Monge-Ampère Equations on Riemannian Manifolds

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1 Introduction

The main purpose of this paper is to study the Dirichlet problem for Monge-Ampère equations on Riemannian manifolds. Let $M^n$ be a smooth Riemannian manifold of dimension $n \geq 2$ and $\Omega \subset M^n$ a compact domain with smooth boundary $\partial \Omega$. We shall consider the classical solvability of the problem

$$
\left( g^{-1} \det(\nabla u) \right)^{1/n} = \psi(x,u,\nabla u) \quad \text{in} \quad \Omega, \quad u = \varphi \quad \text{on} \quad \partial \Omega,
$$

where $g_{ij}$ denotes the metric of $M^n$, $g = \det(g_{ij}) > 0$ and $\varphi \in C^\infty(\partial \Omega)$, $\psi > 0$ is $C^\infty$ with respect to $(x,z,p) \in \bar{\Omega} \times \mathbb{R} \times T_xM$. $T_xM$ denotes the tangent space at $x \in M$.

Monge-Ampère equations arise naturally from some problems in differential geometry. The Dirichlet problem in Euclidean space $\mathbb{R}^n$ has been widely investigated. In this case the solvability has been reduced to the existence of strictly convex subsolutions by Caffarelli, Nirenberg and Spruck [2] and independently by Krylov [8] for strictly convex domains $\Omega$ in $\mathbb{R}^n$. More recently, Spruck and the first author [7] treated the problem for non-convex domains in $\mathbb{R}^n$ as well as on $\mathbb{S}^n$ in connection with the geometric problem of finding hypersurfaces in $\mathbb{R}^{n+1}$ of constant Gauss curvature with prescribed boundary. In this paper we extend some of the known results in $\mathbb{R}^n$ to arbitrary Riemannian Manifolds. Our main result is the following analogue of Theorem 0.3 of [7].

Theorem 1.1. Assume that

$$
\psi(x,z,p) \text{ is a convex function with respect to } p \in T_xM.
$$
Then there exists a locally strictly convex solution of (1.1) in $C^\infty(\bar{\Omega})$, provided that there exists a locally strictly convex strict subsolution $u \in C^2(\bar{\Omega})$ to (1.1), that is,

$$
\left( g^{-1} \det(\nabla^2 u) \right)^{1/n} \geq \psi(x,u,\nabla u) + \delta_0 \quad \text{in} \quad \Omega,
$$

for some $\delta_0 > 0$ and $u = \varphi$ on $\partial \Omega$. Furthermore the solution is unique provided $\psi_z \geq 0$.

A function $u$ is called locally strictly convex in $\bar{\Omega}$ if the Hessian $\nabla^2 u$ is positive definite everywhere in $\bar{\Omega}$. The next result deals with the limiting case $\delta_0 = 0$ in (1.3), i.e., the function $u$ is merely a subsolution.

**Theorem 1.2.** Under conditions (1.2) and (1.3) with $\delta_0 = 0$, Problem (1.1) admits a solution belonging to $C^\infty(\Omega) \cap C^{0,1}(\bar{\Omega})$. The solution is unique provided $\psi_z \geq 0$.

We should remark here that it is not known to us whether this solution has better regularity up to the boundary, even for $M = \mathbb{R}^n$ with the flat metric.

The proof of Theorem 1.2 relies on the following interior estimate for the second derivatives, which may be regarded as an extension of the Pogorelov interior estimate ([10]).

**Theorem 1.3.** Let $u \in C^4(\Omega) \cap C^1(\bar{\Omega})$ be a locally strictly convex solution of (1.1). Assume that there exists a locally strictly convex function $v \in C^2(\bar{\Omega})$ with $v = \varphi$ on $\partial \Omega$. Then

$$
|\nabla^2 u(x)| \leq \frac{C}{(\text{dist}_{M^n}(x,\partial \Omega))^N}, \quad \text{for} \quad x \in \Omega,
$$

where $C$ and $N$ are constants depending on $n$, $\Omega$, $\|u\|_{C^1(\bar{\Omega})}$, and $\|v\|_{C^2(\bar{\Omega})}$.

As a corollary of Theorem 1.1 we obtain the following result for convex domains, which extends Theorem 1.1 of [2] to all Riemannian manifolds.

**Theorem 1.4.** Let $\Omega$ be a strictly convex domain in the sense that there is a locally strictly convex function $f \in C^\infty(\bar{\Omega})$ with $f|_{\partial \Omega} = 0$, and $\psi(x,u,\nabla u) \equiv \psi(x)$. Then (1.1) admits a unique locally strictly convex solution in $C^\infty(\bar{\Omega})$.

