SECOND ORDER ESTIMATES FOR HESSIAN TYPE FULLY NONLINEAR ELLIPTIC EQUATIONS ON RIEMANNIAN MANIFOLDS

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ABSTRACT. We derive *a priori* estimates for second order derivatives of solutions to a wide calss of fully nonlinear elliptic equations on Riemannian manifolds. The equations we consider naturally appear in geometric problems and other applications such as optimal transportation. There are some fundamental assumptions in the literature to ensure the equations to be elliptic and that one can apply Evans-Krylov theorem once the C^2 estimates are derived. However, in previous work one needed extra assumptions which are more technical in nature to overcome various difficulties. In this paper we are able to remove most of these technical assumptions. Indeed, we derive the estimates under conditions which are almost optimal, and prove existence results for the Dirichlet problem which are new even for bounded domains in Euclidean space. Moreover, our methods can be applied to other types of nonlinear elliptic and parabolic equations, including those on complex manifolds.

1. INTRODUCTION

In the study of fully nonlinear elliptic or parabolic equations, *a priori* C^2 estimates are crucial to the question of existence and regularity of solutions. Such estimates are also important in applications. In this paper we are concerned with second derivative estimates for solutions of the Dirichlet problem for equations of the form

$$f(\lambda(\nabla^2 u + A[u])) = \psi(x, u, \nabla u) \tag{1.1}$$

on a Riemannian manifold (M^n, g) of dimension $n \geq 2$ with smooth boundary ∂M, with boundary condition

$$u = \varphi \quad \text{on } \partial M, \tag{1.2}$$

where f is a symmetric smooth function of n variables, $\nabla^2 u$ is the Hessian of u, $A[u] = A(x, u, \nabla u)$ a $(0, 2)$ tensor which may depend on u and ∇u, and

$$\lambda(\nabla^2 u + A[u]) = (\lambda_1, \ldots, \lambda_n)$$

denotes the eigenvalues of $\nabla^2 u + A[u]$ with respect to the metric g.

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Following the pioneer work of Caffarelli, Nirenberg and Spruck [4] we assume f to be defined in an open, convex, symmetric cone $\Gamma \subset \mathbb{R}^n$ with vertex at the origin,

$$\Gamma_n \equiv \{\lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0\} \subseteq \Gamma \neq \mathbb{R}^n$$

and to satisfy the standard structure conditions in the literature:

$$f_i \equiv \frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma,\ 1 \leq i \leq n, \tag{1.3}$$

$$f \text{ is a concave function in } \Gamma, \tag{1.4}$$

and

$$\sup_{\partial \Gamma} f \equiv \sup_{\lambda_0 \in \partial \Gamma} \limsup_{\lambda \to \lambda_0} f(\lambda) \leq 0. \tag{1.5}$$

We shall call a function $u \in C^2(\bar{M})$ *admissible* if $\lambda(\nabla^2 u + A[u]) \in \Gamma$. By (1.3), equation (1.1) is elliptic for admissible solutions.

When $A = 0$ or $A(x)$, the Dirichlet problem (1.1)-(1.2) in $\mathbb{R}^n$ for was first studied by Ivochkina [30] and Caffarelli, Nirenberg and Spruck [4], followed by work in [37], [51], [12], [46], [48], and [9], etc. Li [37] and Urbas [49] studied equation (1.1) with $A = g$ on closed Riemannian manifolds; see also [14] where the Dirichlet problem was treated for $A = \kappa u g$ (κ is constant).

A critical issue to solve the Dirichlet problem for equation (1.1), is to derive *a priori* C^2 estimates for admissible solutions. By conditions (1.3) and (1.5), equation (1.1) becomes uniformly elliptic once C^2 estimates are established, and one therefore obtains global $C^{2,\alpha}$ estimates using Evans-Krylov theorem which crucially relies on the concavity condition (1.4). From this point of view, conditions (1.3)-(1.5) are fundamental to the classical solvability of equation (1.1).

In this paper we shall primarily focus on deriving *a priori* estimates for second order derivatives. In order to state our main results let us introduce some notations; see also [15].

For $\sigma > 0$ let $\Gamma^\sigma = \{\lambda \in \Gamma : f(\lambda) > \sigma\}$ which we assume to be nonempty. By assumptions (1.3) and (1.4) the boundary of Γ^σ, $\partial \Gamma^\sigma = \{\lambda \in \Gamma : f(\lambda) = \sigma\}$ is a smooth, convex and noncompact hypersurface in $\mathbb{R}^n$. For $\lambda \in \Gamma$ we use $T_\lambda = T_\lambda \partial \Gamma^{f(\lambda)}$ to denote the tangent plane at λ to the level surface $\partial \Gamma^{f(\lambda)}$.

The following new condition is essential to our work in this paper:

$$\partial \Gamma^\sigma \cap T_\lambda \partial \Gamma^{f(\lambda)} \text{ is nonempty and compact, } \forall\, \sigma > 0,\ \lambda \in \Gamma^\sigma. \tag{1.6}$$

Throughout the paper we assume $\bar{M} := M \cup \partial M$ is compact and $A[u]$ is smooth on $\bar{M}$ for $u \in C^\infty(\bar{M})$, $\psi \in C^\infty(T^*\bar{M} \times \mathbb{R})$ (for convenience we shall write $\psi = \psi(x, z, p)$ for $(x, p) \in T^*\bar{M}$ and $z \in \mathbb{R}$ though), $\psi > 0$, $\varphi \in C^\infty(\partial M)$. Note that for fixed $x \in \bar{M}$, $z \in \mathbb{R}$ and $p \in T^*_x M$,

$$A(x, z, p) : T^*_x M \times T^*_x M \to \mathbb{R}$$

is a symmetric bilinear map. We shall use the notation

$$A^{\xi\eta}(x,\cdot,\cdot) := A(x,\cdot,\cdot)(\xi,\eta), \quad \xi,\eta \in T_x^*M$$

and, for a function $v \in C^2(M)$, $A[v] := A(x,v,\nabla v)$, $A^{\xi\eta}[v] := A^{\xi\eta}(x,v,\nabla v)$.

Theorem 1.1. *Assume, in addition to* (1.3)-(1.6)*, that*

$$-\psi(x,z,p) \text{ and } A^{\xi\xi}(x,z,p) \text{ are concave in } p, \tag{1.7}$$

$$-\psi_z,\ A_z^{\xi\xi} \geq 0, \quad \forall\, \xi \in T_xM. \tag{1.8}$$

and that there exists an admissible subsolution $\underline{u} \in C^2(\bar{M})$ satisfying

$$\begin{cases} f(\lambda(\nabla^2\underline{u} + A[\underline{u}])) \geq \psi(x,\underline{u},\nabla\underline{u}) & \text{in } \bar{M}, \\ \underline{u} = \varphi & \text{on } \partial M. \end{cases} \tag{1.9}$$

Let $u \in C^4(\bar{M})$ be an admissible solution of equation (1.1) *with $u \geq \underline{u}$ on $\bar{M}$. Then*

$$\max_{\bar{M}} |\nabla^2 u| \leq C_2\big(1 + \max_{\partial M} |\nabla^2 u|\big) \tag{1.10}$$

where $C_2 > 0$ depends on $|u|_{C^1(\bar{M})}$ and $|\underline{u}|_{C^2(\bar{M})}$. In particular, if M is closed, i.e. $\partial M = \emptyset$, then

$$|\nabla^2 u| \leq C_2 \text{ on } M. \tag{1.11}$$

Suppose that u also satisfies the boundary condition (1.2) *and that*

$$\sum f_i(\lambda)\lambda_i \geq 0, \ \forall\, \lambda \in \Gamma. \tag{1.12}$$

Then there exists $C_3 > 0$ depending on $|u|_{C^1(\bar{M})}$, $|\underline{u}|_{C^2(\bar{M})}$ and $|\varphi|_{C^4(\partial M)}$ such that

$$\max_{\partial M} |\nabla^2 u| \leq C_3. \tag{1.13}$$

The assumption (1.6) excludes linear elliptic equations but is satisfied by a very general class of functions f. In particular, Theorem 1.1 applies to $f = \sigma_k^{\frac{1}{k}}$, $k \geq 2$ and $f = (\sigma_k/\sigma_l)^{\frac{1}{k-l}}$, $1 \leq l < k \leq n$ in the Garding cone

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, \ \forall\ 1 \leq j \leq k\},$$

where σ_k is the k-th elementary symmetric function; see [15]. It is also straightforward to verify that the function $f = \log P_k$ satisfies assumptions (1.3)-(1.6) where

$$P_k(\lambda) := \prod_{i_1 < \cdots < i_k} (\lambda_{i_1} + \cdots + \lambda_{i_k}), \quad 1 \leq k \leq n$$

defined in the cone

$$\mathcal{P}_k := \{\lambda \in \mathbb{R}^n : \lambda_{i_1} + \cdots + \lambda_{i_k} > 0\}.$$

When both A and ψ are independent of u and ∇u, Theorem 1.1 was proved by the first author [15] under the weaker assumption that (1.6) holds for all $\lambda \in \partial\Gamma^\sigma$, which improves previous results due to Caffarelli, Nirenberg and Spruck [4], Li [37], Trudinger [46], Urbas [49] and the first author [12], etc. Clearly, the two conditions are equivalent if f is homogeneous or more generally $f(t\lambda) = h(t)f(\lambda)$, $\forall\, t > 0$ in Γ for some positive function h.

Some of the major difficulties in deriving the estimates (1.10) and (1.13) are caused by the presence of curvature and lack of good globally defined functions on general Riemannian manifolds, and by the arbitrary geometry of boundary. As in [15], we make crucial use of the subsolution in both estimates to overcome the difficulties. (We shall also use key ideas from [4], [37], [46], [49], etc. all of which contain significant contributions to the subject.) However, the proofs in the current paper are far more delicate than those in [15], especially for the boundary estimates. The core of our approach is the following inequality

Theorem 1.2. *Assume that* (1.3), (1.4) *and* (1.6) *hold. Let K be a compact subset of Γ and $\sup_{\partial\Gamma} f < a \leq b < \sup_{\Gamma} f$. There exist positive constants $\theta = \theta(K, [a, b])$ and $R = R(K, [a, b])$ such that for any $\lambda \in \Gamma^{[a,b]} = \overline{\Gamma^a} \setminus \Gamma^b$, when $|\lambda| \geq R$,*

$$\sum f_i(\lambda)(\mu_i - \lambda_i) \geq \theta + \theta \sum f_i(\lambda) + f(\mu) - f(\lambda), \;\; \forall\, \mu \in K. \tag{1.14}$$

Perhaps the most important contribution of this paper is the new idea introduced in the proof of Theorem 1.2. It will be further developed in our forthcoming work (e.g. [16]). Note that by the concavity of f we always have

$$\sum f_i(\lambda)(\mu_i - \lambda_i) \geq f(\mu) - f(\lambda), \;\; \forall\, \mu, \lambda \in \Gamma.$$

It may also be worthwhile to point out that in Theorem 1.2 the function f is *not* assumed to be *strictly* concave.

