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Convex hypersurfaces of prescribed curvatures

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1. Introduction

For a smooth strictly convex closed hypersurface Σ in $\mathbb{R}^{n+1}$, the Gauss map $\mathbf{n} : \Sigma \to \mathbb{S}^n$ is a diffeomorphism. A fundamental question in classical differential geometry concerns how much one can recover through the inverse Gauss map when some information is prescribed on $\mathbb{S}^n$ ([27]). This question has attracted much attention for more than a hundred years. The most notable example is probably the Minkowski problem of finding a closed convex hypersurface in $\mathbb{R}^{n+1}$ whose Gauss curvature is prescribed as a positive function defined on $\mathbb{S}^n$. This problem has been solved due to the work of Minkowski [18], Alexandrov [1], Lewy [17], Nirenberg [19], Pogorelov [21], [22], Cheng-Yau [6] and others. In particular, the analytic approach of Nirenberg, Pogorelov and Cheng-Yau to the problem has inspired significant development of the theory of Monge-Ampère equations. Besides the Gauss curvature, there are other important Weingarten curvature functions such as, for example, the mean and scalar curvatures. In the 1950s, A. D. Alexandrov [2] and S.-s. Chern [8], [9] raised questions regarding prescribing Weingarten curvatures. So far, a large part of the problem has not received much consideration. Apart from the Gauss curvature case (the Minkowski problem), very little is known except a uniqueness result for the case $n = 2$ (see [2] and [13]).

In this paper, we initiate an investigation of problems in this direction. Specifically, we consider the problem of finding closed, strictly convex hypersurfaces in $\mathbb{R}^{n+1}$ whose Weingarten curvatures is prescribed as a function defined on $\mathbb{S}^n$ in terms of the inverse Gauss map.

We first recall the definition of Weingarten curvatures for hypersurfaces. Let $S_k(\lambda_1, \ldots, \lambda_n)$ be the k^{th} elementary symmetric function normalized so that

$$S_k(1, \ldots, 1) = 1.$$

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For a C^2 hypersurface Σ in $\mathbb{R}^{n+1}$, let $\kappa = (\kappa_1, \ldots, \kappa_n)$ denote the principal curvatures of Σ with respect to its interior normal. The k^{th} Weingarten curvature W_k of Σ is defined as

$$W_k = S_k(\kappa_1, \ldots, \kappa_n), \quad k = 1, \ldots, n.$$

For $k = 1, 2$ and n, W_k corresponds to the mean, scalar and Gauss curvatures, respectively. The following is a precise formulation of our problem.

Problem. Let $1 \leq k < n$ be a fixed integer. For which smooth, positive function ψ on $\mathbb{S}^n$ does there exist a closed, strictly convex hypersurface Σ in $\mathbb{R}^{n+1}$ such that

$$W_k(\mathbf{n}^{-1}(x)) = \psi(x) \quad \text{for all } \ x \in \mathbb{S}^n? \tag{1.1}$$

Here we exclude the case $k = n$ as it corresponds to the well-known Minkowski problem. Our main result may be stated as follows.

THEOREM 1.1. *Assume $\psi \in C^{l,1}(\mathbb{S}^n)$ $(l \geq 1)$ is a positive function. Suppose ψ is invariant under an automorphic group G of $\mathbb{S}^n$ without fixed points*; *i.e., $\psi(g(x)) = \psi(x)$ for all $g \in G$ and $x \in \mathbb{S}^n$. Then there exists a $C^{l+2,\alpha}$ (for all $0 < \alpha < 1$) closed, strictly convex hypersurface Σ in $\mathbb{R}^{n+1}$ satisfying* (1.1).

In particular, we have

COROLLARY 1.2. *Assume $\psi \in C^{l,1}(\mathbb{S}^n)$ $(l \geq 1)$ is even*; *i.e., $\psi(-x) = \psi(x)$ for all $x \in \mathbb{S}^n$. Then there exists a $C^{l+2,\alpha}$ (for all $0 < \alpha < 1$) closed strictly convex hypersurface Σ in $\mathbb{R}^{n+1}$ satisfying* (1.1). *Moreover, after possibly a translation, Σ satisfies*

$$\mathbf{n}^{-1}(-x) = -\mathbf{n}^{-1}(x) \ \text{ for all } \ x \in \mathbb{S}^n.$$

When $n = 2$, the solution is unique up to translations.

There is an outstanding problem in finding necessary geometric obstructions to existence of solutions in the general case. If they were at hand, one would be able to drop the group invariance assumption in Theorem 1.1 as all necessary *a priori* estimates are established in this paper. We note that similar group invariance assumptions were previously used in other geometric problems such as, for example, conformal deformation of scalar curvatures (see Chang-Yang [5], Chang-Gursky-Yang [4] and references therein).

For the Minkowski problem, a necessary and sufficient condition of solvability is known, i.e.,

$$\int_{\mathbb{S}^n} \frac{x}{\psi(x)} = 0. \tag{1.2}$$

This is also a necessary condition for the Christoffel-Minkowski problems of prescribing elementary symmetric functions of principal radii on outer normals (e.g., [22]). One would expect (1.2) to be a necessary or sufficient condition for problem (1.1) as well. However, it turns out not to be the case. Indeed, we will prove

THEOREM 1.3. (a) *For every* $1 \leq k < n$ *and any nonzero real number* m, *there exists a parameter family of closed strictly convex hypersurfaces* (*all are small perturbations of the unit sphere*) *in* $\mathbb{R}^{n+1}$ *satisfying*

$$\int_{\mathbb{S}^n} \frac{x}{(W_k(\mathbf{n}^{-1}(x)))^m} \neq 0. \tag{1.3}$$

(b) *There exists a function* $f \in C^\infty(\mathbb{S}^n)$ *and a constant* $\delta > 0$ *such that for all* $t \in (0, \delta)$, *problem* (1.1) *has no solution for* $\psi := (1 + tf)^{-1}$ *while* (1.2) *is satisfied.*

Another important question is the uniqueness for prescribing Weingarten curvature problems. For the Minkowski problem the uniqueness is known, as a consequence of Brunn-Minkowski inequality. There is also the Alexandrov-Fenchel-Jensen theorem regarding uniqueness for the Christoffel-Minkowski problems. When $1 \leq k < n$, the uniqueness for problem (1.1) still seems open in general, except for $n = 2$. For constant Weingarten curvature hypersurfaces, the uniqueness is known; see [14] for the mean and Gauss curvatures, and Cheng-Yau [7] and Hartman [12] for the general cases. In this paper, we obtain the following local result.

