THE DIRICHLET PROBLEM FOR A CLASS OF FULLY NONLINEAR ELLIPTIC EQUATIONS

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1. Introduction

In the theory of classical solutions of the Dirichlet problem for nonlinear elliptic equations of second order, many existence results are established under certain curvature conditions on the boundary of the underlying domains. In particular, this has been the case for the mean curvature equation

\[(1 + |Du|^2) \Delta u - u_i u_{ij} = nH (1 + |Du|^2)^{3/2}\]

(henceforth \(u_i = \partial u/\partial x_i, u_{ij} = \partial^2 u/\partial x_i \partial x_j, Du = \text{grad} \ u\) for a function \(u\), and summation convention is understood), and the Monge-Ampère equation

\[
\det(u_{ij}) = \psi(x, u, Du) > 0,
\]

which is closely related to the Gauss-Kronecker curvature of hypersurfaces of Euclidean spaces. Let \(\Omega\) be a bounded smooth domain in \(\mathbb{R}^n\). A well known theorem of J. Serrin [13] states that (1.1) admits a solution \(u \in C^\infty(\bar{\Omega})\) with boundary value \(u = \varphi\) on \(\partial \Omega\), for constant \(H\) and arbitrary \(\varphi \in C^\infty(\partial \Omega)\) if and only if the mean curvature of \(\partial \Omega \geq n|H|/(n - 1)\) everywhere. In the case that the underlying domain \(\Omega\) is strictly convex, the Dirichlet problem for Monge-Ampère equation (1.2) has been studied extensively. It was proved by Caffarelli, Nirenberg and Spruck [1], and independently by N. V. Krylov [11] that the solvability reduces to the existence of smooth convex subsolutions. Trudinger and Urbas [18] studied necessary and sufficient conditions for the existence of solutions. However, as far as the author is aware, until the recent papers [9], [8], there had been no existence results on the Dirichlet problem for the Monge-Ampère equations in non-convex domains.

In [8] Spruck and the author proved the following result: Let \(\varphi \in C^\infty(\partial \Omega)\), and \(\psi \in C^\infty(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)\) be a positive function which satisfies the condition that \(\psi^{1/n}(x, z, p)\) is convex in \(p\). Then (1.2) has a locally convex solution \(u \in C^\infty(\bar{\Omega})\) with \(u = \varphi\) on
\( \partial \Omega \), provided there exists a locally convex function \( \underline{u} \in C^\infty(\overline{\Omega}) \) satisfying \( \underline{u} = \varphi \) on \( \partial \Omega \) and

\[
(1.3) \quad \det(u_{ij}) \geq \psi(x, \underline{u}, D\underline{u}) + \delta_0 \quad \text{in} \quad \overline{\Omega}, \quad \text{for some} \ \delta_0 > 0.
\]

This result, with no curvature assumptions on \( \partial \Omega \) being made, has great advantage in applications to geometric problems. In particular, it contains an optimal existence result for graphs over domains in \( \mathbb{R}^n \) of prescribed Gauss curvature. However, perhaps what is more interesting from the viewpoint of nonlinear elliptic theory is the proof of the result. The subsolution \( \underline{u} \) is used in a crucial way in deriving a priori boundary estimates for the second derivatives of solutions. The following question then naturally arises: Do similar results hold for other nonlinear equations?

The purpose of the present paper is to show that the idea of [8] can be used to treat a more general class of fully nonlinear elliptic equations in arbitrary domains. Throughout the paper, let \( f \) be a smooth symmetric function defined in an open convex symmetric cone \( \Gamma \) in \( \mathbb{R}^n \) with vertex at the origin, \( \Gamma \neq \mathbb{R}^n \), and containing the cone \( \Gamma^+ \equiv \{ \lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0 \} \). We consider the Dirichlet problem in a bounded smooth domain \( \Omega \subset \mathbb{R}^n \)

\[
(1.4) \quad F(D^2u) \equiv f(\lambda(D^2u)) = \psi(x, u, Du) \quad \text{in} \quad \overline{\Omega},
\]

\[
\underline{u} = \varphi \quad \text{on} \quad \partial \Omega.
\]

Here \( \lambda(D^2u) = (\lambda_1, \ldots, \lambda_n) \) denotes the eigenvalues of the Hessian matrix \( D^2u = \{u_{ij}\} \). We assume \( \varphi \in C^\infty(\partial \Omega), \psi \in C^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n) \) and, for convenience, \( \psi > 0 \).