In general, without assumption (1.9) the Dirichlet problem for equation (1.1) is not always solvable either if A or ψ is dependent on u and ∇u, or if there is no geometric restrictions to ∂M being imposed.

The recent work of Guan, Ren and Wang [24] shows that the convexity assumption on ψ in ∇u can not be dropped in general from Theorem 1.1. On the other hand, they derived the second order estimates for $f = \sqrt{\sigma_2}$ without the assumption; such estimates are also known to hold for the Monge-Ampère equation ($f = \sigma_n^{1/n}$). It seems an interesting open question whether it is still true for $f = \sigma_k^{1/k}$, $3 \leq k < n$; see [24].

Our motivation to study equation (1.1) comes in part from its natural connection to geometric problems, and the problem of optimal transportation which turns out to be very closely related and to have interesting applications to differential geometry. The potential function of an optimal mass transport satisfies a Monge-Ampère type equation of form (1.1) where $f = \sigma_n^{1/n}$ and A is determined by the cost function. In [39] Ma, Trudinger and Wang introduced the following condition to establish interior regularity for optimal transports: there exists $c_0 > 0$ such that

$$A^{\xi\xi}_{p_k p_l}(x, z, p)\eta_k\eta_l \leq -c_0|\xi|^2|\eta|^2, \;\forall\, \xi, \eta \in T_x M, \; \xi \perp \eta, \tag{1.15}$$

now often referred as the MTW condition. In this paper we derive the following interior estimate which also extends Theorem 2.1 in [47].

Theorem 1.3. *In addition to* (1.3)-(1.5), (1.7) *and* (1.15), *assume that*

$$\lim_{t\to\infty} f(t\mathbf{1}) = +\infty \tag{1.16}$$

where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$. *Let* $\underline{u} \in C^2(\bar{B}_r)$ *be an admissible function and* $u \in C^4(B_r)$ *an admissible solution of* (1.1) *in a geodesic ball* $B_r \subset M$ *of radius* $r > 0$. *Then*

$$\sup_{B_{\frac{r}{2}}} |\nabla^2 u| \leq C_4 \tag{1.17}$$

where C_4 *depends on* r^{-1}, $|u|_{C^1(B_r)}$, $|\underline{u}|_{C^2(B_r)}$, *and other known data.*

Remark 1.4. The function $\underline{u} \in C^2(\bar{B}_r)$ does not have to be a subsolution.

Equations of form (1.1) appear in many interesting geometric problems. These include the Minkowski problem ([40], [41], [42], [7]) and its generalizations proposed by Alexandrov [1] and Chern [8] (see also [17]), the Christoffel-Minkowski problem (cf. eg. [23]), and the Alexandrov problem of prescribed curvature measure (cf, e.g. [22], [20]), which are associated with equation (1.1) on $\mathbb{S}^n$ for $f = \sigma_n^{\frac{1}{n}}$, $(\sigma_n/\sigma_l)^{\frac{1}{n-l}}$ or $\sigma_k^{\frac{1}{k}}$ and $A = ug$. Another classical example is the Darboux equation

$$\det(\nabla^2 u + g) = K(x)(-2u - |\nabla u|^2) \det g \tag{1.18}$$

on a positively curved surface (M^2, g), which appears in isometric embedding, e.g. the Weyl problem ([40], [21], [29]). In [27] Guan and Wang studied the Monge-Ampère type equation on $\mathbb{S}^n$

$$\det\Big(\nabla^2 u - \frac{u^2 + |\nabla u|^2}{2u} g\Big) = K(x)\Big(\frac{u^2 + |\nabla u|^2}{2u}\Big)^n \det g \tag{1.19}$$

which arises from reflector antenna designs in engineering, while the Schouten tensor equation

$$\sigma_k\Big(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + S_g\Big) = \psi(x) e^{-2ku} \tag{1.20}$$

(where S_g is the Schouten tensor of (M^n, g)) introduced by Viaclovsky [50] is connected with a natural fully nonlinear version of the Yamabe problem and has very interesting applications in conformal geometry; see for instance [5], [10], [26], [28], [35], [43] and references therein.

Interior second order estimates were obtained in [27] for equation (1.19) and in [25] for equation (1.20); see also [6] for a simplified proof. The reader is also referred to [42] for the classical Pogorelov estimate for the Monge-Ampère equation, and to [9], [44] for its generalizations to the Hessian equation ($f = \sigma_k^{1/k}$ in (1.1)) and equations of prescribed curvature, respectively.

The rest of this paper is organized as follows. In Section 2 we present a brief review of some elementary formulas and a consequence of Theorem 1.2; see Proposition 2.2. In Section 3 we give a proof of Theorem 1.2. In Section 4 we prove the maximum principle (1.10) in Theorem 1.1 and derive the interior estimate (1.17), while the boundary estimate (1.13) is established in Section 5. Section 6 is devoted to the gradient estimates. Finally in Section 7 we state some existence results for the Dirichlet problem (1.1)-(1.2) based on the gradient and second order estimates in Theorem 1.1 and Section 6, which can be proved using the standard method of continuity and degree theory.

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2. Preliminaries

Throughout the paper let ∇ denote the Levi-Civita connection of (M^n, g). The curvature tensor is defined by

$$R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z.$$

Under a local frame $e_1, \ldots, e_n$ on M^n we denote $g_{ij} = g(e_i, e_j)$, $\{g^{ij}\} = \{g_{ij}\}^{-1}$, while the Christoffel symbols Γ^k_{ij} and curvature coefficients are given respectively by $\nabla_{e_i} e_j = \Gamma^k_{ij} e_k$ and

$$R_{ijkl} = g(R(e_k, e_l)e_j, e_i), \quad R^i_{jkl} = g^{im} R_{mjkl}.$$

We shall use the notation $\nabla_i = \nabla_{e_i}$, $\nabla_{ij} = \nabla_i \nabla_j - \Gamma^k_{ij} \nabla_k$, etc.

For a differentiable function v defined on M^n, we usually identify ∇v with its gradient, and use $\nabla^2 v$ to denote its Hessian which is locally given by $\nabla_{ij} v = \nabla_i(\nabla_j v) - \Gamma^k_{ij} \nabla_k v$. We recall that $\nabla_{ij} v = \nabla_{ji} v$ and

$$\nabla_{ijk} v - \nabla_{jik} v = R^l_{kij} \nabla_l v, \tag{2.1}$$

$$\begin{aligned} \nabla_{ijkl} v - \nabla_{klij} v = {} & R^m_{ljk} \nabla_{im} v + \nabla_i R^m_{ljk} \nabla_m v + R^m_{lik} \nabla_{jm} v \\ & + R^m_{jik} \nabla_{lm} v + R^m_{jil} \nabla_{km} v + \nabla_k R^m_{jil} \nabla_m v. \end{aligned} \tag{2.2}$$

Let $u \in C^4(\bar{M})$ be an admissible solution of equation (1.1). For simplicity we shall denote $U := \nabla^2 u + A(x, u, \nabla u)$ and, under a local frame $e_1, \ldots, e_n$,

$$U_{ij} \equiv U(e_i, e_j) = \nabla_{ij} u + A^{ij}(x, u, \nabla u),$$

$$\begin{aligned} \nabla_k U_{ij} \equiv {} & \nabla U(e_i, e_j, e_k) = \nabla_{kij} u + \nabla_k A^{ij}(x, u, \nabla u) \\ \equiv {} & \nabla_{kij} u + \nabla'_k A^{ij}(x, u, \nabla u) + A^{ij}_z(x, u, \nabla u) \nabla_k u \\ & + A^{ij}_{p_l}(x, u, \nabla u) \nabla_{kl} u \end{aligned} \tag{2.3}$$

where $A^{ij} = A^{e_i e_j}$ and $\nabla'_k A^{ij}$ denotes the *partial* covariant derivative of A when viewed as depending on $x \in M$ only, while the meanings of A^{ij}_z and $A^{ij}_{p_l}$, etc are obvious. Similarly we can calculate $\nabla_{kl} U_{ij} = \nabla_k \nabla_l U_{ij} - \Gamma^m_{kl} \nabla_m U_{ij}$, etc.

In the rest of this paper we shall always use orthonormal local frames. Write equation (1.1) locally in the form

$$F(U) = \psi(x, u, \nabla u) \tag{2.4}$$

where we identify $U \equiv \{U_{ij}\}$ and F is the function defined by

$$F(B) = f(\lambda(B))$$

for a symmetric matrix B with $\lambda(B) \in \Gamma$; throughout the paper we shall use the notation

$$F^{ij} = \frac{\partial F}{\partial B_{ij}}(U), \quad F^{ij,kl} = \frac{\partial^2 F}{\partial B_{ij} \partial B_{kl}}(U).$$

The matrix $\{F^{ij}\}$ has eigenvalues $f_1, \ldots, f_n$ and is positive definite by assumption (1.3), while (1.4) implies that F is a concave function (see [4]). Moreover, when U_{ij} is diagonal so is $\{F^{ij}\}$, and the following identities hold

$$F^{ij} U_{ij} = \sum f_i \lambda_i, \quad F^{ij} U_{ik} U_{kj} = \sum f_i \lambda_i^2$$

where $\lambda(U) = (\lambda_1, \ldots, \lambda_n)$.

The following result can be found in [15] (Proposition 2.19 and Corollary 2.21).

