CONFORMAL METRICS WITH PRESCRIBED CURVATURE FUNCTIONS ON MANIFOLDS WITH BOUNDARY

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Dedicated to Professor Joel Spruck on the occasion of his 60th birthday.

Abstract. We study the Dirichlet problem for a class of fully nonlinear elliptic equations related to conformal deformations of metrics on Riemannian manifolds with boundary. As a consequence we prove the existence of a conformal metric, given its value on the boundary as a prescribed metric conformal to the (induced) background metric, with a prescribed curvature function of the Schouten tensor.

1. Introduction. Let $(\bar{M}^n, g)$, $n \geq 3$, be a compact smooth Riemannian manifold of dimension $n$ with smooth boundary $\partial M$, and let $M = \bar{M} \setminus \partial M$ be the interior of $\bar{M}$. The Schouten tensor of $g$ is defined as

$$S_g = \frac{1}{n-2} \left( \text{Ric}_g - \frac{R_g}{2(n-1)} g \right)$$

where $\text{Ric}_g$ and $R_g$ are the Ricci tensor and scalar curvature, respectively. The Schouten tensor plays an important role in conformal geometry due to the standard decomposition of the curvature tensor

$$\text{Riem}_g = W_g + S_g \odot g$$

where $W_g$ is the Weyl curvature tensor and $\odot$ is usually referred to as the Kulkarni-Nomizu product, and the fact that the Weyl tensor is conformally invariant. It is therefore important to understand the behavior of Schouten tensor under conformal deformations. In this paper we are concerned with finding conformal metrics on $\bar{M}$ with a prescribed symmetric function of the eigenvalues of Schouten tensor.

To state the problem more precisely, let $\sigma_k$ denote the $k$-th elementary symmetric function and recall that the $\sigma_k$-scalar curvature of $g$, introduced in [40], is defined as $\sigma_k(\lambda(g^{-1}S_g))$, where $\lambda(g^{-1}S_g) = (\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of $g^{-1}S_g$ which locally is given by $(g^{-1}S_g)_{ij} = g^{ik}(S_g)_{kj}$. (This is the scalar curvature,
up to a constant factor, when \( k = 1 \). Given a function \( \psi \) defined on \( \bar{M} \) and a Riemannian metric \( h \) on \( \partial M \) which is conformal to \( g|_{\partial M} \), the induced metric on \( \partial M \), we look for a metric \( \tilde{g} \) on \( \bar{M} \) conformal to \( g \) with \( \sigma_k \)-scalar curvature

\[
\sigma_k(\lambda(\tilde{g}^{-1}S_{\tilde{g}})) = \psi \quad \text{on} \ \bar{M},
\]

and

\[
\tilde{g}|_{\partial M} = h.
\]

For closed Riemannian manifolds the problem of finding conformal metrics of constant or prescribed \( \sigma_k \)-scalar curvature, when \( k \geq 2 \), was initiated by Viaclovsky [40]. The problem reduces to solving a fully nonlinear second order partial differential equation which is elliptic when \( \sigma_k \) is restricted to the cone \( \Gamma_k^+ \) in \( \mathbb{R}^n \):

\[
\Gamma_k^+ := \{ \lambda \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0, \ \forall \ 1 \leq j \leq k \}.
\]

(We will use the notation \( \Gamma_k^+(n) \) to indicate the dimension if necessary.) We call \( \tilde{g} \) a \( \Gamma_k^+ \)-metric if \( \lambda(\tilde{g}^{-1}S_{\tilde{g}}) \in \Gamma_k^+ \). Let \( \Gamma_k^+(g) \) denote the collection of \( \Gamma_k^+ \)-metrics conformal to \( g \). So \( g \in \Gamma_k^+(g) \) if and only if \( \lambda(g^{-1}S_g) \in \Gamma_k^+ \). Following [19] we define \( \Gamma_k^- = -\Gamma_k^+ \).

Our main result may be stated as follows.

