_i} \in T^{1,0} M : \sum \xi_i \sigma_i = 0 \}.$$ 

It is enough to establish a positive lower bound for

$$m_0 = \min_{\xi \in T^{1,0} \partial M, |\xi| = 1} \chi_u(\xi, \bar{\xi}).$$

We assume that $m_0$ is attained at a point $p \in \partial M$ and choose regular local coordinates around $p$ as before such that

$$m_0 = \chi_u \left( \frac{\partial}{\partial \bar{z}_1}, \frac{\partial}{\partial z_1} \right) = u_{1\bar{1}}(0) + \chi_{1\bar{1}}(0).$$

One needs to show

$$m_0 = u_{1\bar{1}}(0) + \chi_{1\bar{1}}(0) \geq c_0 > 0. \quad (4.35)$$

By (4.15),

$$u_{1\bar{1}}(0) = u_{1\bar{1}}(0) - (u - u)x_n(0)\sigma_{1\bar{1}}(0). \quad (4.36)$$

We can assume $u_{1\bar{1}}(0) \leq \frac{1}{2}(u_{1\bar{1}}(0) - \chi_{1\bar{1}}(0))$; otherwise we are done. Thus

$$(u - u)x_n(0)\sigma_{1\bar{1}}(0) \geq \frac{1}{2}(u_{1\bar{1}}(0) + \chi_{1\bar{1}}(0)). \quad (4.37)$$

It follows from (4.12) that

$$\sigma_{1\bar{1}}(0) \geq \frac{u_{1\bar{1}}(0) + \chi_{1\bar{1}}(0)}{2K} \geq \frac{c}{2K} \equiv c_1 > 0 \quad (4.38)$$

where $K = \max_{\partial M} |\nabla(u - u)|$.

Let $\delta > 0$ be small enough so that

$$w \equiv \left| -\sigma_{zn} \frac{\partial}{\partial z_1} + \sigma_{z1} \frac{\partial}{\partial z_n} \right|$$

$$= (g_{1\bar{1}}|\sigma_{zn}|^2 - 2\Re\{g_{1\bar{n}}\sigma_{zn}\sigma_{n\bar{1}} + g_{n\bar{n}}|\sigma_{z1}|^2\})^{1/2} > 0 \quad \text{in } M \cap B_\delta(p).$$
Define \( \zeta = \sum \zeta_i \frac{\partial}{\partial \zeta_i} \in T^{1,0}M \) in \( M \cap B_\delta(p) \):

\[
\begin{align*}
\zeta_1 &= -\frac{\sigma_{zn}}{w}, \\
\zeta_j &= 0, \quad 2 \leq j \leq n - 1, \\
\zeta_n &= \frac{\sigma_{z1}}{w}
\end{align*}
\]

and

\[
\Phi = (\varphi_{jk} + \chi_{jk})\zeta_j \tilde{\zeta}_k - (u - \varphi)_{xn} \sigma_{jk} \zeta_j \zeta_k - u_{1i}(0) - \chi_{1i}(0).
\]

Note that \( \zeta \in T^{1,0}\partial M \) on \( \partial M \) and \( |\zeta| = 1 \). By (4.15),

\[
\Phi = (u_{jk} + \chi_{jk})\zeta_j \tilde{\zeta}_k - u_{1i}(0) - \chi_{1i}(0) \geq 0 \quad \text{on } \partial M \cap B_\delta(p) \quad (4.39)
\]

and \( \Phi(0) = 0 \).

Write \( G = \sigma_{ij} \zeta_i \tilde{\zeta}_j \). We have

\[
\begin{align*}
g^{ij} \Phi_{ij} &\leq -g^{ij}(u_{xn} G)_{ij} + C(1 + g^{ij} g_{ij}) \\
&= -G g^{ij} u_{xn,ij} - 2g^{ij} \Re\{u_{xn} G_{j}\} + C(1 + g^{ij} g_{ij}) \\
&\leq g^{ij} u_{yn,ij} + C(1 + g^{ij} g_{ij}) \quad (4.40)
\end{align*}
\]

by (4.10) and (4.11). It follows that

\[
g^{ij} \left[ \Phi - (u_{yn} - \varphi_{yn})^2 \right]_{ij} \leq C(1 + g^{ij} g_{ij}) \quad \text{in } M \cap B_\delta(p). \quad (4.41)
\]

Moreover, by (4.22) and (4.39),

\[
(u_{yn} - \varphi_{yn})^2 - \Phi \leq C|z|^2 \quad \text{on } \partial M \cap B_\delta(p).
\]

Consequently, we may apply Lemma 4.3 to

\[
h = (u_{yn} - \varphi_{yn})^2 - \Phi
\]

to derive \( \Phi_{xn}(0) \geq -C \) which, by (4.38), implies

\[
u_{xn,yn}(0) \leq \frac{C}{\sigma_{1i}(0)} \leq \frac{C}{c_1} \quad (4.42)
\]

In view of (4.31) and (4.42) we have an \textit{a priori} upper bound for all eigenvalues of \( \{u_{ij}(0) + \chi_{ij}(0)\} \). Since \( \det(u_{ij} + \chi_{ij}) \geq \underline{\psi} > 0 \), the eigenvalues of \( \{u_{ij}(0) + \chi_{ij}(0)\} \) must admit a positive lower bound, i.e.,

\[
\min_{\xi \in T^{1,0}_p M, |\xi| = 1} (u_{ij} + \chi_{ij}) \xi_i \xi_j \geq c_0.
\]
Therefore,

\[ m_0 = \min_{\xi \in T_p^1 \partial M, |\xi| = 1} (u_{ij} + \chi_{ij}) \xi_i \xi_j \geq \min_{\xi \in T_p^1 \partial M, |\xi| = 1} (u_{ij} + \chi_{ij}) \xi_i \xi_j \geq c_0. \]

The proof of Proposition 4.4 is complete. \( \square \)

5. Estimates for the real Hessian and higher derivatives

The primary goal of this section is to derive global estimates for the whole (real) Hessian

\[ |\nabla^2 u| \leq C \text{ on } \tilde{M}. \tag{5.1} \]

This is equivalent to

\[ |u_{xi_xj}(p)|, |u_{xi_yj}(p)|, |u_{yi_yj}(p)| \leq C, \quad \forall 1 \leq i, j \leq n, \tag{5.2} \]

in local coordinates \( z = (z_1, \ldots, z_n), z_j = x_j + \sqrt{-1} y_j \) with \( g_{ij}(p) = \delta_{ij} \) for any fixed point \( p \in M \), where the constant \( C \) may depend on \( |u|_{C^1(M)}, \sup M \Delta u, \inf \psi > 0, \) and the curvature and torsion of \( M \) as well as their derivatives. Once this is done we can apply the Evans–Krylov Theorem to obtain global \( C^{2,\alpha} \) estimates.

