A CORRECTION TO “THE DIRICHLET PROBLEM FOR COMPLEX MONGE-AMPERE EQUATIONS AND REGULARITY OF THE PLURI-COMPLEX GREEN FUNCTION”

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I recently learned that the proof of Lemma 3.1 in [1] contains an error. As a result, the proof of Theorem 1.3 in [1] is incomplete. It also affects the proof of Theorem 1.2 which states that the pluri-complex Green function \( g \) for a strongly pseudoconvex domain \( \Omega \subset \mathbb{C}^n \) with a logarithmic pole at a point \( \zeta \in \Omega \) belongs to \( C^{1,\alpha}(\overline{\Omega} - \{\zeta\}) \) for any \( 0 < \alpha < 1 \). Here we present a proof of this result independent of Lemma 3.1 of [1].

**Theorem 1.** Let \( \Omega \) be a smooth bounded strongly pseudoconvex domain in \( \mathbb{C}^n \) and \( \zeta \in \Omega \). Let \( g \) be the pluri-complex Green function for \( \Omega \) with a logarithmic pole at \( \zeta \). Then \( g \in C^{1,\alpha}(\overline{\Omega} - \{\zeta\}) \) for any \( 0 < \alpha < 1 \).

**Proof.** It is known that the pluri-complex Green function \( g \) is the unique weak solution of the problem

\[
\begin{aligned}
& \text{u is pluri-subharmonic in } \Omega - \{\zeta\} \\
& \det(u_{j\bar{k}}) = 0 \quad \text{in } \Omega - \{\zeta\} \\
& u = 0 \quad \text{on } \partial \Omega \\
& u(z) = \log |z - \zeta| + O(1) \quad \text{as } z \to \zeta.
\end{aligned}
\]

We will show that the solution to (1) is in \( C^{1,1}(\overline{\Omega} - \{\zeta\}) \). Without loss of generality, we may assume \( \zeta = 0 \) and \( B_1 = B_1(0) \subset \Omega \).

In [1] we proved that, for each positive \( \varepsilon \leq \varepsilon_0 \) (for some fixed \( \varepsilon_0 \leq \frac{1}{2} \) so that \( B_{2\varepsilon_0} \subset \Omega \)), there exists a unique strictly pluri-subharmonic solution \( u^\varepsilon \in C^\infty(\overline{\Omega}_\varepsilon) \) to the Dirichlet problem

\[
\begin{aligned}
& \det(u_{j\bar{k}}) = \varepsilon \quad \text{in } \Omega_\varepsilon \equiv \Omega - \overline{B}_\varepsilon, \\
& u = u_0 \quad \text{on } \partial \Omega_\varepsilon
\end{aligned}
\]

where \( u_0 \equiv v + \log |z| \in C^\infty(\overline{\Omega} - \{0\}) \) and \( v \) is the unique strictly pluri-subharmonic solution in \( C^\infty(\overline{\Omega}) \) of the Dirichlet problem

\[
\det(v_{j\bar{k}}) = 1 \quad \text{in } \Omega, \quad v = -\log |z| \quad \text{on } \partial \Omega.
\]

By the maximum principle,

\[
\log |z| - C_0 \leq u \leq u^\varepsilon \leq u^{\varepsilon'} \leq \log |z| \quad \text{in } \Omega_\varepsilon \text{ if } \varepsilon' \leq \varepsilon.
\]
Thus the limit
\[ u(z) = \lim_{\varepsilon \to 0} u^\varepsilon(z) \]
exists for all \( z \in \overline{\Omega} - \{0\} \). We need to show that \( u \in C^{1,0}(\overline{\Omega} - \{0\}) \).

