

COMPLEX MONGE-AMPÈRE EQUATIONS ON HERMITIAN MANIFOLDS

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ABSTRACT. We study complex Monge-Ampère equations in Hermitian manifolds, both for the Dirichlet problem and in the case of compact manifolds without boundary. Our main results extend classical theorems of Yau [43] and Aubin [1] in the Kähler case, and those of Caffarelli, Kohn, Nirenberg and Spruck [9] for the Dirichlet problem in \mathbb{C}^n . As an application we study the problem of finding geodesics in the space of Hermitian metrics, generalizing existing results on Donaldson's conjecture [16] in the Kähler case.

Mathematical Subject Classification (2000). 58J05, 58J32, 32W20, 35J25, 53C55.

1. INTRODUCTION

The complex Monge-Ampère equation has close connections with many important problems in complex geometry and analysis. In the framework of Kähler geometry, it goes back at least to Calabi [11] who conjectured that any element in the first Chern class of a compact Kähler manifold is the Ricci form of a Kähler metric cohomologous to the underlying metric. In [16], Donaldson made several conjectures on the space of Kähler metrics which reduce to questions on a special Dirichlet problem for the homogeneous complex Monge-Ampère (*HCMA*) equation; see also the related work of Mabuchi [36] and Semmes [38]. The HCMA equation, which is well defined on general complex manifolds, also arises naturally in other interesting geometric problems. One such example is the work on intrinsic norms by Chern, Levine and Nirenberg [15], Bedford and Taylor [4] and P.-F. Guan [24], [25]. There are also interesting results in the literature that connect the HCMA equation on general complex manifolds with totally real submanifolds; see, e.g. [42], [37], [35], [27] and [28].

In [43], Yau proved fundamental existence theorems of classical solutions for complex Monge-Ampère equations on compact Kähler manifolds and consequently solved

Research of the first author was supported in part by NSF grants.

the Calabi conjecture. Yau's work also shows the existence of Kähler-Einstein metrics on Kähler manifolds with the first Chern number $c_1(M) \leq 0$, confirming another conjecture of Calabi [11] which was independently proved by Aubin [1] in the case $c_1(M) < 0$. The classical solvability of the Dirichlet problem was established by Caffarelli, Kohn, Nirenberg and Spruck [9] for strongly pseudoconvex domains in \mathbb{C}^n . Later on the first author [20] extended their results to general domains under the assumption of existence of subsolutions.

Our primary goal in this paper is to attempt to extend these results to more general geometric settings. We shall consider complex Monge-Ampère equations on Hermitian manifolds. Besides the technical challenges in the analytic aspect which we shall discuss in more details later, our motivation originates from trying to understand the above mentioned Donaldson conjectures when one considers Hermitian metrics, as well as other geometric problems some of which we shall treat in a forthcoming paper [21].

Let us first consider the Dirichlet problem. Let (M^n, ω) be a compact Hermitian manifold of dimension $n \geq 2$ with smooth boundary ∂M , and $\bar{M} = M \cup \partial M$. Given $\psi \in C^\infty(M \times \mathbb{R})$, $\psi > 0$, and $\varphi \in C^\infty(\partial M)$, we seek solutions of the complex Monge-Ampère equation

$$(1.1) \quad \left(\omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u \right)^n = \psi(z, u) \omega^n \text{ in } \bar{M}$$

satisfying the Dirichlet condition

$$(1.2) \quad u = \varphi \text{ on } \partial M.$$

We require

$$(1.3) \quad \omega_u := \omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u > 0$$

so that equation (1.1) is elliptic; we call such functions *admissible*. Set

$$(1.4) \quad \mathcal{H} = \{ \phi \in C^2(\bar{M}) : \omega_\phi > 0 \}.$$

We shall also call \mathcal{H} the *space of Hermitian metrics*. As in the Kähler case, for $u \in \mathcal{H}$, ω_u is a Hermitian form on M and equation (1.1) describes one of its Ricci forms.

From the theory of fully nonlinear elliptic equations, a crucial step in solving equation (1.1) is to derive *a priori* C^2 estimates for admissible solutions. Our first result is an extension of Yau's estimate for Δu [43] and the gradient estimates due to Blocki [7] and P.-F. Guan [26] in the Kähler case.

Theorem 1.1. *Let $u \in \mathcal{H} \cap C^4(M)$ be a solution of equation (1.1). Then there exist positive constants C_1, C_2 depending on $|u|_{C^0(\bar{M})}$ such that*

$$(1.5) \quad \max_{\bar{M}} |\nabla u| \leq C_1 \left(1 + \max_{\partial M} |\nabla u| \right)$$

and

$$(1.6) \quad \max_{\bar{M}} \Delta u \leq C_2 \left(1 + \max_{\partial M} \Delta u \right).$$

More details are given in Propositions 3.2 and 4.3 of the dependence of C_1 and C_2 on ψ and geometric quantities M (torsion and curvatures). Here we only emphasize that these constants do not depend on $\inf \psi$ so the estimates (1.5) and (1.6) apply to the degenerate case ($\psi \geq 0$).

The gradient estimate (1.5) was also proved independently by Xiangwen Zhang [44] who considered more general equations on compact Hermitian manifolds without boundary.

A substantial difficulty in proving (1.6) is to control the extra terms involving third order derivatives which appear due to the nontrivial torsion. To overcome this we construct a special local coordinate system (Lemma 2.1) and make use of some unique properties of the Monge-Ampère operator. See the proof in Section 4 for details.

In order to solve the Dirichlet problem (1.1)-(1.2) we also need estimates for second derivatives on the boundary. For this we shall follow techniques developed in [23], [19], [20] using subsolutions. Our main existence result for the Dirichlet problem may be stated as follows.

Theorem 1.2. *Suppose that there exists $\underline{u} \in C^0(\bar{M})$ with $\omega_{\underline{u}} \geq 0$ in \bar{M} (in weak sense [3]), $\underline{u} = \varphi$ on ∂M and*

$$(1.7) \quad (\omega_{\underline{u}})^n \geq \psi(z, \underline{u}) \omega^n \text{ in } \bar{M}.$$

Assume further that $\underline{u} \in C^2$ in a neighborhood of ∂M (including ∂M). Then the Dirichlet problem (1.1)-(1.2) admits a solution $u \in \mathcal{H} \cap C^\infty(\bar{M})$ with $u \geq \underline{u}$ in \bar{M} .

We note that Theorem 1.1 also applies to compact manifolds without boundary. Deriving the C^0 estimates, however, seems a difficult question for general ω . In the Kähler case, Yau [43] introduced a Moser iteration approach using his C^2 estimate and the Sobolev inequality. His proof was subsequently simplified by Kazdan [30] for $n = 2$, and by Aubin [1] and Bourguignon independently for arbitrary dimension (see e.g. [39] and [41]). Alternative proofs were given by Kolodziej [31] and Blocki [6]

based on the pluripotential theory ([5]) and the L^2 stability of the complex Monge-Ampère operator ([12]). All these proofs seem to heavily rely on the closedness or, equivalently, existence of local potentials of ω and it is not clear to us whether any of them can be extended to the Hermitian case. In this paper we impose the following condition

$$(1.8) \quad \partial\bar{\partial}\omega^k = 0, \quad 1 \leq k \leq n-1$$

which will also enable us to carry out the continuity method as in [43].

Theorem 1.3. *Let (M, ω) be a compact Hermitian manifold without boundary with ω satisfying (1.8). Assume $\psi_u \geq 0$ and that there exists a function $\phi \in C^\infty(M)$ such that*

$$(1.9) \quad \int_M \psi(z, \phi(z)) \omega^n = \text{Vol}(M).$$

Then there exists a solution $u \in \mathcal{H} \cap C^\infty(M)$ of equation (1.1). Moreover the solution is unique, possibly up to a constant.

Under stronger assumptions on ψ condition (1.8) may be removed.

Theorem 1.4. *Let (M, ω) be a compact Hermitian manifold without boundary. If*

$$(1.10) \quad \lim_{u \rightarrow -\infty} \psi(\cdot, u) = 0, \quad \lim_{u \rightarrow +\infty} \psi(\cdot, u) = \infty.$$

and $\psi_u > 0$, then equation (1.1) has a unique solution in $\mathcal{H} \cap C^\infty(M)$.

For applications in complex geometry it is very important to study the degenerate complex Monge-Ampère equation ($\psi \geq 0$ in (1.1)). In general, the optimal regularity in the degenerate case is $C^{1,1}$; see e.g., [2], [18], and there are many challenging open questions. In the forthcoming article [21] we shall focus on the degenerate, especially the homogeneous, Monge-Ampère equation in Hermitian manifolds and applications in geometric problems. In the current paper we shall only prove the following theorem for a special Dirichlet problem.

Theorem 1.5. *Let $M = N \times S$ where N is a compact Hermitian manifold without boundary, $\dim_{\mathbb{C}} N = n-1$, and S is a compact Riemann surface with smooth boundary $\partial S \neq \emptyset$. Let ω be the product Kähler form on M .*

Suppose that $\psi \geq 0$, $\psi^{\frac{1}{n}} \in C^2(\bar{M} \times R)$, and $\phi \in \mathcal{H} \cap C^4(\bar{M})$ satisfying $(\omega_\phi)^n \geq \psi$ on \bar{M} . Then there exists a weak admissible solution $u \in C^{1,\alpha}(\bar{M})$, for all $\alpha \in (0, 1)$

with $\Delta u \in L^\infty(M)$, of the Dirichlet problem

$$(1.11) \quad \begin{cases} (\omega_u)^n = \psi & \text{in } \bar{M}, \\ u = \phi & \text{on } \partial M. \end{cases}$$

Moreover, the solution is unique if $\psi_u \geq 0$, and $u \in C^{1,1}(\bar{M})$ provided that M has nonnegative bisectional curvature.

As an immediate application of Theorem 1.5 we can extend existing results on a conjecture of Donaldson [16] concerning geodesics in the space of Kähler metrics to the Hermitian setting; see Section 8 for details.

The paper is organized as follows. In Section 2 we briefly recall some basic facts and formulas for Hermitian manifolds and the complex Monge-Ampère operator, fixing the notation along the way. We shall also prove in this section the existence of local coordinates with some special properties; see Lemma 2.1. Such local coordinates are crucial to our proof of (1.6) in Section 4, while the gradient estimate (1.5) is derived in Sections 3. Section 5 concerns the boundary estimates for second derivatives. In Section 6 we come back to finish the global estimates for all (real) second derivatives which enable us to apply the Evans-Krylov theorem [17], [34] to obtain $C^{2,\alpha}$ and therefore higher order estimates by the classical Schauder theory. In Section 7 we discuss the C^0 estimates and existence of solutions, completing the proof of Theorems 1.2-1.5. Finally, in Section 8 we extend results on the Donaldson conjecture in the Kähler case to the space of Hermitian metrics.

The authors wish to thank Pengfei Guan and Fangyang Zheng for very helpful conversations and suggestions.

2. PRELIMINARIES

Let M^n be a complex manifold of dimension n and g a Riemannian metric on M . Let J be the induced almost complex structure on M so J is integrable and $J^2 = -\text{id}$. We assume that J is compatible with g , i.e.

$$(2.1) \quad g(u, v) = g(Ju, Jv), \quad u, v \in TM;$$

such g is called a *Hermitian* metric. Let ω be the *Kähler* form of g defined by

$$(2.2) \quad \omega(u, v) = -g(u, Jv).$$

We recall that g is *Kähler* if its Kähler form ω is closed, i.e. $d\omega = 0$.

The complex tangent bundle $T_C M = TM \times \mathbb{C}$ has a natural splitting

$$(2.3) \quad T_C M = T^{1,0} M + T^{0,1} M$$

where $T^{1,0} M$ and $T^{0,1} M$ are the $\pm\sqrt{-1}$ -eigenspaces of J . The metric g is obviously extended \mathbb{C} -linearly to $T_C M$, and

$$(2.4) \quad g(u, v) = 0 \text{ if } u, v \in T^{1,0} M, \text{ or } u, v \in T^{0,1} M.$$

Let ∇ be the Chern connection of g . It satisfies

$$(2.5) \quad \nabla_u(g(v, w)) = g(\nabla_u v, w) + g(v, \nabla_u w)$$

but may have nontrivial torsion. The torsion T and curvature R of ∇ are defined by

$$(2.6) \quad T(u, v) = \nabla_u v - \nabla_v u - [u, v],$$

and

$$(2.7) \quad R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u, v]} w,$$

respectively. Since $\nabla J = 0$ we have

$$T(Ju, Jv) = JT(u, v)$$

and

$$R(u, v)Jw = JR(u, v)w.$$

It follows that

$$(2.8) \quad T(u, v, w) \equiv g(T(u, v), w) = g(T(Ju, Jv), Jw) \equiv T(Ju, Jv, Jw)$$

and

$$(2.9) \quad R(u, v, w, x) \equiv g(R(u, v)w, x) = g(R(u, v)Jw, Jx).$$

Therefore $R(u, v, w, x) = 0$ unless w and x are of different type.