As in Section 4 we shall use covariant derivatives. We start with communication formulas for the fourth order derivatives. From direct computation,

\[
\begin{aligned}
    v_{ijkl} - v_{klij} &= T_{kl} v_{ij\bar{q}}, \\
    v_{ijkl} - v_{ijkl} &= g^{pq} R_{kliq} v_{pj} + g^{pq} R_{kljq} v_{ip}.
\end{aligned} \tag{5.3}
\]

Therefore, by (4.3)–(4.5), (5.3) and (2.16),

\[
\begin{aligned}
    v_{ijkl} - v_{klij} &= (v_{ijkl} - v_{kijl}) + (v_{kijl} - v_{kijl}) + (v_{kijl} - v_{kili}) \\
    &= \nabla_i(-g^{pq} R_{ijk\bar{q}} v_p + T_{ik} v_{pj}) + T_{ij}^p v_{ki\bar{q}} + g^{pq} \nabla_j (R_{ijkl} v_p) \\
    &= T_{ik}^p v_{pj} + T_{ij} v_{k\bar{q}} - T_{ik} T_{ij} v_{pq} + g^{pq} (R_{kliq} v_{pj} - R_{ij\bar{k}q} v_{lp}) \\
    &\quad + g^{pq} (\nabla_j R_{ik\bar{q}} - \nabla_i R_{ij\bar{k}q} + R_{im\bar{k}q} T_{mj}) v_p \tag{5.4}
\end{aligned}
\]

and

\[
\begin{aligned}
    v_{ijkl} - v_{klij} &= v_{ijkl} - v_{kijl} + v_{kijl} - v_{kili} + v_{kijl} - v_{kili} \\
    &= \nabla_i(-g^{pq} R_{ijk\bar{q}} v_p + T_{ik} v_{pj}) + \nabla_j (T_{il}^p v_{kp}) - g^{pq} R_{ij\bar{k}q} v_{pi} - g^{pq} R_{i\bar{j}kq} v_{lp} \\
    &\quad + (\nabla_l T_{ik}^p) v_{pj} + T_{ik}^p v_{pj} + T_{il}^p v_{kp}. \tag{5.5}
\end{aligned}
\]
We now turn to the proof of (5.2). It suffices to derive the following estimate.

**Proposition 5.1.** There exists constant $C > 0$ depending on $|u|_{C^1(M)}$, $\sup_M \Delta u$ and $\inf \psi > 0$ such that

$$\sup_{\tau \in TM, |\tau| = 1} u_{\tau\tau} \leq C.$$  \hfill (5.6)

**Proof.** Suppose that

$$N := \sup_M \left\{ |\nabla u|^2 + A|\chi u|^2 + \sup_{\tau \in TM, |\tau| = 1} u_{\tau\tau} \right\}$$

is achieved at an interior point $p \in M$ and for some unit vector $\tau \in T_p M$, where $A$ is positive constant to be determined. We choose local coordinates $z = (z_1, \ldots, z_n)$ such that $g_{ij} = \delta_{ij}$ and $\{u_{ij} + \chi_{ij}\}$ is diagonal at $p$. Thus $\tau$ can be written in the form

$$\tau = a_j \frac{\partial}{\partial z_j} + b_{\bar{j}} \frac{\partial}{\partial \bar{z}_j}, \quad a_j, b_{\bar{j}} \in \mathbb{C}, \quad \sum a_j b_{\bar{j}} = \frac{1}{2}.$$

Let $\xi$ be a smooth unit vector field defined in a neighborhood of $p$ such that $\xi(p) = \tau$. Then the function

$$Q = u_{\xi\xi} + |\nabla u|^2 + A|\chi u|^2$$

(defined in a neighborhood of $p$) attains it maximum at $p$ where,

$$Q_i = u_{\tau\tau i} + u_k u_{i\bar{k}} + u_{\bar{k}} u_{ki} + 2A(u_{kk} + \chi_{kk})(u_{i\bar{k}} + \chi_{i\bar{k}}) = 0$$  \hfill (5.7)

and

$$0 \geq g^{i\bar{i}} Q_{i\bar{i}} = g^{i\bar{i}}(u_{i\bar{k}} u_{\bar{k}i} + u_{ki} u_{\bar{k}i}) + g^{i\bar{i}}(u_{i\bar{k}} u_{\bar{k}i} + u_{\bar{k}} u_{ki}) + g^{i\bar{i}}(u_{\tau\tau\bar{i}} + 2A g^{i\bar{i}}(u_{\bar{k}i} + \chi_{\bar{k}i}) + (u_{ki} + \chi_{ki})_{\bar{i}} + 2A(u_{kk} + \chi_{kk}) g^{i\bar{i}}(u_{k\bar{k}} + \chi_{k\bar{k}})_{\bar{i}}) \geq (f)_{k\bar{i}} - C \left( 1 + \sum g^{i\bar{i}} \right).$$  \hfill (5.8)