We seek solutions of (1.4) among functions \( u \in C^2(\overline{\Omega}) \) with \( \lambda(D^2u) \in \Gamma \); we call such functions admissible. The equation is assumed to be elliptic at admissible functions, i.e.,

\[
(1.5) \quad f_i \equiv \frac{\partial f}{\partial \lambda_i} > 0 \quad \text{in} \quad \Gamma, \quad 1 \leq i \leq n,
\]

and to satisfy

\[
(1.6) \quad f \text{ is a concave function},
\]

\[
(1.7) \quad \limsup_{\lambda \to \lambda_0} f(\lambda) \leq 0 \quad \text{for every} \ \lambda_0 \in \partial \Gamma.
\]

This type of equation was first studied by Caffarelli, Nirenberg and Spruck. In [3], they studied the Dirichlet problem (1.4) for \( \psi = \psi(x) \) (i.e., \( \psi \) depends only on \( x \)),
in a domain $\Omega$ satisfying the condition that there exists a sufficiently large number $R > 0$ such that, at every point $x \in \partial \Omega$,
\begin{equation}
(1.8) \quad (\kappa_1, \ldots, \kappa_{n-1}, R) \in \Gamma.
\end{equation}
Here $\kappa_1, \ldots, \kappa_{n-1}$ represent the principal curvatures of $\partial \Omega$ (with respect to the interior normal). Their main theorem states that there exists a unique admissible solution $u \in C^\infty(\Omega)$ of (1.4) if (1.5)-(1.8) hold and, for every $C > 0$ and every compact set $K$ in $\Gamma$ there is a number $R = R(C, K)$ such that
\begin{equation}
(1.9) \quad f(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n + R) \geq C \quad \text{for all } \lambda \in K,
\end{equation}
\begin{equation}
(1.10) \quad f(R\lambda) \geq C \quad \text{for all } \lambda \in K.
\end{equation}
Moreover, in case $\phi \equiv$ constant, condition (1.8) is also necessary for existence of admissible solutions.

This result is proved using the continuity method and \textit{a priori} estimates. Condition (1.8) plays an essential role both in the construction of admissible subsolutions and for the estimate of second derivatives at boundary points.

The above theorem was extended to the case $\psi = \psi(x, u)$ by Y. Y. Li [12] who proved the following result. In addition to (1.5)-(1.10), assume that
\begin{equation}
(1.11) \quad \lim \inf_{\lambda \to 0, \lambda \in \Gamma} f(\lambda) > -\infty,
\end{equation}
\begin{equation}
(1.12) \quad \lim_{|\lambda| \to \infty} \sum_i f_i(\lambda) = \infty \quad \text{in } \{ \lambda \in \Gamma : f(\lambda) \geq \delta \}, \quad \text{for any } \delta > 0.
\end{equation}
Then there exists an admissible solution of (1.4) provided that there exists an admissible subsolution.

Condition (1.8) generally is a quite strong restriction. It is shown in [3] that if the positive $\lambda_i$ axes belong to $\partial \Gamma$, and $\Omega$ satisfies (1.8), then $\partial \Omega$ is necessarily connected.

In this paper we will treat problem (1.4) in an arbitrary smooth domain $\Omega$. We emphasize that no curvature restrictions on $\partial \Omega$ will be made. Instead, we assume that there exists an admissible strict subsolution $u \in C^\infty(\overline{\Omega})$ of (1.4), that is, for some $\delta_0 > 0$
\begin{equation}
F(u) \geq \psi(x, u, Du) + \delta \quad \text{in } \overline{\Omega},
\end{equation}
\begin{equation}
(1.13) \quad u = \varphi \quad \text{on } \partial \Omega.
\end{equation}
The function $\psi$ here is allowed to depend on $Du$. We assume that

$$\psi(x,z,p) \text{ is convex in } p$$

and, for technical reasons, that for every fixed $(x,z) \in \overline{\Omega} \times \mathbb{R}$,

$$\inf_p \psi > 0, \quad \inf_p \frac{\partial \psi}{\partial z} > -\infty, \quad \sup_p \frac{|D_x \psi|}{1 + |p|} < \infty.$$ 

We state our main result of this paper.

**Theorem 1.1.** Assume (1.5)-(1.7), (1.9) and (1.12)-(1.15). There exists an admissible solution $u \in C^\infty(\overline{\Omega})$ of (1.4) with $u \geq \underline{u}$. Moreover, the solution is unique if $\psi_z \geq 0$.

A typical example of $f$ is the function $S_k(\lambda) = (\sigma^{(k)}(\lambda))^{1/k}$ defined on $\Gamma_k$, where

$$\sigma^{(k)}(\lambda) = \sum_{i_1 < \ldots < i_k} \lambda_{i_1} \ldots \lambda_{i_k}$$

is the $k$-th elementary symmetric function. The cone $\Gamma_k$ is the connected component of the set $\{ \lambda \in \mathbb{R}^n : \sigma^{(k)}(\lambda) > 0 \}$ containing $\Gamma^+$. It is verified in [3] and [12] that for $2 \leq k \leq n$, $S_k$ satisfies (1.5)-(1.7), (1.9)-(1.12). As a corollary of Theorem 1.1 we have

**Theorem 1.2.** There exists a solution in $C^\infty(\overline{\Omega})$ to the Dirichlet problem

$$\sigma^{(k)}(\lambda(D^2u)) = \eta(x,u,Du) \text{ in } \overline{\Omega}, \quad 2 \leq k \leq n, \quad u = \varphi \text{ on } \partial\Omega,$$

provided that there exists an admissible strict subsolution and that the function $\psi = \eta^{1/k}$ satisfies (1.14)-(1.15).