**Theorem 1.1.** Assume that \( \psi \in C^\infty(\bar{M}), \ \psi > 0, \) and that \( h \in [g|_{\partial M}] \cap C^\infty(\partial M) \). Problem (1.1)-(1.2) then admits a solution \( \tilde{g} \in C^1(\bar{M}) \cap \Gamma_k^+(g) \) with \( \tilde{g}|_{\partial M} = h \) and

\[
\sigma_k(\lambda(\tilde{g}^{-1}S_{\tilde{g}})) \geq \psi \quad \text{on} \ \bar{M}.
\]

Moreover, for any positive integer \( l \), there exists constant \( C \) depending on \( \|	ilde{g}\|_{C^l(\bar{M})}, \ g, \ \lambda(\tilde{g}^{-1}S_{\tilde{g}}), \ n, \ k \) and \( l \) such that

\[
\|	ilde{g}\|_{C^l(\bar{M})} \leq C,
\]

where the norm is taken with respect to the background metric \( g \).

In particular, we have:

**Corollary 1.2.** If \( S_{\tilde{g}} \in \Gamma_k^+ \) then there exists a metric \( g_1 \in C^\infty(\bar{M}) \cap \Gamma_k^+(g) \) with

\[
\sigma_k(\lambda(\tilde{g}_1^{-1}S_{\tilde{g}_1})) = 1 \quad \text{on} \ \bar{M}
\]

and \( g_1 = cg \) on \( \partial M \) for some constant \( c > 0 \).
Under conformal deformations of metric the Schouten tensors are related by the following formula (see [40]): if \( \tilde{g} = e^{-2u}g \) then

\[
S_{\tilde{g}} = \nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g + S_g
\]

where (and throughout the paper) \( \nabla u \) and \( \nabla^2 u \) denote the gradient and Hessian of \( u \) with respect to the metric \( g \). Consequently, equation (1.1) is equivalent to the second order partial differential equation for \( u \) on \((\bar{M}, g)\)

\[
\sigma_k \left( \lambda \left( g^{-1} \left[ \nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g + S_g \right] \right) \right) = \psi(x)e^{-2ku}.
\]

This is a fully nonlinear equation of similar type to the Hessian equations on which there is a huge amount of literature ([4], [7], [12], [13], [23], [26], [30], [37], [38], [39], [42], etc.). In order to study the Dirichlet problem for (1.6), as usual one needs to derive \( C^2 \) a priori estimates up to boundary for admissible solutions, from which one may apply the Evans-Krylov theorem ([8], [25]) to obtain \( C^{2, \alpha} \) estimates. For closed manifolds, global and local versions of \( C^1 \) and \( C^2 \) estimates that depend on \( C^0 \)-bounds were established by Viaclovsky [41] and Guan-Wang [16], respectively. Our main concern here is the boundary estimates for the second derivatives.

The problem of finding conformal metrics in \( \Gamma_k^1[g] \) of constant \( \sigma_k \)-curvature on closed manifolds, known as the \( \sigma_k \)-Yamabe problem for \( k \geq 2 \), has attracted enormous interest since the work of Viaclovsky [40]. This can be viewed as a fully nonlinear version of the Yamabe problem \( (k = 1) \) which was solved by Aubin [2] and Schoen [33], with important contributions from Trudinger [35]. In [41] Viaclovsky showed that the existence of solution reduces to establishment of \( C^0 \) a priori bounds; he proved such bounds for \( k = n \) under a geometric condition. Subsequently, Chang-Gursky-Yang [5], [6] obtained \( C^0 \) bounds for \( n = 4 \) and \( k = 2 \), in connection with their work on Paneitz operator (see also [20] and [22]), and Guan-Wang [17] and Li-Li [27] independently proved the existence of solution for locally conformally flat manifolds. More recently, the problem was solved for general manifolds, by Gursky-Viaclovsky [21] for \( k > \frac{n}{2} \), based on a result in [15], and by Sheng-Trudinger-Wang [34] for \( k \leq \frac{n}{2} \) assuming a variational structure condition, while Ge-Wang [10] independently obtained a proof for \( k = 2 \) and \( n > 8 \).