Differentiating Eq. (2.27) twice (using covariant derivatives), by (4.4) and (5.4) we have

$$g^{i\bar{i}} u_{k\bar{k}i} = (f)_{k} + g^{i\bar{i}} \left( R_{i\bar{k}i} u_{\bar{i}} - T^l_{ik} u_{\bar{i}l} - (\chi_{i\bar{i}})_{k} \right) \geq (f)_{k} - C \left( 1 + \sum g^{i\bar{i}} \right)$$  \hfill (5.9)

and

$$g^{i\bar{i}} u_{k\bar{k}i} \geq g^{i\bar{i}} g^{j\bar{j}} |u_{ijk}|^2 + g^{i\bar{i}} \left( T^l_{ik} u_{\bar{i}l} + T^l_{ik} u_{i\bar{k}} \right) + (f)_{k\bar{k}} - C \sum g^{i\bar{i}} \geq (f)_{k\bar{k}} - C \left( 1 + \sum g^{i\bar{i}} \right).$$  \hfill (5.10)
Note that
\[ u_{\tau \tau i} = a_k u_{k l i} + 2a_k b_j u_{k l i} + b_k b_j u_{k l i}. \]

Using the formulas in (5.3)–(5.5) we obtain
\[ g^{ij} u_{\tau \tau i} \geq g^{ij} u_{\tau \tau i} - C g^{ij} |T^l_{ik} u_{ik} l - C \left( 1 + \sum_{k,l} |u_{kl}| \right) \sum g^{ij}. \tag{5.11} \]

From (5.9)–(5.11), (5.8) and the inequality
\[ 2 g^{ij} (u_{ki} + \chi_{ki}) (u_{k i} + \chi_{k i}) \geq g^{ij} u_{ki} u_{k i} - g^{ij} (\chi_{k j}) (\chi_{j k}), \tag{5.12} \]
we see that
\[ g^{ij} u_{ki} u_{k i} + (A - C) g^{ij} u_{ki} u_{k i} \leq C \left( 1 + A + \sum_{k,l} |u_{kl}| \right) \left( 1 + \sum g^{ij} \right). \tag{5.13} \]

We now need the nondegeneracy of Eq. (2.27) which implies that there is \( \Lambda > 0 \) depending on \( \sup_M \Delta u \) and \( \inf \psi > 0 \) such that
\[ \Lambda^{-1} \{ g_{i j} \} \leq \{ g_{ij} \} \leq \Lambda \{ g_{ij} \} \]
and therefore,
\[ \sum g^{ij} \leq n \Lambda, \quad g^{ij} u_{ki} u_{k i} \geq \frac{1}{\Lambda} \sum_{i,k} |u_{kl}|^2. \tag{5.14} \]

Plugging these into (5.13) and choosing \( A \) large we derive
\[ \sum_{i,k} |u_{ki}|^2 \leq C. \]

Consequently, \( u_{\tau \tau}(p) \leq C \). Finally,
\[ \sup_{q \in M} \sup_{\tau \in T_{q M}, |\tau| = 1} u_{\tau \tau} \leq u_{\tau \tau}(p) + 2 \sup_M (|\nabla u|^2 + A |\chi u|^2). \]

This completes the proof of (5.6). \( \Box \)

By the Evans–Krylov Theorem (see [25,46,47]) we derive the \( C^{2, \alpha} \) estimates
\[ |u|_{C^{2, \alpha}(M)} \leq C. \tag{5.15} \]
Here we note that the complex Monge–Ampère operator \( u_{ij} \mapsto \log \det (u_{ij} + \chi_{ij}) \), when viewed as a differential operator of real variables, is concave with respect to the real Hessian matrix of \( u \), so the conditions of the Evans–Krylov Theorem are satisfied. The higher order regularity and estimates now follow from the classical Schauder theory for elliptic linear equations.

**Remark 5.2.** When \( M \) is a Kähler manifold, Proposition 5.1 was recently proved by Blocki [10]. He observed that the estimate (5.6) does not depend on \( \inf \psi \) when \( M \) has nonnegative bisectional curvature. This is clearly also true in the Hermitian case.

**Remark 5.3.** An alternative approach to the \( C^{2,\alpha} \) estimate (5.15) is to use (3.2) and the boundary estimate (4.1) (in place of (5.1)) and apply an extension of the Evans–Krylov Theorem; see Theorem 7.3, p. 126 in [17] which only requires \( C^{1,\alpha} \) bounds for the solution. This was pointed out to us by Pengfei Guan to whom we wish to express our gratitude.

### 6. \( C^0 \) estimates and existence

In this section we complete the proof of Theorems 1.1 and 1.3–1.5 using the estimates established in previous sections. We shall consider separately the Dirichlet problem and the case of manifolds without boundary. In each case we need first to derive \( C^0 \) estimates; the existence of solutions then can be proved by the continuity method or combined with degree arguments.

#### 6.1. Compact manifolds without boundary

The \( C^0 \) estimate was established by Yau [71] for solutions of the complex Monge–Ampère equation on compact Kähler manifolds without boundary. The reader is referred to [58,64] for an approach which simplifies Yau’s original proof. Here we shall follow [58] to avoid the use of Green’s function for the Chern Laplacian operator.

Let \((M, \omega)\) be a compact Hermitian manifold without boundary. Replacing \( \chi \) by \( \chi \phi \) for \( \phi \in \mathcal{H}_\chi \cap C^\infty (M) \) if necessary, we shall assume \( \chi \geq \epsilon \omega \). Let \( u \in C^4 (M) \) be an admissible solution of Eq. (2.27) and satisfy

\[
\int_M u \omega^n = 0. \tag{6.1}
\]

We write

\[
\tilde{\chi} = \sum_{k=1}^{\infty} \chi^{k-1} \wedge (\chi u)^{n-k}.
\]

Multiply the identity \((\chi u)^n - \chi^n = \frac{\sqrt{-1}}{2} \partial \bar{\partial} u \wedge \tilde{\chi}\) by \( u |u|^p \), \( p \geq 0 \), and integrate over \( M \),

\[
\int_M u |u|^p [(\chi u)^n - \chi^n] = \frac{\sqrt{-1}}{2} \int_M u |u|^p \partial \bar{\partial} u \wedge \tilde{\chi} = \frac{(p+1)\sqrt{-1}}{2} \int_M |u|^p \partial u \wedge \bar{\partial} u \wedge \tilde{\chi} + \frac{\sqrt{-1}}{2} \int_M u |u|^p \bar{\partial} u \wedge \partial \tilde{\chi}
\]
\[= \frac{2(p+1)\sqrt{-1}}{(p+2)^2} \int_M \partial(u|u|^p) \wedge \bar{\partial}(u|u|^p) \wedge \bar{\chi} \]
\[= \frac{\sqrt{-1}}{2(p+2)} \int_M |u|^{p+2} \bar{\partial} \bar{\partial} \bar{\chi}. \quad (6.2)\]