Proposition 2.1 ([15])**.** *Suppose f satisfies* (1.3). *There is $c_0 > 0$ and an index r such that*

$$\sum_{l<n} F^{ij} U_{il} U_{lj} \geq c_0 \sum_{i \neq r} f_i \lambda_i^2. \tag{2.5}$$

If in addition f satisfies (1.4) *and* (1.12) *then for any index r,*

$$\sum f_i |\lambda_i| \leq \epsilon \sum_{i \neq r} f_i \lambda_i^2 + C\Big(1 + \frac{1}{\epsilon} \sum f_i\Big). \tag{2.6}$$

Let $\mathcal{L}$ be the linear operator locally defined by

$$\mathcal{L} v := F^{ij} \nabla_{ij} v + (F^{ij} A^{ij}_{p_k} - \psi_{p_k}) \nabla_k v, \quad v \in C^2(M) \tag{2.7}$$

where $A^{ij}_{p_k} \equiv A^{ij}_{p_k}[u] \equiv A^{ij}_{p_k}(x, u, \nabla u)$, $\psi_{p_k} \equiv \psi_{p_k}[u] \equiv \psi_{p_k}(x, u, \nabla u)$. The following is a consequence of Theorem 1.2.

Proposition 2.2. *There exist uniform positive constants R, θ such that*

$$\mathcal{L}(\underline{u} - u) \geq \theta \sum F^{ii} + \theta \quad \textit{whenever } |\lambda(U)| \geq R. \tag{2.8}$$

Proof. For any $x \in M$, choose a smooth orthonormal local frame $e_1, \ldots, e_n$ about x such that $\{U_{ij}(x)\}$ is diagonal. From Lemma 6.2 in [4] and Theorem 1.2 we see that there exist positive constants R, θ such that when $|\lambda(U)| \geq R$,

$$F^{ii}(\underline{U}_{ii} - U_{ii}) \geq F(\underline{U}) - F(U) + \theta \sum F^{ii} + \theta \tag{2.9}$$

where $\underline{U} = \{\underline{U}_{ij}\} = \{\nabla_{ij}\underline{u} + A^{ij}[\underline{u}]\}$. By (1.7) and (1.8) we have

$$\begin{aligned} A^{ii}_{p_k} \nabla_k(\underline{u} - u) &\geq A^{ii}(x, u, \nabla \underline{u}) - A^{ii}(x, u, \nabla u) \\ &\geq A^{ii}(x, \underline{u}, \nabla \underline{u}) - A^{ii}(x, u, \nabla u) \end{aligned}$$

and

$$\begin{aligned} -\psi_{p_k} \nabla_k(\underline{u} - u) &\geq -\psi(x, u, \nabla \underline{u}) + \psi(x, u, \nabla u) \\ &\geq -\psi(x, \underline{u}, \nabla \underline{u}) + \psi(x, u, \nabla u). \end{aligned}$$

Thus (2.8) follows from (2.9). □

3. Proof of Theorem 1.2

In this section we prove Theorem 1.2. Throughout the section we assume (1.3), (1.4) and (1.6) hold. To give the reader some idea about the proof, we shall first prove the following simpler version of Theorem 1.2.

Theorem 3.1. *Let $\mu \in \Gamma$ and $\sup_{\partial\Gamma} f < \sigma < \sup_{\Gamma} f$. There exist positive constants θ, R such that for any $\lambda \in \partial\Gamma^{\sigma}$, when $|\lambda| \geq R$,*

$$\sum f_i(\lambda)(\mu_i - \lambda_i) \geq \theta_{\mu} + f(\mu) - f(\lambda). \tag{3.1}$$

Recall that for $\sigma \in (\sup_{\partial\Gamma} f, \sup_{\Gamma} f)$ we have $\Gamma^{\sigma} := \{f > \sigma\} \neq \emptyset$ and by assumptions (1.3) and (1.4), $\partial\Gamma^{\sigma}$ is a smooth convex noncompact complete hypersurface contained in Γ. Let $\mu, \lambda \in \partial\Gamma^{\sigma}$. By the convexity of $\partial\Gamma^{\sigma}$, the open segment

$$(\mu, \lambda) \equiv \{t\mu + (1-t)\lambda : 0 < t < 1\}$$

is either completely contained in or does not intersect with $\partial\Gamma^{\sigma}$. Therefore,

$$f(t\mu + (1-t)\lambda) - \sigma > 0, \ \ \forall\, 0 < t < 1$$

by condition (1.3), unless $(\mu, \lambda) \subset \partial\Gamma^{\sigma}$.

For $\lambda \in \Gamma$ we shall use T_{λ} and ν_{λ} to denote the tangent plane and unit normal vector at λ to $\partial\Gamma^{f(\lambda)}$, respectively. Note that $\nu_{\lambda} = Df(\lambda)/|Df(\lambda)|$.

Proof of Theorem 3.1. We divide the proof into two cases: (**a**) $f(\mu) \geq \sigma$ and (**b**) $f(\mu) < \sigma$. For the first case we use ideas from [15] where the case $f(\mu) = \sigma$ is proved. For case (**b**) we introduce some new ideas which will be used in the proof of Theorem 1.2.

Case (**a**) $f(\mu) \geq \sigma$. By assumption (1.6) there is $R_0 > 0$ such that $T_\mu \cap \partial\Gamma^\sigma$ is contained in the ball $B_{R_0}(0)$. By the convexity of $\partial\Gamma^\sigma$, there exists $\delta > 0$ such that for any $\lambda \in \partial\Gamma^\sigma$ with $|\lambda| \geq 2R_0$, the open segment from μ and λ

$$(\mu, \lambda) \equiv \{t\mu + (1-t)\lambda : 0 < t < 1\}$$

intersects the level surface $\partial\Gamma^{f(\mu)}$ at a unique point η with $|\eta - \mu| > 2\delta$. Since $\nu_\mu \cdot (\eta - \mu)/|\eta - \mu|$ has a uniform positive lower bound (independent of $\lambda \in \partial\Gamma^\sigma$ with $|\lambda| \geq 2R_0$) and the level hypersurface $\partial\Gamma^{f(\mu)}$ is smooth, the point $\mu + \delta|\eta - \mu|^{-1}(\eta - \mu)$ has a uniform positive distance from $\partial\Gamma^{f(\mu)}$, and therefore

$$f(\mu + \delta|\eta - \mu|^{-1}(\eta - \mu)) \geq f(\mu) + \theta_\mu$$

for some uniform constant $\theta > 0$. By the concavity of f,

$$\begin{aligned} \sum f_i(\lambda)(\mu_i - \lambda_i) &\geq \sum f_i(\lambda)(\eta_i - \lambda_i) + \sum f_i(\eta)(\mu_i - \eta_i) \\ &\geq f(\eta) - f(\lambda) + \sup_{0 \leq t \leq 1} f(t\mu + (1-t)\eta) - f(\mu) \\ &\geq f(\mu + \delta|\eta - \mu|^{-1}(\eta - \mu)) - f(\lambda) \\ &\geq \theta + f(\mu) - f(\lambda). \end{aligned} \tag{3.2}$$

We now assume $f(\mu) < \sigma$. Assumption (1.6) implies (see Lemma 3.3 below) that there is $\mu_0 \in \partial\Gamma^\sigma$ such that $\mathrm{dist}(\mu_0, T_\mu) = \mathrm{dist}(\partial\Gamma^\sigma, T_\mu)$ and therefore T_{μ_0} is parallel to T_μ. By assumption (1.6) again there is $R_0 > 0$ such that $T_{\mu_0} \cap \partial\Gamma^{f(\mu)}$ is contained in the ball $B_{R_0}(0)$. By the convexity of $\partial\Gamma^\sigma$ we have for any $\lambda \in \partial\Gamma^\sigma$ with $|\lambda| \geq 2R_0$,

$$\nu_{\mu_0} \cdot \nu_\lambda \leq \max_{\zeta \in \partial\Gamma^\sigma,\, |\zeta| = 2R_0} \nu_{\mu_0} \cdot \nu_\zeta \equiv \beta < 1. \tag{3.3}$$

To see this one can consider the Gauss map $G : \partial\Gamma^\sigma \to \mathbb{S}^n$; the geodesic on $\mathbb{S}^n$ from $G(\mu_0) = \nu_{\mu_0}$ to $G(\lambda) = \nu_\lambda$ must intersect the image of $\partial\Gamma^\sigma \cap \partial B_{R_0}(0)$.

Since $\partial\Gamma^{f(\mu)}$ is smooth, there exists $\delta > 0$ such that

$$\mathrm{dist}(\partial B_\delta^\beta(\mu), \partial\Gamma^{f(\mu)}) > 0$$

where $\partial B_\delta^\beta(\mu) = \{\zeta \in \partial B_\delta(\mu) : \ \nu_\mu \cdot (\zeta - \mu)/\delta \geq \sqrt{1 - \beta^2}\}$. Therefore,

$$\theta \equiv \inf_{\zeta \in \partial B_\delta^\beta(\mu)} f(\zeta) - f(\mu) > 0. \tag{3.4}$$

For any $\lambda \in \partial\Gamma^\sigma$ with $|\lambda| \geq 2R_0$, let P be the 2-plane through μ spanned by ν_μ and ν_λ (translated to μ), and L the line through μ and parallel to $P \cap T_\lambda$. From (3.3) we see that L intersects $\partial B_\delta^\beta(\mu)$ at a unique point ζ. By the concavity of f,

$$\sum f_i(\lambda)(\mu_i - \lambda_i) = \sum f_i(\lambda)(\zeta_i - \lambda_i) \geq f(\zeta) - f(\lambda) \geq \theta + f(\mu) - f(\lambda). \tag{3.5}$$

This completes the proof for case (**b**). □

A careful examination of the above proof for case (**b**) shows that it actually works for case (**a**) too (with some obvious modifications). In what follows we shall extend the ideas to give a proof of Theorem 1.2.

We need the following lemma from [15].

Lemma 3.2 ([15]). *Let $\mu \in \partial\Gamma^\sigma$. Then for any $t > 0$, the part of $\partial\Gamma^\sigma$*

$$\Sigma_t = \Sigma_t(\mu) := \{\lambda \in \partial\Gamma^\sigma : (\lambda - \mu)\cdot \nu_\mu \le t\}$$

is a convex cap with smooth compact boundary on the plane $t\nu_\mu + T_\mu\partial\Gamma^\sigma$.

Lemma 3.3. *The set*

$$L^\sigma(\mu) = \Big\{\zeta \in \partial\Gamma^\sigma : \nu_\mu\cdot(\zeta - \mu) = \min_{\lambda\in\partial\Gamma^\sigma} \nu_\mu\cdot(\lambda-\mu)\Big\}$$

is nonempty and compact. Moreover, $\nu_\lambda = \nu_\mu$ for all $\lambda \in L^\sigma(\mu)$.