THEOREM 1.4. *Let* $1 \leq k < n$. *There exists a constant* $\delta \in (0, 1)$ *such that for all functions* $\psi \in C^{2,\alpha}(\mathbb{S}^n)$ *with* $\|1 - \psi\|_{C^{1,1}(\mathbb{S}^n)} < \delta$, *problem* (1.1) *admits either no solution or a unique solution up to translation.*

While the Minkowski problem is connected with Monge-Ampère equations, the resulting differential equation for problem (1.1), when $1 \leq k < n$, is a Hessian quotient equation on $\mathbb{S}^n$. This type of fully nonlinear equations has been studied by Caffarelli, Nirenberg and Spruck [3], Krylov [16], Trudinger [25] and others for the Dirichlet problem on bounded domains in $\mathbb{R}^n$. In a different context, the Hessian quotient plays an important role in recent work of Huisken-Sinestrari [15].

This article is organized as follows. In Section 2 we reformulate equation (1.1) in terms of the supporting function on $\mathbb{S}^n$ and establish *a priori* estimates for admissible solutions. In Section 3 we consider an auxiliary equation and use a degree theory approach to prove Theorem 1.1. In Section 4 we first prove part (a) of Theorem 1.3 by constructing strictly convex hypersurfaces satisfying (1.3). We then prove part (b) of Theorem 1.3 and Theorem 1.4 with the aid of the *a priori* estimates established in Section 2.

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2. *A priori* estimates for the supporting function

Let Σ be a closed strictly convex hypersurface in $\mathbb{R}^{n+1}$ and $\mathbf{n}(y)$ the unit outer normal vector to Σ at $y \in \Sigma$. The Gauss map $\mathbf{n}$ then is a diffeomorphism from Σ onto $\mathbb{S}^n$. Let $Y = \mathbf{n}^{-1} : \mathbb{S}^n \to \Sigma \subset \mathbb{R}^{n+1}$ denote its inverse Gauss map. For convenience we may assume the origin of $\mathbb{R}^{n+1}$ is contained in the interior of Σ. The supporting function u of Σ is defined as

$$u(x) = x \cdot Y(x), \;\; x \in \mathbb{S}^n \subset \mathbb{R}^{n+1}.$$

Let $\nabla^2 u$ denote the Hessian of u and σ the standard metric of $\mathbb{S}^n$. The Hessian matrix

$$A = \nabla^2 u + u\sigma$$

contains all information of curvatures of Σ. It is well known (e.g., see [6]) that the principal radii $r_i = \frac{1}{\kappa_i}(1 \leq i \leq n)$ of curvature of Σ are the eigenvalues of $\nabla^2 u + u\sigma$.

Let $\mathcal{K}$ denote the collection of all $n \times n$ positive definite symmetric matrices and

$$\Gamma^+ = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n | \lambda_i > 0\}.$$

For each $A \in \mathcal{K}$, let $\lambda(A) = (\lambda_1, \ldots, \lambda_n)$ denote the eigenvalues of A. We define

$$F(A) \equiv (S_{n,n-k}(\lambda(A)))^{\frac{1}{k}}, \quad A \in \mathcal{K} \tag{2.1}$$

where $S_{n,l} = S_n/S_l$ for $0 \leq l \leq n$ (for convenience we take $S_0 = 1$). Note that

$$S_k(\kappa_1, \ldots, \kappa_n) = [S_{n,n-k}(\kappa_1^{-1}, \ldots, \kappa_n^{-1})]^{-1}.$$

Consequently, if Σ is a solution of problem (1.1), its supporting function u satisfies the following partial differential equation on $\mathbb{S}^n$:

$$F(\nabla^2 u + u\sigma) = (S_{n,n-k}(r[u]))^{\frac{1}{k}} = \varphi \quad \text{on } \mathbb{S}^n, \tag{2.2}$$

where $\varphi = \psi^{-\frac{1}{k}}$ and $r[u] = (r_1, \ldots, r_n)$ denotes the eigenvalues of $\nabla^2 u + u\sigma$.

We call a function $v \in C^2(\mathbb{S}^n)$ admissible if $\nabla^2 v + v\sigma$ is positive definite. If u is an admissible solution of (2.2), we can recover a strictly convex hypersurface Σ that solves (1.1) by

$$Y(x) = u(x)x + \nabla u, \quad x \in \mathbb{S}^n \tag{2.3}$$

(see also [6]) so that u is the supporting function of Σ. Therefore, solving problem (1.1) is equivalent to finding an admissible solution of (2.2).

We now proceed to derive *a priori* estimates for admissible solutions of (2.2). The main estimates we obtain in this section are for general cases. The *group invariance* assumption will be only needed when we make use of degree theory in the proof of Theorem 1.1.

When $k = n$, equation (2.2) is a Monge-Ampère type. In general, (2.2) is a Hessian quotient equation on $\mathbb{S}^n$. In the work of Cheng-Yau [6] and Pogorelov [22] on the Minkowski problem, a crucial step is to estimate the diameters of strictly convex hypersurfaces in terms of upper and lower bounds of their Gauss curvature. Cheng-Yau [6] obtained explicit bounds for the inner and outer radii of the convex body, which now is called Cheng-Yau lemma. However, similar estimates do not hold if the Gauss curvature is replaced by other Weingarten curvatures without further regularity assumptions on the curvature function (e.g., a convex perturbation of a long cylinder with caps at ends). Here we will first use the special structure of equation (2.2) to derive positive lower and upper bounds for principal curvatures of the convex hypersurface under a $C^{1,1}$ regularity assumption on its k^{th} Weingarten curvature. Then we apply Cheng-Yau's lemma to obtain C^0 bounds. A similar idea was used by Yau [26].

In the rest of this section, we assume $u \in C^4(\mathbb{S}^n)$ is an admissible solution of (2.2). We stress that the estimates to be derived below are independent of the group invariance assumption.