In local coordinates (z_1, \dots, z_n) , by (2.4)

$$(2.10) \quad g\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) = 0, \quad g\left(\frac{\partial}{\partial \bar{z}_j}, \frac{\partial}{\partial \bar{z}_k}\right) = 0$$

since

$$(2.11) \quad J\frac{\partial}{\partial z_j} = \sqrt{-1}\frac{\partial}{\partial z_j}, \quad J\frac{\partial}{\partial \bar{z}_j} = -\sqrt{-1}\frac{\partial}{\partial \bar{z}_j}.$$

We write

$$(2.12) \quad g_{j\bar{k}} = g\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}\right), \quad \{g^{i\bar{j}}\} = \{g_{i\bar{j}}\}^{-1}.$$

That is, $g^{i\bar{j}}g_{k\bar{j}} = \delta_{ik}$. The Kähler form ω is then given by

$$(2.13) \quad \omega = \frac{\sqrt{-1}}{2} g_{j\bar{k}} dz_j \wedge d\bar{z}_k.$$

The Christoffel symbols Γ_{jk}^l are defined by

$$\nabla_{\frac{\partial}{\partial z_j}} \frac{\partial}{\partial z_k} = \Gamma_{jk}^l \frac{\partial}{\partial z_l}.$$

Recall that by (2.5) and (2.10),

$$(2.14) \quad \begin{cases} \nabla_{\frac{\partial}{\partial z_j}} \frac{\partial}{\partial \bar{z}_k} = \nabla_{\frac{\partial}{\partial \bar{z}_j}} \frac{\partial}{\partial z_k} = 0, \\ \nabla_{\frac{\partial}{\partial \bar{z}_j}} \frac{\partial}{\partial \bar{z}_k} = \Gamma_{\bar{j}\bar{k}}^{\bar{l}} \frac{\partial}{\partial \bar{z}_l} = \overline{\Gamma_{jk}^l} \frac{\partial}{\partial \bar{z}_l} \end{cases}$$

and

$$(2.15) \quad \Gamma_{jk}^l = g^{l\bar{m}} \frac{\partial g_{k\bar{m}}}{\partial z_j}.$$

For the torsion and curvature we use the standard notion

$$T_{ijk} = T\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right), \text{ etc.}$$

and

$$R_{i\bar{j}k\bar{l}} = R\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}, \frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l}\right), \text{ etc.}$$

Obviously,

$$T_{ijk} = T_{i\bar{j}k} = T_{i\bar{j}\bar{k}} = 0, T_{i\bar{j}k} = \frac{\partial g_{j\bar{k}}}{\partial z_i} - \frac{\partial g_{i\bar{k}}}{\partial z_j}$$

and

$$(2.16) \quad T_{ij}^k \equiv g^{k\bar{l}} T_{ij\bar{l}} = \Gamma_{ij}^k - \Gamma_{ji}^k = g^{k\bar{l}} \left(\frac{\partial g_{j\bar{l}}}{\partial z_i} - \frac{\partial g_{i\bar{l}}}{\partial z_j} \right).$$

From (2.9) and (2.15) it follows that

$$(2.17) \quad R_{i\bar{j}kl} = R_{ijkl} = 0,$$

$$(2.18) \quad R_{i\bar{j}k\bar{l}} = -g_{m\bar{l}} \frac{\partial \Gamma_{ik}^m}{\partial \bar{z}_j} = -\frac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial \bar{z}_j} + g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z_i} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}_j}$$

and

$$(2.19) \quad R_{ijk\bar{l}} = g_{m\bar{l}} \left(\frac{\partial \Gamma_{jk}^m}{\partial z_i} - \frac{\partial \Gamma_{ik}^m}{\partial z_j} + \Gamma_{iq}^m \Gamma_{jk}^q - \Gamma_{jq}^m \Gamma_{ik}^q \right).$$

By (2.18) and (2.16) we have

$$(2.20) \quad R_{i\bar{j}k\bar{l}} - R_{k\bar{j}i\bar{l}} = g_{m\bar{l}} \frac{\partial T_{ki}^m}{\partial \bar{z}_j} = g_{m\bar{l}} \nabla_{\bar{j}} T_{ki}^m,$$

which also follows from the general Bianchi identity.

The traces of the curvature tensor

$$(2.21) \quad R_{k\bar{l}} = g^{i\bar{j}} R_{i\bar{j}k\bar{l}} = -g^{i\bar{j}} \frac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial \bar{z}_j} + g^{i\bar{j}} g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z_i} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}_j}$$

and

$$(2.22) \quad S_{i\bar{j}} \equiv g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = -g^{k\bar{l}} \frac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial \bar{z}_j} + g^{k\bar{l}} g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z_i} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}_j}$$

are called the *first* and *second* Ricci tensors, respectively. Note that

$$(2.23) \quad S_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \det g_{k\bar{l}}.$$

The following special local coordinates will be crucial to our proof of the *a priori* estimates for Δu in Theorem 1.1.

Lemma 2.1. *Around a point $p \in M$ there exist local coordinates such that, at p ,*

$$(2.24) \quad g_{i\bar{j}} = \delta_{ij}, \quad \frac{\partial g_{i\bar{i}}}{\partial z_j} = 0, \quad \forall i, j.$$

Proof. Let (z_1, z_2, \dots, z_n) be a local coordinate system around p such that $z_i(p) = 0$ for $i = 1, \dots, n$ and

$$g_{i\bar{j}}(p) := g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right) = \delta_{ij}.$$

Define new coordinates (w_1, w_2, \dots, w_n) by

$$(2.25) \quad w_r = z_r + \sum_{m \neq r} \frac{\partial g_{r\bar{r}}}{\partial z_m}(p) z_m z_r + \frac{1}{2} \frac{\partial g_{r\bar{r}}}{\partial z_r}(p) z_r^2, \quad 1 \leq r \leq n.$$

We have

$$(2.26) \quad \tilde{g}_{i\bar{j}} := g\left(\frac{\partial}{\partial w_i}, \frac{\partial}{\partial \bar{w}_j}\right) = \sum_{r,s} g_{r\bar{s}} \frac{\partial z_r}{\partial w_i} \frac{\partial \bar{z}_s}{\partial \bar{w}_j}.$$

It follows that

$$(2.27) \quad \frac{\partial \tilde{g}_{i\bar{j}}}{\partial w_k} = \sum_{r,s} g_{r\bar{s}} \frac{\partial^2 z_r}{\partial w_i \partial w_k} \frac{\partial \bar{z}_s}{\partial \bar{w}_j} + \sum_{r,s,p} \frac{\partial g_{r\bar{s}}}{\partial z_p} \frac{\partial z_p}{\partial w_k} \frac{\partial z_r}{\partial w_i} \frac{\partial \bar{z}_s}{\partial \bar{w}_j}.$$

Differentiate (2.25) with respect to w_i and w_k . We see that, at p ,

$$\frac{\partial z_r}{\partial w_i} = \delta_{ri}, \quad \frac{\partial^2 z_r}{\partial w_i \partial w_k} = - \sum_{m \neq r} \frac{\partial g_{r\bar{r}}}{\partial z_m} \left(\frac{\partial z_m}{\partial w_i} \frac{\partial z_r}{\partial w_k} + \frac{\partial z_m}{\partial w_k} \frac{\partial z_r}{\partial w_i} \right) - \frac{\partial g_{r\bar{r}}}{\partial z_r} \frac{\partial z_r}{\partial w_i} \frac{\partial z_r}{\partial w_k}.$$

Plugging these into (2.27), we obtain at p ,

$$(2.28) \quad \begin{cases} \frac{\partial \tilde{g}_{i\bar{i}}}{\partial w_k} = \frac{\partial g_{i\bar{i}}}{\partial z_k} - \frac{\partial g_{i\bar{i}}}{\partial z_k} = 0, \quad \forall i, k, \\ \frac{\partial \tilde{g}_{i\bar{j}}}{\partial w_j} = \frac{\partial g_{i\bar{j}}}{\partial z_j} - \frac{\partial g_{j\bar{j}}}{\partial z_i} = T_{ji}^j, \quad \forall i \neq j, \\ \frac{\partial \tilde{g}_{i\bar{j}}}{\partial w_k} = \frac{\partial g_{i\bar{j}}}{\partial z_k}, \quad \text{otherwise.} \end{cases}$$

Finally, switching w and z gives (2.24). \square

Remark 2.2. If, in place of (2.25), we define

$$(2.29) \quad w_r = z_r + \sum_{m \neq r} \frac{\partial g_{m\bar{r}}}{\partial z_r}(p) z_m z_r + \frac{1}{2} \frac{\partial g_{r\bar{r}}}{\partial z_r}(p) z_r^2, \quad 1 \leq r \leq n,$$

then under the new coordinates (w_1, w_2, \dots, w_n) ,

$$(2.30) \quad \begin{cases} \frac{\partial \tilde{g}_{i\bar{j}}}{\partial w_j}(p) = 0, \quad \forall i, j, \\ \frac{\partial \tilde{g}_{i\bar{i}}}{\partial w_k}(p) = T_{ki}^i, \quad \forall i \neq k, \\ \frac{\partial \tilde{g}_{i\bar{j}}}{\partial w_k}(p) = \frac{\partial g_{i\bar{j}}}{\partial z_k}(p), \quad \text{otherwise.} \end{cases}$$

The following lemma and its proof can be found in [40].

Lemma 2.3. *Around a point $p \in M$ there exist local coordinates such that, at p ,*

$$(2.31) \quad \begin{cases} g_{i\bar{j}} = \delta_{ij}, \\ \frac{\partial g_{i\bar{j}}}{\partial z_k} + \frac{\partial g_{k\bar{j}}}{\partial z_i} = 0. \end{cases}$$

Consequently, $T_{ij}^k = 2 \frac{\partial g_{j\bar{k}}}{\partial z_i}$ at p .

Remark 2.4. In general it is impossible to find local coordinates satisfying both (2.24) and (2.31) simultaneously.

Let $\Lambda^{p,q}$ denote differential forms of type (p, q) on M . The exterior differential d has a natural decomposition $d = \partial + \bar{\partial}$ where

$$\partial : \Lambda^{p,q} \rightarrow \Lambda^{p+1,q}, \quad \bar{\partial} : \Lambda^{p,q} \rightarrow \Lambda^{p,q+1}.$$

Recall that $\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$ and, by the Stokes theorem

$$\int_M \partial\alpha = \int_{\partial M} \alpha, \quad \forall \alpha \in \Lambda^{n-1,n}.$$

A similar formula holds for $\bar{\partial}$.

For a function $u \in C^2(M)$, $\partial\bar{\partial}u$ is given in local coordinates by

$$(2.32) \quad \partial\bar{\partial}u = \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j.$$

We use $\nabla^2 u$ to denote the *Hessian* of u :

$$(2.33) \quad \nabla^2 u(X, Y) \equiv \nabla_Y \nabla_X u = Y(Xu) - (\nabla_Y X)u, \quad X, Y \in TM.$$

By (2.14) we see that

$$(2.34) \quad \nabla_{\frac{\partial}{\partial z_i}} \nabla_{\frac{\partial}{\partial \bar{z}_j}} u = \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}.$$

Consequently, the Laplacian of u with respect to the Chern connection is

$$(2.35) \quad \Delta u = g^{i\bar{j}} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j},$$

or equivalently,

$$(2.36) \quad \Delta u \omega^n = \frac{\sqrt{-1}}{2} \partial\bar{\partial}u \wedge \omega^{n-1}.$$

Integrating (2.36) (by parts), we obtain

$$(2.37) \quad \begin{aligned} \frac{2}{\sqrt{-1}} \int_M \Delta u \omega^n &= \int_M \partial\bar{\partial}u \wedge \omega^{n-1} \\ &= \int_{\partial M} \bar{\partial}u \wedge \omega^{n-1} + \int_M \bar{\partial}u \wedge \partial\omega^{n-1} \\ &= \int_{\partial M} (\bar{\partial}u \wedge \omega^{n-1} + u\partial\omega^{n-1}) + \int_M u\partial\bar{\partial}\omega^{n-1}. \end{aligned}$$

Finally, in the following sections where we derive the *a priori* estimates we shall consider the slightly more general equation

$$(2.38) \quad \det \left(\mu\omega + \frac{\sqrt{-1}}{2} \partial\bar{\partial}u \right)^n = \psi(z, u)\omega^n$$

where μ is a given smooth function on M and may also depend on u and ∇u . We shall abuse the notation to write

$$\omega_u := \mu\omega + \frac{\sqrt{-1}}{2}\partial\bar{\partial}u \in \Lambda^{1,1},$$

and $u \in \mathcal{H}$ means $\omega_u > 0$. In local coordinates equation (2.38) takes the form

$$(2.39) \quad \det \left(\mu g_{i\bar{j}} + \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} \right) = \psi(z, u) \det g_{i\bar{j}}.$$

In the current paper we assume $\mu = \mu(z, u)$ and $|\mu| > 0$ in $\bar{M} \times \mathbb{R}$. In [21] we shall consider other cases including $\mu = 0$.

3. GRADIENT ESTIMATES

In this section we derive the gradient estimate (1.5) for a solution $u \in \mathcal{H} \cap C^3(M)$ of (2.38). Throughout this and next sections we shall use ordinary derivatives. For convenience we write in local coordinates,

$$u_i = \frac{\partial u}{\partial z_i}, \quad u_{\bar{i}} = \frac{\partial u}{\partial \bar{z}_i}, \quad u_{i\bar{j}} = \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}, \quad g_{i\bar{j}k} = \frac{\partial g_{i\bar{j}}}{\partial z_k}, \quad g_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l}, \quad \text{etc,}$$

and

$$\mathfrak{g}_{i\bar{j}} = u_{i\bar{j}} + \mu g_{ij}, \quad \{\mathfrak{g}_{i\bar{j}}\} = \{\mathfrak{g}_{i\bar{j}}\}^{-1}.$$

We first present some calculation. In local coordinates,

$$(3.1) \quad |\nabla u|^2 = g^{k\bar{l}} u_k u_{\bar{l}}.$$

We have

$$(3.2) \quad (|\nabla u|^2)_i = g^{k\bar{l}} (u_{k\bar{i}} u_{\bar{l}} + u_k u_{i\bar{l}}) - g^{k\bar{b}} g^{a\bar{l}} g_{a\bar{b}i} u_k u_{\bar{l}}$$

$$(3.3) \quad \begin{aligned} (|\nabla u|^2)_{i\bar{j}} &= g^{k\bar{l}} (u_{k\bar{j}} u_{i\bar{l}} + u_{i\bar{j}k} u_{\bar{l}} + u_k u_{i\bar{j}\bar{l}} + u_{k\bar{i}} u_{\bar{l}j}) \\ &\quad - g^{k\bar{b}} g^{a\bar{l}} g_{a\bar{b}i} (u_{k\bar{j}} u_{\bar{l}} + u_k u_{\bar{l}j}) - g^{k\bar{b}} g^{a\bar{l}} g_{a\bar{b}j} (u_{k\bar{i}} u_{\bar{l}} + u_k u_{i\bar{l}}) \\ &\quad + [(g^{k\bar{q}} g^{p\bar{b}} g^{a\bar{l}} + g^{k\bar{b}} g^{a\bar{q}} g^{p\bar{l}}) g_{a\bar{b}i} g_{p\bar{q}j} - g^{k\bar{b}} g^{a\bar{l}} g_{a\bar{b}i\bar{j}}] u_k u_{\bar{l}}. \end{aligned}$$

Here we have used the formula

$$(g^{k\bar{l}})_i = -g^{k\bar{b}} g^{a\bar{l}} g_{a\bar{b}i}.$$

Let p be a fixed point. We may assume that (2.31) holds at p and

$$(3.4) \quad \{u_{i\bar{j}}\} \text{ is diagonal.}$$

Thus

$$(3.5) \quad (|\nabla u|^2)_i = u_i u_{\bar{i}\bar{i}} + (u_{ki} - g_{k\bar{l}} u_l) u_{\bar{k}}$$

and

$$(3.6) \quad (|\nabla u|^2)_{\bar{i}\bar{i}} = u_{\bar{i}\bar{i}}^2 + u_{i\bar{i}k} u_{\bar{k}} + u_{i\bar{i}\bar{k}} u_k + \sum_k |u_{ki} - g_{k\bar{l}} u_l|^2 - 2\Re\{g_{\bar{l}i} u_{\bar{i}\bar{i}} u_{\bar{l}}\} + (g_{\bar{l}i} g_{p\bar{k}\bar{i}} - g_{\bar{l}\bar{k}i}) u_k u_{\bar{l}}.$$