Proof. If $f(\mu) < \sigma$ this follows from Lemma 3.2 while if $f(\mu) > \sigma$ we see this directly from assumption (1.6). □

Lemma 3.4. *If $R' \ge R > \max\{|\lambda| : \lambda \in L^\sigma(\mu)\}$ then $\beta_{R'}(\mu,\sigma) \le \beta_R(\mu,\sigma) < 1$.*

Proof. We first note that $\nu_\mu = \nu_{\mu^\sigma}$ and therefore $\beta_R(\mu,\sigma) = \beta_R(\mu^\sigma,\sigma)$ for $\mu^\sigma \in L^\sigma(\mu)$. For any $\lambda \in \partial\Gamma^\sigma \setminus L^\sigma(\mu)$ we have $0 < \nu_\mu\cdot\nu_\lambda < 1$ since $\nu_\mu, \nu_\lambda \in \Gamma_n$ and $\nu_\mu \ne \nu_\lambda$ by the convexity of $\partial\Gamma^\sigma$. Therefore $\beta_R < 1$ for $R > \max\{|\mu^\sigma| : \mu^\sigma \in L^\sigma(\mu)\}$ by compactness (of $\partial\Gamma^\sigma \cap \partial B_R$). To see that β_R is non-increasing in R we consider the Gauss map $G : \partial\Gamma^\sigma \to \mathbb{S}^n$. Suppose $\lambda \in \partial\Gamma^\sigma$ and $|\lambda| > R > \max\{|\mu^\sigma| : \mu^\sigma \in L^\sigma(\mu)\}$. Then the geodesic on $\mathbb{S}^n$ from $G(L^\sigma(\mu))$ to $G(\lambda)$ must intersect the image of $\partial\Gamma^\sigma \cap \partial B_R(0)$, that is, it contains a point $G(\lambda_R)$ for some $\lambda_R \in \partial\Gamma^\sigma \cap \partial B_R(0)$. Consequently,

$$\nu_\mu\cdot\nu_\lambda \le \nu_\mu\cdot\nu_{\lambda_R} \le \beta_R.$$

So Lemma 3.4 holds. □

Lemma 3.5. *Let K be a compact subset of Γ and $\sup_{\partial\Gamma} f < a \le b < \sup_\Gamma f$. Then*

$$R_0(K,[a,b]) \equiv \sup_{\mu\in K}\sup_{a\le\sigma\le b}\max_{\lambda\in L^\sigma(\mu)} |\lambda| < \infty \tag{3.6}$$

and

$$0 < \beta_{R'}(K,[a,b]) \le \beta_R(K,[a,b]) < 1, \;\; \forall\, R' \ge R > R_0(K,[a,b]), \tag{3.7}$$

where

$$\beta_R(K,[a,b]) \equiv \sup_{\mu\in K}\beta_R(\mu,[a,b]) \equiv \sup_{\mu\in K}\sup_{a\le\sigma\le b}\beta_R(\mu,\sigma).$$

Proof. We first show that for any $\mu \in \Gamma$,

$$R_0(\mu,[a,b]) \equiv \sup_{a\le\sigma\le b}\max_{\lambda\in L^\sigma(\mu)}|\lambda| < \infty. \tag{3.8}$$

We consider two cases: (**i**) $f(\mu) \le a$ and (**ii**) $f(\mu) \ge b$; the case $a \le f(\mu) \le b$ follows obviously. In case (**i**) the set

$$L^{[a,b]}(\mu) \equiv \bigcup_{a \le \sigma \le b} L^\sigma(\mu)$$

is contained in the region bounded by $\Sigma_t(\mu)$ and T_{μ^b}, where $t = \nu_\mu \cdot (\mu^b - \mu)$ for any $\mu^b \in L^b(\mu)$ which is nonempty by Lemma 3.3, while in case (**ii**) it is in the (bounded) subregion of Γ^b cut by T_μ. So in both cases (3.8) holds.

Next, we show by contradiction that

$$\beta_R(\mu, [a,b]) \equiv \sup_{a \le \sigma \le b} \beta_R(\mu, \sigma) < 1, \quad \forall\, R > R_0(\mu, [a,b]). \tag{3.9}$$

Suppose for each integer $k \ge 1$ there exists $\lambda_k \in \Gamma$ with $|\lambda_k| = R$ and $a \le f(\lambda_k) \le b$ such that

$$\nu_\mu \cdot \nu_{\lambda_k} \ge 1 - \frac{1}{k}.$$

Then by compactness we obtain a point $\lambda \in \Gamma$ with $a \le f(\lambda) \le b$, $\nu_\mu \cdot \nu_\lambda = 1$ and $|\lambda| = R > R_0(\mu, f(\lambda))$. This contradicts Lemma 3.4 from which we also see that $\beta_R(\mu, [a,b])$ is nonincreasing in R for $R > R_0(\mu, [a,b])$.

Suppose now that for each integer $k \ge 1$ there exists $\mu_k \in K$, $\sigma_k \in [a,b]$ and $\lambda_k \in L^{\sigma_k}(\mu_k)$ with $|\lambda_k| \ge k$. By the compactness of K we may assume $\mu_k \to \mu \in K$ as $k \to \infty$. Since $\nu_{\mu_k} \cdot \nu_{\lambda_k} = 1$, and $\partial\Gamma^{\sigma_k}$ is smooth we have

$$\lim_{k\to\infty} \nu_\mu \cdot \nu_{\lambda_k} = 1 + \lim_{k\to\infty} (\nu_\mu - \nu_{\mu_k}) \cdot \nu_{\lambda_k} = 1.$$

This contradicts (3.9) and the monotonicity of $\beta_R(\mu, [a,b])$, proving (3.6). The proof of (3.7) is now obvious. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. The proof below is a straightforward modification of the second part of the proof of Theorem 3.1; we include it here for completeness and the reader's convenience.

Let $\epsilon = \frac{1}{2}\text{dist}(K, \partial\Gamma)$ and

$$K^\epsilon = \{\mu - t\mathbf{1} : \mu \in K,\ 0 \le t \le \epsilon\}$$

where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$. Then K^ϵ is a compact subset of Γ. Let $R = 2R_0(K^\epsilon, [a,b])$, and $\beta = \beta_R(K^\epsilon, [a,b])$. By Lemma 3.5, $0 < \beta < 1$ and therefore, since f is smooth and $Df \ne 0$ everywhere, by the compactness of K^ϵ there exists $\delta > 0$ depending on β and bounds on the (principal) curvatures of $\partial\Gamma^{f(\mu)}$ for all $\mu \in K^\epsilon$, such that

$$\inf_{\mu \in K^\epsilon} \text{dist}(\partial B^\beta_\delta(\mu), \partial\Gamma^{f(\mu)}) > 0$$

where $\partial B_\delta^\beta(\mu)$ denotes the spherical cap as in the proof of Theorem 3.1. Consequently,

$$\theta \equiv \frac{1}{2} \inf_{\mu \in K^\epsilon} \inf_{\zeta \in \partial B_\delta^\beta(\mu)} (f(\zeta) - f(\mu)) > 0. \tag{3.10}$$

Next, for any $\lambda \in \Gamma^{[a,b]} = \overline{\Gamma^a} \setminus \Gamma^b$ with $|\lambda| \geq R$ and $\mu \in K^\epsilon$, as in the proof of Theorem 3.1 let P be the 2-plane through μ spanned by ν_μ and ν_λ (translated to μ), and L the line through μ and parallel to $P \cap T_\lambda$. Since $\nu_\mu \cdot \nu_\lambda \leq \beta < 1$ by Lemma 3.5, L intersects $\partial B_\delta^\beta(\mu)$ at a unique point ζ, and therefore by the concavity of f,

$$\sum f_i(\lambda)(\mu_i - \lambda_i) = \sum f_i(\lambda)(\zeta_i - \lambda_i) \geq f(\zeta) - f(\lambda) \geq 2\theta + f(\mu) - f(\lambda). \tag{3.11}$$

Finally, since K is compact, by the continuity of f there exists $\epsilon_0 \in (0, \epsilon]$ such that

$$f(\mu - t\mathbf{1}) \geq f(\mu) - \theta, \quad \forall\, \mu \in K,\ 0 \leq t \leq \epsilon_0.$$

Combining this with (3.11) gives (1.14) for $\theta(K, [a,b]) = \min\{\epsilon_0, \theta\}$. □

4. Interior and global estimates for second derivatives

In this section we derive the estimates for second derivatives in Theorem 1.1 and Theorem 1.3.

Let

$$W(x) = \max_{\xi \in T_x M, |\xi|=1} (\nabla_{\xi\xi} u + A^{\xi\xi}(x, u, \nabla u)) e^\phi$$

where ϕ is a function to be determined, and assume

$$W(x_0) = \max_{\bar{M}} W$$

for an interior point $x_0 \in M$. Choose a smooth orthonormal local frame $e_1, \ldots, e_n$ about x_0 such that $\nabla_{e_i} e_j = 0$ and U_{ij} is diagonal at x_0. We assume

$$U_{11}(x_0) \geq \cdots \geq U_{nn}(x_0)$$

so $W(x_0) = U_{11}(x_0) e^{\phi(x_0)}$.