PROPOSITION 2.1. *There exist constants* $c_0, C_0 > 0$ *depending only on* n, $\inf \varphi$ *and* $\|\varphi\|_{C^{1,1}(\mathbb{S}^n)}$, *such that*

$$c_0\sigma \le \nabla^2 u + u\sigma \le C_0\sigma \quad \text{on } \mathbb{S}^n. \tag{2.4}$$

Proof. Write

$$H = \operatorname{trace}(\nabla^2 u + u\sigma) = nu + \Delta u.$$

We first estimate H from above. Assume the maximum value of H is achieved at a point $x_0 \in \mathbb{S}^n$ and choose an orthonormal local frame $e_1, \ldots, e_n$ about x_0 such that $u_{ij}(x_0)$ is diagonal. Denote

$$w_{ij} = u_{ij} + \delta_{ij} u,$$

and

$$F^{ij} = \frac{\partial F}{\partial w_{ij}}(\{w_{ij}\}).$$

For the standard metric on $\mathbb{S}^n$,

$$H_{ii} = \Delta w_{ii} - n w_{ii} + H. \tag{2.5}$$

By our assumption the matrix $\{w_{ij}\}$ is positive definite and hence so is $\{F^{ij}\}$. It follows that at x_0, since $\{H_{ij}\} \le 0$ and $\{F^{ij}\}$ is also diagonal,

$$0 \ge F^{ii} H_{ii} = F^{ii} \Delta(w_{ii}) - nF^{ii} w_{ii} + H \sum F^{ii}.$$

Since F is homogeneous of degree one,

$$F^{ii} w_{ii} = \varphi.$$

Next, applying the Laplace operator to equation (2.2), we obtain

$$F^{ii} \Delta(w_{ii}) \ge \Delta\varphi.$$

Here we have used the fact that F is concave. We also have the inequality (see [24])

$$\sum_i F^{ii} \ge 1. \tag{2.6}$$

Combining these inequalities, we see that

$$H \le C (\sum F^{ii})^{-1} \le C.$$

This proves the upper bound in (2.4).

On the other hand, by equation (2.2) and the Newton-Maclaurin inequality,

$$S_n(r[u]) = \varphi^k S_{n-k}(r[u]) \ge \varphi^k (S_n(r[u]))^{\frac{n-k}{n}},$$

and hence

$$S_n(r[u]) \ge c_1 \varphi^n$$

for some constant $c_1 > 0$. Since each of the eigenvalues of $\nabla^2 u + u\sigma$ is bounded from above by a uniform constant, this gives the lower bound in (2.4). $\square$

It follows from Proposition 2.1 that equation (2.2) is uniformly elliptic with respect to admissible solutions. Suppose u is the supporting function of a strictly convex hypersurface Σ. Then by Proposition 2.1, all principal curvatures of Σ are bounded above and below from zero. In particular, the Gauss curvature of Σ admits a positive lower bound and an upper bound as well, which depend only on the k^{th} curvature of Σ. It follows from Cheng-Yau's lemma that the interior of Σ contains a ball whose radius depends only on the k^{th} Weingarten curvature of Σ. After a translation, we may assume the Steiner point of Σ is the origin (that is, u is orthogonal to $\mathrm{Span}(x_1, \ldots, x_{n+1})$). Cheng-Yau's lemma therefore implies a bound for u from above. By Proposition 2.1 we then obtain bounds for the second derivatives, which in turn yields an *a priori* gradient bound for u as ∇u must vanish at some point on $\mathbb{S}^n$. We thus have:

Proposition 2.2. *Suppose u is the supporting function of a strictly convex hypersurface Σ with the origin as its Steiner point. Then there exists a constant $C_1 > 0$ depending only on n, $\inf \varphi$ and $\|\varphi\|_{C^{1,1}(\mathbb{S}^n)}$ such that*

$$\|u\|_{C^2(\mathbb{S}^n)} \leq C_1. \tag{2.7}$$

By the Evans-Krylov and Schauder theory (see, for example, [11]), we obtain $C^{2,\alpha}$ and higher order estimates from the C^2 estimates in Proposition 2.2 and the uniform ellipticity which is guaranteed by Proposition 2.2.

Theorem 2.3. *For each integer $l \geq 1$ and $0 < \alpha < 1$, there exists a constant K depending only on n, l, α, $\min \varphi$ and $\|\varphi\|_{C^{l,1}(\mathbb{S}^n)}$ such that*

$$\|u\|_{C^{l+2,\alpha}(\mathbb{S}^n)} \leq K \tag{2.8}$$

for all nonnegative admissible solutions u of (2.2).

We next list some simple facts about automorphic groups on $\mathbb{S}^n$ which we will use later. For the reader's convenience (also partially because we were not able to find an appropriate reference), we include brief proofs.

Proposition 2.4. *Let G be an automorphic group on $\mathbb{S}^n$. Then*

(i) *G has no fixed points if and only if there is no nontrivial invariant functions under G in the linear span K_1 of $x_1, \ldots, x_{n+1}$;*

(ii) *if G does not have fixed points on $\mathbb{S}^n$ then any invariant function under G is orthogonal to K_1;*

(iii) *no orbit of G is contained strictly in an open hemisphere provided that G does not have fixed points.*

Proof. (i) Suppose G has a fixed point $a \in \mathbb{S}^n$, i.e., $g(a) = a$ for all $g \in G$. Then

$$a \cdot g(x) = g(a) \cdot g(x) = a \cdot x \quad \text{for all} \;\; g \in G, x \in \mathbb{S}^n.$$

Thus the function $v \in \text{Span}(x_1, \ldots, x_{n+1})$ defined by $v(x) = a \cdot x$, $x \in \mathbb{S}^n$ is invariant under G. Conversely, suppose $c \in \mathbb{R}^{n+1}$, $c \neq 0$, such that

$$c \cdot g(x) = c \cdot x, \quad \text{for all} \;\; g \in G, x \in \mathbb{S}^n.$$

Let $a = c/\|c\| \in \mathbb{S}^n$. Then

$$a \cdot g(a) = \|a\|^2 = 1, \quad \text{for all} \;\; g \in G.$$

This implies $g(a) = a$ for all $g \in G$.

(ii) Suppose u is a function invariant under G. We decompose u as a series of spherical harmonic functions in K_j, $j = 1, 2, \ldots$, where K_j is the space of spherical harmonic functions of degree j. G acts invariantly on K_j. Thus each component (in K_j) of u is also invariant under G by the uniqueness of decomposition. In particular, if G has no nontrivial invariant function in K_1, the component of u in K_1 must be 0 and, therefore, u must be orthogonal to K_1.

(iii) Suppose there is a point $x^0 \in S^n$ such that its orbit $G(x^0)$ is contained in an open hemisphere. Let C be one of the smallest closed spherical caps containing $G(x^0)$. We may assume C is bounded by a horizontal plane (below the center of $\mathbb{S}^n$). We then claim the north pole p is a fixed point of G. This can be seen as follows. Suppose $g(p) \neq p$ for some $g \in G$. Then $G(x^0)$ is contained in the intersection of C and $g(C)$. Note that $g(C)$ is congruent but not identical to C since $\operatorname{dist}(g(p), g(C)) = \operatorname{dist}(p, C)$. It is easy to see now that $G(x^0)$ is contained in a strictly smaller cap. $\square$

Under the assumption that u is invariant under the automorphic group G which has no fixed points on $\mathbb{S}^n$, we may obtain bounds for u directly from Proposition 2.1 by some elementary methods. In the following, c_0, C_0 are as in Proposition 2.1.