Lemma 3.1. *Let ϕ be a function such that $e^\phi |\nabla u|^2$ attains its maximum at an interior point p where, in local coordinates, (2.31) and (3.4) hold. Then, at p ,*

$$(3.7) \quad \begin{aligned} |\nabla u|^2 \sum \mathbf{g}^{i\bar{i}} \phi_{\bar{i}\bar{i}} &\leq -2 \sum \Re\{\mathbf{g}^{i\bar{i}} u_i u_{\bar{i}\bar{i}} \phi_{\bar{i}}\} + 2|\nabla f| |\nabla u| - 2|\nabla u|^2 f_u \\ &- |\nabla u|^2 \left(\inf_{j,l} R_{j\bar{j}l\bar{l}} - \sup_{j,l} |T_{l\bar{j}}^j|^2 \right) \sum \mathbf{g}^{i\bar{i}} \\ &+ 2\Re\{(\mu)_k u_{\bar{k}}\} \sum \mathbf{g}^{i\bar{i}}. \end{aligned}$$

Proof. In the proof all calculations are done at p where since $e^\phi |\nabla u|^2$ attains its maximum,

$$(3.8) \quad \frac{(|\nabla u|^2)_i}{|\nabla u|^2} + \phi_i = 0, \quad \frac{(|\nabla u|^2)_{\bar{i}}}{|\nabla u|^2} + \phi_{\bar{i}} = 0$$

and

$$(3.9) \quad \frac{(|\nabla u|^2)_{\bar{i}\bar{i}}}{|\nabla u|^2} - \frac{|(|\nabla u|^2)_i|^2}{|\nabla u|^4} + \phi_{\bar{i}\bar{i}} \leq 0.$$

By (3.5) and (3.8),

$$(3.10) \quad |(|\nabla u|^2)_i|^2 = \sum_k |u_k|^2 |u_{ki} - g_{k\bar{l}} u_l|^2 - 2|\nabla u|^2 \Re\{u_i u_{\bar{i}\bar{i}} \phi_{\bar{i}}\} - |u_i|^2 u_{\bar{i}\bar{i}}^2.$$

Differentiating equation (2.39) we have

$$(3.11) \quad \begin{aligned} \mathbf{g}^{i\bar{j}} (u_{i\bar{j}k} u_{\bar{k}} + u_k u_{i\bar{j}\bar{k}}) &= 2|\nabla u|^2 f_u + 2\Re\{f_{z_k} u_{\bar{k}} + g^{i\bar{j}} g_{i\bar{j}k} u_{\bar{k}}\} \\ &- 2\Re\{\mathbf{g}^{i\bar{j}} (\mu g_{i\bar{j}k} + (\mu)_k g_{i\bar{j}}) u_{\bar{k}}\}. \end{aligned}$$

Note that by (2.31),

$$(3.12) \quad \sum_{i,l} g_{i\bar{i}l} u_{\bar{l}} - \mu \mathbf{g}^{i\bar{i}} g_{i\bar{i}l} u_{\bar{l}} = \mathbf{g}^{i\bar{i}} u_{i\bar{i}} g_{i\bar{i}l} u_{\bar{l}} = -\mathbf{g}^{i\bar{i}} u_{i\bar{i}} g_{\bar{l}i} u_{\bar{l}}.$$

From (3.6) and (3.11) we see that

$$\begin{aligned}
 \mathbf{g}^{i\bar{i}}(|\nabla u|^2)_{i\bar{i}} &= \mathbf{g}^{i\bar{i}}|u_{i\bar{i}} - 2g_{l\bar{i}i}u_{\bar{l}}|^2 + \sum_{i,k} \mathbf{g}^{i\bar{i}}|u_{ki} - g_{k\bar{i}i}u_{\bar{l}}|^2 \\
 (3.13) \quad &+ \mathbf{g}^{i\bar{i}}(g_{l\bar{p}i}g_{p\bar{k}i} - g_{l\bar{k}i})u_k u_{\bar{l}} - 4\mathbf{g}^{i\bar{i}}|g_{l\bar{i}i}u_{\bar{l}}|^2 \\
 &+ 2|\nabla u|^2 f_u + 2\Re\{f_{z_k}u_{\bar{k}}\} - 2\Re\{(\mu)_k u_{\bar{k}}\} \sum \mathbf{g}^{i\bar{i}}.
 \end{aligned}$$

Combining (3.9), (3.10) and (3.13), we obtain

$$\begin{aligned}
 0 &\geq |\nabla u|^2 \left(\inf_{i,l} \{g_{l\bar{p}i}g_{p\bar{l}i} - g_{l\bar{i}i}\} - 4 \sup_{i,l} |g_{l\bar{i}i}|^2 \right) \sum \mathbf{g}^{i\bar{i}} \\
 (3.14) \quad &+ 2|\nabla u|^2 f_u - 2|\nabla f||\nabla u| - 2\Re\{(\mu)_k u_{\bar{k}}\} \sum \mathbf{g}^{i\bar{i}} \\
 &+ 2\Re \sum \{\mathbf{g}^{i\bar{i}}u_i u_{i\bar{i}}\phi_{\bar{i}}\} + |\nabla u|^2 \sum \mathbf{g}^{i\bar{i}}\phi_{i\bar{i}}.
 \end{aligned}$$

This proves (3.7). \square

Proposition 3.2. *There exists $C > 0$ depending on*

$$\sup_M |u|, \inf_M (\psi^{\frac{1}{n}})_u, \sup_M |\nabla \psi^{\frac{1}{n}}|, \inf_M \frac{1}{|\mu|} \left(\inf_{j,l} R_{j\bar{j}l\bar{l}} - \sup_{j,l} |T_{l\bar{j}}^j|^2 \right),$$

and

$$\sup_M \{|\mu|^{-1} + |\nabla \log |\mu|| + (\log |\mu|)_u\}$$

such that

$$(3.15) \quad \max_M |\nabla u| \leq C(1 + \max_{\partial M} |\nabla u|).$$

Proof. Let $L = \inf_M u$ and $\phi = Ae^{\nu(L-u)}$ where $A > 0$ to be determined later and $\nu = \mu/|\mu|$. We have

$$(3.16) \quad e^{\nu(u-L)} \Re\{\mathbf{g}^{i\bar{i}}u_i u_{i\bar{i}}\phi_{\bar{i}}\} = -\nu A |\nabla u|^2 + |\mu|A \sum \mathbf{g}^{i\bar{i}}u_i u_{\bar{i}}$$

and

$$\begin{aligned}
 e^{\nu(u-L)} \sum \mathbf{g}^{i\bar{i}}\phi_{i\bar{i}} &= A\mathbf{g}^{i\bar{i}}(u_i u_{\bar{i}} - \nu u_{i\bar{i}}) \\
 (3.17) \quad &= A\mathbf{g}^{i\bar{i}}u_i u_{\bar{i}} - n\nu A + |\mu|A \sum \mathbf{g}^{i\bar{i}} \\
 &\geq A|\nabla u|^2 \min_i \mathbf{g}^{i\bar{i}} + |\mu|A \sum \mathbf{g}^{i\bar{i}} - n\nu A \\
 &\geq nA(|\mu|^{\frac{n-1}{n}} \psi^{-\frac{1}{n}} |\nabla u|^{\frac{2}{n}} - \nu).
 \end{aligned}$$

By Lemma 3.1, at an interior point where $e^\phi|\nabla u|^2$ achieves its maximum we have

$$\begin{aligned}
(3.18) \quad & A e^{\nu(L-u)} |\nabla u|^2 \left(|\mu|^{\frac{n-1}{n}} \psi^{-\frac{1}{n}} |\nabla u|^{\frac{2}{n}} - 3\nu + \frac{(n-1)|\mu|}{n} \sum \mathfrak{g}^{i\bar{i}} \right) \\
& \leq 2|\nabla f| |\nabla u| - 2|\nabla u|^2 f_u + 2(\mu_u |\nabla u|^2 + |\nabla \mu| |\nabla u|) \sum \mathfrak{g}^{i\bar{i}} \\
& \quad - |\nabla u|^2 \left(\inf_{j,l} R_{j\bar{j}l\bar{l}} - \sup_{j,l} |T_{lj}^j|^2 \right) \sum \mathfrak{g}^{i\bar{i}}.
\end{aligned}$$

Choose A such that

$$\frac{n-1}{n} |A| \geq \sup_M \frac{e^{u-L}}{|\mu|} \left(1 + 2\mu_u - \inf_{j,l} R_{j\bar{j}l\bar{l}} + \sup_{j,l} |T_{lj}^j|^2 \right).$$

We obtain

$$|\nabla u| \leq 2|\nabla \log |\mu|| e^{u-L}$$

or

$$A|\mu|^{\frac{n-1}{n}} |\nabla u|^{\frac{n+2}{n}} - 3\psi^{\frac{1}{n}} |\nabla u| + 2n e^{u-L} ((\psi^{\frac{1}{n}})_u |\nabla u| - |\nabla \psi^{\frac{1}{n}}|) \leq 0.$$

This gives the estimate in (3.15). \square

4. GLOBAL ESTIMATES FOR Δu

In this section we derive the estimate (1.6). We wish to include the degenerate case $\psi \geq 0$. So we shall still assume $\psi > 0$ but the estimates will not depend on the lower bound of ψ . We shall follow the notations in Section 3 and use ordinary derivatives.

Throughout Sections 4-6 we assume u is a solution of (1.1) in $\mathcal{H} \cap C^4(M)$. In local coordinates,

$$\Delta u = g^{k\bar{l}} u_{k\bar{l}}.$$

Therefore,

$$(4.1) \quad (\Delta u)_i = g^{k\bar{l}} u_{k\bar{l}i} - g^{k\bar{b}} g^{a\bar{l}} g_{a\bar{b}i} u_{k\bar{l}}$$

$$\begin{aligned}
(4.2) \quad (\Delta u)_{i\bar{i}} &= g^{k\bar{l}} u_{k\bar{l}i\bar{i}} - g^{k\bar{b}} g^{a\bar{l}} (g_{a\bar{b}i} u_{k\bar{l}} + g_{a\bar{b}i} u_{k\bar{l}}) \\
&\quad + [(g^{k\bar{q}} g^{p\bar{b}} g^{a\bar{l}} + g^{k\bar{b}} g^{a\bar{q}} g^{p\bar{l}}) g_{a\bar{b}i} g_{p\bar{q}\bar{i}} - g^{k\bar{b}} g^{a\bar{l}} g_{a\bar{b}i\bar{i}}] u_{k\bar{l}}.
\end{aligned}$$

Lemma 4.1. *Assume that (2.24) and (3.4) hold at $p \in M$. Then at p ,*

$$(4.3) \quad \begin{aligned} \mathfrak{g}^{\bar{i}\bar{i}}(\Delta u)_{\bar{i}\bar{i}} &\geq \mathfrak{g}^{\bar{i}\bar{i}} \mathfrak{g}^{j\bar{j}} |u_{\bar{i}\bar{j}j} + (\mu g_{\bar{i}\bar{j}})_j|^2 + \Delta(f) - 2\mathfrak{g}^{\bar{i}\bar{i}} \Re\{T_{ik}^i(\mu)_{\bar{k}}\} \\ &\quad - (|\mu||T|^2 + \Delta(\mu)) \sum \mathfrak{g}^{\bar{i}\bar{i}} - \mu \mathfrak{g}^{\bar{i}\bar{i}} (S_{\bar{i}\bar{i}} - R_{\bar{i}\bar{i}}) \\ &\quad + \inf_{j,k} R_{j\bar{j}k\bar{k}} \left((\Delta u + n\mu) \sum \mathfrak{g}^{\bar{i}\bar{i}} - n^2 \right). \end{aligned}$$

Proof. By (4.1) and (4.2),

$$(4.4) \quad (\Delta u)_i = u_{k\bar{k}i} - g_{k\bar{k}i} u_{k\bar{k}},$$

$$(4.5) \quad \begin{aligned} (\Delta u)_{\bar{i}\bar{i}} &= u_{\bar{i}\bar{i}k\bar{k}} - 2\Re\{u_{k\bar{j}i} g_{j\bar{k}\bar{i}}\} + (g_{k\bar{p}i} g_{p\bar{k}\bar{i}} + g_{p\bar{k}i} g_{k\bar{p}\bar{i}} - g_{k\bar{k}i\bar{i}}) u_{k\bar{k}} \\ &= u_{\bar{i}\bar{i}k\bar{k}} - 2\Re\{u_{k\bar{j}i} g_{j\bar{k}\bar{i}}\} + (\mathfrak{g}_{k\bar{k}} - \mu)(g_{p\bar{k}i} g_{k\bar{p}\bar{i}} + R_{\bar{i}\bar{i}k\bar{k}}). \end{aligned}$$

Differentiating equation (2.39) twice we obtain

$$(4.6) \quad \begin{aligned} \mathfrak{g}^{\bar{i}\bar{i}} u_{\bar{i}\bar{i}k\bar{k}} &= \mathfrak{g}^{\bar{i}\bar{i}} \mathfrak{g}^{j\bar{j}} |u_{\bar{i}\bar{j}k} + (\mu g_{\bar{i}\bar{j}})_k|^2 + (f)_{k\bar{k}} + g_{\bar{i}\bar{i}k\bar{k}} - g_{\bar{i}\bar{j}k} g_{j\bar{i}\bar{k}} - \mathfrak{g}^{\bar{i}\bar{i}} (\mu g_{\bar{i}\bar{i}})_{k\bar{k}} \\ &= \mathfrak{g}^{\bar{i}\bar{i}} \mathfrak{g}^{j\bar{j}} |u_{\bar{i}\bar{j}k} + (\mu g_{\bar{i}\bar{j}})_k|^2 + (f)_{k\bar{k}} - R_{k\bar{k}\bar{i}\bar{i}} - (\mu)_{k\bar{k}} \sum \mathfrak{g}^{\bar{i}\bar{i}} \\ &\quad + \mu \mathfrak{g}^{\bar{i}\bar{i}} (R_{k\bar{k}\bar{i}\bar{i}} - |g_{\bar{i}\bar{j}k}|^2) - 2\mathfrak{g}^{\bar{i}\bar{i}} \Re\{(\mu)_k g_{\bar{i}\bar{i}k}\}. \end{aligned}$$

From (2.24) we have,

$$(4.7) \quad \sum_{j,k} u_{k\bar{j}i} g_{j\bar{k}\bar{i}} = \sum_{j \neq k} [u_{\bar{i}\bar{j}k} + (\mu g_{\bar{i}\bar{j}})_k] g_{j\bar{k}\bar{i}} - \mu \sum_{j \neq k} g_{\bar{i}\bar{j}k} g_{j\bar{k}\bar{i}} - \sum_k (\mu)_k g_{i\bar{k}\bar{i}}.$$