The function $\log U_{11} + \phi$ attains its maximum at x_0 where

$$\frac{\nabla_i U_{11}}{U_{11}} + \nabla_i \phi = 0, \tag{4.1}$$

$$\frac{\nabla_{ii} U_{11}}{U_{11}} - \Big(\frac{\nabla_i U_{11}}{U_{11}}\Big)^2 + \nabla_{ii}\phi \leq 0. \tag{4.2}$$

By simple calculation,

$$(\nabla_i U_{11})^2 \leq (\nabla_1 U_{1i})^2 + C U_{11}^2, \tag{4.3}$$

$$\nabla_{ii} U_{11} \geq \nabla_{11} U_{ii} + \nabla_{ii} A^{11} - \nabla_{11} A^{ii} - C U_{11}. \tag{4.4}$$

Differentiating equation (2.4) twice, we obtain at x_0,

$$F^{ii}\nabla_k U_{ii} = \nabla_k\psi + \psi_u\nabla_k u + \psi_{p_j}\nabla_{kj}u, \tag{4.5}$$

and, by (4.1),

$$\begin{aligned} F^{ii}\nabla_{11}U_{ii} + & F^{ij,kl}\nabla_1 U_{ij}\nabla_1 U_{kl} \\ & \geq \psi_{p_j}\nabla_{11j}u + \psi_{p_l p_k}\nabla_{1k}u\nabla_{1l}u - CU_{11} \\ & \geq \psi_{p_j}\nabla_j U_{11} + \psi_{p_1p_1}U_{11}^2 - CU_{11} \\ & = -U_{11}\psi_{p_j}\nabla_j\phi + \psi_{p_1p_1}U_{11}^2 - CU_{11}. \end{aligned} \tag{4.6}$$

Next,

$$\begin{aligned} F^{ii}(\nabla_{ii}A^{11} - \nabla_{11}A^{ii}) \geq & F^{ii}(A^{11}_{p_j}\nabla_{iij}u - A^{ii}_{p_j}\nabla_{11j}u) \\ & + F^{ii}(A^{11}_{p_ip_i}U_{ii}^2 - A^{ii}_{p_1p_1}U_{11}^2) - CU_{11}\sum F^{ii} \\ \geq & U_{11}F^{ii}A^{ii}_{p_j}\nabla_j\phi - CU_{11}\sum F^{ii} - CU_{11} \\ & - C\sum_{i\geq 2} F^{ii}U_{ii}^2 - U_{11}^2\sum_{i\geq 2} F^{ii}A^{ii}_{p_1p_1}. \end{aligned} \tag{4.7}$$

By (4.2), (4.4), (4.6) and (4.7) we obtain

$$\mathcal{L}\phi \leq U_{11}\sum_{i\geq 2} F^{ii}A^{ii}_{p_1p_1} - \psi_{p_1p_1}U_{11} + E + \frac{C}{U_{11}}F^{ii}U_{ii}^2 + C\sum F^{ii} + C \tag{4.8}$$

where

$$E = \frac{1}{U_{11}^2}F^{ii}(\nabla_i U_{11})^2 + \frac{1}{U_{11}}F^{ij,kl}\nabla_1 U_{ij}\nabla_1 U_{kl}.$$

Let

$$\phi = \frac{\delta|\nabla u|^2}{2} + b\eta$$

where b, δ are undetermined constants, $0 < \delta < 1 \leq b$, and η is a C^2 function which may depend on u but not on its derivatives. We have

$$\nabla_i\phi = \delta\nabla_j u\nabla_{ij}u + b\nabla_i\eta = \delta\nabla_i u U_{ii} - \delta\nabla_j u A^{ij} + b\nabla_i\eta \tag{4.9}$$

$$\begin{aligned} \nabla_{ii}\phi & = \delta\nabla_{ij}u\nabla_{ij}u + \delta\nabla_j u\nabla_{iij}u + b\nabla_{ii}\eta \\ & \geq \frac{\delta}{2}U_{ii}^2 - C\delta + \delta\nabla_j u\nabla_{iij}u + b\nabla_{ii}\eta. \end{aligned} \tag{4.10}$$

By (2.1), (2.3), and (4.5) we see that

$$\begin{aligned} F^{ii}\nabla_j u\nabla_{iij}u & \geq F^{ii}\nabla_j u(\nabla_j U_{ii} - \nabla_j A^{ii}) - C|\nabla u|^2\sum F^{ii} \\ & \geq (\psi_{p_l} - F^{ii}A^{ii}_{p_l})\nabla_j u\nabla_{jl}u - C|\nabla u|^2\sum F^{ii} - C|\nabla u|^2. \end{aligned} \tag{4.11}$$

It follows that

$$\mathcal{L}\phi \geq b\mathcal{L}\eta + \frac{\delta}{2} F^{ii} U_{ii}^2 - C \sum F^{ii} - C. \tag{4.12}$$

We now go on to prove (1.10) in Theorem 1.1. Let $\eta = \underline{u} - u$ so by (4.9),

$$(\nabla_i \phi)^2 \leq C\delta^2 (1 + U_{ii}^2) + 2b^2 (\nabla_i \eta)^2 \leq C\delta^2 U_{ii}^2 + Cb^2. \tag{4.13}$$

Next we estimate E. For fixed $0 < s \leq 1/3$ let

$$J = \{i : U_{ii} \leq -sU_{11}\}, \quad K = \{i : U_{ii} > -sU_{11}\}.$$

We have

$$\begin{aligned} -F^{ij,kl} \nabla_1 U_{ij} \nabla_1 U_{kl} &\geq \sum_{i \neq j} \frac{F^{ii} - F^{jj}}{U_{jj} - U_{ii}} (\nabla_1 U_{ij})^2 \\ &\geq 2 \sum_{i \geq 2} \frac{F^{ii} - F^{11}}{U_{11} - U_{ii}} (\nabla_1 U_{i1})^2 \\ &\geq \frac{2}{(1+s)U_{11}} \sum_{i \in K} (F^{ii} - F^{11}) (\nabla_1 U_{i1})^2 \\ &\geq \frac{2(1-s)}{(1+s)U_{11}} \sum_{i \in K} (F^{ii} - F^{11}) ((\nabla_i U_{11})^2 - CU_{11}^2/s). \end{aligned} \tag{4.14}$$

The first inequality in (4.14) is a consequence of an inequality due to Andrews [2] and Gerhardt [11]; it was also included in an original version of [4]. By (4.14), (4.1) and (4.13) we obtain

$$\begin{aligned} E &\leq \frac{1}{U_{11}^2} \sum_{i \in J} F^{ii} (\nabla_i U_{11})^2 + C \sum_{i \in K} F^{ii} + \frac{CF^{11}}{U_{11}^2} \sum_{i \in K} (\nabla_i U_{11})^2 \\ &\leq \sum_{i \in J} F^{ii} (\nabla_i \phi)^2 + C \sum F^{ii} + CF^{11} \sum (\nabla_i \phi)^2 \\ &\leq Cb^2 \sum_{i \in J} F^{ii} + C\delta^2 \sum F^{ii} U_{ii}^2 + C \sum F^{ii} + C(\delta^2 U_{11}^2 + b^2) F^{11}. \end{aligned} \tag{4.15}$$

It follows from (4.8), (4.12) and (4.15) that

$$\begin{aligned} b\mathcal{L}\eta \leq \Big(C\delta^2 - \frac{\delta}{2} + \frac{C}{U_{11}} \Big) F^{ii} U_{ii}^2 + Cb^2 \sum_{i \in J} F^{ii} + C \sum F^{ii} \\ + C(\delta^2 U_{11}^2 + b^2) F^{11} + C. \end{aligned} \tag{4.16}$$

By Proposition 2.2 there exist uniform positive constants θ, R satisfying

$$\mathcal{L}(\underline{u} - u) \geq \theta \sum F^{ii} + \theta$$

provided that $U_{11}(x_0) > R$ and hence, when b is sufficiently large,

$$\Big(C\delta^2 - \frac{\delta}{2} + \frac{C}{U_{11}}\Big)F^{ii}U_{ii}^2 + Cb^2\sum_{i\in J}F^{ii} + C(\delta^2U_{11}^2 + b^2)F^{11} \geq 0.$$

This implies a bound $U_{11}(x_0) \leq C$ as otherwise the first term would be negative for δ chosen sufficiently small, and $|U_{ii}| \geq sU_{11}$ for $i \in J$. The proof of (1.10) in Theorem 1.1 is therefore complete.

We now turn to the interior estimate (1.17) in Theorem 1.3. Following [25] we choose a cutoff function $\zeta \in C_0^\infty(B_r)$ such that

$$0 \leq \zeta \leq 1,\ \zeta|_{B_{\frac{r}{2}}} \equiv 1,\ |\nabla\zeta| \leq \frac{C_r}{\sqrt{\zeta}},\ |\nabla^2\zeta| \leq C_r \tag{4.17}$$

where C_r is a constant depending on r.

Let $\eta = \underline{u} - u + \log\zeta$. By (4.1) and (4.13),

$$\begin{aligned} E \leq F^{ii}(\nabla_i\phi)^2 &\leq C\delta^2F^{ii}U_{ii}^2 + \frac{Cb^2}{\zeta^2}F^{ii}(\nabla_i\zeta)^2 + Cb^2\sum F^{ii} \\ &\leq C\delta^2F^{ii}U_{ii}^2 + \frac{Cb^2}{\zeta}\sum F^{ii}. \end{aligned} \tag{4.18}$$

Under the MTW condition (1.15) we have

$$A^{ii}_{p_1p_1} \leq -c_0 < 0.$$

It therefore follows from (4.8), (4.12) and (4.18) that

$$\begin{aligned} b\mathcal{L}\eta \leq & \Big(\frac{C}{U_{11}} + C\delta^2 - \frac{\delta}{2}\Big)F^{ii}U_{ii}^2 - c_0U_{11}\sum_{i\geq 2}F^{ii} \\ & + Cb^2\Big(\frac{1}{\zeta} + \frac{1}{U_{11}}\Big)\sum F^{ii} + C. \end{aligned} \tag{4.19}$$

From the assumption $\underline{u} \in C^2(\overline{B}_r)$ is admissible so for $B > 0$ sufficiently large,

$$\lambda(Bg + \underline{U}) \in \Gamma_n \ \text{ in } \overline{B}_r$$

and therefore,

$$F(2Bg + \underline{U}) \geq F(Bg) \ \text{ in } \overline{B}_r.$$

By the concavity of F,

$$\begin{aligned} F^{ii}(\underline{U}_{ii} - U_{ii}) &\geq F(2Bg + \underline{U}) - F(U) - 2B\sum F^{ii} \\ &\geq F(Bg) - 2B\sum F^{ii} - \psi(x, u, \nabla u). \end{aligned} \tag{4.20}$$

For $B > 0$ sufficiently large we have

$$\begin{aligned}\mathcal{L}\eta \geq\, & \mathcal{L}(\underline{u} - u) - \frac{C}{\zeta}\sum F^{ii} \\ \geq\, & F^{ii}(\underline{U}_{ii} - U_{ii}) - \frac{C}{\zeta}\sum F^{ii} - C \\ \geq\, & F(Bg) - \Big(2B + \frac{C}{\zeta}\Big)\sum F^{ii} - C.\end{aligned} \tag{4.21}$$

From (4.19), (4.21) we derive a bound $\zeta(x_0)U_{11}(x_0) \leq C$ which yields $W(x_0) \leq C$ when we fix δ small and B large. This proves

$$|\nabla^2 u| \leq \frac{C}{\zeta} \text{ in } B_r.$$

The proof of (1.17) in Theorem 1.3 is complete.