The Monge-Ampère equation corresponds to $\sigma^{(n)}$. Also note that $\Gamma_n = \Gamma^+$ and in this case (1.8) simply means $\Omega$ is a strictly convex domain.

It would be interesting to see whether Theorem 1.1 still holds when $\delta_0$ in (1.13) vanishes, i.e., $\underline{u}$ is merely a subsolution. Using approximation and the gradient bounds (independent of $\delta_0$) derived in Section 2 below, we obtain from Theorem 1.1 a weak solution in the viscosity sense.

**Corollary 1.3.** Under the assumptions of Theorem 1.1 with $\delta_0 = 0$ in (1.13), (1.4) admits a Lipschitz viscosity solution which is also unique if $\psi_z \geq 0$. 
The interested reader is referred to [5] for an excellent exposition of the basic theory of viscosity solutions and further references. We remark that for the Monge-Ampère equation, the solution (obtained in Corollary 1.3) is actually smooth inside the domain; see [7] for details.

Another natural class of nonlinear elliptic equations of interest consists of the prescribed curvature equations of the form

\[ f(\kappa[u]) = \psi(x, u), \]

where \( f \) is as in (1.4), and \( \kappa[u] = (\kappa_1, \ldots, \kappa_n) \) denotes the principal curvatures of the graph of \( u \). In particular, the cases \( f = \sigma^{(1)} \) and \( f = \sigma^{(n)} \) correspond to the mean curvature and the Gauss curvature equations, respectively.

The Dirichlet problem for general equations of type (1.17) has been treated by Caffarelli-Nirenberg-Spruck [4] for strictly convex domains and constant boundary data, and by Trudinger [16] in the viscosity sense. In [10] Ivochkina considered the case \( f = \sigma^{(k)} \) and proved the solvability of the Dirichlet problem in domains whose boundaries satisfy certain curvature conditions. Her result extends that of Serrin mentioned at the beginning. Related results can also be found in the work of Trudinger [15], [17]. We expect that results similar to Theorem 1.1 hold for this class of equations; we hope to study this problem in future work. We note also that the case of mean curvature is a borderline and possibly limiting case.

Fully nonlinear elliptic equations have received considerable study since the beginning of the eighties. Among the major recent works are the contributions of L. C. Evans [6], N. V. Krylov [11], Caffarelli-Kohn-Nirenberg-Spruck [2] and Trudinger [14] who have discovered \( C^{2,\alpha} \) estimates from \( C^2 \) bounds for fully nonlinear uniformly elliptic equations of second order with nonlinearity being concave (or convex) in \( D^2 u \). This result is fundamental to our proof of the existence part in Theorem 1.1. In Section 2 we first make use of degree theory and reduce the proof to establishing the estimate

\[ \|u\|_{C^2(\Omega)} \leq C, \]

for all admissible solutions \( u \) of (1.4) with \( u \geq u_0 \). Then we derive the \( C^1 \) bounds. The second derivative estimates are established in Section 3. These estimates at the boundary points comprise the main work of this paper. The uniqueness in Theorem 1.1 is a direct consequence of the maximum principle.
We conclude this section with some remarks concerning our assumptions. It is shown in [3] that (1.6) implies that the function $F$ is concave in $D^2 u$. Our proof will heavily rely on this fact. Condition (1.7) implies that, once (1.18) is established, the set of $\lambda(u_{ij})$ for all admissible solutions $u$ of (1.4) with $u \geq \bar{u}$ is contained in a compact subset of $\Gamma$, and therefore, (1.4) is uniformly elliptic at any admissible solution $u \geq \bar{u}$. This enables us to extend (1.18) directly to $C^{2,\alpha}$ estimates through elliptic machinery. Condition (1.7) also enters at a critical step in the proof of (1.18). The assumptions in (1.15) purely arise from the technique we employ for gradient bounds. The growth rate condition of $D_x \psi$ in $p$ does not seem very natural in the sense of conditions such as (1.13), (1.14) and perhaps may be improved (we will not, however, pursue the optimal growth condition as it is not the main concern of this paper). On the other hand, if one assumes in addition that, for any fixed $(x, z) \in \Omega \times \mathbb{R}$,

\begin{equation}
\inf_{p} \psi_z > 0, \sup_{p} |D_x \psi| < \infty,
\end{equation}

one can replace (1.12) by a weaker condition:

\begin{equation}
\lim_{|\lambda| \to \infty} \sum_i f_i(1 + \lambda_i^2) = \infty \text{ in } \{ \lambda \in \Gamma : f(\lambda) \geq \delta \}, \text{ for any } \delta > 0.
\end{equation}

This will be clearly seen from the proof. Finally we note that (1.10) is not assumed in Theorem 1.1.