For manifolds with boundary, Li-Li [28] studied the problem of finding conformal metrics of constant \( \sigma_k \)-scalar curvature and constant mean curvature on the boundary. The boundary value problem (1.1)-(1.2), for \( k = n \), was considered by Schnürer [32] who proved the existence of a solution in \( C^\infty(M) \cap C^0(\bar{M}) \), which is smooth in the interior of \( M \) but only continuous up to boundary, by an approximation approach under some additional assumptions; this appears to be the only work in this direction. In the following sections we will deal with
equation (1.6) directly and prove the existence of a solution that is smooth up to boundary. In fact we will consider the following slightly more general equation

\[(1.7) \quad F(W[u]) := \sigma^{1/k}_k(\lambda(g^{-1}W[u])) = \psi(x,u,\nabla u)\]

for

\[(1.8) \quad W[u] := \nabla^2 u + sdu \otimes du - \frac{t}{2}|\nabla u|^2g + A,\]

where \(s, t \in \mathbb{R}, \ t \geq 0\), and \(A\) is a smooth symmetric \((0,2)\) tensor on \(\bar{M}\). Accordingly, a function \(v \in C^2(\bar{M})\) is called admissible if \(\lambda(g^{-1}W[v]) \in \Gamma^+_k\).

In Section 2 we derive a priori boundary estimates for the second derivatives of admissible solutions of (1.7). The global estimates for the gradient and second derivatives are established in Section 3 where we emphasize the case \(t = 0\) since for \(t > 0\) one can appeal to [41] or [19] with minor modifications. For completeness we nevertheless include the case \(t > 0\) in our calculations. Technically, for the global estimates the case \(t > 0\) is easier because of the sign of the gradient term containing \(t\) in (1.8), while for the boundary estimates in Section 2 the case \(t = 0\) is slightly simpler, and requires less regularity on the subsolution (which may be assumed to be in \(C^2(\bar{M})\) instead of \(C^3(\bar{M})\)). Finally in Section 4 we prove an existence result, Theorem 4.1 which implies Theorem 1.1, for the Dirichlet problem for equation (1.7).

We list some well known properties of the function \(\sigma^{1/k}_k(\lambda)\) in \(\Gamma^+_k\) that we will need in our proof. These include

\[(1.9) \quad f_i := \frac{\partial \sigma^{1/k}_k}{\partial \lambda_i} > 0 \text{ in } \Gamma^+_k, \quad 1 \leq i \leq n,\]

which implies the ellipticity of (1.7) for admissible solutions,

\[(1.10) \quad \sum_i f_i \geq 1 \text{ in } \Gamma^+_k,\]

\[(1.11) \quad \sigma^{1/k}_k > 0 \text{ in } \Gamma^+_k; \quad \sigma^{1/k}_k = 0 \text{ on } \partial \Gamma^+_k,\]

\[(1.12) \quad \sigma^{1/k}_k \text{ is a concave function in } \Gamma^+_k,\]

which implies the concavity of the function \(F\) defined in (1.7), and for every \(C > 0\) and every compact set \(E\) in \(\Gamma^+_k\) there exists \(R = R(E,C) > 0\) such that

\[(1.13) \quad \sigma^{1/k}_k(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n + R) \geq C, \quad \forall \lambda = (\lambda_1, \ldots, \lambda_n) \in E.\]
as well as the following that will be needed only when $t = 0$ in (1.8) in deriving the global $C^1$ and $C^2$ estimates in Section 3,

\begin{equation}
\tag{1.14}
f_j(\lambda) \geq \mu_0 \left( 1 + \sum_i f_i \right) \quad \text{if } \lambda_j < 0, \quad \forall \lambda \in \Gamma_{\psi,\bar{\psi}}.
\end{equation}

and, for $k > 1$,

\begin{equation}
\tag{1.15}
\lim_{|\lambda| \to \infty} \sum_i f_i = +\infty, \quad \forall \lambda \in \Gamma_{\psi,\bar{\psi}}
\end{equation}

for any positive constants $\psi < \bar{\psi}$, where

$$
\Gamma_{\psi,\bar{\psi}} = \{ \lambda \in \Gamma : \frac{\psi}{k} \leq \frac{1}{k}(\lambda) \leq \bar{\psi} \},
$$

and $\mu_0$ is a uniform positive constant depending on $\psi, \bar{\psi}$.