LEMMA 2.5. *Let γ be a geodesic on $\mathbb{S}^n$ with the arc length parametrization and write $u(s) = u(\gamma(s))$. Then, for all $s \in [0, \frac{\pi}{2}]$,*

$$c_0(1 - \cos s) \leq u(s) - u(0)\cos s - u'(0)\sin s \leq C_0(1 - \cos s).$$

Proof. For $0 \leq s < \frac{\pi}{2}$, set $h = u/\cos s$. Then

$$(h' \cos^2 s)' = u'' \cos s - u\,(\cos s)'' = (u'' + u)\cos s.$$

By Proposition 2.1 we obtain

$$c_0 \cos s \leq (h' \cos^2 s)' \leq C_0 \cos s, \quad 0 \leq s < \frac{\pi}{2}.$$

It follows that

$$c_0 \sin s \leq h'(s)\cos^2 s - h'(0) \leq C_0 \sin s, \quad 0 \leq s < \frac{\pi}{2}.$$

Integrating this again we obtain the desired inequalities. $\square$

COROLLARY 2.6. *Suppose u is invariant under an automorphic group G which has no fixed points on $\mathbb{S}^n$. Then*

$$c_0 \leq u \leq C_0 \quad \text{on } \mathbb{S}^n.$$

Proof. Suppose u achieves its minimum and maximum at $x_0, y_0 \in \mathbb{S}^n$, respectively. Thus $\nabla u(x_0) = 0$ and $\nabla u(y_0) = 0$. Moreover, by Proposition 2.1, we have $u(x_0) \leq C_0$ and $u(y_0) \geq c_0$ since $\nabla^2 u(x_0)$ is positive semi-definite and $\nabla^2 u(y_0)$ is negative semi-definite. Let s_0 be the distance between x_0 and y_0

on $\mathbb{S}^n$. Since u is invariant under G and by Proposition 2.4, the orbit of any point on $\mathbb{S}^n$ is not contained in an open hemisphere, replacing x_0 by $g(x_0)$ for some $g \in G$ if necessary, we may assume $s_0 \leq \frac{\pi}{2}$. By Lemma 2.5 we have

$$u(x_0) - u(y_0)\cos s_0 \geq c_0(1 - \cos s_0)$$

and

$$u(y_0) - u(x_0)\cos s_0 \geq C_0(1 - \cos s_0)$$

from which follows that $c_0 \leq u(x_0) \leq u(y_0) \leq C_0$. □

Remark 2.7. In the general case, we derive from Lemma 2.5 that

$$2c_0 \leq \min_{\mathbb{S}^n} u + \max_{\mathbb{S}^n} u \leq 2C_0.$$

This gives an upper bound for u provided that $u \geq 0$ on $\mathbb{S}^n$.

Remark 2.8. If u is invariant under an automorphic group without fixed points, its Steiner point is the origin since u is orthogonal to the linear span of $x_1, \ldots, x_{n+1}$ by Proposition 2.4. Corollary 2.6 also follows from a result of Schneider [23].

3. Existence via degree theory

With the estimates derived in the last section, it would be natural to use continuity methods to obtain a solution for equation (2.2). Unfortunately, while the closeness follows from the estimates, the openness is difficult to establish due to the lack of geometric obstructions. Instead, we will approach the problem using degree theory, which is the only place we need the *group invariance* assumption.

We first consider some auxiliary equations of the form

$$F(\nabla^2 u + v\sigma) = \frac{u}{v}\varphi \quad \text{on } \mathbb{S}^n. \tag{3.1}$$

Let $v \in C^2(\mathbb{S}^n)$, $v > 0$ and set

$$\mathcal{A}[v] = \{u \in C^2(\mathbb{S}^n) : \nabla^2 u + v\sigma > 0\}.$$

Our goal here is to find a unique solution of (3.1) in $\mathcal{A}[v]$. We first need to derive *a priori* C^2 estimates for solutions of (3.1) in $\mathcal{A}[v]$.

LEMMA 3.1. *For any function $u \in C^2(\mathbb{S}^n)$,*

$$\|u\|^2_{C^1(\mathbb{S}^n)} \leq 4\|u\|_{C^0(\mathbb{S}^n)}\|u\|_{C^2(\mathbb{S}^n)}. \tag{3.2}$$

Proof. Letting $U = u - \min_{\mathbb{S}^n} u$, we have $U \geq 0$ on $\mathbb{S}^n$. Suppose $\max|\nabla U| = |\nabla U(p)| = e_1 U(p)$ at some point $p \in \mathbb{S}^n$ and with a unit vector field e_1. Let γ be the great circle on $\mathbb{S}^n$ which is tangential to e_1 at p, parametrized by the arc-length s with $\gamma(0) = p$. We write $U(s) = U(\gamma(s))$. This then reduces the problem to the one-dimensional case: we only have to show that

$$(U'(0))^2 \leq 2(\max U)(\max U'').$$

By Taylor's expansion,

$$0 \leq U(s) \leq U(0) + U'(0)s + (\max U'')\frac{s^2}{2} \quad \text{for all } s \in \mathbb{R}.$$

Note that we may assume $U(0) > 0$ (otherwise, $U'(0) = 0$ since U is nonnegative). Taking $s = -2\frac{U(0)}{U'(0)}$, we obtain

$$-U(0) + 2(\max U'')\frac{U^2(0)}{|U'(0)|^2} \geq 0.$$

Thus

$$|U'(0)|^2 \leq 2(\max U'')U(0).$$

This proves Lemma 3.1. □

In Proposition 3.2 and Corollary 3.3 below, let $u \in C^4(\mathbb{S}^n) \cap \mathcal{A}[v]$ be a solution of (3.1).

Proposition 3.2. *There exists a constant* $c_1 > 0$ *depending only on* $\min v$, $\min \varphi$, $\|v\|_{C^2(\mathbb{S}^n)}$ *and* $\|\varphi\|_{C^2(\mathbb{S}^n)}$ *such that*

$$\frac{1}{c_1} \leq u \leq c_1 \quad \textit{and} \quad |\nabla^2 u| \leq c_1 \quad \textit{on } \mathbb{S}^n. \tag{3.3}$$

Proof. At a point on $\mathbb{S}^n$ where u achieves its maximum value we have

$$v \geq F(\nabla^2 u + v\sigma) = \frac{u}{v}\varphi$$

since $\nabla^2 u$ is negative semi-definite. The maximum value of u is thus controlled by $\max v$ and $\min \varphi$. Similarly, at a point where the minimum value of u occurs,

$$v \leq F(\nabla^2 u + v\sigma) = \frac{u}{v}\varphi.$$

Therefore u is bounded from below by a positive constant depending only on $\min v$ and $\max \varphi$.