2. Beginning of proof

In this section we first explain how to make use of degree theory and a priori estimates to prove the existence in Theorem 1.1. To start with, we note that $F(u_{ij}) > \delta_0 > 0$ in $\Omega$. So by the implicit function theorem we can find an admissible function $u^0 \in C^\infty(\overline{\Omega})$ with $u^0 = \bar{u}$ on $\partial \Omega$, and satisfying

\begin{equation}
0 < F(u_{ij}^0) \leq F(u_{ij}) - \epsilon_0 \text{ in } \overline{\Omega}, \text{ for some } 0 < \epsilon_0 < \delta_0.
\end{equation}

The maximum principle then implies $u^0 > \bar{u}$ in $\Omega$. For each $t$ in $0 \leq t \leq 1$ we wish to find an admissible solution $u^t \in C^\infty(\overline{\Omega})$, $u^t \geq \bar{u}$, of

\begin{equation}
F(u_{ij}^t) = \psi^t(x, u^t, D u^t) \text{ in } \overline{\Omega}, \quad u^t = \phi \text{ on } \partial \Omega,
\end{equation}

where $\psi^t = t\psi + (1-t)F(u_{ij}^0)$; $u^1$ then is the desired solution of (1.4) in Theorem 1.1. We note that (2.2) satisfies all assumptions in Theorem 1.1. For example, $\bar{u}$ is an
strict subsolution of (2.2) for all \( t \), i.e.,
\[
F(u_{ij}) \geq \psi'(x, u, Du) + \epsilon_0, \quad 0 \leq t \leq 1.
\]

As in [1, Section 7] and [12, Section 4], the existence for (2.2) can be proved using degree theory and the \textit{a priori} estimate
\[
\|u^t\|_{C^{2,\alpha}(\Omega)} \leq C, \quad \text{independent of } t
\]
for all admissible solutions \( u^t \geq u \) of (2.2).

It is shown in [2] and [11] for uniformly elliptic fully nonlinear second order equations which are concave in the second derivatives, the \( C^{2,\alpha} \) norm for solutions can be estimated using \textit{a priori} \( C^2 \) bounds. Thus as remarked at the end of Section 1 it suffices for us to derive
\[
\|u^t\|_{C^2(\Omega)} \leq C, \quad \text{independent of } t.
\]

Ignoring \( t \) let us assume \( u \geq u \) to be an admissible solution of (1.4) in \( C^\infty(\Omega) \) and derive (1.18) to finish our proof of Theorem 1.1. First note that, since \( \Gamma \) is a symmetric cone which is not \( \mathbb{R}^n \) and contains \( \Gamma^+ \),
\[
\sum \lambda_i > 0 \quad \text{for all } \lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma.
\]
It follows from the maximum principle that \( u \leq u \leq h \) in \( \Omega \), where \( h \) is the harmonic function in \( \Omega \) with \( h = \varphi \) on \( \partial \Omega \). Consequently,
\[
|u| \leq K \text{ in } \Omega \text{ and } |Du| \leq K \text{ on } \partial \Omega.
\]
Moreover, it follows from (1.15) that for some uniform constants \( \psi_0, C_0 > 0 \),
\[
\psi(x, u, Du) \geq \psi_0 > 0,
\]
\[
\psi_z(x, u, Du) \geq -C_0, \quad |D_x \psi(x, u, Du)| \leq C_0(1 + |Du|).
\]

In the rest of the paper, \( L \) will denote the linear operator defined by
\[
Lw = F^{ij}w_{ij} - \psi_{p_i}w_i - cw, \quad \text{for } w \in C^2(\Omega),
\]
where
\[
F^{ij} = \frac{\partial F}{\partial u_{ij}}(D^2u), \quad \psi_{p_i} = \psi_{p_i}(x, u, Du)
\]
and
\[
c = \max_{x \in \Omega} \max_{|z| \leq 2K} |\psi_z(x, z, Du)|, \quad K \text{ as in (2.5)}.
\]
Lemma 2.1. There exists $\epsilon > 0$, independent of $\delta_0$ in (1.13), such that

\begin{equation}
L(u - u) \leq \frac{\psi_0}{2} - \epsilon \sum_i F^{ii}.
\end{equation}

Proof. Consider $w = u - \frac{\epsilon}{2} |x|^2$. Since $u$ is a subsolution, for $\epsilon$ small enough, $w$ is still admissible and satisfies

$$F(D^2 w) \geq \psi(x, u, Du) - \frac{\psi_0}{2}.$$ 

By the concavity of $F$ we have

$$F(D^2 w) \leq F(D^2 u) + F^{ij}(w_{ij} - u_{ij}),$$

and hence

\begin{equation}
F^{ij}(u_{ij} - u_{ij}) \leq \frac{\psi_0}{2} - \epsilon \sum_i F^{ii} + \psi(x, u, Du) - \psi(x, u, Du).
\end{equation}