The last term vanishes under assumption (1.8) which implies \(\bar{\partial} \bar{\partial} \bar{\chi} = 0\). Since \(\chi > 0\) and \(\chi u \geq 0\), we see that \(\chi^{k-1} \wedge (\chi u)^{n-k} \geq 0\) for all \(k\). Therefore,

\[\int_M \left| \nabla (u|u|^p) \right|^2 \omega^n = \frac{\sqrt{-1}}{2} \int_M \partial(u|u|^p) \wedge \bar{\partial}(u|u|^p) \wedge \chi^{n-1} \]
\[\leq \frac{\sqrt{-1}}{2e^{n-1}} \int_M \partial(u|u|^p) \wedge \bar{\partial}(u|u|^p) \wedge \chi^n \]
\[\leq \frac{\sqrt{-1}}{2e^{n-1}} \int_M \partial(u|u|^p) \wedge \bar{\partial}(u|u|^p) \wedge \bar{\chi} \]
\[= \frac{(p+2)^2}{4(p+1)e^{n-1}} \int_M u|u|^p(\psi \omega^n - \chi^n) \]
\[\leq C \int_M |u|^{p+1} \omega^n. \quad (6.3)\]

Here in the last step we assume \(\psi\) does not depend on \(u\).

From this we can use the Sobolev inequality and Moser iteration method as in [58,64] to derive

\[\sup_M |u| \leq C \left(1 + |u|_{L^2(\omega)}^2\right).\]

Next, following Siu [58], by the Poincaré–Sobolev inequality, (6.1) and (6.3) with \(p = 0\) we obtain

\[|u|_{L^2(\omega)}^2 \leq C|\nabla u|_{L^2(\omega)}^2 \leq C|u|_{L^1(\omega)} \leq C|u|_{L^2(\omega)}.\]

This proves \(|u|_{L^2(\omega)} \leq C\) and therefore \(\sup_M |u| \leq C\).

If \(\psi\) depends on \(u\) and satisfies (1.10), a bound for \(\sup_M |u|\) follows directly from Eq. (2.27) by the maximum principle. Indeed, suppose \(u(p) = \max_M u\) for some \(p \in M\). Then \(\{u_{ij}(p)\} \leq 0\) and, therefore

\[\det \chi_{ij} \geq \det(u_{ij} + \chi_{ij}) = \psi(p, u(p)) \det g_{ij}.\]

This implies an upper bound \(u(p) \leq C\) by (1.10). That \(\min_M u \geq -C\) follows from a similar argument.
Proof of Theorem 1.3. We first consider the case that $\psi$ does not depend on $u$. By assumption (1.8) we see that
\[ \int_M (\chi u)^n = \int_M \chi^n. \]
Therefore,
\[ \int_M \chi^n = \int_M \psi \omega^n \]
is a necessary condition for the existence of admissible solutions, and that the linearized operator,
\[ v \mapsto g^{i\bar{j}} v_{i\bar{j}}, \]
of Eq. (1.1) is self-adjoint (with respect to the volume form $(\chi u)^n$). So the continuity method proof of Yau [71] works to give a unique admissible solution $u \in \mathcal{H} \cap C^{2,\alpha}(M)$ of (1.1) satisfying (6.1). The smoothness of $u$ follows from the Schauder regularity theory.

For the general case under the assumption $\psi u \geq 0$, one can still follow the proof in [71]; we omit it here.

Proof of Theorem 1.4. The uniqueness follows easily from the assumption $\psi u > 0$ and the maximum principle. For the existence we make use of the continuity method. We may assume $\chi > 0$. For $0 \leq s \leq 1$ consider
\[ (\chi u)^n = \psi^s(z, u) \omega^n \text{ in } M \] (6.4)
where $\psi^s(z, u) = (1-s)e^\alpha \eta + s \psi(z, u)$ and $\eta$ is defined by $\chi^n = \eta \omega^n$. Set
\[ S := \{ s \in [0, 1]: \text{Eq. (6.4) is solvable in } \mathcal{H}_\chi \cap C^{2,\alpha}(M) \} \]
and let $u^s \in \mathcal{H}_\chi \cap C^{2,\alpha}(M)$ be the unique solution of (6.4) for $s \in S$. Obviously $S \neq \emptyset$ as $0 \in S$ with $u^0 = 0$. Moreover, by the $C^{2,\alpha}$ estimates we see that $S$ is closed. We need to show that $S$ is also open in and therefore equal to $[0, 1]$; $u^1$ is then the desired solution.

Let $s \in S$ and let $\Delta^s$ denote the Laplace operator of $(M, \chi u^s)$. In local coordinates,
\[ \Delta^s v = g^{i\bar{j}} v_{i\bar{j}} = \tilde{g}^{i\bar{j}} \partial_i \overline{\partial_j} v \]
where $\{ \tilde{g}^{i\bar{j}} \} = \{ g_{i\bar{j}}^s \}^{-1}$ and $g_{i\bar{j}}^s = \chi_{i\bar{j}} + u^s_{i\bar{j}}$. Note that $\Delta^s - \psi^s u$, where $\psi^s u = \psi^s(\cdot, u^s)$, is the linearized operator of Eq. (6.4) at $u^s$. We wish to prove that for any $\phi \in C^\alpha(M, \chi u^s)$ there exists a unique solution $v \in C^{2,\alpha}(M, \chi u^s)$ to the equation
\[ \Delta^s v - \psi^s u \cdot v = \phi, \] (6.5)
which implies by the implicit function theorem that $S$ contains a neighborhood of $s$ and hence is open in $[0, 1]$, completing the proof.