We observe that in this case strict subsolutions can be easily constructed from $f$. This Theorem was first proved by Caffarelli, Nirenberg and Spruck in [2] for $M = \mathbb{R}^n$. Effort to extend the results in [2] to general Riemannian manifolds had been made by
Corona in [4] where some existence results were obtained for two dimensional Riemannian manifolds with nonnegative curvature under a further hypothesis on the existence of a supersolution. Our results hold in all dimensions and without any restrictions on the curvature of $M$. Theorem 1.1 and Theorem 1.2, which extend the corresponding results in [7] and [6], require neither any convexity of the domain $\Omega$, nor the existence of supersolutions. After we essentially completed our work, we received the preprint of A. Atallah and C. Zuily [1] where they also established Theorem 1.4. Their technique is different from ours in deriving a priori estimates for the double normal second derivatives on the boundary. Our proof, which is based on an idea of Trudinger [11], is much simpler. More general fully nonlinear elliptic equations of Monge-Ampère type on compact manifolds without boundary have been studied by the second author in [9].

In Section 2 we shall derive $C^{2,\alpha}$ a priori estimates for the desired solutions of (1.1). Once such estimates are established, one can apply degree theory to prove the existence result in Theorem 1.1 as in [2] (see [6] for some modifications needed of the argument). The uniqueness part follows from the maximum principle. Theorem 1.3 and Theorem 1.2 are proved in Section 3.

In conclusion of this introduction we recall some formulae for commuting covariant derivatives on $M^n$. Throughout the paper, $\nabla$ denotes the covariant differentiation on $M^n$. Let $e_1, \ldots, e_n$ be a local frame on $M^n$. We use the notation $\nabla_i = \nabla_{e_i}$, $\nabla_{ij} = \nabla_i \nabla_j$, etc. For a differentiable function $v$ defined on $M^n$, $\nabla v$ denotes the gradient, and $\nabla^2 v$ the Hessian which is given by

$$
\nabla_{ij} v = \nabla_i (\nabla_j v) - (\nabla_i e_j) v.
$$

We recall that $\nabla_{ij} v = \nabla_{ji} v$ and

\begin{align}
(1.5) \quad &\nabla_{ijk} v - \nabla_{ikj} v = R_{kji}^l \nabla_l v, \\
(1.6) \quad &\nabla_{ijkl} v - \nabla_{ikjl} v = R_{i}^{lm} R_{lkj}^n \nabla_m v + \nabla_i R_{lkj}^n \nabla_m v, \\
(1.7) \quad &\nabla_{ijkl} v - \nabla_{jikl} v = R_{kji}^m \nabla_l m v + R_{lji}^m \nabla_m v.
\end{align}

Finally, from (1.6) and (1.7) we obtain

\begin{align}
(1.8) \quad &\nabla_{ijkl} v - \nabla_{klij} v = R_{lki}^m \nabla_l m v + \nabla_i R_{lkj}^m \nabla_m v + R_{lkj}^m \nabla_m v \\
&+ R_{jki}^m \nabla_l m v + R_{jli}^m \nabla_m v + \nabla_k R_{jli}^m \nabla_m v.
\end{align}
2 A priori estimates

We denote by $A$ the collection of admissible functions:

$$A = \{ v \in C^2(\bar{\Omega}) : \{ \nabla_{ij} v \} > 0, \ v \geq u \ \text{in} \ \bar{\Omega} \ \text{and} \ v = u \ \text{on} \ \partial\Omega \}.$$ 

In this section we shall establish the following a priori estimates.

**Theorem 2.1.** Assume (1.2) holds and $\delta_0 > 0$ in (1.3). Let $u \in C^{4,\alpha}(\bar{\Omega})$ be a solution of the Dirichlet problem (1.1) in $A$. Then

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leq C.$$  \hfill (2.1)

**Proof.** First we note that for any $v \in A$ we have

$$\max_{\bar{\Omega}} |\nabla v| = \max_{\partial\Omega} |\nabla v|$$

and, since $\Delta v > 0$, the maximum principle yields $u \leq v \leq h$ (where $h$ is the harmonic function in $\Omega$ with $h = \varphi$ on $\partial\Omega$) and thus an a priori bound for the gradient $\nabla v$ on $\partial\Omega$. It follows that

$$\|v\|_{C^1(\bar{\Omega})} \leq C_0, \ \text{for all} \ v \in A.$$  \hfill (2.2)