Since $\nabla^2 u + v\sigma$ is positive definite, to estimate $|\nabla^2 u|$ we only need to derive an upper bound for Δu. Assume the maximum value of Δu is achieved at a point x_0 and choose an orthonormal local frame about x_0 such that $u_{ij}(x_0)$

is diagonal. We have at x_0, since $\{(\Delta u)_{ij}\} \leq 0$ and F^{ij} is also diagonal,

$$\begin{aligned}
0 &\geq F^{ii}(\Delta u)_{ii} \\
&= F^{ii}(\Delta(u_{ii}) + 2\Delta u - 2nu_{ii}) \\
&= F^{ii}\Delta(u_{ii} + v) - 2nF^{ii}(u_{ii} + v) + (2\Delta u + 2nv - \Delta v)\sum F^{ii} \\
&\geq \Delta(\frac{u}{v}\varphi) - 2n\frac{u}{v}\varphi + (2\Delta u + 2nv - \Delta v)\sum F^{ii} \\
&\geq \frac{\Delta u}{v}\varphi + 2\nabla u \nabla(\frac{\varphi}{v}) + u\Delta(\frac{\varphi}{v}) + (\Delta u - \Delta v + nv)\sum F^{ii}
\end{aligned}$$

by concavity of F. Now (3.3) follows from Lemma 3.1. □

Corollary 3.3. *Let $u \in C^4(\mathbb{S}^n) \cap \mathcal{A}[v]$ be a solution of* (3.1). *There exists a constant $c_2 > 0$ such that*

$$\frac{1}{c_2}\sigma \leq \nabla^2 u + v\sigma \leq c_2\sigma \quad \textit{on } \mathbb{S}^n. \tag{3.4}$$

Proof. We only have to derive the lower bound. Note that $F(\nabla^2 u + v\sigma)$ is bounded below from zero and the eigenvalues of $(\nabla^2 u + v\sigma)$ are bounded from above by Proposition 3.2. As in the proof of Proposition 2.1, this implies a positive lower bound for the product of the eigenvalues of $\nabla^2 u + v\sigma$, which in turn implies a positive lower bound for all eigenvalues of $\nabla^2 u + v\sigma$. □

Theorem 3.4. *Assume v, $\varphi \in C^4(\mathbb{S}^n)$, $v > 0$, $\varphi > 0$ on $\mathbb{S}^n$. Then there exists a unique solution $u \in C^{5,\alpha}(\mathbb{S}^n) \cap \mathcal{A}[v]$ of* (3.1), *where $0 < \alpha < 1$. Furthermore,*

$$\|u\|_{C^{5,\alpha}(\mathbb{S}^n)} \leq C$$

for some constant C depending only on n, α, $\min v$, $\min \varphi$, $\|v\|_{C^4(\mathbb{S}^n)}$ and $\|\varphi\|_{C^4(\mathbb{S}^n)}$.

Proof. We will show that for $0 \leq t \leq 1$ the equation

$$F(\nabla^2 u + v^t\sigma) = \frac{u}{v^t}\varphi^t \quad \text{on } \mathbb{S}^n \tag{3.5}$$

has a unique smooth solution in $\mathcal{A}[v^t]$ with appropriate *a priori* estimates, where $v^t = tv + (1-t)$ and $\varphi^t = t\varphi + (1-t)$. By a standard comparison argument (see, for example, the proof of Theorem 17.1 in [11]) we see that the solution of (3.5) in $\mathcal{A}[v^t]$, if it exists, is unique for each $t \in [0,1]$. Set

$$\mathcal{T} = \{s \in [0,1] : (3.5) \text{ is solvable in } C^{5,\alpha}(\mathbb{S}^n) \cap \mathcal{A}[v^t] \text{ for all } t \in [0,s]\}.$$

For $t \in \mathcal{T}$, let $u^t \in C^{5,\alpha}(\mathbb{S}^n) \cap \mathcal{A}[v^t]$ be the solution of (3.5). We note that $0 \in \mathcal{T}$ and $u^0 = 1$. By Proposition 3.2 and Corollary 3.3, equation (3.5) is uniformly elliptic at u^t and

$$\|u^t\|_{C^2(\mathbb{S}^n)} \leq C, \quad \text{independent of } t.$$

By the Evans-Krylov theorem and the classical Schauder estimates we obtain

$$\|u^t\|_{C^{5,\alpha}(\mathbb{S}^n)} \leq C, \quad \text{independent of } t. \tag{3.6}$$

This implies that $\mathcal{T}$ is closed in $[0,1]$.

Next, let $\mathcal{L}_t$ be the linearized operator of $u \mapsto F(\nabla^2 u + v^t\sigma) - \frac{u}{v^t}\varphi^t$ at u^t; that is,

$$\mathcal{L}_t\rho = F^{ij}(\nabla^2 u^t + v^t\sigma)\rho_{ij} - \frac{\varphi^t}{v^t}\rho, \quad \text{for } \rho \in C^2(\mathbb{S}^n).$$

By the maximum principle,

$$\mathcal{L}_t : C^{5,\alpha}(\mathbb{S}^n) \to C^{3,\alpha}(\mathbb{S}^n)$$

is one-to-one. Thus, $\mathcal{L}_t$ is invertible if and only if its index, $\operatorname{ind}(\mathcal{L}_t)$, is equal to zero. By the Fredholm alternative and the regularity theory, $\mathcal{L}_0 : H^2 \to L^2$ is invertible. Then the regularity result shows $\mathcal{L}_0 : C^{k+2,\alpha} \to C^{k,\alpha}$ is invertible. The linear operator $\mathcal{L}_0$, which is given by

$$\mathcal{L}_0\rho = \Delta\rho - \rho, \quad \text{for } \rho \in C^2(\mathbb{S}^n),$$

is invertible from $C^{5,\alpha}(\mathbb{S}^n)$ onto $C^{3,\alpha}(\mathbb{S}^n)$ and hence $\operatorname{ind}(\mathcal{L}_0) = 0$. Consequently, $\operatorname{ind}(\mathcal{L}_t) = 0$ and $\mathcal{L}_t$ is invertible for all $t \in \mathcal{T}$, as the index is homotopy invariant. By the implicit function theorem, $\mathcal{T}$ is open in $[0,1]$ and thus $\mathcal{T} = [0,1]$. $\square$

We are now in a position to solve (2.2) and prove Theorem 1.1.