There are many other useful formulas and properties that $\sigma_k$ satisfies; for example see [4], [7], [9], [14], [24], [30], [31], [36] where one can also find the proofs of the above listed. But for our purpose (1.9)-(1.15) will be sufficient.

We end this Introduction by remarking that similar results to Theorem 1.1 hold for the equation

\begin{equation}
\tag{1.16}
\sigma_k(\lambda(g^{-1} S^r_\tau g)) = \psi \quad \text{on } \bar{M},
\end{equation}

where

$$
S^r_\tau g = \frac{1}{n-2} \left( \text{Ric}_g - \frac{\tau R_g}{2(n-1)} g \right), \quad \tau < 1.
$$

This equation has been studied by Gursky-Viaclovsky [19] on closed manifolds both for the case $S^r_\tau g \in \Gamma^+_k$ and for $S^r_\tau g \in \Gamma^-_k$; see also [29] where a flow approach to the negative cone case is presented. In both cases it reduces to an equation of the form

\begin{equation}
\tag{1.17}
\sigma_k^{1/k} \left( \lambda \left( g^{-1} \left[ \nabla^2 u + r(\Delta u) g + s du \otimes du - \frac{t}{2} |\nabla u|^2 g + A \right] \right) \right) = \psi(x, u),
\end{equation}

where $s, t \in \mathbb{R}$ and $r > 0$. Because of the term $r(\Delta u)g$, equation (1.17) is more similar to the Laplace equation and much stronger results hold. In fact, one does not need the assumption of a subsolution in proving the boundary estimates for second derivatives, and only (1.9)-(1.12) are needed in all the estimates. In particular, Theorem 1.1 holds for (1.17) in the negative cone $\Gamma^-_k$ without assumption (1.3), as long as $S^r_\tau g \in \Gamma^-_k$. The details will be discussed in a forthcoming paper in which we are mainly concerned with complete conformal metrics. A consequence from the paper is that on a convex domain in $\mathbb{R}^n$ there exists a complete
conformally flat metric $g$ with Ricci tensor $\text{Ric}_g \in \Gamma_k^-$ such that

$$(1.18) \quad \sigma_k(\lambda - g^{-1} \text{Ric}_g)) = 1.$$ 

2. Boundary estimates for second derivatives. An important part in solving the Dirichlet problem for equation (1.7) is to establish a priori boundary estimates for the second derivatives of admissible solutions. In this section we will focus on this task. Throughout this section we assume $\psi$ to be positive and smooth, and that $u \in C^3(M)$ is an admissible subsolution of (1.7) with respect to the Dirichlet boundary condition

$$(2.1) \quad u = \varphi \quad \text{on } \partial M$$

where $\varphi \in C^\infty(\partial M)$, i.e., $u$ satisfies

$$(2.2) \quad \begin{cases}  F(W[u]) \geq \psi(x, u, \nabla u) & \text{on } \bar{M}, \\  u = \varphi & \text{on } \partial M. \end{cases}$$

**Theorem 2.1.** Let $u \in C^3(\bar{M})$ be an admissible solution of (1.7)-(2.1) with $u \geq u$ in $\bar{M}$, and suppose $t \geq 0$. Then

$$(2.3) \quad |\nabla^2 u| \leq C \quad \text{on } \partial M$$

where constant $C > 0$ depends on $n$, $\|u\|_{C^1(\bar{M})}$, $\|u\|_{C^3(\bar{M})}$, $\lambda \langle g^{-1} W[u] \rangle$, $\psi$ and its first derivatives.