Therefore,

$$\psi_0 \equiv \inf_{x \in \bar{\Omega}} \inf_{v \in A} \psi(x,v,\nabla v) > 0.$$  \hfill (2.3)

The rest of the proof is devoted to the a priori estimates for the second derivatives. We shall first derive a bound on the boundary

$$|\nabla^2 u| \leq C_1 \ \text{on} \ \partial\Omega.$$  \hfill (2.4)

After that we shall take up the global estimate

$$|\nabla^2 u| \leq C_2 \ \text{in} \ \bar{\Omega}.$$  \hfill (2.5)

The desired $C^{2,\alpha}$ estimate (2.1) then follows from the results of Evans [5], Krylov [8] and Caffarelli, Kohn, Nirenberg and Spruck [2], [3].

(a) Bounds for $|\nabla^2 u|$ on $\partial\Omega$. About a point $x_0 \in \partial\Omega$, let $e_1, \ldots, e_n$ be a local orthonormal frame on $M^n$ obtained by parallel translation of a local orthonormal frame
on \( \partial \Omega \) and the interior unit normal vector field to \( \partial \Omega \) along the geodesics perpendicular to \( \partial \Omega \) on \( M^n \). We assume \( e_n \) is the parallel translation of the unit normal field on \( \partial \Omega \) and set

\[
B_{\alpha\beta} = \langle \nabla_\alpha e_\beta, e_n \rangle, \quad 1 \leq \alpha, \beta \leq n - 1.
\]

On \( \partial \Omega \) we have \( u - \bar{u} = 0 \), so

\[
\nabla_\alpha (u - \bar{u}) = 0 \quad \text{for} \quad \alpha < n, \quad \text{and}
\]

\[
\nabla_{\alpha\beta} (u - \bar{u}) = \nabla_\alpha (\nabla_\beta (u - \bar{u})) - \sum_i \langle \nabla_\alpha e_\beta, e_i \rangle \nabla_i (u - \bar{u})
\]

\[
= -B_{\alpha\beta} \nabla_n (u - \bar{u}) \quad \text{for} \quad \alpha, \beta < n.
\]

It follows that

\[
|\nabla_{\alpha\beta} u| \leq C, \quad \text{on} \quad \partial \Omega \quad \text{for} \quad \alpha, \beta < n.
\]

In order to estimate the derivatives \( \nabla_{na} u \) for \( \alpha \leq n \), we need to utilize the assumption that \( u \) is a strict subsolution in construction of barrier functions. Consider the following linear operator

\[
L = F^{ij} \nabla_{ij} - \psi_{p_i}(x, u, \nabla u) \nabla_i - c,
\]

where \( \{F^{ij}\} \) is the inverse matrix of

\[
\begin{pmatrix}
\frac{n \nabla_{ij} u}{(\det(\nabla_{ij} u))^{1/n}}
\end{pmatrix}
\]

and

\[
c \equiv \max_{x \in \Omega} \max_{\bar{u} \leq z \leq u} |\psi_z(x, z, Du)| < \infty.
\]

For fixed \( \alpha \leq n \), differentiating Equation (1.1) and using the formula for commuting the covariant derivatives, we find

\[
|L \nabla_\alpha (u - \bar{u})| \leq C(1 + \sum_i F^{ii}).
\]

Now consider the distance function

\[
d(x) \equiv \text{dist}_{M^n}(x, x_0),
\]

in a small subdomain

\[
\Omega_\delta = \{x \in \Omega : d(x) < \delta\};
\]
by choosing \( \delta \) small enough we may assume \( d \) is smooth in \( \Omega_\delta \) and, since \( \nabla_{ij}d^2(x_0) = 2\delta_{ij} \),

\[
\{\delta_{ij}\} \leq \{\nabla_{ij}d^2\} \leq 3\{\delta_{ij}\} \quad \text{in} \quad \Omega_\delta.
\]

(2.9)

The following lemma is a direct extension of Lemma 2.2(i) of [7]. For the reader’s convenience we include its proof.