Using the convexity of $\psi$ in $p$ we find

$$\psi(x, u, Du) - \psi(x, u, Du) \leq \psi_z(x, z_0, Du)(u - u) + \psi_p(x, u, Du)(u_j - u_j),$$

for some $z_0$ with $u \leq z_0 \leq u$. Thus (2.8) follows from (2.9). \hfill \Box

We now apply Lemma 2.1 to derive an \textit{a priori} gradient bound

\begin{equation}
|Du| \leq C \text{ on } \Omega.
\end{equation}

Set $v = u - u$ and consider the function $|Du|(1 + v)^{-1}$. Suppose this function achieves its maximum $M_0$ at a point $x_0 \in \Omega$. It suffices to estimate $M_0$.

If $x_0 \in \partial\Omega$, then we have a bound for $M_0$ from (2.5). So we consider the case $x_0 \in \Omega$. We may assume $u_1(x_0) = |Du(x_0)| > 0$. The function $\log u_1 - \log(1 + v)$ then achieves a local maximum at $x_0$. Thus at that point,

\begin{equation}
\frac{u_{1i}}{u_1} - \frac{v_i}{1 + v} = 0 \quad \text{for every } i = 1,\ldots, n
\end{equation}

and, since the matrix $\{F^{ij}\}$ is positive definite,

\begin{equation}
F^{ij} \frac{u_{1ij}}{u_1} - F^{ij} \frac{v_{ij}}{1 + v} \leq 0.
\end{equation}

Differentiating (1.4) with respect to $x_k$ we find

$$F^{ij} u_{ijk} = \psi_{x_k} + \psi_z u_k + \psi_p u_{jk}.$$ 

Plug this with $k = 1$ into (2.12) and make use of (2.11) to derive

$$\frac{-1}{1 + v} L v + \psi_z + \frac{\psi_{x_1}}{u_1} \leq c.$$
It follows from (2.7) and Lemma 2.1 that
\[ \sum_i f_i \equiv \sum_i F^{ii} \leq C. \]
In view of (1.12) and (2.6) this implies \( |u_{11}(x_0)| \leq C \) and hence yields a bound for \( u_1(x_0) \) by (2.11); (2.10) is thus established.

**Remark.** Consider the function \( \frac{1}{2} |Du|^2 - v \). Suppose it attains its maximum value at a point \( x_0 \in \Omega \), then at \( x_0 \) we have
\[ u_k u_{ki} - v_i = 0, \quad F^{ij}(u_k u_{kij} + u_{ki} u_{kj}) - F^{ij} v_{ij} \leq 0. \]
Thus (note that \( F^{ij} u_{ki} u_{kj} \equiv \sum_i f_i \lambda_i^2 \)),
\[ \sum_i f_i \lambda_i^2 - L v + \psi_x u_k + \psi_z |Du|^2 \leq C. \]
Using the Cauchy-Schwarz inequality, one sees that a bound for the gradient can be derived from assumptions (1.19), (1.20) and Lemma 2.1.

3. **A priori estimates for second derivatives**

In this section we establish *a priori* estimates in \( \Omega \) for the second derivatives of \( u \) in two steps. First we show how to derive
\[ (3.1) \quad |u_{ij}| \leq C \quad \text{in } \overline{\Omega}, \]
if we know bounds for \( u_{ij} \) on \( \partial \Omega \). Then we come back to estimate \( u_{ij} \) on \( \partial \Omega \).

**Step 1.** Let us first assume we have a bound
\[ (3.2) \quad |u_{ij}| \leq C \quad \text{on } \partial \Omega. \]
In order to derive (3.1), it suffices to estimate
\[ M = \max_{x \in \Pi} \max_{|\xi| = 1} \frac{\partial^2 u}{\xi^2} v + b - \frac{1}{2} |Du|^2, \]
where \( v = u - \underline{\bar{u}} \) and \( b = 1 + \max_{\Pi} |Du|^2 \). If the maximum occurs on \( \partial \Omega \), then \( M \) is estimated via (3.2). So consider the case that \( M \) is achieved at some point \( x_0 \) in \( \Omega \) and for some direction \( \xi \). By rotating the coordinates, we may assume \( \xi = (1, 0, \ldots, 0) \) and \( u_{ij}(x_0) = 0 \) for \( i \neq j \). It suffices to establish a uniform upper bound for \( u_{11}(x_0) \).
The function \( \log u_{11} - \log(v + b - \frac{1}{2}|Du|^2) \) then attains a maximum at \( x_0 \). Hence at that point, since \( u_{ij} \) is diagonal,

\[
\frac{u_{11i}}{u_{11}} - \frac{v_i - u_i u_{ii}}{v + b - \frac{1}{2}|Du|^2} = 0, \quad \text{for all } i,
\]

\[
\frac{u_{11ii}}{u_{11}} - \frac{v_{ii} - u_{ii}^2 - u_j u_{jjii}}{v + b - \frac{1}{2}|Du|^2} \leq 0, \quad \text{for all } i.
\]