The proof follows a standard approach, using the Lax–Milgram theorem and the Fredholm alternative. For completeness we include it here.
Let $\gamma > 0$ and define a bilinear form on the Sobolev space $H^1(M, \chi_{us})$ by

$$B[v, w] := \int_M \left[ (\nabla v + v \text{tr} \tilde{T}, \nabla w)_{\chi_{us}} + (\gamma + \psi_u^s v) w \right](\chi_{us})^n$$

$$= \int_M \left[ g^{ij} (v_i + v \tilde{T}_{ik}) w_j + (\gamma + \psi_u^s v) w \right](\chi_{us})^n$$  \hspace{1cm} (6.6)

where $\tilde{T}$ denotes the torsion of $\chi_{us}$ and $\text{tr} \tilde{T}$ its trace. In local coordinates,

$$\text{tr} \tilde{T} = \tilde{T}_{ik}^k dz_i = g^{kj} (\chi_{ij}^k - \chi_{kj}^i) dz_i$$

so it only depends on the second derivatives of $u$.

It is clear that for $\gamma > 0$ sufficiently large $B$ satisfies the Lax–Milgram hypotheses, i.e.,

$$|B[v, w]| \leq C \|v\|_{H^1} \|w\|_{H^1}$$  \hspace{1cm} (6.7)

by the Schwarz inequality, and

$$B[v, v] \geq c_0 \|v\|_{H^1}^2, \quad \forall v \in H^1(M, \chi_{us})$$  \hspace{1cm} (6.8)

where $c_0$ is a positive constant independent of $s \in [0, 1]$ since $\psi_u > 0$, $|u|^2_{C^2(M)} \leq C$ and $M$ is compact. By the Lax–Milgram theorem, for any $\phi \in L^2(M, \chi_{us})$ there is a unique $v \in H^1(M, \chi_{us})$ satisfying

$$B[v, w] = \int_M \phi w(\chi_{us})^n, \quad \forall w \in H^1(M, \chi_{us}).$$  \hspace{1cm} (6.9)

On the other hand,

$$B[v, w] = \int_M (-\Delta u + \psi_u^s v + \gamma v) w(\chi_{us})^n$$  \hspace{1cm} (6.10)

by integration by parts. Thus $v$ is a weak solution to the equation

$$L_\gamma v := \Delta u - \psi_u^s v - \gamma v = \phi.$$  \hspace{1cm} (6.11)

We write $v = L_\gamma^{-1} \phi$.

By the Sobolev embedding theorem the linear operator

$$K := \gamma L_\gamma^{-1} : L^2(M, \chi_{us}) \to L^2(M, \chi_{us})$$

is compact. Note also that $v \in H^1(M, \chi_{us})$ is a weak solution of Eq. (6.5) if and only if

$$v - Kv = \zeta$$  \hspace{1cm} (6.12)
where \( \zeta = L^{-1}_\gamma \phi \). Indeed, (6.5) is equivalent to
\[
v = L^{-1}_\gamma (\gamma v + \phi) = \gamma L^{-1}_\gamma v + L^{-1}_\gamma \phi.
\] (6.13)
Since the solution of Eq. (6.5), if exists, is unique, by the Fredholm alternative equation (6.12) is uniquely solvable for any \( \zeta \in L^2(M, \chi_{\omega'}) \). Consequently, for any \( \phi \in L^2(M, \chi_{\omega'}) \) there exists a unique solution \( v \in H^1(M, \chi_{\omega'}) \) to Eq. (6.5). By the regularity theory of linear elliptic equations, \( v \in C^{2,\alpha}(M, \chi_{\omega'}) \) if \( \phi \in C^{\alpha}(M, \chi_{\omega'}) \). This completes the proof. \( \square \)

6.2. The Dirichlet problem

We now turn to the proof of Theorem 1.1. Let
\[
\mathcal{A}_u = \{ v \in \mathcal{H}_\chi : v \geq u \text{ in } M, v = u \text{ on } \partial M \}.
\]
By the maximum principle, \( v \leq h \) on \( \bar{M} \) for all \( v \in \mathcal{A}_u \) where \( h \) satisfies \( \Delta h + \text{tr} \chi = 0 \) in \( M \) and \( h = u \) on \( \partial M \). Therefore we have \( C^0 \) bounds for solutions of the Dirichlet problem (1.1)–(1.2) in \( \mathcal{A}_u \). The proof of existence of such solutions then follows that of Theorem 1.1 in [29]; so is omitted here.

Proof of Theorem 1.5. As we only assume \( \psi \geq 0 \), Eq. (1.1) is degenerate. So we need to approximate it by nondegenerate equations.

For \( \varepsilon > 0 \), let \( \psi^\varepsilon \) be a smooth function such that
\[
\sup \left\{ \psi - \varepsilon, \frac{\varepsilon}{2} \right\} \leq \psi^\varepsilon \leq \sup \left\{ \psi, \varepsilon \right\}
\]
and consider the approximation problem
\[
\begin{aligned}
(\chi u)^n &= \psi^\varepsilon \omega^n \quad \text{in } \bar{M}, \\
u &= \varphi \quad \text{on } \partial M.
\end{aligned}
\] (6.14)
Note that \( u \) is a subsolution of (6.14) when \( 0 < \varepsilon \leq \varepsilon \) where \( \varepsilon > 0 \) satisfies \( \chi u \geq \varepsilon \omega \). By Theorem 1.1 there is a unique solution \( u^\varepsilon \in C^{2,\alpha}(\bar{M}) \) of (6.14) with \( u^\varepsilon \geq \phi \) on \( \bar{M} \) for \( \varepsilon \in (0, \varepsilon] \).