Proof of Theorem 1.1. Let G be an automorphic group without fixed points on $\mathbb{S}^n$ and consider the Banach space

$$\mathcal{B} = \{w \in C^4(\mathbb{S}^n) : w(g(x)) = w(x) \text{ for all } g \in G \text{ and } x \in \mathbb{S}^n\}.$$

Assume $\varphi \in \mathcal{B}$, $\varphi > 0$ on $\mathbb{S}^n$. For $w \in \mathcal{B}$ and $0 \leq t \leq 1$, we write $v = e^w$ and denote by u^t the unique solution of (3.5) in $\mathcal{A}[v^t]$ with $\varphi^t = t\varphi + (1-t)$, as in the proof of Theorem 3.4. From the uniqueness we see that $\log u^t \in \mathcal{B}$. By Theorem 3.4 the map

$$T_t : \mathcal{B} \to \mathcal{B}, \;\; w \mapsto \log u^t \tag{3.7}$$

is compact. Moreover, according to Corollary 2.6 and Theorem 2.3 there exists no solution of

$$w - T_t w = 0 \tag{3.8}$$

on the boundary of

$$\mathcal{B}_R = \{w \in \mathcal{B} : \; \|w\|_{C^5} < R\}$$

in $\mathcal{B}$ when R is sufficiently large. Consequently, the degree $\deg(I - T_t, \mathcal{B}_R, 0)$ is well defined and independent of t. When $t = 0$, if w satisfies $T_0 w = w$, then

e^w is the supporting function of a unit sphere (see, e.g., Cheng-Yau [7] and Hartman [12]). As $w \in \mathcal{B}$, by Proposition 2.4, $w = 0$. That is, $T_0 w = w$ has a unique solution $w = 0$. So the fixed point of T_0 is isolated and

$$\deg(I - T_0) = \deg(I - T_0, B_\delta(0), 0)$$

for any small $\delta > 0$. Let $\tilde{T}_0 v = e^{T_0(\log v)}$. Now,

$$\deg(I - T_0, B_\delta(0), 0) = \deg(I - \tilde{T}_0, B_\delta(1), 0).$$

Next, let us look at the derivative $(I - \tilde{T}_0)'$. As $\tilde{T}_0 v$ satisfies

$$F(\nabla^2(\tilde{T}_0 v) + v\sigma) = \frac{\tilde{T}_0 v}{v},$$

the linearized operator at v of $\tilde{T}_0$ satisfies

$$F^{ij}(\nabla^2(\tilde{T}_0 v) + v\sigma)((\tilde{T}'_{0,v}\rho)_{ij} + \rho\delta_{ij}) = \frac{\tilde{T}'_{0,v}\rho}{v} - \frac{\rho\tilde{T}_0 v}{v^2}.$$

At $v = 1$, we have $\tilde{T}_0 v = 1$, $(\tilde{T}_0 v)_{ij} + v\delta_{ij} = \delta_{ij}$ and $F^{ij} = \delta_{ij}$. We see that

$$\Delta(\tilde{T}'_{0,1}\rho) + n\rho = \tilde{T}'_{0,1}\rho - \rho.$$

That is,

$$(\Delta - 1)(\tilde{T}'_{0,1}\rho) = -(n+1)\rho.$$

This yields

$$\tilde{T}'_{0,1} = (n+1)(1 - \Delta)^{-1}.$$

If $(I - \tilde{T}'_{0,1})\rho = 0$, ρ satisfies $\Delta\rho = -n\rho$. That is, $\rho \in \mathrm{Span}\{x_1, \ldots, x_{n+1}\}$. On the other hand, as $\rho \in \mathcal{B}$, ρ is orthogonal to the span of $x_1, \ldots, x_{n+1}$ by Proposition 2.4. We must have $\rho = 0$. Therefore, $I - \tilde{T}'_{0,1}$ is injective in $\mathcal{B}$. By the standard degree theory (see [20]),

$$\deg(I - \tilde{T}_0, B_\delta(1), 0) = (-1)^\beta,$$

where β is the number of eigenvalues of $\tilde{T}'_{0,1}$ which are greater than one. Let us calculate β. If $\gamma > 1$ and

$$\tilde{T}'_{0,1}\rho = \gamma\rho,$$

we then have

$$\Delta\rho = (1 - \frac{n+1}{\gamma})\rho.$$

Note that $1 - \frac{n+1}{\gamma} > -n$. We must have $1 - \frac{n+1}{\gamma} = 0$, that is $\gamma = n+1$, as 0 is the only eigenvalue of Δ greater than $-n$. We conclude that $\beta = 1$ and

$$\deg(I - T_0) = -1.$$

Thus, (3.8) has a solution for each $0 \le t \le 1$; the one corresponding to $t = 1$ is then an admissible solution of (2.2).

This completes the existence part of Theorem 1.1. Finally, the regularity in Theorem 1.1 follows from Theorem 2.3. □

4. Proof of Theorems 1.3 and 1.4

We start with some calculation. Let $v \in C^\infty(\mathbb{S}^n)$ and consider $u_t = 1 + tv$. For $t > 0$ small, u_t is a supporting function of some smooth strictly convex hypersurface, and

$$S_n(\nabla^2 u_t + u_t\sigma) = \sum_{i=1}^{n} \frac{n!}{i!(n-i)!} S_i\, t^i.$$

Here, and in the rest of this section, we write $S_i = S_i(\nabla^2 v + v\sigma)$. It follows that

$$\int_{\mathbb{S}^n} x_j S_i d\sigma = 0, \quad \text{for all } 1 \le j \le n+1, \ 1 \le i \le n \tag{4.1}$$

since

$$\int_{\mathbb{S}^n} x_j S_n(\nabla^2 u_t + u_t\sigma) d\sigma = 0, \quad \text{for all } 1 \le j \le n+1$$

for all $t > 0$ sufficiently small.