Differentiating (1.4) twice, using the concavity of \( F \) and the convexity of \( \psi \) in \( p \), one finds

\[
F^{ii} u_{ij} u_j \geq \psi_p u_j u_{jj} - C,
\]

\[
F^{ii} u_{ii11} \geq (\psi)_11 \geq \psi_p u_{jj11} - C(1 + u_{11}).
\]

Multiplying (3.4) by \( u_{11} F^{ii} \) and taking sum, with the aid of (3.3) and the above inequalities, we obtain

\[
F^{ii} u_{ii}^2 - Lv \leq C.
\]

This and (2.8) then yield

\[
F^{ii} u_{ii}^2 + \sum_i F^{ii} \leq C.
\]

Consequently, a bound for \( u_{11}(x_0) \) follows from (1.20).

**Step 2.** To estimate the second derivatives of \( u \) at an arbitrary point on \( \partial \Omega \), we may assume the point is the origin of \( \mathbb{R}^n \) and that the positive \( x_n \) axis is in direction of the interior normal to \( \partial \Omega \) at 0. Near 0, \( \partial \Omega \) locally can be represented as a graph

\[
x_n = \rho(x') = \frac{1}{2} \sum_{\alpha, \beta < n} B_{\alpha\beta} x_\alpha x_\beta + O(|x'|^3), \quad x' = (x_1, \ldots, x_{n-1}).
\]

From \( (u - \underline{u})(x', \rho(x')) = 0 \) it follows that

\[
(u - \underline{u})_{\alpha\beta}(0) = -(u - \underline{u})_\alpha(0) B_{\alpha\beta}, \quad \alpha, \beta < n
\]

and hence

\[
|u_{\alpha\beta}(0)| \leq C, \quad \alpha, \beta < n.
\]

To estimate \( u_{\alpha n}(0) \) we need make use of the assumption \( \delta_0 > 0 \) in (1.13). First we note that in the proof of Lemma 2.1, by requiring \( \epsilon \) depend on \( \delta_0 \), we may replace \( (-\psi_0) \) by \( \delta_0 \). Consequently, we obtain the following key lemma which is a direct extension of Lemma 2.2 in [8].
Lemma 3.1. There exists a uniform positive constant $\epsilon_1$ such that

\begin{equation}
L(u - u) \leq -\epsilon_1 \left(1 + \sum F^{ii}\right) \text{ in } \Omega.
\end{equation}

With the aid of Lemma 3.1 we can use an upper barrier of the form

\[ w = A(u - u) + B|x|^2 \text{ in } \Omega \cap B_\sigma(0), \]

with $A \gg B$ large and $\sigma > 0$ small, to estimate $u_{an}(0)$, $\alpha < n$. By subtracting a linear function we may assume $u(0) = u_\alpha(0) = 0$, $\alpha < n$. For $\alpha < n$ set

\[ T = \frac{\partial}{\partial x_\alpha} + \sum \beta B_{\alpha\beta} \left( x_\beta \frac{\partial}{\partial x_n} - x_n \frac{\partial}{\partial x_\beta}\right). \]

As in [3] we find

\[ |T(u - u)| \leq C|x|^2 \text{ on } \partial(\Omega \cap B_\sigma(0)), \]

\[ |LT(u - u)| \leq C \left(1 + \sum F^{ii}\right) \text{ in } \Omega \cap B_\sigma(0). \]

In view of Lemma 3.1, if we choose $A \gg B \gg 1$, then

\[ L(w \pm T(u - u)) \leq 0 \text{ in } \Omega \cap B_\sigma(0), \]

\[ w \geq \pm T(u - u) \text{ on } \partial(\Omega \cap B_\sigma(0)). \]

Thus by the maximum principle $w \geq |T(u - u)|$ in $\Omega \cap B_\sigma(0)$. Consequently,

\begin{equation}
|u_{an}(0)| \leq w_n(0) + |u_{an}(0)| \leq C, \quad \alpha < n.
\end{equation}

It remains to prove

\begin{equation}
u_{an}(0) \leq C.
\end{equation}

For this we will modify the argument in [3], using a key idea from [8]. Let $\Gamma'$ denote the projection to $\lambda' = (\lambda_1, \ldots, \lambda_{n-1})$ of $\Gamma$. From now on derivatives are computed at 0.

Lemma 3.2. The distance of $\lambda'(u_{\alpha\beta}) \in \Gamma'$ from $\partial \Gamma'$, the boundary of $\Gamma'$ in $\mathbb{R}^{n-1}$, is greater than some uniform positive constant $c_0$.