By the estimates in Section 3 we have
\[
|u^\varepsilon|_{C^1(\bar{M})} \leq C_1, \quad \sup_M \Delta u^\varepsilon \leq C_2 \left( 1 + \sup_{\partial M} \Delta u^\varepsilon \right), \quad \text{independent of } \varepsilon.
\] (6.15)
On the boundary \( \partial M \), the estimates in Section 4 for the pure tangential and mixed tangential-normal second derivatives are independent of \( \varepsilon \), i.e.,
\[
|u^\varepsilon_\xi|, |u^\varepsilon_\eta| \leq C_3, \quad \forall \xi, \eta \in T \partial M, \ |\xi|, |\eta| = 1, \quad \text{independent of } \varepsilon.
\] (6.16)
where \( v \) is the unit normal to \( \partial M \). For the estimate of the double normal derivative \( u^\varepsilon_{\xi\eta} \), note that \( \partial M = N \times \partial S \) and \( T_C \partial M = T N \); this is the only place we need the assumption \( M = N \times S \) so Theorem 1.5 actually holds for local product spaces. Thus,
\[ \chi_{\xi \bar{\xi}} + u^\varepsilon_{\xi \bar{\xi}} = \chi_{\xi \bar{\xi}} + u_{\xi \bar{\xi}} \geq c_0, \quad \forall \xi \in T_C \partial M = TN, \ |\xi| = 1, \] 

(6.17)

where \( c_0 \) depends only on \( u \). Therefore,

\[ |u^\varepsilon_{\nu \nu}| \leq C, \quad \text{independent of } \varepsilon \text{ on } \partial M. \] 

(6.18)

Finally, from \( \sup_M |\Delta u^\varepsilon| \leq C \) we see that \( |u^\varepsilon|_{C^1,\alpha(\tilde{M})} \) is bounded for any \( \alpha \in (0, 1) \). Taking a convergent subsequence we obtain a solution \( u \in C^1,\alpha(\tilde{M}) \) of (1.1). By Remark 5.2, \( u \in C^{1,1}(\tilde{M}) \) when \( M \) has nonnegative bisectional curvature. \( \square \)

7. Proof of Theorem 1.2

By a theorem of Harvey and Wells [41] (see also [52]) there exists a strictly plurisubharmonic function \( \rho \in C^3(\bar{N}) \), where \( N \subseteq M \) is a neighborhood of \( X \), such that \( \rho^{-1}(0) = X, \rho = 1 \) on \( \partial N \) and \( \nabla \rho \neq 0 \) on \( \bar{N} \setminus X \). Let \( \mu = a\rho \). We can fix \( a > 2 \) sufficiently large so that \( \chi \mu \geq 2 \omega \) in \( \bar{N} \).

For \( 0 < \varepsilon \leq 1 \) denote \( M_{\varepsilon} = \{ u < \varepsilon \} \) and let \( u^\varepsilon \in C^3(\bar{N}) \) be a function such that \( u^\varepsilon = u \) in \( M_{\varepsilon}/2 \), \( u^\varepsilon \in C^\infty(\bar{M}_{a/2} \setminus M_{\varepsilon}) \) and \( u^\varepsilon \rightarrow u \) in \( C^3(\bar{N}) \) as \( \varepsilon \rightarrow 0 \). We denote \( M_{\varepsilon,\delta} = [u^\varepsilon < \delta] \). Given \( 0 < \delta < 1 \) we see that for all \( \varepsilon \) sufficiently small, \( M_{\varepsilon}/2 \subseteq M_{\varepsilon,\delta} \subseteq M_{\varepsilon,1} \subseteq M_{a/2} \) and \( \nabla u^\varepsilon \neq 0 \), \( \chi u^\varepsilon \geq \omega \) on \( \tilde{M}_{\varepsilon,1} \setminus M_{\varepsilon,\delta}. \)

We now consider the following Dirichlet problem

\[
\begin{align*}
\begin{cases}
(\chi u^\varepsilon)^n &= \delta \omega^n & \text{in } \tilde{M}_{\varepsilon,1} \setminus M_{\varepsilon,\delta}, \\
u &= u^\varepsilon & \text{on } \partial(\tilde{M}_{\varepsilon,1} \setminus M_{\varepsilon,\delta}).
\end{cases}
\end{align*}
\] 

(7.1)

Note that \( u^\varepsilon \) is a subsolution of (7.1) and \( \partial(M_{\varepsilon,1} \setminus M_{\varepsilon,\delta}) \) is smooth. By Theorem 1.1 there exists a unique solution \( u^{\varepsilon,\delta} \in H^x \cap C^\infty(\tilde{M}_{\varepsilon,1} \setminus M_{\varepsilon,\delta}) \) to problem (7.1). It follows from the maximum principle that \( u^\varepsilon \leq u^{\varepsilon,\delta} \leq 1 \) in \( \tilde{M}_{\varepsilon,1} \setminus M_{\varepsilon,\delta} \).

By (the proof of) Proposition 3.1,

\[
\max_{\tilde{M}_{\varepsilon,1} \setminus M_{\varepsilon,\delta}} \left| \nabla u^{\varepsilon,\delta} \right| \leq C \left( 1 + \max_{\partial(\tilde{M}_{\varepsilon,1} \setminus M_{\varepsilon,\delta})} \left| \nabla u^{\varepsilon,\delta} \right| \right)
\] 

(7.2)

where \( C \) depends on \( |u^\varepsilon|_{C^3} \). Since \( u^\varepsilon \rightarrow u \) in \( C^3(\bar{N}) \) as \( \varepsilon \rightarrow 0 \), we see that \( C \) can be chosen uniformly in \( \varepsilon \). Consequently, there exists a sequence \( \varepsilon_k \rightarrow 0 \) such that \( u^{\varepsilon_k,\delta} \) converges to a function \( u^\delta \in C^{0,1}(M_1 \setminus M_\delta) \) as \( k \) tends to infinity. Moreover, \( u^\delta \) is an admissible weak solution (see [4]) of the problem

\[
\begin{align*}
\begin{cases}
(\chi u^\delta)^n &= \delta \omega^n & \text{in } \tilde{M}_1 \setminus M_\delta, \\
u^\delta &= u & \text{on } \partial(M_1 \setminus M_\delta),
\end{cases}
\end{align*}
\] 

(7.3)

and

\[
\max_{M_1 \setminus M_\delta} \left| \nabla u^\delta \right| \leq C \left( 1 + \max_{\partial(M_1 \setminus M_\delta)} \left| \nabla u^\delta \right| \right).
\] 

(7.4)
Obviously, $|\nabla u^\delta| \leq |\nabla u| \leq C$ on $\partial M_1$ where $C$ is independent of $\delta$. We wish to show that

$$|\nabla u^\delta| \leq C \quad \text{on } \partial M_\delta, \text{ independent of } \delta$$

and therefore

$$\max_{M_1 \setminus M_\delta} |\nabla u^\delta| \leq C, \quad \text{independent of } \delta. \quad (7.6)$$