**Lemma 2.2.** There is a uniform positive constant \( \epsilon_1 \) such that

\[
\mathcal{L}(u - u) \geq \epsilon_1 (1 + \sum_i F^{ii}) \quad \text{in} \quad \Omega_\delta.
\]

**Proof.** Consider the function \( w = u - \epsilon d^2 \); in view of (2.9), for \( \epsilon > 0 \) small enough,

\[
(\det(\nabla_{ij}w))^\frac{1}{n} \geq (\det(\nabla_{ij}u))^\frac{1}{n} - \frac{\delta_0}{2} \geq \psi(x, u, \nabla u) + \frac{\delta_0}{2}.
\]

(2.10)

By the concavity of the operator \( (\det(\cdot))^\frac{1}{n} \) and the convexity of \( \psi(\cdot, \cdot, p) \) in \( p \) we then find

\[
F^{ij} \nabla_{ij}(w - u) \geq (\det(\nabla_{ij}w))^\frac{1}{n} - (\det(\nabla_{ij}u))^\frac{1}{n} \\
\quad \geq \psi(x, u, \nabla u) - \psi(x, u, \nabla u) + \frac{\delta_0}{2} \\
\quad \geq \sum_i \psi_{pi}(x, u, \nabla u) \nabla_i(u - u) + \psi_z(x, z, \nabla u) (u - u) + \frac{\delta_0}{2},
\]

for some \( z = z(x) \) with \( u \leq z \leq u \).

Using Lemma 2.2 we may employ a barrier function in \( \Omega_\delta \) of the form

\[
v = A(u - u) + Bd^2,
\]

(2.11)

to estimate \( \nabla_{\alpha n}u \). For the mixed normal tangential derivatives \( \nabla_{\alpha n}u(x_0) \), \( \alpha < n \), we may choose \( A \gg B \gg 1 \) such that

\[
\mathcal{L}(v \pm \nabla_{\alpha}(u - u)) \leq 0 \quad \text{in} \quad \Omega_\delta
\]

by (2.8), and

\[
v \pm \nabla_{\alpha}(u - u) \geq 0 \quad \text{on} \quad \partial \Omega_\delta,
\]

since \( \nabla_{\alpha}(u - u) = 0 \) on \( \partial \Omega \cap \Omega_\delta \), and \( |\nabla_{\alpha}(u - u)| \leq C \) in \( \Omega \). It follows from the maximum principle that

\[
v \geq |\nabla_{\alpha}(u - u)| \quad \text{in} \quad \Omega_\delta.
\]
Consequently,

\begin{equation}
|\nabla_{n\alpha}u(x_0)| \leq \nabla_nv(x_0) + |\nabla_{n\alpha}u(x_0)| \leq C, \quad \alpha < n.
\end{equation}

It remains to estimate the double normal derivative $\nabla_{nn}u$. Since $u$ is locally convex, it suffices to derive an upper bound

\begin{equation}
\nabla_{nn}u \leq C \quad \text{on} \quad \partial\Omega.
\end{equation}

Here we make use of an important idea of Trudinger [11]. For $x \in \partial\Omega$ let

$$
\lambda(x) = \min_{|\xi|=1, \xi \in T_x(\partial\Omega)} \nabla_{\xi\xi}u(x),
$$

and assume that $\lambda(x)$ is minimized at $x_0 \in \partial\Omega$ with $\xi = e_1(x_0)$, that is

$$
\nabla_{11}u(x_0) \leq \nabla_{\xi\xi}u(x)
$$

for all $x \in \partial\Omega$ and any unit vector $\xi \in T_x(\partial\Omega)$. As in [2], (2.13) will follow from Equation (1.1) if we can prove

\begin{equation}
\nabla_{11}u(x_0) \geq c_0 > 0
\end{equation}

for some uniform constant $c_0$. We may assume $\nabla_{11}u(x_0) < \frac{1}{2}\nabla_{11}u(x_0)$, since otherwise we are done as $\nabla_{11}u(x_0) \geq c_1 > 0$ for some uniform $c_1 > 0$. From (2.6) we have

\begin{equation}
\nabla_{11}u = \nabla_{11}u - B_{11}\nabla_n(u - \bar{u}) \quad \text{on} \quad \partial\Omega.
\end{equation}

It follows that

$$
B_{11}(x_0)\nabla_n(u - \bar{u})(x_0) \geq \frac{1}{2}\nabla_{11}u(x_0) \geq \frac{c_1}{2}
$$

and for $x \in \partial\Omega$, since $\nabla_{11}u|_{\partial\Omega}$ is minimized at $x_0$,

$$
B_{11}(x)\nabla_n(u - \bar{u})(x) \leq \nabla_{11}u(x) - \nabla_{11}u(x_0) + B_{11}(x_0)\nabla_n(u - \bar{u})(x_0)
$$

Because $B_{11}$ is smooth near $\partial\Omega$ and $0 < \nabla_n(u - \bar{u}) \leq C$, we must have

\begin{equation}
B_{11} \geq c_2 > 0 \quad \text{on} \quad \Omega_\delta
\end{equation}

for some uniform $c_2 > 0$, if $\delta$ is chosen sufficiently small. Therefore,

\begin{equation}
\nabla_n(u - \bar{u})(x) \leq \Psi(x) \quad \text{for} \quad x \in \Omega_\delta \cap \partial\Omega \quad \text{and} \quad \nabla_n(u - \bar{u})(x_0) = \Psi(x_0)
\end{equation}
where $\Psi(x) = B_{11}^{-1}(x)[\nabla_{11}u(x) - \nabla_{11}u(x_0) + B_{11}(x_0)\nabla_n(u - u)(x_0)]$.