We first note that (3.10) will follow from Lemma 3.2. For if $u_{nn}$ is very large then from Lemma 1.2 of [3], $\lambda'(u_{ij})$ is close to $\lambda'(u_{\alpha\beta})$ and so its distance to $\partial \Gamma'$ is greater than $c_0/2$. Thus (by (3.7)) for some constant $M$, $(\lambda'(u_{ij}), M)$ belongs to a compact set in $\Gamma$. But our assumption (1.9) then implies a bound on $\lambda_n$ and hence (3.10).
Proof of Lemma 3.2. Without loss of generality, we may assume that at 0, \( u = u = 0, u_\alpha = u_\alpha = 0, \alpha < n, \) and \( u_n = 0. \) From (3.6) we then have

\[
(3.11) \quad u_{\alpha\beta} = u_{\alpha\beta} - u_n B_{\alpha\beta}. 
\]

We observe that \( \lambda'(u_{\alpha\beta}) \) belongs to \( \Gamma'. \) To see this let \( E_{nn} \) denote the \( n \times n \) matrix whose \( (n, n) \) entry is one and all others are zero. Since \( \lambda(u_{ij}) \in \Gamma, \lambda(u_{ij} + aE_{nn}) \in \Gamma \) for any \( a > 0. \) By Lemma 1.2 of [3], \( \lambda'(u_{ij} + aE_{nn}) \in \Gamma' \) when \( a \) is sufficiently large.

Let \( t_0 \) be the first positive number such that \( \lambda'(\sigma_{\alpha\beta}) \in \partial \Gamma' \), where

\[
(3.12) \quad \sigma = \sum_{\alpha,\beta<n} (u_{\alpha\beta} - t_0 B_{\alpha\beta}) x_\alpha x_\beta. 
\]

In view of (3.11), if \( t_0 = +\infty \) then we are done. So let us assume \( t_0 < +\infty \) and derive a bound

\[
(3.13) \quad u_n \leq t_0 - \eta 
\]

for some uniform \( \eta > 0, \) from which Lemma 3.2 follows.

By rotating coordinates \( x_1, \ldots, x_{n-1}, \) we may assume \( \{\sigma_{\alpha\beta}\} \) to be diagonal with \( \sigma_1 \leq \ldots \leq \sigma_{n-1,n-1}. \) As in [3], one can find a support plane to the cone \( \Gamma' \) at \( \lambda'(\sigma_{\alpha\beta}), \) i.e., there exists a fixed \( \mu' = (\mu_1, \ldots, \mu_{n-1}) \in \mathbb{R}^{n-1} \) with

\[
\mu_1 \geq \cdots \geq \mu_{n-1} \geq 0, \quad \sum_{\alpha<n} \mu_\alpha = 1, \quad \sum_{\alpha<n} \mu_\alpha \sigma_{\alpha\alpha} = 0 
\]

such that

\[
(3.14) \quad \Gamma' \text{ lies in } \{ \lambda' \in \mathbb{R}^{n-1} : \mu' \cdot \lambda' > 0 \}. 
\]

Let \( S \) be a surface represented locally near 0 by

\[
x_n = \rho(x') - \frac{\tau}{2} |x'|^2 \quad \text{with } 0 < \tau \text{ small,}
\]

and let \( d(x) \) denote the distance of \( x \in \Omega \) to \( S. \) At the origin

\[
d_{\alpha\beta} = \tau \delta_{\alpha\beta} - B_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq n - 1; \quad d_{in} = 0, \quad 1 \leq i \leq n.
\]

Rewrite

\[
B_{\alpha\beta} - \tau \delta_{\alpha\beta} = \frac{1}{t_0} (\omega_{\alpha\beta} - \sigma_{\alpha\beta}), 
\]

where

\[
\omega_{\alpha\beta} = u_{\alpha\beta} - t_0 \tau \delta_{\alpha\beta}. 
\]
We see for fixed positive $\tau$ sufficiently small, $\lambda'(\omega_{\alpha\beta})$ lies in some compact subset of $\Gamma'$. Hence by Lemma 6.2 of [3], we obtain

\[
\sum_{\alpha<n} \mu_{\alpha} d_{\alpha \alpha} \leq -a < 0,
\]

for some fixed positive constant $a$; $\tau$ is so fixed.