By Cauchy-Schwarz inequality,

$$(4.8) \quad 2 \sum_{j \neq k} |\Re\{[u_{\bar{i}\bar{j}k} + (\mu g_{\bar{i}\bar{j}})_k] g_{j\bar{k}\bar{i}}\}| \leq \sum_{j \neq k} \mathfrak{g}^{j\bar{j}} |u_{\bar{i}\bar{j}k} + (\mu g_{\bar{i}\bar{j}})_k|^2 + \sum_{j \neq k} \mathfrak{g}_{j\bar{j}} |g_{k\bar{j}i}|^2.$$

Finally, combining (4.5), (4.6), (4.7), (4.8) and

$$|g_{\bar{i}\bar{j}k}|^2 + |g_{k\bar{j}i}|^2 - 2\Re\{g_{\bar{i}\bar{j}k} g_{j\bar{k}\bar{i}}\} = |g_{k\bar{j}i} - g_{\bar{i}\bar{j}k}|^2 = |T_{ik}^j|^2$$

we derive

$$(4.9) \quad \begin{aligned} \mathfrak{g}^{\bar{i}\bar{i}}(\Delta u)_{\bar{i}\bar{i}} &\geq \mathfrak{g}^{\bar{i}\bar{i}} \mathfrak{g}^{j\bar{j}} |u_{\bar{i}\bar{j}j} + (\mu g_{\bar{i}\bar{j}})_j|^2 + \Delta(f) + (\mathfrak{g}^{\bar{i}\bar{i}} \mathfrak{g}_{j\bar{j}} - 1) R_{\bar{i}\bar{i}j\bar{j}} \\ &\quad - \mu \sum_k \mathfrak{g}^{\bar{i}\bar{i}} (R_{\bar{i}\bar{i}k\bar{k}} - R_{k\bar{k}\bar{i}\bar{i}}) - \mu \sum_{j,k} \mathfrak{g}^{\bar{i}\bar{i}} |T_{ik}^j|^2 \\ &\quad - 2\mathfrak{g}^{\bar{i}\bar{i}} \Re\{T_{ik}^i(\mu)_{\bar{k}}\} - \Delta(\mu) \sum \mathfrak{g}^{\bar{i}\bar{i}}. \end{aligned}$$

which gives (4.3). \square

Lemma 4.2. *Suppose $e^\phi(n\mu + \Delta u)$ achieves its maximum at an interior point $p \in M$ where (2.24) and (3.4) hold. Then, at p ,*

$$(4.10) \quad \begin{aligned} (n\mu + \Delta u)\mathbf{g}^{i\bar{i}}\phi_{i\bar{i}} + 2\mathbf{g}^{i\bar{i}}\Re\{\phi_i\bar{\lambda}_i\} \\ \leq - (n\mu + \Delta u)\inf_{j,k} R_{j\bar{j}k\bar{k}} \sum \mathbf{g}^{i\bar{i}} - \Delta(f) \\ + n^2 \inf_{j,k} R_{j\bar{j}k\bar{k}} + A \sum \mathbf{g}^{i\bar{i}} \end{aligned}$$

where $\lambda_i = (n-1)(\mu)_i - \mu T_{ji}^j$ and

$$(4.11) \quad A = |\mu| \sup_k |R_{k\bar{k}} - S_{k\bar{k}}| + |\mu||T|^2 + |\nabla(\mu)||T| + \Delta(\mu) - n \inf_k (\mu)_{k\bar{k}}.$$

Proof. Since $e^\phi(n\mu + \Delta u)$ achieves its maximum at p ,

$$(4.12) \quad \frac{(n\mu + \Delta u)_i}{n\mu + \Delta u} + \phi_i = 0, \quad \frac{(n\mu + \Delta u)_{\bar{i}}}{n\mu + \Delta u} + \phi_{\bar{i}} = 0,$$

$$(4.13) \quad \frac{(n\mu + \Delta u)_{i\bar{i}}}{n\mu + \Delta u} - \frac{|(n\mu + \Delta u)_i|^2}{(n\mu + \Delta u)^2} + \phi_{i\bar{i}} \leq 0.$$

Note that

$$(n\mu + \Delta u)_i = \sum_j (u_{i\bar{j}} + \mu g_{i\bar{j}})_j + \lambda_i$$

by (4.4) and (2.24). We have by (4.12),

$$(4.14) \quad \begin{aligned} |(n\mu + \Delta u)_i|^2 &= \sum_j |(u_{i\bar{j}} + \mu g_{i\bar{j}})_j|^2 + 2\Re\{(n\mu + \Delta u)_i\bar{\lambda}_i\} - |\lambda_i|^2 \\ &= \sum_j |(u_{i\bar{j}} + \mu g_{i\bar{j}})_j|^2 - 2(n\mu + \Delta u)\Re\{\phi_i\bar{\lambda}_i\} - |\lambda_i|^2. \end{aligned}$$

By Cauchy-Schwarz inequality,

$$(4.15) \quad \begin{aligned} \sum_{i,j} \mathbf{g}^{i\bar{i}} |(u_{i\bar{j}} + \mu g_{i\bar{j}})_j|^2 &= \sum_i \mathbf{g}^{i\bar{i}} \left| \sum_j \mathbf{g}_{j\bar{j}}^{1/2} \mathbf{g}_{j\bar{j}}^{-1/2} (u_{i\bar{j}} + \mu g_{i\bar{j}})_j \right|^2 \\ &\leq (n\mu + \Delta u) \mathbf{g}^{i\bar{i}} \mathbf{g}^{j\bar{j}} |u_{i\bar{j}j} + (\mu g_{i\bar{j}})_j|^2. \end{aligned}$$

From (4.13), (4.3), (4.14) and (4.15) we derive (4.10). \square

Let $\phi = e^{\eta(u)}$ with $\eta \geq 0$, $\mu\eta' < 0$, and $\eta'' \geq 0$. We have

$$(4.16) \quad \phi_i = e^\eta \eta' u_i, \quad \phi_{i\bar{i}} = e^\eta [\eta' u_{i\bar{i}} + (\eta'' + \eta'^2) |u_i|^2].$$

Therefore,

$$(4.17) \quad 2\mathbf{g}^{i\bar{i}} \Re\{\phi_i \bar{\lambda}_i\} = 2e^\eta \eta' \mathbf{g}^{i\bar{i}} \Re\{u_i \bar{\lambda}_i\} \geq -e^\eta \mathbf{g}^{i\bar{i}} (|\lambda_i|^2 + \eta'^2 |u_i|^2).$$

Suppose now that both ψ and μ are independent of u . Plugging (4.16) and (4.17) into (4.10), we see that

$$(4.18) \quad \begin{aligned} 0 &\geq (n\mu + \Delta u) \left(e^{-\eta} \inf_{j,k} R_{j\bar{j}k\bar{k}} - \mu\eta' \right) \sum \mathfrak{g}^{i\bar{i}} \\ &\quad + \eta'^2 (n\mu + \Delta u - 1) \sum \mathfrak{g}^{i\bar{i}} |u_i|^2 + n\eta' (n\mu + \Delta u) \\ &\quad - n^2 e^{-\eta} \inf_{j,k} R_{j\bar{j}k\bar{k}} + e^{-\eta} \Delta f - C_3 \sum \mathfrak{g}^{i\bar{i}} \end{aligned}$$

where

$$(4.19) \quad C_3 = A + n^2 |\nabla \mu|^2 + n^2 \mu^2 |T|^2$$

and A is given in (4.11).

Following Yau [43] we shall make use of the inequality

$$(4.20) \quad \left(\sum \mathfrak{g}^{i\bar{i}} \right)^{n-1} \geq \frac{\sum \mathfrak{g}_{i\bar{i}}}{\mathfrak{g}_{1\bar{1}} \cdots \mathfrak{g}_{n\bar{n}}} = \frac{n\mu + \Delta u}{\det(\mu g_{i\bar{j}} + u_{i\bar{j}})} = \frac{n\mu + \Delta u}{\psi}.$$

Choosing $\eta = A(U - \nu u)$ where $\nu = \mu/|\mu|$, $U = \sup_M \nu u$ and $A > 0$ is a constant such that

$$(4.21) \quad |\mu|A + e^{-\eta} \inf_{j,k} R_{j\bar{j}k\bar{k}} \geq 2,$$

we see from (4.18) and (4.20) that

$$(4.22) \quad (n\mu + \Delta u)^{\frac{n}{n-1}} + n\nu A \psi^{\frac{1}{n-1}} (n\mu + \Delta u) + \psi^{\frac{1}{n-1}} \Delta f - n^2 e^{-\eta} \psi^{\frac{1}{n-1}} \inf_{j,k} R_{j\bar{j}k\bar{k}} \leq 0$$

provided that

$$(4.23) \quad n\mu + \Delta u \geq 1 + C_3.$$

This gives us a bound $(n\mu + \Delta u)(0) \leq C$ which depends on $|u|_{C^0(\bar{M})}$, $|\psi^{\frac{1}{n-1}}|_{C^2(M)}$, $|\mu|_{C^2(M)}$ and geometric quantities of (M, g) . Finally,

$$(4.24) \quad \sup_M (n\mu + \Delta u) \leq e^{\phi(0) - \inf_M \phi} (n\mu + \Delta u)(0) \leq C.$$

We have therefore proved the following.

Proposition 4.3. *Suppose both μ and ψ are independent of u . Then*

$$(4.25) \quad \max_{\bar{M}} \Delta u \leq C(1 + \max_{\partial M} \Delta u)$$

where $C > 0$ depends on

$$|u|_{C^0(\bar{M})}, |\psi^{\frac{1}{n-1}}|_{C^2(\bar{M})}, |\mu|_{C^2(\bar{M})}, \sup_M \frac{1}{|\mu|}, \inf_{j,k} R_{j\bar{j}k\bar{k}}, \sup_k |R_{k\bar{k}} - S_{k\bar{k}}|, |T|^2.$$

If ψ and μ depend also on u the estimate (4.25) still holds with C depending in addition on $\sup_M |\nabla u|$. Indeed, in places of (4.18) we have

$$(4.26) \quad \begin{aligned} 0 &\geq (n\mu + \Delta u) \left(e^{-\eta} \inf_{j,k} R_{j\bar{j}k\bar{k}} - (n+1)e^{-\eta} |\mu_u| - \mu\eta' \right) \sum \mathfrak{g}^{i\bar{i}} \\ &\quad + (n\eta' + e^{-\eta} f_u)(n\mu + \Delta u) \\ &\quad - n^2 e^{-\eta} \inf_{j,k} R_{j\bar{j}k\bar{k}} - C'_3 \sum \mathfrak{g}^{i\bar{i}} \end{aligned}$$

where C'_3 depends on C_3 , $|u|_{C^1(\bar{M})}$, as well as the derivatives of $\psi^{\frac{1}{n-1}}$ and μ . This again will give a bound for $(n\mu + \Delta u)(0)$ and therefore (4.24).

5. BOUNDARY ESTIMATES FOR SECOND DERIVATIVES

In this section we derive *a priori* estimates for second derivatives (the *real* Hessian) on the boundary

$$(5.1) \quad \max_{\partial M} |\nabla^2 u| \leq C.$$

In order to track the dependence on the curvature and torsion of the estimates we shall use covariant derivatives in this section. So we begin with a brief review of formulas for changing the orders of covariant derivatives which we shall also need in Section 6.

In local coordinates $z = (z_1, \dots, z_n)$, $z_j = x_j + \sqrt{-1}y_j$, we shall use notations such as

$$v_i = \nabla_{\frac{\partial}{\partial z_i}} v, \quad v_{ij} = \nabla_{\frac{\partial}{\partial z_j}} \nabla_{\frac{\partial}{\partial z_i}} v, \quad v_{x_i} = \nabla_{\frac{\partial}{\partial x_i}} v, \quad \text{etc.}$$

Recall that

$$(5.2) \quad v_{i\bar{j}} - v_{\bar{j}i} = 0, \quad v_{ij} - v_{ji} = T_{ij}^l v_l.$$

By straightforward calculations,

$$(5.3) \quad \begin{cases} v_{i\bar{j}k} - v_{i\bar{k}j} = \overline{T_{jk}^l} v_{i\bar{l}}, \\ v_{i\bar{j}k} - v_{ik\bar{j}} = -g^{l\bar{m}} R_{k\bar{j}i\bar{m}} v_l, \\ v_{ijk} - v_{ikj} = g^{l\bar{m}} R_{jkim} v_l + T_{jk}^l v_{il}. \end{cases}$$

Therefore,

$$\begin{aligned}
 (5.4) \quad v_{i\bar{j}k} - v_{k\bar{i}j} &= (v_{i\bar{j}k} - v_{ik\bar{j}}) + (v_{ik\bar{j}} - v_{k\bar{i}j}) \\
 &= -g^{l\bar{m}}R_{k\bar{j}i\bar{m}}v_l + T_{ik}^l v_{l\bar{j}} + \nabla_{\bar{j}}T_{ik}^l v_l \\
 &= -g^{l\bar{m}}R_{i\bar{j}k\bar{m}}v_l + T_{ik}^l v_{l\bar{j}}
 \end{aligned}$$

by (2.20), and

$$\begin{aligned}
 (5.5) \quad v_{ijk} - v_{kij} &= (v_{ijk} - v_{ikj}) + (v_{ikj} - v_{kij}) \\
 &= g^{l\bar{m}}R_{jki\bar{m}}v_l + T_{jk}^l v_{il} + T_{ik}^l v_{lj} + \nabla_j T_{ik}^l v_l.
 \end{aligned}$$

Since

$$\frac{\partial}{\partial x_k} = \frac{\partial}{\partial z_k} + \frac{\partial}{\partial \bar{z}_k}, \quad \frac{\partial}{\partial y_k} = \sqrt{-1} \left(\frac{\partial}{\partial z_k} - \frac{\partial}{\partial \bar{z}_k} \right),$$

we see that

$$(5.6) \quad \begin{cases} v_{z_i x_j} - v_{x_j z_i} = v_{ij} - v_{ji} = T_{ij}^l v_l, \\ v_{z_i y_j} - v_{y_j z_i} = \sqrt{-1}(v_{ij} - v_{ji}) = \sqrt{-1}T_{ij}^l v_l, \end{cases}$$

$$\begin{aligned}
 (5.7) \quad v_{z_i \bar{z}_j x_k} - v_{x_k z_i \bar{z}_j} &= (v_{i\bar{j}k} + v_{i\bar{j}\bar{k}}) - (v_{k\bar{i}j} + v_{k\bar{i}\bar{j}}) \\
 &= (v_{i\bar{j}k} - v_{k\bar{i}j}) + (v_{i\bar{j}\bar{k}} - v_{i\bar{k}\bar{j}}) \\
 &= -g^{l\bar{m}}R_{i\bar{j}k\bar{m}}v_l + T_{ik}^l v_{l\bar{j}} + \overline{T_{jk}^l} v_{i\bar{l}},
 \end{aligned}$$

and, similarly,

$$\begin{aligned}
 (5.8) \quad v_{z_i z_{\bar{j}} y_k} - v_{y_k z_i z_{\bar{j}}} &= \sqrt{-1}((v_{i\bar{j}k} - v_{i\bar{j}\bar{k}}) - (v_{k\bar{i}j} - v_{k\bar{i}\bar{j}})) \\
 &= \sqrt{-1}((v_{i\bar{j}k} - v_{k\bar{i}j}) - (v_{i\bar{j}\bar{k}} - v_{i\bar{k}\bar{j}})) \\
 &= \sqrt{-1}(-g^{l\bar{m}}R_{i\bar{j}k\bar{m}}v_l + T_{ik}^l v_{l\bar{j}} - \overline{T_{jk}^l} v_{i\bar{l}}).
 \end{aligned}$$