We observe that $\Psi$ is smooth in $\Omega_\delta$. Thus by (2.8), (2.17) and Lemma 2.2 we may choose $A \gg B \gg 1$ such that

$$v + \Psi - \nabla_n(u - u) \geq 0 \text{ on } \partial\Omega_\delta,$$

$$\mathcal{L}(v + \Psi - \nabla_n(u - u)) \leq 0 \text{ in } \Omega_\delta,$$

where $v$ as in (2.11). As before, the maximum principle then yields

$$v + \Psi - \nabla_n(u - u) \geq 0 \text{ in } \Omega_\delta.$$ 

Consequently, since $v + \Psi - \nabla_n(u - u) = 0$ at $x_0$, we have

$$\nabla_{nn}u(x_0) \leq C.$$

This shows that the eigenvalues of $\{\nabla_{ij}u(x_0)\}$ are all bounded (and all positive). On the other hand, Equation (1.1) says the product of these eigenvalues are bounded below from zero by a uniform positive constant ($\psi_0$ in (2.3)). Thus each of them must be bounded below from zero. In particular, we obtain the estimate (2.14) which in turn implies (2.13). This establishes (2.4).

(b) **Bounds for $|\nabla^2 u|$ on $\overline{\Omega}$.** Set

$$W = \max_{x \in \overline{\Omega}} \max_{|\xi|=1, \xi \in T_{x_0} \mathbb{R}^n} \nabla_\xi \xi u \exp \left\{ \frac{a}{2} |\nabla u|^2 - b(u - u) \right\},$$

where $a, b$ are positive constants to be determined later. In order to establish (2.5) it suffices to derive a bound for $W$.

If the maximum $W$ occurs on $\partial\Omega$, then $W$ can be estimated via (2.4) and the $C^1$ estimate (2.2). So we may assume $W$ is achieved at a point $x_0 \in \Omega$ and for some unit vector $\xi \in T_{x_0} \mathbb{R}^n$. Choose a smooth orthonormal local frame $e_1, \ldots, e_n$ about $x_0$ such that $e_1(x_0) = \xi$ and $\{\nabla_{ij}u(x_0)\}$ is diagonal and set

$$\lambda_i = \nabla_{ii}u(x_0) > 0 \text{ for } i = 1, \ldots, n.$$ 

We need only estimate $\lambda_1$ from above.
At the point \( x_0 \) where the function \( \log \nabla_{11}u + \frac{a}{2} \nabla u^2 - b(u - u) \) (defined near \( x_0 \)) attains its maximum, we have for each \( i = 1, \ldots, n \),

\[
\frac{\nabla_{i11}u}{\lambda_i} + a \nabla_i u \nabla_{i1}u - b \nabla_i(u - u) = 0,
\]

and

\[
0 \geq \frac{\nabla_{i1i}u}{\lambda_i} - \frac{(\nabla_{i11}u)^2}{\lambda_i^2} + a(\nabla_{ii}u)^2 + a \sum_j \nabla_j u \nabla_{ijj}u - b \nabla_{ii}(u - u).
\]

Rewrite Equation (1.1) as

\[
\log \det \nabla_{ij}u = n \log \psi
\]

and differentiate it twice to obtain at \( x_0 \),

\[
\sum_i \frac{\nabla_{jii}u}{\lambda_i} = n \nabla_j(\log \psi) \quad \text{for all } j,
\]

\[
\sum_i \frac{\nabla_{11i}u}{\lambda_i} - \sum_{i,j} \frac{(\nabla_{1ij}u)^2}{\lambda_i \lambda_j} = n \nabla_{11}(\log \psi).
\]