**Lemma 2.** There exists a constant \( C_1 \) independent of \( \varepsilon \) such that
\[ \| \nabla u^\varepsilon \| \leq C_1 \text{ on } \partial \Omega \quad \text{and} \quad \| \nabla u^\varepsilon \| \leq \frac{C_1}{\varepsilon} \text{ on } \partial B_\varepsilon. \]

**Proof.** Since \( u \leq u^\varepsilon \leq 0 \) in \( \Omega \) and \( u^\varepsilon = 0 \) on \( \partial \Omega \), we have
\[ \| \nabla u^\varepsilon \| = u^\varepsilon_{\nu} \leq u_{\nu} = \| \nabla u \| \text{ on } \partial \Omega. \]
This proves the first inequality in (4). To prove the second one, let \( \tilde{u}(z) = u^\varepsilon(\varepsilon z) - \log \varepsilon \) and \( \tilde{u}(z) = u(\varepsilon z) - \log \varepsilon = v(\varepsilon z) + \log |z| \) for \( z \in \overline{B_2} - B_1 \). Note that
\[ \det(\tilde{u}_{jk}) = \varepsilon^{2n} \det(u_{jk}) = \varepsilon^{2n+1}. \]
Let \( \tilde{h} \) be the harmonic function on \( \overline{B_2} - B_1 \) with \( \tilde{h} = \log 2 \) on \( \partial B_2 \) and \( \tilde{h}(z) = v(\varepsilon z) \) on \( \partial B_1 \). Then \( \tilde{u} \leq \tilde{u} \leq \hat{h} \) on \( \overline{B_2} - B_1 \) by the maximum principle, since \( \Delta \tilde{u} \geq 0 \) in \( B_2 - B_1 \), \( \tilde{u} \leq \tilde{h} \) on \( \partial B_2 \) and \( \hat{u} = \hat{h} \) on \( \partial B_1 \). Consequently,
\[ \| \nabla \tilde{u} \| \leq C_1 \text{ on } \partial B_1. \]
This implies the second inequality in (4) as \( \nabla u^\varepsilon(z) = \frac{1}{\varepsilon} \nabla \tilde{u}(\frac{z}{\varepsilon}). \) \( \square \)

**Lemma 3.** Let \( u \) be a strictly pluri-subharmonic \( C^3 \) function with \( \det u_{jk} = \text{constant} \). Let \( \{u^k\} = \{u_{jk}\}^{-1} \). Then, for any constant \( a \geq 0 \),
\[ u^{ij}(e^{au}\| \nabla u \|^2)_{ij} \geq 0. \]

**Proof.** We verify this by direct calculation. First,
\[ \| \nabla u \|^2 = \sum_k (u_ku_k) = \sum_k (u_{ki}u_k + u_ku_{ki}) \]
\[ (\| \nabla u \|^2)_{ij} = \sum_k (u_{ki}u_k + u_ku_{ki})j = \sum_k (u_{kij}u_k + u_{kij}u_k + u_ku_{kj} + u_ku_{jk}). \]
Since \( \det u_{jk} \) is constant, we see that
\[ u^{ij}(u_{kij}u_k + u_ku_{ij}) = 0 \]
and therefore
\[ u^{ij}(\| \nabla u \|^2)_{ij} = u^{ij}u_{kij} + \sum_k u_{kk}. \]
We also note that
\[ u^{ij}(\| \nabla u \|^2)_{ij} = \| \nabla u \|^2 + u^{ij}u_{ki}u_ku_j. \]
By Cauchy-Schwarz inequality,
\[ 2a|\text{Re}\{u^\overline{i}u_k^\overline{j}u_j\}| \leq a^2|\nabla u|^2u^\overline{i}u_j^\overline{j} + u^\overline{i}u_k^\overline{j}u_j. \]

Finally,
\[
e^{-au^\overline{i}u_j^\overline{j}}(e^{au}|\nabla u|^2)_{ij} = |\nabla u|^2u^\overline{i}u_j^\overline{j}(au_j + a^2u_iu_j)
+ 2a\text{Re}\{u^\overline{j}(|\nabla u|^2)_{ij}\} + u^\overline{j}(|\nabla u|^2)_{ij} = a(n + 2)|\nabla u|^2 + a^2|\nabla u|^2u^\overline{i}u_j
+ 2a\text{Re}\{u^\overline{j}u_k^\overline{k}u_j\} + u^\overline{j}u_k^\overline{k}u_j + \sum_k u_{kk} \geq a(n + 2)|\nabla u|^2 + \sum_k u_{kk} \geq 0.
\]

This proves Lemma 3. \(\square\)

It follows from Lemma 2 and Lemma 3 by the maximum principle that
\[
|\nabla u^\varepsilon| \leq C_1e^{-u^\varepsilon} \quad \text{on } \overline{\Omega_e}.
\]