For convenience we set

$$t_{2k-1} = x_k, \quad t_{2k} = y_k, \quad 1 \leq k \leq n-1; \quad t_{2n-1} = y_n, \quad t_{2n} = x_n.$$

By (5.7), (5.8) and the identity

$$(5.9) \quad \mathfrak{g}^{i\bar{j}}T_{ki}^l u_{l\bar{j}} = T_{ki}^i - \mu \mathfrak{g}^{i\bar{j}}T_{ki}^l g_{l\bar{j}}$$

we obtain for all $1 \leq \alpha \leq 2n$,

$$\begin{aligned}
 (5.10) \quad |\mathfrak{g}^{i\bar{j}}(u_{t_\alpha i\bar{j}} - u_{i\bar{j}t_\alpha})| &= |\mathfrak{g}^{i\bar{j}}u_{t_\alpha i\bar{j}} - (f)_{t_\alpha}| \\
 &\leq 2|T| + (|\mu||T| + |R| + |\nabla u||\nabla T|) \sum \mathfrak{g}^{i\bar{j}}g_{i\bar{j}}.
 \end{aligned}$$

We also record here the following identity which we shall use later: for a function η ,

$$(5.11) \quad \begin{aligned} \mathfrak{g}^{i\bar{j}} \eta_i u_{x_n \bar{j}} &= \mathfrak{g}^{i\bar{j}} \eta_i (2u_{n\bar{j}} + \sqrt{-1} u_{y_n \bar{j}}) \\ &= 2\eta_n - 2\mu \mathfrak{g}^{i\bar{j}} \eta_i g_{n\bar{j}} + \sqrt{-1} \mathfrak{g}^{i\bar{j}} \eta_i u_{y_n \bar{j}}. \end{aligned}$$

We now start to derive (5.1). We assume

$$(5.12) \quad |u| + |\nabla u| \leq K \text{ in } \bar{M}.$$

Set

$$(5.13) \quad \underline{\psi} \equiv \min_{|u| \leq K, z \in \bar{M}} \psi(z, u) > 0, \quad \bar{\psi} \equiv \max_{|u| \leq K, z \in \bar{M}} \psi(z, u).$$

Let σ be the distance function to ∂M . Note that $|\nabla \sigma| = \frac{1}{2}$ on ∂M . There exists $\delta_0 > 0$ such that σ is smooth and $\nabla \sigma \neq 0$ in

$$M_{\delta_0} := \{z \in M : \sigma(z) < \delta_0\},$$

which we call the δ_0 -neighborhood of ∂M . We can therefore write

$$(5.14) \quad u - \underline{u} = h\sigma, \text{ in } M_{\delta_0}$$

where h is a smooth function.

Consider a boundary point $p \in \partial M$. We choose local coordinates $z = (z_1, \dots, z_n)$, $z_j = x_j + iy_j$, around p in a neighborhood which we assume to be contained in M_{δ_0} such $\frac{\partial}{\partial x_n}$ is the interior normal direction to ∂M at p where we also assume $g_{i\bar{j}} = \delta_{ij}$; (Here and in what follows we identify p with $z = 0$.) for later reference we call such local coordinates *regular* coordinate charts.

By (5.14) we have

$$(u - \underline{u})_{x_n} = h_{x_n} \sigma + h \sigma_{x_n}$$

and

$$(u - \underline{u})_{j\bar{k}} = h_{j\bar{k}} \sigma + h \sigma_{j\bar{k}} + 2\Re\{h_j \sigma_{\bar{k}}\}.$$

Since $\sigma = 0$ on ∂M and $\sigma_{x_n}(0) = 2|\nabla \sigma| = 1$, we see that

$$(u - \underline{u})_{x_n}(0) = h(0)$$

and

$$(5.15) \quad (u - \underline{u})_{j\bar{k}}(0) = (u - \underline{u})_{x_n}(0) \sigma_{j\bar{k}}(0) \quad j, k < n.$$

Similarly,

$$(5.16) \quad (u - \underline{u})_{t_\alpha t_\beta}(0) = -(u - \underline{u})_{x_n}(0) \sigma_{t_\alpha t_\beta}, \quad \alpha, \beta < 2n.$$

It follows that

$$(5.17) \quad |u_{t_\alpha t_\beta}(0)| \leq C, \quad \alpha, \beta < 2n$$

where C depends on $|u|_{C^1(\bar{M})}$, $|\underline{u}|_{C^1(\bar{M})}$, and the principal curvatures of ∂M .

To estimate $u_{t_\alpha x_n}(0)$ for $\alpha \leq 2n$, we will follow [20] and employ a barrier function of the form

$$(5.18) \quad v = (u - \underline{u}) + t\sigma - N\sigma^2,$$

where t, N are positive constants to be determined. Recall that $\underline{u} \in C^2$ and $\omega_{\underline{u}} > 0$ in a neighborhood of ∂M . We may assume that there exists $\epsilon > 0$ such that $\omega_{\underline{u}} > \epsilon\omega$ in M_{δ_0} . Locally, this gives

$$(5.19) \quad \{\underline{u}_{j\bar{k}} + \mu g_{j\bar{k}}\} \geq \epsilon \{g_{j\bar{k}}\}.$$

The following is the key ingredient in our argument.

Lemma 5.1. *For N sufficiently large and t, δ sufficiently small,*

$$\begin{aligned} \mathfrak{g}^{i\bar{j}} v_{i\bar{j}} &\leq -\frac{\epsilon}{4} \left(1 + \sum \mathfrak{g}^{i\bar{j}} g_{i\bar{j}} \right) \quad \text{in } \Omega_\delta, \\ v &\geq 0 \quad \text{on } \partial\Omega_\delta \end{aligned}$$

where $\Omega_\delta = M \cap B_\delta$ and B_δ is the (geodesic) ball of radius δ centered at p .

Proof. This lemma was first proved in [20] for domains in \mathbb{C}^n . For completeness we include the proof here with minor modifications. By (5.19) we have

$$(5.20) \quad \mathfrak{g}^{i\bar{j}}(u_{i\bar{j}} - \underline{u}_{i\bar{j}}) = \mathfrak{g}^{i\bar{j}}(u_{i\bar{j}} + \mu g_{i\bar{j}} - \underline{u}_{i\bar{j}} - \mu g_{i\bar{j}}) \leq n - \epsilon \sum \mathfrak{g}^{i\bar{j}} g_{i\bar{j}}.$$

Obviously,

$$\mathfrak{g}^{i\bar{j}} \sigma_{i\bar{j}} \leq C_1 \sum \mathfrak{g}^{i\bar{j}} g_{i\bar{j}}$$

for some constant $C_1 > 0$ under control. Thus

$$\mathfrak{g}^{i\bar{j}} v_{i\bar{j}} \leq n + \{C_1(t + N\sigma) - \epsilon\} \sum \mathfrak{g}^{i\bar{j}} g_{i\bar{j}} - 2N \mathfrak{g}^{i\bar{j}} \sigma_i \sigma_{\bar{j}} \quad \text{in } \Omega_\delta.$$

Let $\lambda_1 \leq \dots \leq \lambda_n$ be the eigenvalues of $\{u_{i\bar{j}} + \mu g_{i\bar{j}}\}$ (with respect to $\{g_{i\bar{j}}\}$). We have $\sum \mathfrak{g}^{i\bar{j}} g_{i\bar{j}} = \sum \lambda_k^{-1}$ and

$$(5.21) \quad \mathfrak{g}^{i\bar{j}} \sigma_i \sigma_{\bar{j}} \geq \frac{1}{2\lambda_n}$$

since $|\nabla\sigma| \equiv \frac{1}{2}$ where σ is smooth. By the arithmetic-geometric mean-value inequality,

$$\frac{\epsilon}{4} \sum \mathbf{g}^{i\bar{j}} g_{i\bar{j}} + \frac{N}{\lambda_n} \geq \frac{n\epsilon}{4} (N\lambda_1^{-1} \cdots \lambda_n^{-1})^{\frac{1}{n}} \geq \frac{n\epsilon N^{\frac{1}{n}}}{4\psi^{\frac{1}{n}}} \geq c_1 N^{\frac{1}{n}}$$

for some constant $c_1 > 0$ depending on the upper bound of ψ .

We now fix $t > 0$ sufficiently small and N large so that $c_1 N^{1/n} \geq 1 + n + \epsilon$ and $C_1 t \leq \frac{\epsilon}{4}$. Consequently,

$$\mathbf{g}^{i\bar{j}} v_{i\bar{j}} \leq -\frac{\epsilon}{4} \left(1 + \sum \mathbf{g}^{i\bar{j}} g_{i\bar{j}} \right) \quad \text{in } \Omega_\delta$$

if we require δ to satisfy $C_1 N \delta \leq \frac{\epsilon}{4}$ in Ω_δ .

On $\partial M \cap B_\delta$ we have $v = 0$. On $M \cap \partial B_\delta$,

$$v \geq t\sigma - N\sigma^2 \geq (t - N\delta)\sigma \geq 0$$

if we require, in addition, $N\delta \leq t$. □

Remark 5.2. For the real Monge-Ampère equations, Lemma 5.1 was proved in [19] both for domains in \mathbb{R}^n and in general Riemannian manifolds, improving earlier results in [29], [23] and [22].

Lemma 5.3. *Let $w \in C^2(\overline{\Omega_\delta})$. Suppose that w satisfies*

$$\mathbf{g}^{i\bar{j}} w_{i\bar{j}} \geq -C_1 \left(1 + \sum \mathbf{g}^{i\bar{j}} g_{i\bar{j}} \right) \quad \text{in } \Omega_\delta$$

and

$$w \leq C_0 \rho^2 \quad \text{on } B_\delta \cap \partial M, \quad w(0) = 0$$

where ρ is the distance function to the point p (where $z = 0$) on ∂M . Then $w_\nu(0) \leq C$, where ν is the interior unit normal to ∂M , and C depends on ϵ^{-1} , C_0 , C_1 , $|w|_{C^0(\overline{\Omega_\delta})}$, $|u|_{C^1(\bar{M})}$ and the constants N , t and δ determined in Lemma 5.1.

Proof. By Lemma 5.1, $Av + B\rho^2 - w \geq 0$ on $\partial\Omega_\delta$ and

$$\mathbf{g}^{i\bar{j}} (Av + B\rho^2 - w)_{i\bar{j}} \leq 0 \quad \text{in } \Omega_\delta$$

when $A \gg B$ and both are sufficiently large. By the maximum principle,

$$Av + B\rho^2 - w \geq 0 \quad \text{in } \overline{\Omega_\delta}.$$

Consequently,

$$Av_\nu(0) - w_\nu(0) = D_\nu(Av + B\rho^2 - w)(0) \geq 0$$

since $Av + B\rho^2 - w = 0$ at the origin. □

We next apply Lemma 5.3 to estimate $u_{t_\alpha x_n}(0)$ for $\alpha < 2n$. For fixed $\alpha < 2n$, we write $\eta = \sigma_{t_\alpha}/\sigma_{x_n}$ and define

$$\mathcal{T} = \nabla_{\frac{\partial}{\partial t_\alpha}} - \eta \nabla_{\frac{\partial}{\partial x_n}}.$$

We wish to apply Lemma 5.3 to

$$w = (u_{y_n} - \varphi_{y_n})^2 \pm \mathcal{T}(u - \varphi).$$

By (5.12),

$$|\mathcal{T}(u - \varphi)| + (u_{y_n} - \varphi_{y_n})^2 \leq C \text{ in } \Omega_\delta.$$

On ∂M since $u - \varphi = 0$ and \mathcal{T} is a tangential differential operator, we have

$$\mathcal{T}(u - \varphi) = 0 \text{ on } \partial M \cap B_\delta$$

and, similarly,

$$(5.22) \quad (u_{y_n} - \varphi_{y_n})^2 \leq C\rho^2 \text{ on } \partial M \cap B_\delta.$$

We compute next

$$(5.23) \quad \mathfrak{g}^{i\bar{j}}(\mathcal{T}u)_{i\bar{j}} = \mathfrak{g}^{i\bar{j}}(u_{t_\alpha i\bar{j}} + \eta u_{x_n i\bar{j}}) + \mathfrak{g}^{i\bar{j}}\eta_{i\bar{j}}u_{x_n} + 2\mathfrak{g}^{i\bar{j}}\Re\{\eta_i u_{x_n \bar{j}}\}.$$

By (5.10) and (5.11),

$$(5.24) \quad |\mathfrak{g}^{i\bar{j}}(u_{t_\alpha i\bar{j}} + \eta u_{x_n i\bar{j}})| \leq |\mathcal{T}(f)| + C_1(|T| + |R| + |\nabla T|) \left(1 + \sum \mathfrak{g}^{i\bar{j}}g_{i\bar{j}}\right)$$

and

$$(5.25) \quad 2|\mathfrak{g}^{i\bar{j}}\Re\{\eta_i u_{x_n \bar{j}}\}| \leq \mathfrak{g}^{i\bar{j}}u_{y_n i}u_{y_n \bar{j}} + C_2 \left(1 + \sum \mathfrak{g}^{i\bar{j}}g_{i\bar{j}}\right)$$

where C_1 and C_2 are independent of the curvature and torsion. Applying (5.10) again, we derive

$$(5.26) \quad \begin{aligned} \mathfrak{g}^{i\bar{j}}[(u_{y_n} - \varphi_{y_n})^2]_{i\bar{j}} &= 2\mathfrak{g}^{i\bar{j}}(u_{y_n} - \varphi_{y_n})_i(u_{y_n} - \varphi_{y_n})_{\bar{j}} \\ &\quad + 2(u_{y_n} - \varphi_{y_n})\mathfrak{g}^{i\bar{j}}(u_{y_n} - \varphi_{y_n})_{i\bar{j}} \\ &\geq \mathfrak{g}^{i\bar{j}}u_{y_n i}u_{y_n \bar{j}} - 2\mathfrak{g}^{i\bar{j}}\varphi_{y_n i}\varphi_{y_n \bar{j}} \\ &\quad + 2(u_{y_n} - \varphi_{y_n})\mathfrak{g}^{i\bar{j}}(u_{y_n i\bar{j}} - \varphi_{y_n i\bar{j}}) \\ &\geq \mathfrak{g}^{i\bar{j}}u_{y_n i}u_{y_n \bar{j}} - |(f)_{y_n}| - C_3 \\ &\quad - C_4(1 + |T| + |R| + |\nabla T|) \sum \mathfrak{g}^{i\bar{j}}g_{i\bar{j}}. \end{aligned}$$

Finally, combining (5.23)-(5.26) we obtain

$$(5.27) \quad \mathfrak{g}^{i\bar{j}}[(u_{y_n} - \varphi_{y_n})^2 \pm \mathcal{T}(u - \varphi)]_{i\bar{j}} \geq -C \left(1 + |Df| + \sum \mathfrak{g}^{i\bar{j}}g_{i\bar{j}}\right) \text{ in } \Omega_\delta$$

where $C = C_0(1 + |R| + |T| + |\nabla T|)$ with C_0 independent of the curvature and torsion.