Using (1.5) we write for \( i \geq 2 \),

\[
(\nabla_{i11}u)^2 = (\nabla_{i11}u)^2 - 2 \nabla_{i11}u \nabla_{i1}u R_{11}^i - (\nabla_{i1}u R_{11}^i)^2,
\]

and combine with (2.18) to derive

\[
\sum_{i,j} \frac{(\nabla_{1ij}u)^2}{\lambda_i \lambda_j} - \sum_i \frac{(\nabla_{i11}u)^2}{\lambda_i^2} \geq -Ca - Cb \sum_i \frac{1}{\lambda_i}.
\]

Now multiplying (2.19) by \( \lambda_1 \lambda_i^{-1} \), from (1.8), (2.21) and (2.23) we find with \( f = n \log \psi \),

\[
0 \geq \nabla_{11}f - C(1 + a) - C(1 + b + \lambda_i) \sum_i \frac{1}{\lambda_i}
\]

\[
+ a\lambda_i^2 + a\lambda_1 \sum_{i,j} \frac{\nabla_{j}u \nabla_{ijj}u}{\lambda_i} - b\lambda_1 \sum_i \frac{\nabla_{ii}(u - u)}{\lambda_i}.
\]

Since \( u \) is locally strictly convex, we see at \( x_0 \),

\[
\sum_i \frac{\nabla_{ii}(u - u)}{\lambda_i} \leq n - c_1 \sum_i \frac{1}{\lambda_i},
\]

for some uniform constant \( c_1 > 0 \) depending only on \( \nabla^2 u \). Using (1.5), (2.18) and (2.20) we calculate

\[
\nabla_{11}f \geq \sum_j f_{p_j} \nabla_{11j}u - C(1 + \lambda_i^2)
\]

\[
\geq -a\lambda_i \sum_j f_{p_j} \nabla_{j}u \nabla_{jj}u - C(1 + b\lambda_1 + \lambda_i^2),
\]

\[9\]
\[ \sum_{i,j} \frac{\nabla_j u \nabla_{ij} u}{\lambda_i} \geq \sum_j \nabla_j u \nabla_j f - C \sum_i \frac{1}{\lambda_i} \]

\[ \geq \sum_j f_{p_j} \nabla_j u \nabla_{jj} - C \sum_i \frac{1}{\lambda_i}. \]

Thus

\[ \nabla_{11} f + a \lambda_1 \sum_{i,j} \frac{\nabla_j u \nabla_{ij} u}{\lambda_i} \geq -C(1 + (a + b)\lambda_1 + \lambda_1^2) - C a \lambda_1 \sum_i \frac{1}{\lambda_i} \]

and from (2.24),

\[ 0 \geq \left[ (a - C)\lambda_1^2 - C(a + b)\lambda_1 - C \right] + \left[ (bc_1 - Ca)\lambda_1 - C(1 + b) \right] \sum_i \frac{1}{\lambda_i}. \]

Consequently, from (2.26) we have either

\[ (a - C)\lambda_1^2 - C(a + b)\lambda_1 - C \leq 0, \quad \text{or} \quad (bc_1 - Ca)\lambda_1 - C(1 + b) \leq 0. \]

Choosing \( b \gg a \gg 1 \) then yields the desired upper bound for \( \lambda_1 \). This completes the proof of Theorem 2.1.

\[ \square \]

**Remark.** If from part (b) of the above proof it is easy to see that the following result holds for Monge-Ampère type equations on manifolds without boundary.

**Theorem 2.3.** Let \( M^n \) be a smooth compact Riemannian manifold without boundary. Let \( u \in C^4(M) \) be a solution of the Monge-Ampère equation

\[ (g^{-1} \det(g_{ij} + \nabla_{ij} u))^\frac{1}{2} = \psi(x, u, \nabla u), \quad \{g_{ij} + \nabla_{ij} u\} > 0 \quad \text{on} \ M, \]

where \( \psi > 0 \) is a smooth function. Then

\[ |\nabla^2 u| \leq C \equiv C(n, M, \psi, |u|_{C^0}). \]

Note that hypothesis (1.2) is not needed here. In order to derive (2.28) we may follow part (b) to estimate the quantity

\[ W = \max_{x \in \Omega} \max_{|\xi|=1, \xi \in T_x M^n} \nabla_{\xi_i} u \exp \left( \frac{a}{2} |\nabla u|^2 - bu \right), \]

\( a, b \) being positive constants to be determined. Then we have to estimate the gradient \( \nabla u \) in terms of \( |u|_{C^0} \). To this end, we consider the function \( w = u + \frac{1}{2} |\nabla u|^2 \). At a point \( x_0 \in M \) where \( w \) attains its maximum value, we have

\[ 0 = \nabla_i w = (g_{ij} + \nabla_{ij} u)g^{ik} \nabla_k u \quad \text{for} \ i = 1, \ldots, n. \]

Since \( \{g_{ij} + \nabla_{ij} u\} > 0 \), it follows that \( |\nabla u(x_0)| = 0 \) and thus \( w(x_0) = u(x_0) \). Now \( |\nabla u(x)|^2 \leq 2(u(x_0) - u(x)) \leq 4 \max_M |u| \) for all \( x \in M \).
3 Interior estimates for second derivatives

In this section we first prove Theorem 1.3 by construction of a suitable test function. Then we give a proof of Theorem 1.2 using approximation based on Theorem 1.1 and Theorem 1.3.