For a fixed k $(1 \le k < n)$, by straightforward calculation we see that

$$S_{n,k}(\nabla^2 u_t + u_t\sigma) = 1 + a_1 t + a_2 t^2 + a_3 t^3 + O(t^4) \tag{4.2}$$

where

$$\begin{aligned} a_1 &= (n-k)S_1, \\ a_2 &= \frac{n-k}{2}[(n+k-1)S_2 - 2kS_1^2], \\ a_3 &= \frac{k(n-k)}{2}[2kS_1^3 - (n+2k-2)S_1S_2] + aS_3, \end{aligned} \tag{4.3}$$

for some constant a depending only on k and n. From this we compute, for any $m \in \mathbb{R}$, the coefficients of the Taylor expansion

$$[S_{n,k}(\nabla^2 u_t + u_t\sigma)]^m = 1 + b_1 t + b_2 t^2 + b_3 t^3 + O(t^4) \tag{4.4}$$

to obtain

$$\begin{aligned} b_1 &= m(n-k)S_1, \\ b_2 &= \frac{m(n-k)}{2}[(n+k-1)S_2 + (m(n-k) - n - k)S_1^2] \end{aligned} \tag{4.5}$$

and, when $m = \frac{n+k}{n-k}$,

$$b_3 = \frac{nk(n+k)}{6}(3S_1S_2 - 2S_1^3) + bS_3 \tag{4.6}$$

where b is a constant. We are now in a position to prove the following result which implies part (a) of Theorem 1.3.

PROPOSITION 4.1. *For every integer k, $1 \le k < n$, and any $m \in \mathbb{R}$, $m \neq 0$, there exists $v \in C^\infty(\mathbb{S}^n)$ such that the function $u_t = 1 + tv$ satisfies*

$$\int_{\mathbb{S}^n} x\,[S_{n,k}(\nabla^2 u_t + u_t\sigma)]^m d\sigma \neq 0 \tag{4.7}$$

for all $t > 0$ sufficiently small.

Proof. We use the spherical coordinates on $\mathbb{S}^n$

$$\begin{aligned} x_1 &= \cos\theta_1, \\ x_j &= \sin\theta_1 \cdots \sin\theta_{j-1}\cos\theta_j, \quad 1 < j \le n, \\ x_{n+1} &= \sin\theta_1 \cdots \sin\theta_{n-1}\sin\theta_n, \\ d\sigma_{\mathbf{S^n}} &= \sin^{n-1}\theta_1 \sin^{n-2}\theta_2 \cdots \sin\theta_{n-1} d\theta_1 \cdots d\theta_n, \end{aligned} \tag{4.8}$$

where $0 \le \theta_j \le \pi,\ \ 1 \le j \le n-1;\ \ 0 \le \theta_n \le 2\pi$. Let

$$g(x) = \eta(\cos^2\theta_1)\cdots\eta(\cos^2\theta_{n-1})(\cos 2\theta_n + \sin 3\theta_n) \tag{4.9}$$

where η is a smooth cut-off function satisfying $0 \le \eta \le 1$; $\eta(t) = 1$ if $|t| < \frac{1}{2}$ and $\eta(t) = 0$ if $|t| > \frac{3}{4}$. One finds that

$$\int_{\mathbb{S}^n} x_j g(x) = 0, \quad \text{for all } \ 1 \le j \le n+1, \quad \int_{\mathbb{S}^n} x_{n+1} g^2(x) \neq 0. \tag{4.10}$$

Note that the linear elliptic operator L defined by $L(v) = S_1(\nabla^2 v + v\sigma)$ is self-adjoint with kernel $K_1 = \mathrm{Span}(x_1, \ldots, x_{n+1})$. As g is orthogonal to the kernel of L, there exists $v \in C^\infty(\mathbb{S}^n)$ satisfying the equation

$$S_1(\nabla^2 v + v\sigma) = g \ \text{ on } \mathbb{S}^n. \tag{4.11}$$

By (4.10), we see from (4.1)–(4.5) that $u_t = 1 + tv$ satisfies (4.7) for all $t > 0$ sufficiently small, provided that $m \neq \frac{n+k}{n-k}$.

Turning to the case $m = \frac{n+k}{n-k}$, we take $v = x_1^l$ where $l > 1$ is an odd integer. For $t > 0$ sufficiently small, the function $u_t = 1 + tv$ then is the supporting function of a surface of revolution. For convenience we write $\theta = \theta_1$ and, therefore, $x_1 = \cos\theta$, $0 \le \theta \le \pi$. Using a formula in [10] with some simplification, we obtain

$$\begin{aligned} S_1 &= \tfrac{1-l}{n}(n\cos^2\theta - l\sin^2\theta)\cos^{l-2}\theta, \\ S_2 &= \tfrac{(1-l)^2}{n}(n\cos^2\theta - 2l\sin^2\theta)\cos^{2l-2}\theta. \end{aligned}$$

It follows that

$$3S_1S_2 - 2S_1^3 = \frac{(1-l)^3}{n^3}(n^3\cos^6\theta - 3n^2 l\cos^4\theta\sin^2\theta + 2l^3\sin^6\theta)\cos^{3l-6}\theta.$$

We calculate

$$\begin{aligned}\int_{\mathbb{S}^n} x_1(3S_1S_2 - 2S_1^3)d\sigma &= c_1\int_0^\pi (3S_1S_2 - 2S_1^3)\sin^{n-1}\theta\cos\theta d\theta\\ &= c_2\int_0^\pi (n^3\cos^6\theta - 3n^2 l\cos^4\theta\sin^2\theta + 2l^3\sin^6\theta)\cos^{3l-5}\theta\sin^{n-1}\theta d\theta\\ &= n^2c_2\int_0^\pi (n\cos^{3l+1}\theta\sin^{n-1}\theta - 3l\cos^{3l-1}\theta\sin^{n+1}\theta)d\theta\\ &\quad +2l^3c_2\int_0^\pi \cos^{3l-5}\theta\sin^{n+5}\theta d\theta\\ &= 2l^3c_2\int_0^\pi \cos^{3l-5}\theta\sin^{n+5}\theta d\theta < 0\end{aligned}$$

since

$$\int_0^\pi (n\cos^{3l+1}\theta\sin^{n-1}\theta - 3l\cos^{3l-1}\theta\sin^{n+1}\theta)d\theta = \cos^{3l}\theta\sin^n\theta\Big|_0^\pi = 0$$

and $l > 1$ is an odd integer, where c_1 is a positive constant (equal to the volume of $\mathbb{S}^{n-1}$) and $c_2 = \frac{c_1(1-l)^3}{n^3} < 0$. From (4.1)–(4.6) it follows that u_t satisfies (4.7) for all $t > 0$ sufficiently small. □