Consequently, we may apply Lemma 5.3 to $w = (u_{y_n} - \varphi_{y_n})^2 \pm \mathcal{T}(u - \varphi)$ to obtain

$$(5.28) \quad |u_{t_\alpha x_n}(0)| \leq C, \quad \alpha < 2n.$$

By (5.6) we also have

$$(5.29) \quad |u_{x_n t_\alpha}(0)| \leq C, \quad \alpha < 2n.$$

It remains to establish the estimate

$$(5.30) \quad |u_{x_n x_n}(0)| \leq C.$$

Since we have already derived

$$(5.31) \quad |u_{t_\alpha t_\beta}(0)|, |u_{t_\alpha x_n}(0)|, |u_{x_n t_\alpha}(0)| \leq C, \quad \alpha, \beta < 2n,$$

it suffices to prove

$$(5.32) \quad 0 \leq \mu + u_{n\bar{n}}(0) = \mu + u_{x_n x_n}(0) + u_{y_n y_n}(0) \leq C.$$

Expanding $\det(u_{i\bar{j}} + \mu g_{i\bar{j}})$, we have

$$(5.33) \quad \det(u_{i\bar{j}}(0) + \mu g_{i\bar{j}}) = a(u_{n\bar{n}}(0) + \mu) + b$$

where

$$a = \det(u_{\alpha\bar{\beta}}(0) + \mu g_{\alpha\bar{\beta}})|_{\{1 \leq \alpha, \beta \leq n-1\}}$$

and b is bounded in view of (5.31). Since $\det(u_{i\bar{j}} + \mu g_{i\bar{j}})$ is bounded, we only have to derive an *a priori* positive lower bound for a , which is equivalent to

$$(5.34) \quad \sum_{\alpha, \beta < n} u_{\alpha\bar{\beta}}(0) \xi_\alpha \bar{\xi}_\beta \geq c_0 |\xi|^2, \quad \forall \xi \in \mathbb{C}^{n-1}$$

for a uniform constant $c_0 > 0$.

Proposition 5.4. *There exists $c_0 = c_0(\underline{\psi}^{-1}, \varphi, \underline{u}) > 0$ such that (5.34) holds.*

Proof. Let $T_C \partial M \subset T_C M$ be the complex tangent bundle of ∂M and

$$T^{1,0} \partial M = T^{1,0} M \cap T_C \partial M = \left\{ \xi \in T^{1,0} M : d\sigma(\xi) = 0 \right\}.$$

In local coordinates,

$$T^{1,0} \partial M = \left\{ \xi = \xi_i \frac{\partial}{\partial z_i} \in T^{1,0} M : \sum \xi_i \sigma_i = 0 \right\}.$$

It is enough to establish a positive lower bound for

$$m_0 = \min_{\xi \in T^{1,0}\partial M, |\xi|=1} \omega_u(\xi, \bar{\xi}).$$

We assume that m_0 is attained at a point $p \in \partial M$ and choose regular local coordinates around p as before such that

$$m_0 = \omega_u\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial \bar{z}_1}\right) = u_{1\bar{1}}(0) + \mu.$$

One needs to show

$$(5.35) \quad m_0 = u_{1\bar{1}}(0) + \mu \geq c_0 > 0.$$

By (5.15),

$$(5.36) \quad u_{1\bar{1}}(0) = \underline{u}_{1\bar{1}}(0) - (u - \underline{u})_{x_n}(0)\sigma_{1\bar{1}}(0).$$

We can assume $u_{1\bar{1}}(0) + \mu \leq \frac{1}{2}(\underline{u}_{1\bar{1}}(0) + \mu)$; otherwise we are done. Thus

$$(5.37) \quad (u - \underline{u})_{x_n}(0)\sigma_{1\bar{1}}(0) \geq \frac{1}{2}(\underline{u}_{1\bar{1}}(0) + \mu).$$

It follows from (5.12) that

$$(5.38) \quad \sigma_{1\bar{1}}(0) \geq \frac{\underline{u}_{1\bar{1}}(0) + \mu}{2C} \geq \frac{\epsilon}{2C} \equiv c_1 > 0$$

where $C = \max_{\partial M} |\nabla(u - \underline{u})|$.

Let $\delta > 0$ be small enough so that

$$\begin{aligned} w &\equiv \left| -\sigma_{z_n} \frac{\partial}{\partial z_1} + \sigma_{z_1} \frac{\partial}{\partial z_n} \right| \\ &= (g_{1\bar{1}}|\sigma_{z_n}|^2 - 2\Re\{g_{1\bar{n}}\sigma_{z_n}\sigma_{\bar{z}_1}\} + g_{n\bar{n}}|\sigma_{z_1}|^2)^{\frac{1}{2}} > 0 \text{ in } M \cap B_\delta(p). \end{aligned}$$

Define $\zeta = \sum \zeta_i \frac{\partial}{\partial z_i} \in T^{1,0}M$ in $M \cap B_\delta(p)$:

$$\begin{cases} \zeta_1 = -\frac{\sigma_{z_n}}{w}, \\ \zeta_j = 0, \quad 2 \leq j \leq n-1, \\ \zeta_n = \frac{\sigma_{z_1}}{w} \end{cases}$$

and

$$\Phi = (\varphi_{j\bar{k}} + \mu g_{j\bar{k}})\zeta_j \bar{\zeta}_k - (u - \varphi)_{x_n} \sigma_{j\bar{k}} \zeta_j \bar{\zeta}_k - u_{1\bar{1}}(0) - \mu.$$

Note that $\zeta \in T^{1,0}\partial M$ on ∂M and $|\zeta| = 1$. By (5.15),

$$(5.39) \quad \Phi = (u_{j\bar{k}} + \mu g_{j\bar{k}})\zeta_j \bar{\zeta}_k - u_{1\bar{1}}(0) - \mu \geq 0 \text{ on } \partial M \cap B_\delta(p)$$

and $\Phi(0) = 0$.

Write $G = \sigma_{i\bar{j}} \zeta_i \bar{\zeta}_j$. We have

$$\begin{aligned}
\mathfrak{g}^{i\bar{j}} \Phi_{i\bar{j}} &\leq -\mathfrak{g}^{i\bar{j}} (u_{x_n} G)_{i\bar{j}} + C \left(1 + \sum u^{j\bar{j}}\right) \\
(5.40) \quad &= -G \mathfrak{g}^{i\bar{j}} u_{x_n i\bar{j}} - 2\mathfrak{g}^{i\bar{j}} \Re \{u_{x_n i} G_{j\bar{j}}\} + C \left(1 + \sum \mathfrak{g}^{i\bar{j}} g_{i\bar{j}}\right) \\
&\leq \mathfrak{g}^{i\bar{j}} u_{y_n i} u_{y_n j} + C \left(1 + \sum \mathfrak{g}^{i\bar{j}} g_{i\bar{j}}\right)
\end{aligned}$$

by (5.10) and (5.11). It follows that

$$(5.41) \quad \mathfrak{g}^{i\bar{j}} [\Phi - (u_{y_n} - \varphi_{y_n})^2]_{i\bar{j}} \leq C \left(1 + \sum \mathfrak{g}^{i\bar{j}} g_{i\bar{j}}\right) \text{ in } M \cap B_\delta(p).$$

Moreover, by (5.22) and (5.39),

$$(u_{y_n} - \varphi_{y_n})^2 - \Phi \leq C|z|^2 \text{ on } \partial M \cap B_\delta(p).$$

Consequently, we may apply Lemma 5.3 to

$$h = (u_{y_n} - \varphi_{y_n})^2 - \Phi$$

to derive $\Phi_{x_n}(0) \geq -C$ which, by (5.38), implies

$$(5.42) \quad u_{x_n x_n}(0) \leq \frac{C}{\sigma_{1\bar{1}}(0)} \leq \frac{C}{c_1}.$$

In view of (5.31) and (5.42) we have an *a priori* upper bound for all eigenvalues of $\{u_{i\bar{j}} + \mu g_{i\bar{j}}\}$ at p . Since $\det(u_{i\bar{j}} + \mu g_{i\bar{j}}) \geq \underline{\psi} > 0$, the eigenvalues of $\{u_{i\bar{j}} + \mu g_{i\bar{j}}\}$ at p must admit a positive lower bound, i.e.,

$$\min_{\xi \in T_p^{1,0} M, |\xi|=1} (u_{i\bar{j}} + \mu g_{i\bar{j}}) \xi_i \bar{\xi}_j \geq c_0.$$

Therefore,

$$m_0 = \min_{\xi \in T_p^{1,0} \partial M, |\xi|=1} (u_{i\bar{j}} + \mu g_{i\bar{j}}) \xi_i \bar{\xi}_j \geq \min_{\xi \in T_p^{1,0} M, |\xi|=1} (u_{i\bar{j}} + \mu g_{i\bar{j}}) \xi_i \bar{\xi}_j \geq c_0.$$

The proof of Proposition 5.4 is complete. \square

We have therefore established (5.1).

6. ESTIMATES FOR THE REAL HESSIAN AND HIGHER DERIVATIVES

The primary goal of this section is to derive global estimates for the whole (real) Hessian

$$(6.1) \quad |\nabla^2 u| \leq C \text{ on } \bar{M}.$$

This is equivalent to

$$(6.2) \quad |u_{x_i x_j}(p)|, |u_{x_i y_j}(p)|, |u_{y_i y_j}(p)| \leq C, \quad \forall 1 \leq i, j \leq n$$

in local coordinates $z = (z_1, \dots, z_n)$, $z_j = x_j + \sqrt{-1}y_j$ with $g_{i\bar{j}}(p) = \delta_{ij}$ for any fixed point $p \in M$, where the constant C may depend on $|u|_{C^1(M)}$, $\sup_M \Delta u$, $\inf \psi > 0$, and the curvature and torsion of M as well as their derivatives. Once this is done we can apply the Evans-Krylov Theorem to obtain global $C^{2,\alpha}$ estimates.

As in Section 5 we shall use covariant derivatives. We start with communication formulas for the fourth order derivatives. From direct computation,

$$(6.3) \quad \begin{cases} v_{i\bar{j}k\bar{l}} - v_{i\bar{j}l\bar{k}} = \overline{T_{kl}^q} v_{i\bar{j}q}, \\ v_{i\bar{j}k\bar{l}} - v_{i\bar{j}l\bar{k}} = g^{p\bar{q}} R_{k\bar{l}i\bar{q}} v_{pj} + g^{p\bar{q}} R_{k\bar{l}j\bar{q}} v_{ip}. \end{cases}$$

Therefore, by (5.3), (5.4), (5.5), (6.3) and (2.20),

$$(6.4) \quad \begin{aligned} v_{i\bar{j}k\bar{l}} - v_{k\bar{l}i\bar{j}} &= (v_{i\bar{j}k\bar{l}} - v_{k\bar{l}i\bar{j}}) + (v_{k\bar{l}i\bar{j}} - v_{k\bar{l}j\bar{i}}) + (v_{k\bar{l}i\bar{j}} - v_{k\bar{l}i\bar{j}}) \\ &= \nabla_{\bar{l}}(-g^{p\bar{q}} R_{i\bar{j}k\bar{q}} v_p + T_{ik}^p v_{p\bar{j}}) + \overline{T_{jl}^q} v_{ki\bar{q}} + g^{p\bar{q}} \nabla_{\bar{j}}(R_{i\bar{l}k\bar{q}} v_p) \\ &= g^{p\bar{q}}(R_{k\bar{l}i\bar{q}} v_{p\bar{j}} - R_{i\bar{j}k\bar{q}} v_{p\bar{l}}) + T_{ik}^p v_{p\bar{j}\bar{l}} + \overline{T_{jl}^q} v_{ki\bar{q}} \\ &\quad + g^{p\bar{q}}(\nabla_{\bar{j}} R_{i\bar{l}k\bar{q}} - \nabla_{\bar{l}} R_{i\bar{j}k\bar{q}}) v_p \end{aligned}$$

and

$$(6.5) \quad \begin{aligned} v_{i\bar{j}k\bar{l}} - v_{k\bar{l}i\bar{j}} &= v_{i\bar{j}kl} - v_{k\bar{l}ij} + v_{k\bar{l}ij} - v_{k\bar{l}i\bar{j}} + v_{k\bar{l}i\bar{j}} - v_{k\bar{l}i\bar{j}} \\ &= \nabla_{\bar{l}}(-g^{p\bar{q}} R_{i\bar{j}k\bar{q}} v_p + T_{ik}^p v_{p\bar{j}}) - g^{p\bar{q}} R_{l\bar{j}k\bar{q}} v_{pi} - g^{p\bar{q}} R_{l\bar{j}i\bar{q}} v_{kp} \\ &\quad + \nabla_{\bar{j}}(g^{p\bar{q}} R_{i\bar{l}k\bar{q}} v_p + T_{il}^p v_{kp}) \\ &= -g^{p\bar{q}} R_{i\bar{j}k\bar{q}} v_{pl} - g^{p\bar{q}} R_{i\bar{j}l\bar{q}} v_{kp} - g^{p\bar{q}} R_{l\bar{j}k\bar{q}} v_{pi} \\ &\quad - g^{p\bar{q}}[(\nabla_{\bar{l}} R_{i\bar{j}k\bar{q}}) + (\nabla_{\bar{j}} R_{i\bar{l}k\bar{q}})] v_p \\ &\quad + [(\nabla_{\bar{l}} T_{ik}^p) + g^{p\bar{q}} R_{i\bar{l}k\bar{q}}] v_{p\bar{j}} \\ &\quad + T_{ik}^p v_{p\bar{j}\bar{l}} + T_{il}^p v_{kp\bar{j}}. \end{aligned}$$

Turning to the proof of (6.2), it suffices to prove the following.