**Proof of Theorem 1.3.** Without loss of generality, we may assume \( v \geq u \) in \( \Omega \). For if otherwise we can apply Theorem 1.1 using \( v \) as a strict subsolution to find a locally strictly convex function \( w \in C^\infty(\bar{\Omega}) \) satisfying

\[
\det(w_{ij}) = \epsilon_0 \text{ in } \Omega, \quad w = \varphi \text{ on } \partial \Omega
\]

with \( 0 < \epsilon_0 < \min_{x \in \Omega}\{\psi(x, u(x), Du(x)), \psi(x, v, Dv(x))\} \). By the maximum principle, we have \( w \geq u \) in \( \Omega \). Now we may replace \( v \) by \( w \).

Next, let \( h \) be the harmonic function in \( \Omega \) with \( h = \varphi \) on \( \partial \Omega \). Since \( \Delta v > 0 \) in \( \Omega \), where \( \Delta \) is the Laplace-Beltrami operator, there exists a uniform constant \( \epsilon_0 > 0 \) such that

\[
(h - v)(x) \geq \epsilon_0 \text{dist}_{M^n}(x, \partial \Omega) \quad \text{for } x \in \Omega.
\]

For \( t > 0 \) consider the function \( v^t \equiv v + t(h - v) \); we may choose \( t > 0 \) sufficiently small so that \( v^t \) is still locally strictly convex. Write \( \eta \equiv v^t - u \) and set

\[
W = \max_{x \in \Omega} \max_{|\xi| = 1, \xi \in T_x M^n} \eta^N \nabla \xi u \left(e^{\frac{a}{2} |\nabla u|^2}\right),
\]

where \( a \) and \( N \) are positive constants to be determined later. We shall first derive a bound for \( W \). Since \( \eta = 0 \) on \( \partial \Omega \), \( W \) is achieved at some interior point \( x_0 \in \Omega \) and for some unit vector \( \xi \in T_{x_0} M^n \). We may choose a smooth orthonormal local frame \( e_1, \ldots, e_n \) about \( x_0 \) such that \( e_1(x_0) = \xi \) and \( \left\{ \nabla_i u(x_0) \right\} \) is diagonal. Set

\[
\lambda_i = \nabla_i u(x_0) > 0 \quad \text{for } i = 1, \ldots, n.
\]

The function \( N \log \eta + \log \nabla_{11} u + \frac{a}{2} |\nabla u|^2 \) (defined near \( x_0 \)) then attains a maximum at \( x_0 \) where for all \( i \),

\[
N \frac{\eta_i}{\eta} + \frac{\nabla_i u}{\lambda_i} + a \lambda_i \nabla_i u = 0,
\]

and

\[
0 \geq N \frac{\nabla_i \eta}{\eta} - N \left( \frac{\nabla_i \eta}{\eta} \right)^2 + \frac{\nabla_{i11} u}{\lambda_i} - \frac{\left( \nabla_{i11} u \right)^2}{\lambda_i^2} + a \lambda_i^2 + a \sum_j \nabla_j u \nabla_{ijj} u.
\]
From (3.2) we have for $i \geq 2$,

$$
N \left( \frac{\nabla_i \eta}{\eta} \right)^2 = \frac{(\nabla_{i11} u)^2}{N \lambda_i^2} - \frac{2a \lambda_i \nabla_i u \nabla_i \eta}{\eta} - \frac{(a \lambda_i \nabla_i u)^2}{N}.
$$

We use (2.22) and (3.2) to find when $N \geq 1$,

$$
\sum_{i,j} \left( \frac{\nabla_{ij} u}{\lambda_i \lambda_j} \right)^2 - \frac{(\nabla_{111} u)^2}{\lambda_1^2} - \left( 1 + \frac{1}{N} \right) \sum_{i>1} \left( \frac{\nabla_{i11} u}{\lambda_i \lambda_i} \right)^2 \geq -Ca - \frac{CN}{\eta} \sum_i \frac{1}{\lambda_i}.
$$