**Lemma 4.** There exists a constant \(C_2\) independent of \(\varepsilon\) such that
\[
|\nabla^2 u^\varepsilon| \leq C_2 \quad \text{on } \partial\Omega \quad \text{and} \quad |\nabla^2 u^\varepsilon| \leq \frac{C_2}{\varepsilon^2} \quad \text{on } \partial B_e.
\]

**Proof.** The first estimate in (7) may be proved as in [1]. We only prove the second one here. Let \(\tilde{u}, \tilde{\overline{u}}\) and \(\tilde{h}\) be as in the proof of Lemma 2. It suffices to show that
\[
|\nabla^2 \tilde{u}| \leq C_2 \quad \text{on } \partial B_1.
\]

For a fixed point \(z^0 \in \partial B_1\), we may assume \(z^0 = (0, \ldots, 1)\), i.e., the coordinates of \(z^0\) are \(x_j = y_j = 0, 1 \leq j \leq n - 1, x_n = 1\) and \(y_n = 0\). Since \(\tilde{u}(z) = v(\varepsilon z)\) on \(\partial B_1\) and \(|\nabla \tilde{u}| \leq C_1\), it is trivial to obtain a bound for the pure tangential second order derivatives at \(z^0\)
\[
|\tilde{u}_{x_i x_k}|, |\tilde{u}_{x_i y_j}|, |\tilde{u}_{y_j y_l}| \leq C, \quad 1 \leq i, k \leq n - 1, 1 \leq j, l \leq n.
\]

To estimate the mixed tangential normal derivatives we need the following analogue of Lemma 2.1 of [1].

**Lemma 5.** Let \(U_\delta = (B_2 - B_1) \cap B_\delta(z^0)\) and \(w = (\tilde{u} - \tilde{\overline{u}}) + t(\tilde{h} - \tilde{\overline{u}}) - Nd^2\), where \(d\) is the distance function from \(\partial B_1\), and \(t, N\) are positive constants. For \(N\) sufficiently large and \(t, \delta\) sufficiently small, we have
\[
\tilde{u}^{i\overline{k}}w_{j\overline{k}} \leq -\frac{1}{64}(1 + \sum \hat{u}^{kk}) \quad \text{in } U_\delta, \quad v \geq 0 \quad \text{on } \partial U_\delta.
\]
Proof. We first note that this does not follow from Lemma 2.1 of [1] as \( \tilde{\bar{u}} \) is not uniformly positive definite in \( \varepsilon \). In order to prove (10) we have to make use of a special property of \( \tilde{\bar{u}} \). Since \( \tilde{\bar{u}}(z) = v(\varepsilon z) + \log |z| \) and \( v \) is plurisubharmonic, we see that

\[
\tilde{\bar{u}}_{j\bar{k}} \geq \tilde{\bar{u}}_{j\bar{k}}(\log |z|) \geq \frac{1}{16} \sum_{k=1}^{n-1} \tilde{u}_{k\bar{k}} \quad \text{in } U_{\delta}
\]

for \( \delta \) sufficiently small. It follows that

\[
\tilde{\bar{u}}_{j\bar{k}}(\tilde{\bar{u}} - \tilde{\bar{u}}) \leq \frac{1}{16} \sum_{k=1}^{n-1} \tilde{u}_{k\bar{k}} \quad \text{in } U_{\delta}
\]

when \( \delta \) is sufficiently small. The rest of the proof is similar to that of Lemma 2.1 in [1] and therefore omitted. \( \square \)

Returning to the proof of Lemma 4, as in [1] we may derive a bound for the mixed tangential normal derivatives at \( z^0 \) with the aid of Lemma 5

\[
|\tilde{u}_{x_kx_n}|, \quad |\tilde{u}_{x_ny_j}| \leq C, \quad 1 \leq k \leq n - 1, 1 \leq j \leq n.
\]

It remains to establish an estimate for the pure normal second order derivative

\[
|\tilde{u}_{n\bar{n}}(z^0)| \leq C.
\]

Because of (9) and (11) it suffices to prove

\[
|\tilde{u}_{n\bar{n}}(z^0)| \leq C.
\]