Proposition 6.1. *There exists constant $C > 0$ depending on $|u|_{C^1(\bar{M})}$, $\sup_M \Delta u$ and $\inf \psi > 0$ such that*

$$(6.6) \quad \sup_{\tau \in TM, |\tau|=1} u_{\tau\tau} \leq C.$$

Proof. Let

$$N := \sup_M \left\{ |\nabla u|^2 + A|\omega_u|^2 + \sup_{\tau \in TM, |\tau|=1} u_{\tau\tau} \right\}$$

where A is positive constant to be determined, and assume that it is achieved at an interior point $p \in M$ and for some unit vector $\tau \in T_p M$. We choose local coordinates $z = (z_1, \dots, z_n)$ such that $g_{i\bar{j}} = \delta_{ij}$ and $\{u_{i\bar{j}}\}$ is diagonal at p . Thus τ can be written in the form

$$\tau = a_j \frac{\partial}{\partial z_j} + b_j \frac{\partial}{\partial \bar{z}_j}, \quad a_j, b_j \in \mathbb{C}, \quad \sum a_j b_j = \frac{1}{2}.$$

Let ξ be a smooth unit vector field defined in a neighborhood of p such that $\xi(p) = \tau$. Then the function

$$Q = u_{\xi\xi} + |\nabla u|^2 + A|\omega_u|^2$$

(defined in a neighborhood of p) attains its maximum at p where, therefore

$$(6.7) \quad Q_i = u_{\tau\tau i} + u_k u_{i\bar{k}} + u_{\bar{k}} u_{ki} + 2A(u_{k\bar{k}} + \mu)(u_{k\bar{k}i} + (\mu)_i) = 0$$

and

$$(6.8) \quad \begin{aligned} 0 &\geq \mathfrak{g}^{i\bar{i}} Q_{i\bar{i}} = \mathfrak{g}^{i\bar{i}}(u_{k\bar{i}} u_{i\bar{k}} + u_{ki} u_{\bar{k}\bar{i}}) + \mathfrak{g}^{i\bar{i}}(u_k u_{i\bar{k}\bar{i}} + u_{\bar{k}} u_{ki\bar{i}}) \\ &\quad + \mathfrak{g}^{i\bar{i}} u_{\tau\tau i\bar{i}} + 2A \mathfrak{g}^{i\bar{i}}(u_{k\bar{i}} + (\mu)_i)(u_{l\bar{k}\bar{i}} + (\mu)_{\bar{i}}) \\ &\quad + 2A(u_{k\bar{k}} + \mu) \mathfrak{g}^{i\bar{i}}(u_{k\bar{k}i\bar{i}} + (\mu)_{i\bar{i}}). \end{aligned}$$

Differentiating equation (2.39) twice (using covariant derivatives), by (5.4) and (6.4) we obtain

$$(6.9) \quad \mathfrak{g}^{i\bar{i}} u_{k\bar{i}\bar{i}} = (f)_k + \mathfrak{g}^{i\bar{i}} R_{i\bar{i}k\bar{l}} u_l - \mathfrak{g}^{i\bar{i}} T_{ik}^l u_{l\bar{i}} - (\mu)_k \sum \mathfrak{g}^{i\bar{i}} \geq (f)_k - C \sum \mathfrak{g}^{i\bar{i}}$$

and

$$(6.10) \quad \begin{aligned} \mathfrak{g}^{i\bar{i}} u_{k\bar{k}i\bar{i}} &\geq \mathfrak{g}^{i\bar{i}} \mathfrak{g}^{j\bar{j}} u_{i\bar{j}k} u_{j\bar{i}\bar{k}} + \mathfrak{g}^{i\bar{i}} (T_{ik}^p u_{p\bar{i}\bar{k}} + \overline{T_{ik}^p} u_{ki\bar{p}}) \\ &\quad + (f)_{k\bar{k}} - C \sum \mathfrak{g}^{i\bar{i}} \geq (f)_{k\bar{k}} - C \left(1 + \sum \mathfrak{g}^{i\bar{i}} \right). \end{aligned}$$

(Here we used (5.4) again for the last inequality.)

Note that

$$u_{\tau\tau i\bar{i}} = a_k a_l u_{kl i\bar{i}} + 2a_k b_l u_{k\bar{l} i\bar{i}} + b_k b_l u_{\bar{k}\bar{l} i\bar{i}}.$$

Using the formulas in (6.3), (6.4) and (6.5) we obtain

$$\begin{aligned}
 \mathfrak{g}^{i\bar{i}} u_{\tau\tau i\bar{i}} &\geq \mathfrak{g}^{i\bar{i}} u_{i\bar{i}\tau\tau} - C \mathfrak{g}^{i\bar{i}} |T_{ik}^l u_{l\bar{i}k}| - C \left(1 + \sum_{k,l} |u_{kl}|\right) \sum \mathfrak{g}^{i\bar{i}} \\
 (6.11) \qquad &\geq (f)_{\tau\tau} - C \mathfrak{g}^{i\bar{i}} u_{l\bar{i}k} u_{i\bar{l}k} - C \left(1 + \sum_{k,l} |u_{kl}|\right) \sum \mathfrak{g}^{i\bar{i}}.
 \end{aligned}$$

Plugging (6.9), (6.10), (6.11) into (6.8) and using the inequality

$$\begin{aligned}
 (6.12) \qquad &2\mathfrak{g}^{i\bar{i}} (u_{k\bar{l}i} + (\mu)_i)(u_{l\bar{k}\bar{i}} + (\mu)_{\bar{i}}) \\
 &\geq \mathfrak{g}^{i\bar{i}} u_{k\bar{l}i} u_{l\bar{k}\bar{i}} - \mathfrak{g}^{i\bar{i}} (\mu)_i (\mu)_{\bar{i}} \geq \mathfrak{g}^{i\bar{i}} u_{k\bar{l}i} u_{l\bar{k}\bar{i}} - C \sum \mathfrak{g}^{i\bar{i}},
 \end{aligned}$$

we see that

$$(6.13) \qquad \mathfrak{g}^{i\bar{i}} u_{ki} u_{k\bar{i}} + (A - C) \mathfrak{g}^{i\bar{i}} u_{k\bar{l}i} u_{l\bar{k}\bar{i}} - C \left(1 + A + \sum_{k,l} |u_{kl}|\right) \left(1 + \sum \mathfrak{g}^{i\bar{i}}\right) \leq 0.$$

We now need the nondegeneracy of equation (2.39) which implies that there is $\Lambda > 0$ depending on $\sup_M \Delta u$ and $\inf \psi > 0$ such that

$$\Lambda^{-1} \{g_{i\bar{j}}\} \leq \{\mathfrak{g}_{i\bar{j}}\} \leq \Lambda \{g_{i\bar{j}}\}$$

and, therefore,

$$(6.14) \qquad \begin{cases} \sum \mathfrak{g}^{i\bar{i}} \leq n\Lambda, \\ \mathfrak{g}^{i\bar{i}} u_{ki} u_{k\bar{i}} \geq \frac{1}{\Lambda} \sum_{i,k} |u_{ki}|^2. \end{cases}$$

Plugging these into (6.13) and choosing A large we derive

$$\sum_{i,k} |u_{ki}|^2 \leq C.$$

Consequently we have a bound $u_{\tau\tau}(p) \leq C$. Finally,

$$\sup_{q \in M} \sup_{\tau \in T_q M, |\tau|=1} u_{\tau\tau} \leq u_{\tau\tau}(p) + 2 \sup_M (|\nabla u|^2 + A|\omega_u|^2).$$

This completes the proof of (6.6). \square

We can now appeal to the Evans-Krylov Theorem ([17], [32], [33]) for $C^{2,\alpha}$ estimates

$$(6.15) \qquad |u|_{C^{2,\alpha}(M)} \leq C.$$

Higher order regularity and estimates now follow from the classical Schauder theory for elliptic linear equations.

Remark 6.2. When M is a Kähler manifold, Proposition 6.1 was recently proved by Blocki [8]. He observed that the estimate (6.6) does not depend on $\inf \psi$ when M has nonnegative bisectional curvature. This is clearly also true in the Hermitian case.

Remark 6.3. An alternative approach to the $C^{2,\alpha}$ estimate (6.15) is to use (1.6) and the boundary estimate (5.1) (in place of (6.1)) and apply an extension of the Evans-Krylov Theorem; see Theorem 7.3, page 126 in [14] which only requires $C^{1,\alpha}$ bounds for the solution. This was pointed out to us by Pengfei Guan to whom we wish to express our gratitude.

7. C^0 ESTIMATES AND EXISTENCE

In this section we complete the proof of Theorem 1.2-1.4 using the estimates established in previous sections. We shall consider separately the Dirichlet problem and the case of manifolds without boundary. In each case we need first to derive C^0 estimates; the existence of solutions then can be proved by the continuity method, possibly combined with degree arguments.

7.1. Compact manifolds without boundary. For the C^0 estimate on compact manifolds without boundary, we follow the argument in [39], [41] which simplifies the original proof of Yau [43].

Let M be a compact Hermitian manifold without boundary and u an admissible solution of equation (2.39), $\sup_M u = -1$. In this case we assume $\mu > 0$. (When $\mu < 0$ equation (2.39) does not have solutions by the maximum principle.) We write

$$\chi = \sum_{k=0}^{n-1} (\mu\omega)^k \wedge (\omega_u)^{n-1-k}.$$

Multiply the identity $(\omega_u)^n - (\mu\omega)^n = \frac{\sqrt{-1}}{2} \partial\bar{\partial}u \wedge \chi$ by $(-u)^p$ and integrate over M ,

$$\begin{aligned} \int_M (-u)^p [(\omega_u)^n - (\mu\omega)^n] &= \frac{\sqrt{-1}}{2} \int_M (-u)^p \partial\bar{\partial}u \wedge \chi \\ (7.1) \quad &= \frac{p\sqrt{-1}}{2} \int_M (-u)^{p-1} \partial u \wedge \bar{\partial}u \wedge \chi + \frac{\sqrt{-1}}{2} \int_M (-u)^p \bar{\partial}u \wedge \partial\chi \\ &= \frac{2p\sqrt{-1}}{(p+1)^2} \int_M \partial(-u)^{\frac{p+1}{2}} \bar{\partial}(-u)^{\frac{p+1}{2}} \wedge \chi - \frac{\sqrt{-1}}{2(p+1)} \int_M (-u)^{p+1} \partial\bar{\partial}\chi. \end{aligned}$$

We now assume that $\partial\bar{\partial}(\mu\omega)^k = 0$, for all $1 \leq k \leq n-1$, which implies $\partial\bar{\partial}\chi = 0$, and that ψ does not depend on u . Since $\mu\omega > 0$ and $\omega_u \geq 0$, we see that $(\mu\omega)^k \wedge (\omega_u)^{n-1-k} \geq 0$ for each k . Therefore,

$$\begin{aligned}
 \int_M |\nabla(-u)^{\frac{p+1}{2}}|^2 \omega^n &= \frac{\sqrt{-1}}{2} \int_M \partial(-u)^{\frac{p+1}{2}} \bar{\partial}(-u)^{\frac{p+1}{2}} \wedge \omega^{n-1} \\
 &\leq \frac{\sqrt{-1}}{2 \inf \mu^{n-1}} \int_M \partial(-u)^{\frac{p+1}{2}} \bar{\partial}(-u)^{\frac{p+1}{2}} \wedge \chi \\
 &= \frac{(p+1)^2}{2p \inf \mu^{n-1}} \int_M (-u)^p (\psi - \mu^n) \omega^n \\
 &\leq C \int_M (-u)^{p+1} \omega^n.
 \end{aligned}
 \tag{7.2}$$

After this we can derive a bound for $\inf u$ by the Moser iteration method, following the argument in [41].

If ψ depends on u and satisfies (1.10), a bound for $\sup_M |u|$ follows directly from equation (2.39) by the maximum principle. Indeed, suppose $u(p) = \max_M u$ for some $p \in M$. Then $\{u_{i\bar{j}}(p)\} \leq 0$ and, therefore

$$\mu^n \det g_{i\bar{j}} \geq \det(u_{i\bar{j}} + \mu g_{ij}) = \psi(p, u(p)) \det g_{i\bar{j}}.$$

This implies an upper bound $u(p) \leq C$ by (1.10). That $\min_M u \geq -C$ follows from a similar argument.

Proof of Theorem 1.3. We first consider the case that ψ does not depend on u . By assumption (1.8) we see that

$$\int_M (\psi - 1) \omega^n = 0$$

is a necessary condition for the existence of admissible solutions, and that the linearized operator, $v \mapsto \mathbf{g}^{i\bar{j}} v_{i\bar{j}}$, of equation (1.1) is self-adjoint. So the continuity method proof in [43] works to give a unique admissible solution $u \in \mathcal{H} \cap C^{2,\alpha}(M)$ of (1.1) satisfying

$$\int_M u \omega^n = 0.$$

The smoothness of u follows from the Schauder regularity theory.

For the general case under the assumption $\psi_u \geq 0$, one can still follow the proof of Yau [43]. So we omit it here. \square

Proof of Theorem 1.4. The uniqueness follows easily from the assumption $\psi_u > 0$ and the maximum principle. For the existence we make use of the continuity method. For $0 \leq s \leq 1$ consider

$$(7.3) \quad (\omega_u)^n = \psi^s(z, u)\omega^n \text{ in } M$$

where $\psi^s(z, u) = (1 - s)e^u + s\psi(z, u)$. Set

$$S := \{s \in [0, 1] : \text{equation (7.3) is solvable in } \mathcal{H} \cap C^{2,\alpha}(M)\}$$

and let $u^s \in \mathcal{H} \cap C^{2,\alpha}(M)$ be the unique solution of (7.3) for $s \in S$. Obviously $S \neq \emptyset$ as $0 \in S$ with $u^0 = 0$. Moreover, by the $C^{2,\alpha}$ estimates we see that S is closed. We need to show that S is also open in and therefore equal to $[0, 1]$; u^1 is then the desired solution.

Let $s \in S$ and let Δ^s denote the Laplace operator of (M, ω_{u^s}) . In local coordinates,

$$\Delta^s v = \mathbf{g}^{i\bar{j}} v_{i\bar{j}} = \mathbf{g}^{i\bar{j}} \partial_i \bar{\partial}_{\bar{j}} v$$

where $\{\mathbf{g}^{i\bar{j}}\} = \{\mathbf{g}_{i\bar{j}}^s\}^{-1}$ and $\mathbf{g}_{i\bar{j}}^s = g_{i\bar{j}} + u_{i\bar{j}}^s$. Note that $\Delta^s - \psi_u^s$, where $\psi_u^s = \psi_u^s(\cdot, u^s)$, is the linearized operator of equation (7.3) at u^s . We wish to prove that for any $\phi \in C^\alpha(M, \omega_{u^s})$ there exists a unique solution $v \in C^{2,\alpha}(M, \omega_{u^s})$ to the equation

$$(7.4) \quad \Delta^s v - \psi_u^s v = \phi,$$

which implies by the implicit function theorem that S contains a neighborhood of s and hence is open in $[0, 1]$, completing the proof.

The proof follows a standard approach, using the Lax-Milgram theorem and the Fredholm alternative. For completeness we include it here.