Before we start our proof of Theorem 2.1, let us first fix some notation. When we consider a point $x_0$ on $\partial M$ we shall use a special orthonormal local frame $e_1, \ldots, e_n$ about $x_0$ obtained from parallel transports, along geodesics perpendicular to $\partial M$, of an orthonormal local frame $e_1, \ldots, e_n-1$ on $\partial M$ and the interior unit normal vector field $e_n$ to $\partial M$. We shall call $e_1, \ldots, e_n$ a principal local frame. Let $\rho(x)$ denote the distance from $x \in M$ to $x_0$,

$$\rho(x) \equiv \text{dist}_M(x, x_0),$$

and set

$$\Omega_\delta = \{ x \in M: \rho(x) < \delta \}, \quad \delta > 0.$$ 

Since $\nabla_i \rho^2(x_0) = 2 \delta_i$, for $\delta > 0$ sufficiently small we may assume $\rho$ is a smooth function in $\Omega_\delta$ and

$$(2.4) \quad \{ \delta_{ij} \} \leq \{ \nabla_i \rho \} \leq 3 \{ \delta_{ij} \} \quad \text{in } \Omega_\delta.$$
Since $\partial M$ is smooth, we may also assume the distance function

$$d_{\partial M}(x) = \text{dist}_M(x, \partial M)$$

to the boundary $\partial M$ to be smooth in $\Omega_\delta$. Consider the linearized operator $L$ which is locally defined by

$$L v = F^{ij}(\nabla_i v + 2s \nabla_i u \nabla_j v - t \nabla_i u \nabla_j v \delta_{ij}) - \psi_{pi}(x,u,\nabla u) \nabla_i v, \quad v \in C^2(\Omega_\delta)$$

where

$$F^{ij} = \frac{\partial F}{\partial W_{ij}}(W[u])$$

and $W_{ij} = W_{ij}[u] = W[u](e_i, e_j)$. It is known ([4]) that $\{F^{ij}\}$ is positive definite with eigenvalues $(f_1, \ldots, f_n)$.

**Lemma 2.2.** For any $B > 0$ there exist positive constants $\mu, \delta$ sufficiently small and $N$ sufficiently large such that the function

$$v = (u - u) - \frac{s(u - u)^2}{2} + \mu d_{\partial M} - \frac{Nd_{\partial M}^2}{2}$$

satisfies

$$L v \leq -B - \beta \sum F^{ii} \quad \text{in} \quad \Omega_\delta, \quad v \geq 0 \quad \text{on} \quad \partial \Omega_\delta$$

where $\beta > 0$ depends on $\lambda(g^{-1}W[u])$.

**Proof.** In this proof we shall write $d = d_{\partial M}$. We calculate

$$L(u - u) = F^{ij}(W_{ij}[u] - W_{ij}[u]) + sF^{ij}(u - u) \nabla_i(u - u) - t \nabla(u - u)^2 \sum F^{ii} - \psi_{pi}(u - u),$$

$$L(u - u)^2 = 2(u - u)L(u - u) + 2F^{ij}(u - u) \nabla_i(u - u) \nabla_j(u - u),$$

and

$$Ld^2 = 2dLd + 2F^{ij} \nabla_i d \nabla_j d.$$ 

One sees that

$$|Ld| \leq C_0 \left(1 + \sum F^{ii}\right)$$
where \( C_0 \) depends on \( \|u\|_{C^1(\Omega)}, \psi \), and geometric quantities of \( \tilde{M} \) and \( \partial M \). Since \( u - \bar{u} \geq 0 \) and \( \{F^{ij}\} \) is positive definite we have

\[
\mathcal{L}v = (1 - s(u - \bar{u}))\mathcal{L}(u - \bar{u}) - sF^{ij}\nabla_i(u - \bar{u})\nabla_j(u - \bar{u}) \\
+ (\mu - Nd)\mathcal{L}d - NF^{ij}\nabla_i d\nabla_j d \\
\leq (1 - s(u - \bar{u}))F^{ij}(W_{ij}[u] - W_{ij}[\bar{u}] - N\nabla_i d\nabla_j d) \\
- \frac{t}{2}(1 - s(u - \bar{u}))|\nabla(u - \bar{u})|^2 \sum F^{ii} + C + C_0(\mu + Nd) \left(1 + \sum F^{ii}\right).
\]