Since \( \tilde{u} - \tilde{\bar{u}} = 0 \) on \( \partial B_1 \),

\[
\tilde{u}_{j\bar{k}}(z^0) = \tilde{\bar{u}}_{j\bar{k}}(z^0) + \frac{1}{2}(\tilde{u} - \tilde{\bar{u}})_{x_n}(z^0) \delta_{j\bar{k}}
\]

and therefore

\[
\sum_{j,k<\bar{n}} \tilde{u}_{j\bar{k}}(z^0)\xi_j\bar{\xi}_k \geq \sum_{j,k<\bar{n}} \tilde{\bar{u}}_{j\bar{k}}(z^0)\xi_j\bar{\xi}_k = |\xi|^2
\]

for any \( \xi = (\xi_1, \ldots, \xi_{n-1}) \in \mathbb{C}^{n-1} \). Finally, solving equation (5) for \( \tilde{u}_{\bar{n}\bar{n}} \) we see that (13) follows from (9), (11) and (14). This completes the proof of (8) and therefore that of Lemma 4. \( \square \)

**Lemma 6.** There exists a constant \( C_3 \) independent of \( \varepsilon \) such that

\[
|w_{jk}^\varepsilon| \leq C_3e^{-2u^\varepsilon} \quad \text{in } \Omega_{\varepsilon}.
\]

**Proof.** It suffice to derive an upper bound

\[
M \equiv \max_{z \in \overline{\Omega}_\varepsilon} \max_{|\xi|=1, \xi \in \mathbb{C}^n} e^{2u^\varepsilon} \sum_{j,k<\bar{n}} w_{jk}^\varepsilon(z)\xi_j\bar{\xi}_k \leq C \quad \text{ independent of } \varepsilon.
\]
We claim that $M$ is achieved on $\partial \Omega_\varepsilon$. Suppose $M$ is achieved at an interior point $z^0$ for some $\xi \in \mathbb{C}^n$. We may assume $\xi = (1, 0, \ldots, 0)$ and $\{u^\varepsilon_{jk}(z^0)\}$ is diagonal. Thus the function $\varphi \equiv 2u^\varepsilon + \log u^\varepsilon_{11}$ attains a maximum value at $z^0$ where, therefore

$$
\sum \frac{\varphi_{kk}}{u^\varepsilon_{kk}} \leq 0.
$$

On the other hand, differentiating equation (1) twice, we obtain

(17) $$
\sum \frac{u^\varepsilon_{1kk}}{u^\varepsilon_{kk}} - \sum \frac{u^\varepsilon_{1j}u^\varepsilon_{1jk}}{u^\varepsilon_{jj}u^\varepsilon_{kk}} = 0
$$

and hence

$$
\sum \frac{\varphi_{kk}}{u^\varepsilon_{kk}} = 2n + \frac{1}{u^\varepsilon_{11}} \sum \left( \frac{u^\varepsilon_{1kk}}{u^\varepsilon_{kk}} - \frac{u^\varepsilon_{1k}u^\varepsilon_{1l}}{u^\varepsilon_{11}u^\varepsilon_{kk}} \right) \geq 2n.
$$

This contradiction shows that $M$ is achieved on $\partial \Omega_\varepsilon$. By Lemma 4, we obtain (16). □

We are now in a position to finish the proof of Theorem 1. Let $K$ be a compact subset of $\overline{\Omega} - \{0\}$. We show that, for $\varepsilon$ sufficiently small so that $K \subset \overline{\Omega}_\varepsilon$,

(18) $$
\|u^\varepsilon\|_{C^{1,\alpha}(K)} \leq C = C(K) \quad \text{independent of } \varepsilon.
$$

First, by (3) and (6) we have

(19) $$
\|u^\varepsilon\|_{C^1(K)} \leq C = C(K) \quad \text{independent of } \varepsilon.
$$

Next, from Lemma 6 we see that

(20) $$
\Delta u^\varepsilon \equiv \sum_{j=1}^n \left( u^\varepsilon_{x_jx_j} + u^\varepsilon_{y_jy_j} \right) \equiv 4 \sum_{j=1}^n u^\varepsilon_{jj} \leq C \quad \text{in } K \text{ independent of } \varepsilon.
$$

Now, (18) follows from (19) and (20) with the aid of the standard regularity theory. This proves that $u \in C^{1,\alpha}(\overline{\Omega} - \{0\})$ and therefore completes the proof of Theorem 1. □

References


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