Let $\gamma > 0$ and define a bilinear form on the Sobolev space $H^1(M, \omega_{u^s})$ by

$$(7.5) \quad \begin{aligned} B[v, w] &:= \int_M [\langle \nabla v + v \operatorname{tr} \tilde{T}, \nabla w \rangle_{\omega_{u^s}} + (\gamma + \psi_u^s)vw](\omega_{u^s})^n \\ &= \int_M [\mathbf{g}^{i\bar{j}}(v_i + v \tilde{T}_{ik}^k)w_{\bar{j}} + (\gamma + \psi_u^s)vw](\omega_{u^s})^n \end{aligned}$$

where \tilde{T} denotes the torsion of ω_{u^s} and $\operatorname{tr} \tilde{T}$ its trace. In local coordinates,

$$\operatorname{tr} \tilde{T} = \tilde{T}_{ik}^k dz_i = \mathbf{g}^{k\bar{j}}(g_{i\bar{j}k} - g_{k\bar{j}i}) dz_i$$

so it only depends on the second derivatives of u .

It is clear that for $\gamma > 0$ sufficiently large B satisfies the Lax-Milgram hypotheses, i.e.,

$$(7.6) \quad |B[v, w]| \leq C \|v\|_{H^1(\omega^s)} \|w\|_{H^1(\omega^s)}$$

by the Schwarz inequality, and

$$(7.7) \quad B[v, v] \geq c_0 \|v\|_{H^1(\omega^s)}^2, \quad \forall v \in H^1(M, \omega_{u^s})$$

where c_0 is a positive constant independent of $s \in [0, 1]$ since $\psi_u > 0$, $|u^s|_{C^2(M)} \leq C$ and M is compact. By the Lax-Milgram theorem, for any $\phi \in L^2(M, \omega_{u^s})$ there is a unique $v \in H^1(M, \omega_{u^s})$ satisfying

$$(7.8) \quad B[v, w] = \int_M \phi w (\omega_{u^s})^n \quad \forall w \in H^1(M, \omega_{u^s}).$$

On the other hand,

$$(7.9) \quad B[v, w] = \int_M (-\Delta^s v + \psi_u^s v + \gamma v) w (\omega_{u^s})^n$$

by integration by parts. Thus v is a weak solution to the equation

$$(7.10) \quad L_\gamma v := \Delta^s v - \psi_u^s v - \gamma v = \phi.$$

We write $v = L_\gamma^{-1} \phi$.

By the Sobolev embedding theorem the linear operator

$$K := \gamma L_\gamma^{-1} : L^2(M, \omega_{u^s}) \rightarrow L^2(M, \omega_{u^s})$$

is compact. Note also that $v \in H^1(M, \omega_{u^s})$ is a weak solution of equation (7.4) if and only if

$$(7.11) \quad v - K v = \zeta$$

where $\zeta = L_\gamma^{-1} \phi$. Indeed, (7.4) is equivalent to

$$(7.12) \quad v = L_\gamma^{-1}(\gamma v + \phi) = \gamma L_\gamma^{-1} v + L_\gamma^{-1} \phi.$$

Since the solution of equation (7.4), if exists, is unique, by the Fredholm alternative equation (7.11) is uniquely solvable for any $\zeta \in L^2(M, \omega_{u^s})$. Consequently, for any $\phi \in L^2(M, \omega_{u^s})$ there exists a unique solution $v \in H^1(M, \omega_{u^s})$ to equation (7.4). By the regularity theory of linear elliptic equations, $v \in C^{2,\alpha}(M, \omega_{u^s})$ if $\phi \in C^\alpha(M, \omega_{u^s})$. This completes the proof. \square

7.2. The Dirichlet problem. We now turn to the proof of Theorem 1.2. Let

$$\mathcal{A}_{\underline{u}} = \{v \in \mathcal{H} : v \geq \underline{u} \text{ in } M, v = \underline{u} \text{ on } \partial M\}.$$

By the maximum principle, $v \leq h$ on \bar{M} for all $v \in \mathcal{A}_{\underline{u}}$ where h satisfies $\Delta h + n = 0$ in M and $h = \underline{u}$ on ∂M . Therefore we have C^0 bounds for solutions of the Dirichlet

problem (1.1)-(1.2) in \mathcal{A}_u . The proof of existence of such solutions then follows that of Theorem 1.1 in [19]; so is omitted here.

Proof of Theorem 1.5. As we only assume $\psi \geq 0$, equation (1.11) is degenerate. So we need to approximate it by nondegenerate equations. Since $\omega_\phi > 0$ and M is compact, there is $\varepsilon_0 > 0$ such that $\omega_\phi \geq \varepsilon_0 \omega$, and therefore $(\omega_\phi)^n \geq \varepsilon_0^n \omega^n$.

For $\varepsilon \in (0, \varepsilon_0]$ let ψ^ε be a smooth function such that

$$\sup \left\{ \psi - \varepsilon, \frac{\varepsilon^n}{2} \right\} \leq \psi^\varepsilon \leq \sup \{ \psi, \varepsilon^n \}$$

and consider the approximating problem

$$(7.13) \quad \begin{cases} (\omega_u)^n = \psi^\varepsilon \omega^n & \text{in } \bar{M}, \\ u = \phi & \text{on } \partial M. \end{cases}$$

Note that ϕ is a subsolution of (7.13) when $0 < \varepsilon \leq \varepsilon_0$. By Theorem 1.2 there is a unique solution $u^\varepsilon \in C^{2,\alpha}(\bar{M})$ of (7.13) with $u^\varepsilon \geq \phi$ on \bar{M} for $\varepsilon \in (0, \varepsilon_0]$.

By Theorem 1.1 it is easy to see that

$$(7.14) \quad |u^\varepsilon|_{C^1(\bar{M})} \leq C_1, \quad \sup_M \Delta u^\varepsilon \leq C_2(1 + \sup_{\partial M} \Delta u^\varepsilon), \quad \text{independent of } \varepsilon.$$

On the boundary ∂M , the estimates in Section 5 for the pure tangential and mixed tangential-normal second derivatives are independent of ε , i.e.,

$$(7.15) \quad |u_{\xi\eta}^\varepsilon|, |u_{\xi\nu}^\varepsilon| \leq C_3, \quad \forall \xi, \eta \in T\partial M, |\xi|, |\eta| = 1 \text{ independent of } \varepsilon.$$

where ν is the unit normal to ∂M . For the estimate of the double normal derivative $u_{\nu\nu}^\varepsilon$, note that $\partial M = N \times \partial S$ and $T_C \partial M = TN$; this is the only place we need the assumption $M = N \times S$ so Theorem 1.5 actually holds for local product spaces. So

$$(7.16) \quad 1 + u_{\xi\xi}^\varepsilon = 1 + \phi_{\xi\xi} \geq c_0 \quad \forall \xi \in T_C \partial M = TN, \quad |\xi| = 1.$$

where c_0 depends only on ϕ . From the proof in Section 5 we see that

$$(7.17) \quad |u_{\nu\nu}^\varepsilon| \leq C, \quad \text{independent of } \varepsilon \text{ on } \partial M.$$

Finally, from $\sup_M |\Delta u^\varepsilon| \leq C$ we see that $|u^\varepsilon|_{C^{1,\alpha}(\bar{M})}$ is bounded for any $\alpha \in (0, 1)$. Taking a convergent subsequence we obtain a solution $u \in C^{1,\alpha}(\bar{M})$ of (1.11) with the desired properties. By Remark 6.2, $u \in C^{1,1}(\bar{M})$ when M has nonnegative bisectional curvature. \square

8. GEODESICS IN THE SPACE OF HERMITIAN METRICS.

Let (M, g) be a compact Hermitian manifold without boundary. The space of Hermitian metrics

$$(8.1) \quad \mathcal{H} = \{\phi \in C^2(M) : \omega_\phi > 0\}$$

is an open subset of $C^2(M)$. The tangent space $T_\phi \mathcal{H}$ of \mathcal{H} at $\phi \in \mathcal{H}$ is naturally identified to $C^2(M)$. Following Mabuchi [36], Semmes [38] and Donaldson [16] who considered the Kähler case, we define

$$(8.2) \quad \langle \xi, \eta \rangle_\phi = \int_M \xi \eta (\omega_\phi)^n, \quad \xi, \eta \in T_\phi \mathcal{H}.$$

Accordingly, the length of a regular curve $\varphi : [0, 1] \rightarrow \mathcal{H}$ is defined to be

$$(8.3) \quad L(\varphi) = \int_0^1 \langle \dot{\varphi}, \dot{\varphi} \rangle_\phi^{\frac{1}{2}} dt.$$

Henceforth $\dot{\varphi} = \partial\varphi/\partial t$ and $\ddot{\varphi} = \partial^2\varphi/\partial t^2$. The geodesic equation takes the form

$$(8.4) \quad \ddot{\varphi} - |\nabla \dot{\varphi}|_\varphi^2 = 0,$$

or in local coordinates

$$(8.5) \quad \ddot{\varphi} - g(\varphi)^{j\bar{k}} \dot{\varphi}_{z_j} \dot{\varphi}_{\bar{z}_k} = 0.$$

Here $\{g(\varphi)^{j\bar{k}}\}$ is the inverse matrix of $\{g(\varphi)_{j\bar{k}}\} = \{g_{j\bar{k}} + \varphi_{j\bar{k}}\}$.

It was observed by Donaldson [16], Mabuchi [36] and Semmes [38] that the geodesic equation (8.4) reduces to a homogeneous complex Monge-Ampère equation in $M \times A$ where $A = [0, 1] \times \mathbb{S}^1$. Let

$$w = z_{n+1} = t + \sqrt{-1}s$$

be a local coordinate of A . We may view a smooth curve φ in \mathcal{H} as a function on $M \times [0, 1]$ and therefore a rotation-invariant function (constant in s) on $M \times A$. Clearly,

$$\dot{\varphi} = \frac{\partial\varphi}{\partial t} = 2 \frac{\partial\varphi}{\partial w} = 2 \frac{\partial\varphi}{\partial \bar{w}}, \quad \ddot{\varphi} = \frac{\partial^2\varphi}{\partial t^2} = 4 \frac{\partial^2\varphi}{\partial w \partial \bar{w}}.$$

Therefore,

$$\begin{aligned}
(8.6) \quad & \det \begin{bmatrix} & & & \varphi_{1\bar{w}} \\ & (g(\varphi)_{j\bar{k}}) & & \vdots \\ & & & \varphi_{n\bar{w}} \\ \varphi_{w\bar{1}} & \cdots & \varphi_{w\bar{n}} & \varphi_{w\bar{w}} \end{bmatrix} \\
&= \frac{1}{4} \det(g(\varphi)_{i\bar{j}}) \cdot \det \begin{bmatrix} & & g(\varphi)^{k\bar{1}}\dot{\varphi}_k \\ & I & \vdots \\ \dot{\varphi}_{\bar{1}} & \cdots & \dot{\varphi}_{\bar{n}} & \ddot{\varphi} \end{bmatrix} \\
&= \frac{1}{4} \det(g(\varphi)_{i\bar{j}}) \cdot (\ddot{\varphi} - g(\varphi)^{j\bar{k}}\dot{\varphi}_{z_j}\dot{\varphi}_{\bar{z}_k}).
\end{aligned}$$

So a geodesic φ in \mathcal{H} satisfies

$$(8.7) \quad (\tilde{\omega}_\varphi)^{n+1} \equiv \left(\tilde{\omega} + \frac{\sqrt{-1}}{2} \partial\bar{\partial}\varphi \right)^{n+1} = 0 \quad \text{in } M \times A$$

where

$$(8.8) \quad \tilde{\omega} = \omega + \frac{\sqrt{-1}}{2} \partial\bar{\partial}|w|^2 = \frac{\sqrt{-1}}{2} \left(\sum_{j,k \leq n} g_{j\bar{k}} dz_j \wedge d\bar{z}_k + dw \wedge d\bar{w} \right)$$

is the lift of ω to $M \times A$.

Conversely, if $\varphi \in C^2(M \times A)$ is a rotation-invariant solution of (8.7) such that

$$(8.9) \quad \varphi(\cdot, w) \in \mathcal{H}, \quad \forall w \in A,$$

then φ is a geodesic in \mathcal{H} .

In the Kähler case, Donaldson [16] conjectured that $\mathcal{H}^\infty \equiv \mathcal{H} \cap C^\infty(M)$ is geodesically convex, i.e., any two functions in \mathcal{H}^∞ can be connected by a smooth geodesic. More precisely,

Conjecture 8.1 (Donaldson [16]). *Let M be a compact Kähler manifold without boundary and $\rho \in C^\infty(M \times \partial A)$ such that $\rho(\cdot, w) \in \mathcal{H}$ for $w \in \partial A$. Then there exists a unique solution φ of the Monge-Ampère equation (8.7) satisfying (8.9) and the boundary condition $\varphi = \rho$.*

The uniqueness was proved by Donaldson [16] as a consequence of the maximum principle. In [13], X.-X. Chen obtained the existence of a weak solution with $\Delta\varphi \in L^\infty(M \times A)$; see also the recent work of Blocki [8] who proved that the solution is in $C^{1,1}(M \times A)$ when M has nonnegative bisectional curvature. As a corollary of Theorem 1.5 these results can be extended to the Hermitian case.

Theorem 8.2. *Let M be a compact Hermitian manifold without boundary. and let $\varphi_0, \varphi_1 \in \mathcal{H} \cap C^4(M)$. There exists a unique (weak) solution $\varphi \in C^{1,\alpha}(M \times A)$, $\forall 0 < \alpha < 1$, with $\tilde{\omega}_\varphi \geq 0$ and $\Delta\varphi \in L^\infty(M \times A)$ of the Dirichlet problem*

$$(8.10) \quad \begin{cases} (\tilde{\omega}_\varphi)^{n+1} = 0 & \text{in } M \times A \\ \varphi = \varphi_0 & \text{on } M \times \Gamma_0, \\ \varphi = \varphi_1 & \text{on } M \times \Gamma_1 \end{cases}$$

where $\Gamma_0 = \partial A|_{t=0}$, $\Gamma_1 = \partial A|_{t=1}$. Moreover, $\varphi \in C^{1,1}(M \times A)$ if M has nonnegative bisectional curvature.

Proof. In order to apply Theorem 1.5 to the Dirichlet problem (8.10) we only need to construct a strict subsolution. This is easily done for the annulus $A = [0, 1] \times \mathbb{S}^1$. Let

$$\underline{\varphi} = (1 - t)\varphi_0 + t\varphi_1 + K(t^2 - t).$$

Since $\varphi_0, \varphi_1 \in \mathcal{H}(\omega)$ we see that $\tilde{\omega}_{\underline{\varphi}} > 0$ and $(\tilde{\omega}_{\underline{\varphi}})^{n+1} \geq 1$ for K sufficiently large. \square

Remark 8.3. By the uniqueness φ is rotation invariant (i.e., independent of s).

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