Multiplying (3.3) by $\lambda_1 \lambda_i^{-1}$, from (1.8), (2.21), (3.4) and (3.5) we see that

$$
0 \geq \frac{N \lambda_1}{\eta} \sum_i \frac{\nabla_i \eta}{\lambda_i} - N \left( \frac{\nabla_1 \eta}{\eta} \right)^2 - \frac{Ca \lambda_1}{\eta} + \nabla_{11} f
$$

$$
- Ca - C \left( \lambda_1 + \frac{N}{\eta} \right) \sum_i \frac{1}{\lambda_i} + a \lambda_i + a \lambda_i \sum_{i,j} \frac{\nabla_j u \nabla_{ij} u}{\lambda_i}
$$

where $f = n \log \psi$. Similarly to (2.25) we have

$$
\nabla_{11} f + a \lambda_1 \sum_{i,j} \frac{\nabla_j u \nabla_{ij} u}{\lambda_i} \geq -C - C \lambda_1 \left( a + \lambda_1 + \frac{N}{\eta} \right) - Ca \lambda_1 \sum_i \frac{1}{\lambda_i}.
$$

We observe that since $v^t$ is locally strictly convex,

$$
\sum_i \frac{\nabla_i \eta}{\lambda_i} = \sum_i \frac{\nabla_i v^t}{\lambda_i} - n \geq c_1 \sum_i \frac{1}{\lambda_i} - n,
$$

for some $c_1 > 0$ depending only on $\nabla^2 v^t$. Now multiply (3.6) by $\eta^2 \exp a |\nabla u|^2$ to obtain

$$
0 \geq (a - C)W^2 - C(a + N)W - CN + \eta^{N-1} e^{\frac{a}{2} |\nabla u|^2} (c_1 NW - CaW - CN) \sum_i \frac{1}{\lambda_i},
$$

which implies either

$$
0 \geq (a - C)W^2 - C(a + N)W - CN \leq 0, \quad \text{or} \quad (c_1 N - Ca)W - CN \leq 0.
$$

Taking $N \gg a \gg 1$ then yields a bound for $W$. Finally, for any $x \in \Omega$ we have

$$
\max_{||\xi||=1} \nabla_{\xi \xi} u(x) \leq W \eta^N \exp \frac{-a}{2} |\nabla u(x)|^2,
$$

and therefore, (1.4) follows from the inequality (3.1).

We now prove Theorem 1.2.
\textbf{Proof of Theorem 1.2.} For each integer $k \geq 1$ set
\begin{equation}
\psi^k(x,z,p) \equiv \psi(x,z,p) - \frac{1}{2k} \psi_0,
\end{equation}
where $\psi_0$ as in (2.3), and consider the Dirichlet problem for
\begin{equation}
\left(g^{-1} \det(\nabla_{ij} u)\right)^{1/n} = \psi^k(x,u,\nabla u) \quad \text{in} \quad \bar{\Omega}, \quad u = \varphi \quad \text{on} \quad \partial \Omega.
\end{equation}
We note that $u$ is a strict subsolution of (3.8). Thus it follows from Theorem 1.1 that (3.8) admits a solution $u^k \in \mathcal{A}$ for each $k \geq 1$. By (2.2) we have a uniform $C^1$ bound
\begin{equation}
\|u^k\|_{C^1(\bar{\Omega})} \leq C_0 \quad \text{independent of } k.
\end{equation}
We do not have global \emph{a priori} estimates independent of $k$ for second derivatives. But we can apply Theorem 1.3 to $u^k$ (with $v = u^1$) to obtain for all $k \geq 2$
\begin{equation}
|\nabla^2 u^k(x)| \leq \frac{C}{(\text{dist}_{M^n}(x, \partial \Omega))^{N}}, \quad \text{for } x \in \Omega.
\end{equation}
Here $C$, $N$ are uniform constants independent of $k$. It follows from the $C^{2,\alpha}$ interior estimate of L. C. Evans [5] that for any subdomain $\Omega' \subset \subset \Omega,$
\begin{equation}
\|\nabla^2 u^k\|_{C^\alpha(\bar{\Omega}')} \leq C, \quad \text{uniformly for } k \geq 2.
\end{equation}
We conclude from (3.9), (3.11) and the standard regularity theory that $u^k$ has a subsequence which converges to a solution $u \in C^\infty(\Omega) \cap C^{0,1}(\Omega)$ of (1.1). \hfill \square
References