Consider an arbitrarily fixed point $p \in X$. Let $v \in T_pM$ be a unit normal vector to $X$, i.e., $v \in N_pX$. Since $X$ is totally real and $\dim X = n$, we see that $Jv \in T_pX$. Through $p$ there exists a complex curve $S = S(p, v) \subset M$ such that $T_pS$ is spanned by $v$ and $Jv$. We may assume $S$ to be a geodesic disk about $p$ of radius $\gamma > 0$ which is independent of $p$ and $v \in N_pX$. Moreover, since $X$ is totally real and $C^3$, we may assume (choosing $\gamma$ sufficiently small) that $X \cap B_\gamma(p)$ is a connected curve for any geodesic disk $B_\gamma(p)$ about $p$ of radius $\gamma' \leq \gamma$. Let $\Gamma = X \cap S$. We see that $\Gamma$ divides $S$ into two components; let $S^+$ denote that one to which $v$ is the interior unit normal and $B^+\gamma(p) = S^+ \cap B_\gamma(p)$.

For $\delta \geq 0$ sufficiently small, let $h^\delta$ be the solution of the problem

$$\begin{cases}
\Delta_S h + \tr(\chi|_S) = 0 & \text{in } S^+_\delta, \\
h = \eta^\delta & \text{on } \partial S^+_\delta,
\end{cases} \quad (7.7)$$

where $\chi|_S$ is the restriction of $\chi$ on $S$, $S^+_\delta = S^+ \cap M_\delta$, and $\eta^\delta$ is a smooth function on $\partial S^+_\delta$ with $\eta^\delta = \delta$ on $\partial S^+_\delta \cap B_{\gamma/2}(p)$ and $\eta^\delta = 1$ on $\partial S^+ \setminus \partial M_\delta$. By the elliptic regularity theory $h^\delta \in C^{2,\alpha}(S^+_\delta \cap B_{\gamma'}(p))$ for all $\gamma' \leq \gamma$ and

$$|h^\delta|_{C^{2,\alpha}(S^+_\delta \cap B_{\gamma'}(p))} \leq C, \quad \text{independent of } \delta. \quad (7.8)$$

Since $h^\delta \geq u^\delta$ on $\partial (S^+ \cap M_\delta)$ and $\Delta u^\delta + \tr(\chi|_S) \geq 0$ in $S^+_\delta$, by the maximum principle $h^\delta \geq u^\delta$ in $S^+ \cap M_\delta$. Consequently,

$$\frac{\partial u}{\partial \bar{n}} \leq \frac{\partial u^\delta}{\partial \bar{n}} \leq \frac{\partial h^\delta}{\partial \bar{n}} \quad \text{on } S^+ \cap \partial M_\delta \quad (7.9)$$

where $\bar{n}$ is the interior unit normal vector field to $S^+ \cap \partial M_\delta$ in $S^+_\delta$.

Note that $\{S(p, v); v \in N_pX, |v| = 1\}$ forms a foliation of a neighborhood of $p$ which contains a geodesic ball about $p$ of a fixed radius (independent of $p$) in $M$. Let $q \in \partial M_\delta$ and $\bar{n}$ be the unit normal vector to $\partial M_\delta$ at $q$. When $\delta$ is sufficiently small, there exists $p \in X$ and a unique $v \in N_pX \subset T_pM, |v| = 1$ such that $q \in S(p, v)$ and $\bar{n} \in T_qS$ and therefore is conormal to $\partial S^+_\delta (p, v)$ at $q$. Consequently, by (7.8) and (7.9),

$$|\nabla u^\delta(q)| = \left| \frac{\partial u^\delta}{\partial \bar{n}}(q) \right| \leq \max \left\{ \left| \frac{\partial u}{\partial \bar{n}}(q) \right|, \left| \frac{\partial h^\delta}{\partial \bar{n}}(q) \right| \right\} \leq C, \quad \text{independent of } q, \delta. \quad (7.10)$$

This proves (7.5) and (7.6).
We observe that if $\delta' < \delta$ then $u^{\delta'} \geq u^\delta$ on $\partial (M_1 \setminus M_\delta)$. By the maximum principle, $u^{\delta'} \geq u^\delta$ in $M_1 \setminus M_\delta$ if $\delta' < \delta$. Therefore $u^\delta$ converges to a function $u$ as $\delta \to 0$ pointwise in $M_1 \setminus X$. From (7.6) we see that $u \in C^{0,1}(\tilde{M}_1 \setminus X)$ and solves (1.7) (in the weak sense of Bedford and Taylor [4]).

Let $p \in X$ and $\nu \in N_p X$, $|\nu| = 1$, and $S = S(p, \nu)$ be as before. Let $h^\delta$ be the solution of problem (7.7) for $\delta \geq 0$. We have $u \leq u^\delta \leq h^\delta \leq h^0$ in $S^\delta$ for all $\delta > 0$. Thus, $u \leq u \leq h^0$ in $S^+$. This shows that $u$ can be extended to $u \in C^{0,1}(M_1)$ with $u = 0$ on $X$.

The proof of Theorem 1.2 is complete.