5. Boundary estimates for second derivatives

In this section we establish the boundary estimate (1.13) in Theorem 1.1. We shall assume that the function $\varphi \in C^4(\partial M)$ is extended to a C^4 function on $\bar{M}$, still denoted φ.

For a point x_0 on ∂M, we shall choose smooth orthonormal local frames $e_1, \dots, e_n$ around x_0 such that when restricted to ∂M, e_n is normal to ∂M. For $x \in \bar{M}$ let $\rho(x)$ and $d(x)$ denote the distances from x to x_0 and ∂M, respectively,

$$\rho(x) \equiv \mathrm{dist}_{M^n}(x, x_0), \quad d(x) \equiv \mathrm{dist}_{M^n}(x, \partial M)$$

and $M_\delta = \{x \in M : \rho(x) < \delta\}$.

Since $u - \underline{u} = 0$ on ∂M we have

$$\nabla_{\alpha\beta}(u - \underline{u}) = -\nabla_n(u - \underline{u})\mathit{\Pi}(e_\alpha, e_\beta), \;\; \forall\, 1 \leq \alpha, \beta < n \;\text{ on } \partial M \tag{5.1}$$

where $\mathit{\Pi}$ denotes the second fundamental form of ∂M. Therefore,

$$|\nabla_{\alpha\beta} u| \leq C, \;\; \forall\, 1 \leq \alpha, \beta < n \;\text{ on } \partial M. \tag{5.2}$$

To proceed we calculate using (4.5) and (2.1) for each $1 \leq k \leq n$,

$$|\mathcal{L}\nabla_k(u - \varphi)| \leq C\Big(1 + \sum f_i|\lambda_i| + \sum f_i\Big), \tag{5.3}$$

and

$$\mathcal{L}|\nabla_k(u - \varphi)|^2 \geq F^{ij}U_{i\beta}U_{j\beta} - C\Big(1 + \sum f_i|\lambda_i| + \sum f_i\Big). \tag{5.4}$$

As in [15] we need the following crucial lemma.

Lemma 5.1. *There exist some uniform positive constants t, δ, ε sufficiently small and N sufficiently large such that the function*

$$v = (u - \underline{u}) + td - \frac{Nd^2}{2} \tag{5.5}$$

satisfies $v \geq 0$ on $\bar{M}_\delta$ and

$$\mathcal{L}v \leq -\varepsilon\Big(1 + \sum F^{ii}\Big) \quad \text{in } M_\delta. \tag{5.6}$$

The proof of Lemma 5.1 is similar to that of Lemma 4.1 in [15] using Proposition 2.2, so we omit it here.

Let

$$\Psi = A_1 v + A_2 \rho^2 - A_3 \sum_{\beta<n} |\nabla_\beta (u - \varphi)|^2. \tag{5.7}$$

For fixed $1 \leq \alpha < n$, we derive using Lemma 5.1 and Proposition 2.1 as in [15]

$$\begin{cases} \mathcal{L}(\Psi \pm \nabla_\alpha (u - \varphi)) \leq 0 \ \text{ in } M_\delta, \\ \Psi \pm \nabla_\alpha (u - \varphi) \geq 0 \quad \text{on } \partial M_\delta \end{cases} \tag{5.8}$$

when $A_1 \gg A_2 \gg A_3 \gg 1$. By the maximum principle we derive $\Psi \pm \nabla_\alpha (u - \varphi) \geq 0$ in M_δ and therefore

$$|\nabla_{n\alpha} u(x_0)| \leq \nabla_n \Psi(x_0) \leq C, \ \ \forall\, \alpha < n. \tag{5.9}$$

The rest of this section is devoted to derive

$$\nabla_{nn} u(x_0) \leq C. \tag{5.10}$$

The idea is similar to that used in [15] but the proof is much more complicated due to the dependence of ψ on u and ∇u. So we shall carry out the proof in detail.

As in [15], following an idea of Trudinger [46] we prove that there are uniform constants c_0, R_0 such that for all $R > R_0$, $(\lambda'[\{U_{\alpha\beta}(x_0)\}], R) \in \Gamma$ and

$$f(\lambda'[\{U_{\alpha\beta}(x_0)\}], R) \geq \psi[u](x_0) + c_0 \tag{5.11}$$

which implies (5.10) by Lemma 1.2 in [4], where $\lambda'[\{U_{\alpha\beta}\}] = (\lambda'_1, \cdots, \lambda'_{n-1})$ denotes the eigenvalues of the $(n-1) \times (n-1)$ matrix $\{U_{\alpha\beta}\}$ $(1 \leq \alpha, \beta \leq n-1)$, and $\psi[u] = \psi(\cdot, u, \nabla u)$.

Let

$$\tilde{m} \equiv \min_{x_0 \in \partial M} \big(\lim_{R \to +\infty} f(\lambda'[\{U_{\alpha\beta}(x_0)\}], R) - \psi(x_0)\big),$$

$$\tilde{c} \equiv \min_{x_0 \in \partial M} \big(\lim_{R \to +\infty} f(\lambda'[\{\underline{U}_{\alpha\beta}(x_0)\}], R) - F(\underline{U}_{ij}(x_0))\big) > 0.$$

We wish to show $\tilde{m} > 0$. Without loss of generality we assume $\tilde{m} < \tilde{c}/2$ (otherwise we are done) and suppose $\tilde{m}$ is achieved at a point $x_0 \in \partial M$. Choose local orthonormal frames around x_0 as before and assume $\nabla_{nn} u(x_0) \geq \nabla_{nn} \underline{u}(x_0)$.

For a symmetric $(n-1)^2$ matrix $\{r_{\alpha\beta}\}$ such that $(\lambda'[\{r_{\alpha\beta}\}], R) \in \Gamma$ when R is sufficientluy large, define

$$\tilde{F}[r_{\alpha\beta}] \equiv \lim_{R\to+\infty} f(\lambda'[\{r_{\alpha\beta}\}], R)$$

Note that $\tilde{F}$ is concave by (1.4). There exists a positive semidefinite matrix $\{\tilde{F}_0^{\alpha\beta}\}$ such that

$$\tilde{F}_0^{\alpha\beta}(r_{\alpha\beta} - U_{\alpha\beta}(x_0)) \geq \tilde{F}[r_{\alpha\beta}] - \tilde{F}[U_{\alpha\beta}(x_0)] \tag{5.12}$$

for any symmetric matrix $\{r_{\alpha\beta}\}$ with $(\lambda'[\{r_{\alpha\beta}\}], R) \in \Gamma$ when R is sufficiently large. In particular,

$$\tilde{F}_0^{\alpha\beta}U_{\alpha\beta} - \psi[u] - \tilde{F}_0^{\alpha\beta}U_{\alpha\beta}(x_0) + \psi[u](x_0) \geq \tilde{F}[U_{\alpha\beta}] - \psi[u] - \tilde{m} \geq 0 \text{ on } \partial M. \tag{5.13}$$

By (5.1) we have on ∂M,

$$U_{\alpha\beta} = \underline{U}_{\alpha\beta} - \nabla_n(u - \underline{u})\sigma_{\alpha\beta} + A^{\alpha\beta}[u] - A^{\alpha\beta}[\underline{u}] \tag{5.14}$$

where $\sigma_{\alpha\beta} = \langle\nabla_\alpha e_\beta, e_n\rangle$; note that $\sigma_{\alpha\beta} = \mathit{\Pi}(e_\alpha, e_\beta)$ on ∂M. It follows that at x_0,

$$\begin{aligned}
\nabla_n(u - \underline{u})\tilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta} &= \tilde{F}_0^{\alpha\beta}(\underline{U}_{\alpha\beta} - U_{\alpha\beta}) + \tilde{F}_0^{\alpha\beta}(A^{\alpha\beta}[u] - A^{\alpha\beta}[\underline{u}]) \\
&\geq \tilde{F}[\underline{U}_{\alpha\beta}] - \tilde{F}[U_{\alpha\beta}] + \tilde{F}_0^{\alpha\beta}(A^{\alpha\beta}[u] - A^{\alpha\beta}[\underline{u}]) \\
&= \tilde{F}[\underline{U}_{\alpha\beta}] - \psi[u] - \tilde{m} + \tilde{F}_0^{\alpha\beta}(A^{\alpha\beta}[u] - A^{\alpha\beta}[\underline{u}]) \\
&\geq \tilde{c} - \tilde{m} + \psi[\underline{u}] - \psi[u] + \tilde{F}_0^{\alpha\beta}(A^{\alpha\beta}[u] - A^{\alpha\beta}[\underline{u}]) \\
&\geq \frac{\tilde{c}}{2} + H[u] - H[\underline{u}]
\end{aligned} \tag{5.15}$$

where $H[u] = \tilde{F}_0^{\alpha\beta}A^{\alpha\beta}[u] - \psi[u]$.

Define

$$\mathit{\Phi} = -\eta\nabla_n(u - \underline{u}) + H[u] + Q$$

where $\eta = \tilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta}$ and

$$Q \equiv \tilde{F}_0^{\alpha\beta}\nabla_{\alpha\beta}\underline{u} - \tilde{F}_0^{\alpha\beta}U_{\alpha\beta}(x_0) + \psi[u](x_0).$$

From (5.13) and (5.14) we see that $\mathit{\Phi}(x_0) = 0$ and $\mathit{\Phi} \geq 0$ on ∂M near x_0.

By (5.3) and assumption (1.7) we have

$$\begin{aligned}
\mathcal{L}H &\leq H_z[u]\mathcal{L}u + H_{p_k}[u]\mathcal{L}\nabla_k u + F^{ij}H_{p_kp_l}[u]\nabla_{ki}u\nabla_{lj}u + C\sum F^{ii} + C \\
&\leq C\sum f_i + C\sum f_i|\lambda_i| + C.
\end{aligned}$$

Therefore,

$$\mathcal{L}\mathit{\Phi} \leq C\sum f_i + C\sum f_i|\lambda_i| + C. \tag{5.16}$$

Consider the function Ψ defined in (5.7). Applying Lemma 5.1 and Proposition 2.1 again for $A_1 \gg A_2 \gg A_3 \gg 1$ we derive

$$\begin{cases} \mathcal{L}(\Psi+\Phi) \leq 0 \ \text{ in } M_\delta, \\ \Psi+\Phi \geq 0 \ \text{ on } \partial M_\delta. \end{cases} \tag{5.17}$$

By the maximum principle, $\Psi+\Phi \geq 0$ in M_δ. Thus $\nabla_n\Phi(x_0) \geq -\nabla_n\Psi(x_0) \geq -C$.