Remark 4.2. In the case $m = \frac{n+k}{n-k}$, u_t constructed in the proof of Proposition 4.1 is the support function of a surface of revolution. A similar construction can also be done for $m \neq \frac{n+k}{n-k}$. It follows from the proof of Proposition 4.1 that the linearized operator L_{u_t} of $S^m_{n,k}$ at u_t is not self-adjoint with respect to the standard metric on $\mathbb{S}^n$. We complement this with the following observation. Suppose w is a positive function defined on $\mathbb{S}^n$ such that

$$\int_{\mathbb{S}^n} x_j w(x)[S_{n,k}(\nabla^2 u + u\sigma)]^m = 0 \tag{4.12}$$

for all $u \in C^\infty(\mathbb{S}^n)$ with $\{\nabla^2 u + u\sigma\} > 0$, where $1 \le j \le n$, $1 \le k < n$ and $m \in \mathbb{R}$, $m \neq 0$ (all are fixed). Then, for any $v \in C^2(\mathbb{S}^n)$, as the function $u_t = 1 + tv$ satisfies (4.12) for all $t > 0$ sufficiently small, we have

$$\int_{\mathbb{S}^n} x_j w(x) S_1(\nabla^2 v + v\sigma) = 0$$

by (4.4) and (4.5). This implies $\Delta(x_j w) + n x_j w = 0$ on $\mathbb{S}^n$. Since the kernel of $\Delta + n$ is the linear span of $x_1, \ldots, x_n$, we see that $w \equiv$ const.

With the aid of the *a priori* estimates established in Section 2, we have the following nonexistence result which proves part (b) of Theorem 1.3.

PROPOSITION 4.3. *Let $1 \le k < n$ and $g \in C^\infty(\mathbb{S}^n)$ satisfy* (4.10). *Then there exists a constant $\delta > 0$ such that for all $0 < t < \delta$ the equation*

$$S_{n,k}(\nabla^2 u + u\sigma) = 1 + t(n-k)g \tag{4.13}$$

does not have any admissible solution.

Proof. Suppose that there exists a sequence of positive numbers $t_l \to 0$ $(l \to \infty)$ and admissible functions $v_l \in S^{2,\alpha}(\mathbb{S}^n)$ satisfying

$$S_{n,k}(\nabla^2 v_l + v_l\sigma) = 1 + t_l(n-k)g, \quad l = 1, 2, \ldots.$$

Let $u_l = 1 + t_l v$ where $v \in C^\infty(\mathbb{S}^n)$ is a solution of (4.11). We may assume that $v \perp K_1$ and $v_l \perp K_1$ for all $l \ge 1$, where K_1 is the span of $x_1, \ldots, x_{n+1}$. Let

$$w_l = \frac{u_l - v_l}{t_l^2}.$$

By (4.2) we see that w_l satisfies

$$L_l w_l = a_2 + O(t_l) \tag{4.14}$$

where a_2 is as in (4.3) and $L_l = a_l^{ij}\nabla_{ij} + c_l$ is an elliptic operator with coefficients

$$a_l^{ij} = \int_0^1 \frac{\partial S_{n,k}}{\partial \lambda_{ij}}(\nabla^2 u_l^s + u_l^s\sigma)ds$$

and $c_l = \sum a_l^{ii}$, where $u_l^s = su_l + (1-s)v_l$. By Proposition 2.1 and Theorem 2.3, L_l is uniformly elliptic and $L_l \to L_0 = \Delta + n$ in the Banach space of bounded linear operators from $C^3(\mathbb{S}^n)$ to $C^1(\mathbb{S}^n)$. We claim that there exists a constant C such that

$$\|w_l\|_{C^0(\mathbb{S}^n)} \le C \quad \text{for all } \ l \ge 1. \tag{4.15}$$

Suppose this is not true. After passing to a subsequence, we may assume

$$\|w_l\|_{C^0(\mathbb{S}^n)} \ge l \quad \text{for all } \ l$$

and let $\tilde{w}_l = w_l/\|w_l\|_{C^0(\mathbb{S}^n)}$. Then

$$L_l \tilde{w}_l = \frac{a_2 + O(t_l)}{\|w_l\|_{C^0(\mathbb{S}^n)}}.$$

We obtain by the standard elliptic estimates,

$$\|\tilde{w}_l\|_{C^3(\mathbb{S}^n)} \le C \quad \text{independent of } l.$$

It follows that there exists a subsequence $\{\tilde{w}_{l_j}\}$ that converges in the $C^{2,\alpha}(\mathbb{S}^n)$ norm to a function $\tilde{w} \in C^{2,\alpha}(\mathbb{S}^n)$. Now,

$$\Delta\tilde{w} + n\tilde{w} = 0 \quad \text{on } \ \mathbb{S}^n,$$

and therefore $\tilde{w} \in K_1$. Since $\tilde{w}_l \perp K_1$ for all $l \geq 1$, we have $\tilde{w} \perp K_1$ and hence $\tilde{w} = 0$. This is a contradiction as $\|\tilde{w}_l\|_{C^0(\mathbb{S}^n)} = 1$ for all $l \geq 1$. Therefore, the claim is true.

Again by the elliptic estimates we obtain from (4.15),

$$\|w_l\|_{C^3(\mathbb{S}^n)} \leq C \quad \text{independent of } l.$$

This implies that a subsequence of $\{w_l\}$ converges to some $w_0 \in C^{2,\alpha}(\mathbb{S}^n)$ (in the C^2 norm). By (4.14),

$$\Delta w_0 + n w_0 = a_2 \quad \text{on } \mathbb{S}^n.$$

Thus $a_2 \perp K_1$, which is a contradiction since

$$\int_{\mathbb{S}^n} x_{n+1} a_2 = -k(n-k) \int_{\mathbb{S}^n} x_{n+1} g^2 \neq 0$$

by (4.3) and (4.10). ∎

Proof of Theorem 1.4. This follows similar lines to those in the proof of Proposition 4.3. Suppose the theorem is not true; then there is a sequence of $g_j \in C^{1,1}(\mathbb{S}^n)$ with $\|1 - g_j\|_{C^{1,1}(\mathbb{S}^n)} = c_j \leq \frac{1}{j}$ such that there are two solutions $u_1^j \perp K_1$ and $u_2^j \perp K_1$ with $u_1^j - u_2^j \neq 0$, for each $j = 1, 2, \ldots$. The function $w^j = \frac{u_1^j - u_2^j}{\|u_1^j - u_2^j\|_{C^0}}$ satisfies an elliptic equation $L_j(w^j) = 0$, for each j with L_j uniformly elliptic. By the uniqueness theorem for equation (2.2) with $\varphi = 1$, we have $L_j \to L_0 = \Delta + n$. Passing to a subsequence, w^j converges to a function w in C^2 with $\Delta w + nw = 0$. This is a contradiction to the facts $\|w\|_{C^0} = 1$ and $w \perp K_1$. ∎

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