8. A Dirichlet problem related to Donaldson conjecture

Let $(M^n, g)$ be a compact Hermitian manifold without boundary and consider the open convex cone in $C^2(M)$:

$$\mathcal{H} = \{ \phi \in C^2(M): \omega_\phi > 0 \}. \tag{8.1}$$

The tangent space $T_\phi \mathcal{H}$ of $\mathcal{H}$ at $\phi \in \mathcal{H}$ is naturally identified to $C^2(M)$. Following [51,57,23] one can define a Riemannian metric on $\mathcal{H}$ using the $L^2$ inner product on $T_\phi \mathcal{H}$ with respect to the volume form of $\omega_\phi$:

$$\langle \xi, \eta \rangle_\phi = \int_M \xi \eta (\omega_\phi)^n, \quad \xi, \eta \in T_\phi \mathcal{H}. \tag{8.2}$$

Accordingly, the length of a regular curve $\varphi : [0, 1] \to \mathcal{H}$ is

$$L(\varphi) = \int_0^1 \langle \dot{\varphi}, \dot{\varphi} \rangle_\varphi^\frac{1}{2} dt. \tag{8.3}$$

Henceforth $\dot{\varphi} = \partial \varphi / \partial t$ and $\ddot{\varphi} = \partial^2 \varphi / \partial t^2$. By straightforward calculation, one derives the geodesic equation

$$\ddot{\varphi} - |\nabla \dot{\varphi} + \dot{\varphi} \text{ tr } T|_\varphi^2 = \frac{1}{2} \dot{\varphi}^2 (\text{div } T - |\text{tr } T|^2), \tag{8.4}$$

or in local coordinates

$$\ddot{\varphi} - g^{jk}_{\varphi} (\psi_{zj} + \dot{\varphi} T_{jl}^l)(\dot{\psi}_{z_k} + \dot{\varphi} T_{kl}^l) = \frac{1}{2} \dot{\varphi}^2 g^{jk}_{\varphi} (\partial_j T_{kl}^l - T_{jl}^l T_{kl}^l). \tag{8.5}$$

Here $\{g^{jk}_{\varphi}\}$ is the inverse matrix of $\{g_{jk} + \varphi_{jk}\}$, and $T_{jk}^l$ is the torsion of $\omega_\varphi$.

When $\omega$ is Kähler, Eq. (8.4) takes the form

$$\ddot{\varphi} - |\nabla \dot{\varphi}|_\varphi^2 = 0. \tag{8.6}$$
It was observed by Donaldson [23] and Semmes [57] that Eq. (8.6) reduces to a homogeneous complex Monge–Ampère equation in $M \times A$ where $A = [0, 1] \times S^1$. Let

$$w = z_{n+1} = t + \sqrt{-1}s$$

be a local coordinate of $A$. If we view a smooth curve in $\mathcal{H}$ as a function on $M \times [0, 1]$ and therefore a rotation-invariant function (constant in $s$) on $M \times A$, then a geodesic $\varphi$ in $\mathcal{H}$ satisfies

$$((\bar{\omega}_\varphi)^{n+1} \equiv (\bar{\omega} + \frac{\sqrt{-1}}{2} \bar{\partial} \partial \varphi)^{n+1} = 0 \text{ in } M \times A)$$

(8.7)

where

$$\bar{\omega} = \omega + \frac{\sqrt{-1}}{2} \bar{\partial} \partial |w|^2 = \frac{\sqrt{-1}}{2} \left( \sum_{j,k \leq n} g_{j\bar{k}} dz_j \wedge d\bar{z}_k + dw \wedge d\bar{w} \right)$$

(8.8)

is the lift of $\omega$ to $M \times A$. Conversely, if $\varphi \in C^2(M \times A)$ is a rotation-invariant solution of (8.7) such that

$$\varphi(\cdot, w) \in \mathcal{H}, \quad \forall w \in A,$$

(8.9)

then $\varphi$ is a geodesic in $\mathcal{H}$.

In the Kähler case, Donaldson [23] conjectured that $\mathcal{H}^\infty \equiv \mathcal{H} \cap C^\infty(M)$ is geodesically convex, i.e., any two functions in $\mathcal{H}^\infty$ can be connected by a smooth geodesic. More precisely,

**Conjecture 8.1.** (See Donaldson [23].) Let $M$ be a compact Kähler manifold without boundary and $\rho \in C^\infty(M \times \partial A)$ such that $\rho(\cdot, w) \in \mathcal{H}$ for $w \in \partial A$. Then there exists a unique solution $\varphi$ of the Monge–Ampère equation (8.7) satisfying (8.9) and the boundary condition $\varphi = \rho$.

The uniqueness was proved by Donaldson [23] as a consequence of the maximum principle. In [15], X.-X. Chen obtained the existence of a weak solution with $\Delta \varphi \in L^\infty(M \times A)$; see also the recent work of Blocki [10] who proved that the solution is in $C^{1,1}(M \times A)$ when $M$ has nonnegative bisectional curvature. As a corollary of Theorem 1.5 these results can be extended to the Hermitian case.

**Theorem 8.2.** Let $M$ be a compact Hermitian manifold without boundary and let $\varphi_0, \varphi_1 \in \mathcal{H} \cap C^4(M)$. There exists a unique (weak) solution $\varphi \in C^{1,\alpha}(M \times A)$, $\forall 0 < \alpha < 1$, with $\bar{\omega}_\varphi \geq 0$ and $\Delta \varphi \in L^\infty(M \times A)$ of the Dirichlet problem

$$\begin{cases}
(\bar{\omega}_\varphi)^{n+1} = 0 & \text{in } M \times A, \\
\varphi = \varphi_0 & \text{on } M \times \Gamma_0, \\
\varphi = \varphi_1 & \text{on } M \times \Gamma_1
\end{cases}$$

(8.10)

where $\Gamma_0 = \partial A|_{t=0}$, $\Gamma_1 = \partial A|_{t=1}$. Moreover, $\varphi \in C^{1,1}(M \times A)$ if $M$ has nonnegative bisectional curvature.
**Proof.** In order to apply Theorem 1.5 to the Dirichlet problem (8.10) we only need to construct a strict subsolution. This is easily done for the annulus \( A = [0, 1] \times S^1 \). Let

\[
\varphi = (1-t)\varphi_0 + t\varphi_1 + K(t^2 - t).
\]

Since \( \varphi_0, \varphi_1 \in \mathcal{H}(\omega) \) we see that \( \tilde{\omega}_{\varphi} > 0 \) and \( (\tilde{\omega}_\varphi)^{n+1} \geq 1 \) for \( K \) sufficiently large. \( \square \)

**Remark 8.3.** By the uniqueness \( \varphi \) is rotation invariant (i.e., independent of \( s \)).

**Note added in proof**

In a more recently paper Tosatti and Weinkove were able to remove condition (1.8) in Theorem 1.3. See V. Tosatti and B. Weinkove, “The complex Monge–Ampère equation on compact Hermitian manifolds”, arXiv:0910.1390.

**References**