Write $u^t = tu + (1-t)\underline{u}$ and

$$H[u^t] = \tilde{F}_0^{\alpha\beta} A^{\alpha\beta}[u^t] - \psi[u^t].$$

We have

$$\begin{aligned} H[u] - H[\underline{u}] &= \int_0^1 \frac{dH[u^t]}{dt} dt \\ &= (u-\underline{u}) \int_0^1 H_z[u^t]dt + \sum \nabla_k(u-\underline{u}) \int_0^1 H_{p_k}[u^t]dt. \end{aligned}$$

Therefore, at x_0,

$$H[u] - H[\underline{u}] = \nabla_n(u-\underline{u}) \int_0^1 H_{p_n}[u^t]dt \tag{5.18}$$

and

$$\begin{aligned} \nabla_n H[u] =& \nabla_n H[\underline{u}] + \sum \nabla_{kn}(u-\underline{u}) \int_0^1 H_{p_k}[u^t]dt \\ &+ \nabla_n(u-\underline{u}) \int_0^1 (H_z[u^t] + \nabla'_n H_{p_n}[u^t] + H_{zp_n}[u^t]\nabla_n u^t)dt \\ &+ \nabla_n(u-\underline{u}) \sum \int_0^1 H_{p_n p_l}[u^t]\nabla_{ln}u^t dt \\ \leq& \nabla_{nn}(u-\underline{u}) \int_0^1 (H_{p_n}[u^t] + tH_{p_n p_n}[u^t]\nabla_n(u-\underline{u}))dt + C \\ \leq& \nabla_{nn}(u-\underline{u}) \int_0^1 H_{p_n}[u^t]dt + C \end{aligned} \tag{5.19}$$

since $H_{p_n p_n} \leq 0$, $\nabla_{nn}(u-\underline{u}) \geq 0$ and $\nabla_n(u-\underline{u}) \geq 0$. It follows that

$$\begin{aligned} \nabla_n\Phi(x_0) \leq& -\eta(x_0)\nabla_{nn}(x_0) + \nabla_n H[u](x_0) + C \\ \leq& \Big(-\eta(x_0) + \int_0^1 H_{p_n}[u^t](x_0)dt\Big)\nabla_{nn}u(x_0) + C. \end{aligned} \tag{5.20}$$

By (5.15) and (5.18),

$$\eta(x_0) - \int_0^1 H_{p_n}[u^t](x_0)dt \geq \frac{\tilde{c}}{2\nabla_n(u-\underline{u})(x_0)} \geq \epsilon_1\tilde{c} > 0 \tag{5.21}$$

for some uniform $\epsilon_1 > 0$. This gives

$$\nabla_{nn}u(x_0) \leq \frac{C}{\epsilon_1 \tilde{c}}. \tag{5.22}$$

So we have an *a priori* upper bound for all eigenvalues of $\{U_{ij}(x_0)\}$. Consequently, $\lambda[\{U_{ij}(x_0)\}]$ is contained in a compact subset of Γ by (1.5), and therefore

$$m_R \equiv f(\lambda[U_{\alpha\beta}(x_0)], R) - \psi[u](x_0) > 0$$

when R is sufficiently large. This proves (5.11) and the proof of (1.13) is complete.

6. The gradient estimates

In this section we consider the gradient estimates. Throughout the section, and in Theorems 6.1-6.3 below in particular, we assume (1.3)-(1.5), (1.7) and the following growth conditions hold

$$\begin{cases} p \cdot \nabla_x A^{\xi\xi}(x,z,p) + |p|^2 A_z^{\xi\xi}(x,z,p) \leq \bar{\psi}_1(x,z)|\xi|^2(1+|p|^{\gamma_1}), \\ \quad p \cdot \nabla_x \psi(x,z,p) + |p|^2 \psi_z(x,z,p) \geq -\bar{\psi}_2(x,z)(1+|p|^{\gamma_2}), \end{cases} \tag{6.1}$$

for some functions $\bar{\psi}_1, \bar{\psi}_2 > 0$ and constants $\gamma_1, \gamma_2 > 0$. Let $u \in C^3(\bar{M})$ be an admissible solution of (1.1).

Theorem 6.1. *Assume, in addition, that* (1.15) *and* (1.16) *hold,* $\gamma_1 < 4$*,* $\gamma_2 = 2$ *in* (6.1)*, and that there is an admissible function* $\underline{u} \in C^2(\bar{M})$*. Then*

$$\max_{\bar{M}} |\nabla u| \leq C_1\big(1 + \max_{\partial M} |\nabla u|\big) \tag{6.2}$$

where C_1 *depends on* $|u|_{C^0(\bar{M})}$ *and* $|\underline{u}|_{C^2(\bar{M})}$.

Proof. Let $w = |\nabla u|$ and ϕ a positive function to be determined. Suppose the function $w\phi^{-a}$ achieves a positive maximum at an interior point $x_0 \in M$ where $a < 1$ is constant. Choose a smooth orthonormal local frame $e_1, \ldots, e_n$ about x_0 such that $\nabla_{e_i} e_j = 0$ at x_0 and $\{U_{ij}(x_0)\}$ is diagonal.

The function $\log w - a \log \phi$ attains its maximum at x_0 where for $i = 1, \ldots, n$,

$$\frac{\nabla_i w}{w} - \frac{a \nabla_i \phi}{\phi} = 0, \tag{6.3}$$

$$\frac{\nabla_{ii} w}{w} + \frac{(a - a^2)|\nabla_i \phi|^2}{\phi^2} - \frac{a \nabla_{ii} \phi}{\phi} \leq 0. \tag{6.4}$$

Next,

$$w \nabla_i w = \nabla_l u \nabla_{il} u$$

and, by (2.1) and (6.3),

$$\begin{aligned} w\nabla_{ii}w &= \nabla_l u\nabla_{iil}u + \nabla_{il}u\nabla_{il}u - \nabla_i w\nabla_i w \\ &= (\nabla_{lii}u + R^k_{iil}\nabla_k u)\nabla_l u + \Big(\delta_{kl} - \frac{\nabla_k u\nabla_l u}{w^2}\Big)\nabla_{ik}u\nabla_{il}u \\ &\geq (\nabla_l U_{ii} - A^{ii}_{p_k}\nabla_{lk}u - A^{ii}_u\nabla_l u - \nabla'_l A^{ii})\nabla_l u - C|\nabla u|^2 \\ &= \nabla_l u\nabla_l U_{ii} - \frac{w^2}{\phi}(aA^{ii}_{p_k}\nabla_k\phi + \phi A^{ii}_u) - \nabla_l u\nabla'_l A^{ii} - Cw^2. \end{aligned} \tag{6.5}$$

By (4.5) and (6.3),

$$\begin{aligned} F^{ii}\nabla_l u\nabla_l U_{ii} &= \nabla_l u\nabla'_l\psi + \psi_u|\nabla u|^2 + \psi_{p_k}\nabla_l u\nabla_{lk}u \\ &= \nabla_l u\nabla'_l\psi + \psi_u|\nabla u|^2 + \frac{aw^2}{\phi}\psi_{p_k}\nabla_k\phi. \end{aligned} \tag{6.6}$$

Let $\phi = (u - \underline{u}) + b > 0$ where $b = 1 + \sup_M(\underline{u} - u)$. By the MTW condition (1.15) we have

$$\begin{aligned} -A^{ii}_{p_k}\nabla_k\phi &= A^{ii}_{p_k}(x, u, \nabla u)\nabla_k(\underline{u} - u) \\ &\geq A^{ii}(x, u, \nabla\underline{u}) - A^{ii}(x, u, \nabla u) + c_0(|\nabla\phi|^2 - |\nabla_i\phi|^2) \\ &\geq A^{ii}(x, \underline{u}, \nabla\underline{u}) - A^{ii}(x, u, \nabla u) + c_0(|\nabla\phi|^2 - |\nabla_i\phi|^2) - C. \end{aligned} \tag{6.7}$$

By (6.4), (6.5) and (6.7),

$$\begin{aligned} 0 \geq\ & \frac{\nabla_l u}{w^2}F^{ii}\nabla_l U_{ii} + \frac{a}{\phi}F^{ii}(\underline{U}_{ii} - U_{ii}) + \frac{ac_0|\nabla\phi|^2}{\phi}\sum F^{ii} \\ &+ \frac{a - a^2 - c_0 a\phi}{\phi^2}F^{ii}|\nabla_i\phi|^2 - F^{ii}A^{ii}_u - \frac{\nabla_l u}{w^2}F^{ii}\nabla'_l A^{ii} - C\sum F^{ii}. \end{aligned} \tag{6.8}$$

Note that for $a \in (0, 1)$,

$$\frac{ac_0|\nabla\phi|^2}{\phi}\sum F^{ii} + \frac{a - a^2 - c_0 a\phi}{\phi^2}F^{ii}|\nabla_i\phi|^2 \geq c'_0|\nabla\phi|^2\sum F^{ii} \tag{6.9}$$

for some $c'_0 > 0$. This is obvious if $c_0\phi < 1$; if $c_0\phi \geq 1$ then

$$\frac{ac_0|\nabla\phi|^2}{\phi}\sum F^{ii} + \frac{a - a^2 - c_0 a\phi}{\phi^2}F^{ii}|\nabla_i\phi|^2 \geq \frac{a - a^2}{\phi^2}|\nabla\phi|^2\sum F^{ii}.$$

By (6.6) and the convexity of $\psi(x, z, p)$ in p,

$$F^{ii}\nabla_l u\nabla_l U_{ii} \geq \nabla_l u\nabla'_l\psi + \psi_u|\nabla u|^2 + \frac{aw^2}{\phi}(\psi(x, u, \nabla u) - \psi(x, u, \nabla\underline{u})). \tag{6.10}$$