On $\partial \Omega$ near 0 we have

\[
u = \varphi = \frac{1}{2} \sum_{\alpha, \beta<n} \varphi_{\alpha\beta} x_{\alpha} x_{\beta} + t_0 \left( \frac{1}{2} \sum_{\alpha, \beta<n} B_{\alpha\beta} x_{\alpha} x_{\beta} - \rho(x') \right) + P(x') + O(|x'|^4),
\]

where $P$ is a homogeneous cubic in $x'$. For $h > 0$ small let

\[D_h = \{ x \in \Omega : |x'| < h, |x_n| < h^2 \}.
\]

In $D_h$, we will employ the barrier function

\[v = w + \eta(C|x'|^2 - x_n),\]

where

\[w = \sigma(x') + t_0 x_n + P(x') + l(x') \left( \frac{\tau}{2} |x'|^2 - d \right) + \frac{M}{2} d^2.
\]

Now following exactly the argument in [3, pp 289-291], we can choose suitable positive constants $C, M$ large, $h, \eta$ small, and a linear function $l(x')$, such that

\[
u \leq v \quad \text{on} \quad \partial D_h,
\]

and at every point in $D_h$,

\[
\lambda(v_{ij}) \notin \{ \lambda \in \Gamma : f(\lambda) \geq \psi_0 \}.
\]

It follows from the maximum principle (see [3, Lemma B]) that $u \leq v$ in $D_h$, which yields (3.13).

\[\square\]

The proof of Theorem 1.1 is thus complete.

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Notes added on December 5, 1995. In his review of this paper, John Urbas points out that condition (1.12), which is used in deriving the a priori gradient estimate (2.10), seems incompatible with (1.6). In the following we derive a gradient bound without assuming (1.12). Instead, we assume that the function $f$ satisfies

\begin{equation}
(3.18) 
  f_i(\lambda) \geq \nu_0 > 0 \quad \text{for any } \lambda \in \Gamma \text{ with } \lambda_i \leq 0.
\end{equation}

We are also able to replace condition (1.15) by the following weaker assumption:

\begin{equation}
(3.19) 
  \inf_p \frac{p \cdot D_x \psi + |p|^2 \psi_z}{(1 + |p|^2)^{\gamma}} > -\infty \quad \text{for some } 0 < \gamma < 2.
\end{equation}

Set $v = u - w$ and

\[ M_0 = \max_{\Omega} we^{av^2}, \quad w = (1 + |Du|^2)^{1/2} \]

where $a > 0$ is a sufficiently small constant to be determined later. It suffices to estimate $M_0$.

Suppose $M_0$ is attained at a point $x_0 \in \bar{\Omega}$. If $x_0 \in \partial \Omega$, then we have a bound for $M_0$ from (2.5). So we consider the case $x_0 \in \Omega$. The function $\log w + av^2$ then achieves a local maximum at $x_0$. Thus at that point,

\begin{equation}
(3.20) 
  \frac{w_i}{w} + 2avv_i = 0 \quad \text{for every } i = 1, \ldots, n
\end{equation}

and, since the matrix $\{F^{ij}\}$ is positive definite,

\begin{equation}
(3.21) 
  F^{ij}w_{ij}w - F^{ij}w_iw_j - 2aF^{ij}v_iv_j + 2avF^{ij}v_{ij} \leq 0.
\end{equation}

We compute

\[ F^{ij}w_{ij} \geq \frac{1}{w}F^{ij}u_{ijk}u_k. \]

As before, we differentiate equation (1.4) with respect to $x_k$ to find

\[ F^{ij}u_{ijk} = \psi_{x_k} + \psi_z u_k + \psi_p u_{jk}. \]

Thus

\[ F^{ij}w_{ij} \geq \frac{1}{w}(\psi_{x_k} u_k + \psi_z |Du|^2) + \psi_p w_j. \]

Combining this inequality and (3.20) with (3.21), we obtain

\[ 2a(1 - 2av^2)F^{ij}v_iv_j + 2av(F^{ij}v_i - \psi_p v_j) + \frac{1}{w^2}(\psi_{x_k} u_k + \psi_z |Du|^2) \leq 0. \]

By the assumptions (1.6) and (1.14) we thus derive

\[ F^{ij}v_{ij} - \psi_p v_j \geq -C. \]
Choosing $\alpha$ such that $4\alpha \max_{\Omega} v^2 \leq 1$ then yields

$$F^{ij}v_i v_j \leq C(1 + w^{2\gamma-2}).$$

(3.22)

Here we have used assumption (3.19). By a rotation of the coordinate system, we may assume $\{u_{ij}(x_0)\}$ is diagonal, and therefore, $F^{ij} = \delta_{ij} f_i$. We may also assume that, at $x_0$, $|u_1| \geq |Du|/n$. Now, assume $|Du(x_0)| \geq 2n \max_{\Omega} |Du|$. Then it follows from (3.20) that

$$u_{11}(x_0) \leq -\alpha v w^2 < 0.$$  

Thus, by (3.18), $F^{11} \geq \nu_0 > 0$. It follows from (3.22) that

$$\nu_0 v_1^2 \leq C(1 + w^{2\gamma-2}).$$

However,

$$|v_1(x_0)| \geq |u_1(x_0)| - |u_1(x_0)| \geq \frac{1}{n} |Du(x_0)| - |D_2u(x_0)| \geq \frac{1}{2n} |Du(x_0)|.$$  

We obtain, at $x_0$,

$$w^2 \leq C(1 + w^{2\gamma-2}),$$

which implies a bound $|w(x_0)| \leq C$.

References


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