Plugging (6.9), (6.10) and (4.20) into (6.8), we derive for B sufficiently large

$$\begin{aligned} 0 \geq\, & \frac{\nabla_l u \nabla'_l \psi}{w^2} + \psi_u + \frac{a}{\phi}\Big(F(Bg) - \psi(x, u, \nabla \underline{u}) - 2B \sum F^{ii}\Big) \\ & + c'_0 |\nabla \phi|^2 \sum F^{ii} - F^{ii} A^{ii}_u - \frac{\nabla_l u}{w^2} F^{ii} \nabla'_l A^{ii} - C \sum F^{ii}. \end{aligned} \tag{6.11}$$

By (6.1) we obtain

$$\begin{aligned} 0 \geq\, & aF(Bg) - a\psi(x, u, \nabla \underline{u}) - C\phi |\nabla u|^{\gamma_2 - 2} \\ & + (c'_0 \phi |\nabla \phi|^2 - C\phi |\nabla u|^{\gamma_1 - 2} - C\phi - 2aB) \sum F^{ii}. \end{aligned} \tag{6.12}$$

Since $\gamma_1 < 4$ and $\gamma_2 = 2$, by (1.16) this yields a bound $|\nabla u(x_0)| \leq C$ if B is chosen sufficiently large. □

Theorem 6.2. *Assume, in addition, that* **(i)** $\psi = \psi(x, p)$, $A = A(x, p)$; **(ii)** (M^n, g) *has nonnegative sectional curvature; and* **(iii)** (1.9), (1.6) (6.1) *hold for* $\gamma_1, \gamma_2 < 2$ *in* (6.1), *and that there exist constants* $K > 0$ *and* $c_1 > 0$ *such that*

$$\psi(x, p) \geq c_1, \ \ \forall\, x \in \bar{M}, p \in T_x \bar{M}; |p| \geq K. \tag{6.13}$$

Then (6.2) *holds.*

Proof. Since (M, g) has nonnegative sectional curvature, in orthonormal local frame,

$$R^k_{iil} \nabla_k u \nabla_l u \geq 0.$$

In the proof of Theorem 6.1, we therefore have in place of (6.5),

$$\begin{aligned} w \nabla_{ii} w & \geq \nabla_{lii} u \nabla_l u + R^k_{iil} \nabla_k u \nabla_l u \\ & \geq \nabla_l u \nabla_l U_{ii} - \frac{w^2}{\phi}(a A^{ii}_{p_k} \nabla_k \phi + \phi A^{ii}_u) - \nabla_l u \nabla'_l A^{ii}. \end{aligned} \tag{6.14}$$

By (6.1), (6.4), (6.14) and (6.10), we obtain

$$\begin{aligned} 0 \geq & - \frac{a}{\phi} \mathcal{L} \phi + \frac{\nabla_l u \nabla'_l \psi}{w^2} - \frac{1}{w^2} F^{ii} \nabla_l u \nabla'_l A^{ii} + \frac{(a - a^2)}{\phi^2} F^{ii} |\nabla_i \phi|^2 \\ \geq\, & \frac{a}{\phi} \mathcal{L}(\underline{u} - u) + c_0 F^{ii} |\nabla_i \phi|^2 - C |\nabla u|^{\gamma_1 - 2} \sum F^{ii} - C |\nabla u|^{\gamma_2 - 2}. \end{aligned} \tag{6.15}$$

Suppose $|\lambda(U(x_0))| \geq R$ for R sufficiently large. As ψ and A are independent of u, by the comparison principle $u \geq \underline{u}$ in M. Consequently, we may apply Proposition 2.2 to derive a bound $|\nabla u(x_0)| \leq C$ from (6.15).

Suppose now that $|\lambda(U(x_0))| \leq R$ and $|\nabla u(x_0)| \geq K$ for K sufficiently large. Then there is $C_2 > 0$ depending on R and K such that

$$C_2^{-1} I \leq \{F^{ij}\} \leq C_2 I.$$

Since $\mathcal{L}(\underline{u} - u) \geq 0$, it follows from (6.15) that

$$c_0 C_2^{-1} |\nabla \phi|^2 - nCC_2 |\nabla u|^{\gamma_1 - 2} - C |\nabla u|^{\gamma_2 - 2} \leq 0.$$

This proves $|\nabla u(x_0)| \leq C$. □

An alternative assumption which is commonly used in deriving gradient estimate is the following

$$f_j(\lambda) \geq \nu_0\Big(1 + \sum f_i(\lambda)\Big) \text{ if } \lambda_j < 0, \ \forall\, \lambda \in \Gamma^\sigma \tag{6.16}$$

for any $\sigma > 0$ where $\nu_0 > 0$ depends on σ; see e.g. [18], [33], [38], [45], and [49].

Theorem 6.3. *Assume, in addition, that* $\gamma_1, \gamma_2 < 4$, (1.12), (6.16) *hold, and that*

$$-\psi_z(x,z,p),\ p \cdot D_p\psi(x,z,p),\ -p \cdot D_pA^{\xi\xi}(x,p)/|\xi|^2 \leq \bar{\psi}(x,z)(1 + |p|^\gamma), \tag{6.17}$$

$$|A^{\xi\eta}(x,z,p)| \leq \bar{\psi}(x,z)|\xi||\eta|(1 + |p|^\gamma), \quad \forall\, \xi, \eta \in T^*\bar{M}; \xi \perp \eta. \tag{6.18}$$

for some function $\bar{\psi} \geq 0$ *and constant* $\gamma \in (0,2)$. *There exists a constant* C_1 *depending on* $|u|_{C^0(\bar{M})}$ *and other known data such that* (6.2) *holds.*

Proof. In the proof of Theorem 6.1 we take $\phi = -u + \sup_M u + 1$. By the concavity of $A^{ii}(x,z,p)$ in p we have

$$A^{ii} = A^{ii}(x,u,\nabla u) \leq A^{ii}(x,u,0) + A^{ii}_{p_k}(x,u,0)\nabla_k u. \tag{6.19}$$

By assumption (1.12),

$$-F^{ii}\nabla_{ii}\phi = F^{ii}\nabla_{ii}u = F^{ii}U_{ii} - F^{ii}A^{ii} \geq -F^{ii}A^{ii} \geq -C(1 + |\nabla u|)\sum F^{ii}. \tag{6.20}$$

It follows from (6.4), (6.5), (6.6), (6.20), (6.1) and (6.17) that for $a < 1$,

$$\begin{aligned} 0 \geq & \frac{(a-a^2)}{\phi^2}F^{ii}|\nabla_i u|^2 + \frac{\nabla_l u \nabla'_l \psi}{w^2} + \psi_u - \frac{a}{\phi}\psi_{p_k}\nabla_k u \\ & + \frac{a}{\phi}F^{ii}A^{ii}_{p_k}\nabla_k u - F^{ii}A^{ii}_u - F^{ii}\frac{\nabla_l u \nabla'_l A^{ii}}{w^2} - C(1 + |\nabla u|)\sum F^{ii} \\ \geq & c_0 F^{ii}|\nabla_i u|^2 - C|\nabla u|^{\gamma-2} - C(1 + |\nabla u| + |\nabla u|^{\gamma-2})\sum F^{ii}. \end{aligned} \tag{6.21}$$

Without loss of generality we assume $\nabla_1 u(x_0) \geq \frac{1}{n}|\nabla u(x_0)| > 0$. Recall that $U_{ij}(x_0)$ is diagonal. By (6.3), (6.19) and (6.18) we derive

$$U_{11} = -\frac{a}{\phi}|\nabla u|^2 + A_{11} + \frac{1}{\nabla_1 u}\sum_{k\geq 2}\nabla_k u A^{1k} \leq -\frac{a}{\phi}|\nabla u|^2 + C(1 + |\nabla u| + |\nabla u|^{\gamma-2}). \tag{6.22}$$

If $U_{11}(x_0) \geq 0$ we obtain a bound $|\nabla u(x_0)| \leq C$ from (6.22). If $U_{11}(x_0) < 0$ then by (6.16),

$$f_1 \geq \nu_0\Big(1 + \sum f_i\Big)$$

and a bound $|\nabla u(x_0)| \leq C$ follows from (6.21). □

7. The Dirichlet Problem

We now turn to existence of solutions to the Dirichlet problem (1.1) and (1.2). We first consider the special case $A = A(x, p)$ and $\psi = \psi(x, p)$.

Theorem 7.1. *Suppose (M^n, g) is a compact Riemannian manifold of nonnegative sectional curvature with smooth boundary ∂M, $A = A(x, p)$ and $\psi = \psi(x, p)$ are smooth, and $\varphi \in C^\infty(\partial M)$. Assume that* (1.3)-(1.5)*,* (1.7)*,* (1.9)*,* (1.12)*,* (6.1) *and* (6.13) *hold for $\gamma_1, \gamma_2 < 2$ in* (6.1)*. Then there exists an admissible solution $u \in C^\infty(\bar{M})$ of equation* (1.1) *satisfying the boundary condition* (1.2)*.*

As A and ψ are assumed to be independent of u in Theorem 7.1, by the maximum principle it is easy to derive the estimate

$$\max_{\bar{M}} |u| + \max_{\partial M} |\nabla u| \leq C. \tag{7.1}$$

By Theorems 1.1 and 6.2 we obtain

$$|u|_{C^2(\bar{M})} \leq C. \tag{7.2}$$

From (1.5) and the fact that $\psi > 0$ we see that equation (1.1) becomes uniformly elliptic for admissible solutions satisfying (7.2). Consequently, the concavity condition (1.4) allows us to apply Evans-Krylov theorem in order to obtain $C^{2,\alpha}$ estimates; and higher order estimates follow from the Schauder theory. Theorem 7.1 may then be proved using the standard continuity method.

Theorem 7.2. *Let (M^n, g) be a compact Riemannian manifold with smooth boundary ∂M. Suppose $A = A(x, p)$, $\psi = \psi(x, z, p)$ are smooth and $\varphi \in C^\infty(\partial M)$. In addition to* (1.3)-(1.5)*,* (1.7)-(1.9)*,* (1.12)*,* (6.1)*,* (6.16) *and* (6.17)*, assume that*

$$|A^{\xi\eta}(x, p)| \leq \bar{\psi}(x)|\xi||\eta|(1 + |p|^\gamma), \quad \forall\, \xi, \eta \in T^*\bar{M}, \xi \perp \eta \tag{7.3}$$

for some function $\bar{\psi} \geq 0$ and constant $\gamma \in (0, 2)$. Then the Dirichlet problem (1.1) *and* (1.2) *admits an admissible solution $u \in C^\infty(\bar{M})$ satisfying $u \geq \underline{u}$ on $\bar{M}$.*

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