Note that \( \lambda(g^{-1}W[u]) \) lies in a compact subset of \( \Gamma^+_k \). So there exists a positive constant \( \beta \) depending only on \( W[u] \) such that \( \lambda(g^{-1}W[u] - 4\beta I) \) still lies in a compact subset of \( \Gamma^+_k \). Since \( |\nabla d| = 1 \), by (1.13) we see that for \( N > 0 \) sufficiently large

\[
F(W_{ij}[u] - 4\beta \delta_{ij} + N\nabla_i d\nabla_j d) \geq \psi_1 + 4B + C \quad \text{in } \Omega_\delta.
\]

By the concavity of \( F \) we thus have,

\[
F^{ij}(W_{ij}[u] - W_{ij}[\bar{u}] + 4\beta \delta_{ij} - N\nabla_i d\nabla_j d) \\
\leq F(W_{ij}[u]) - F(W_{ij}[\bar{u}] - 4\beta \delta_{ij} + N\nabla_i d\nabla_j d) \leq -4B.
\]

Consequently, if we require \( \mu \) and \( \delta \) to satisfy

\[
(\mu + N\delta)C_0 \leq \min\{\beta, B\}
\]

and

\[
s(u - \bar{u}) \leq 1/2 \quad \text{in } \Omega_\delta
\]

then

\[
\mathcal{L}v \leq (1 - s(u - \bar{u}))(-4B - 4\beta \sum F^{ii}) \\
+ (\mu + Nd)C_0 \left(1 + \sum F^{ii}\right) \leq -B - \beta \sum F^{ii}.
\]

Finally, having fixed \( \mu \) and \( N \) we may require \( \delta \leq 2\mu/N \) to ensure \( v \geq 0 \) on \( \Omega_\delta \). \( \square \)

**Lemma 2.3.** Let \( h \in C^2(\Omega_\delta) \). If \( h \leq 0 \) on \( \partial M \), \( h(x_0) = 0 \) and

\[
-\mathcal{L}h \leq C_0 \left(1 + \sum F^{ii}\right) \quad \text{in } \Omega_\delta,
\]

then
then
\begin{equation}
\nabla_n h(x_0) \leq C
\end{equation}
for some constant $C$ depending on $C_0, \beta^{-1}, \|h\|_{C^0(\overline{\Omega}_0)}$ and $\|u\|_{C^1(\overline{M})}$, where $\beta$ is as in Lemma 2.2.

**Proof.** By Lemma 2.2 we can choose $A \gg B \gg 1$ such that $A v + B \rho^2 - h \geq 0$ on $\partial \Omega_k$ and $\mathcal{L}(A v + B \rho^2 - h) \leq 0$ in $\Omega_k$. It follows from the maximum principle that $A v + B \rho^2 - h \geq 0$ in $\Omega_k$. Consequently,
\begin{equation}
\nabla_n (A v + B \rho^2 - h) (x_0) \geq 0
\end{equation}
since $A v + B \rho^2 - h = 0$ at $x_0$. This gives (2.8).

We are now ready for the proof of Theorem 2.1.

**Proof of Theorem 2.1.** Let $x_0$ be an arbitrary point on $\partial M$. Since $u - u = 0$ on $\partial M$ we have
\begin{equation}
\nabla_{\alpha\beta}(u - u)(x_0) = -\nabla_n (u - u)(x_0) \Pi(e_\alpha, e_\beta), \quad 1 \leq \alpha, \beta \leq n - 1
\end{equation}
where $\nabla_{\alpha\beta} = \nabla_{e_\alpha} \nabla_{e_\beta}$ and $\Pi$ denotes the second fundamental form of $\partial M$. We therefore have the estimate for the pure tangential second order derivatives:
\begin{equation}
|\nabla_{\alpha\beta} u(x_0)| \leq C, \quad 1 \leq \alpha, \beta \leq n - 1
\end{equation}
Next, differentiating equation (1.7) with respect to $e_k$ we find for each $k = 1, \ldots, n$,
\begin{equation}
|\mathcal{L} \nabla_k (u - u)| \leq C \left( 1 + \sum_i F_{i\bar{i}} \right).
\end{equation}
Applying Lemma 2.3 to $h = \pm \nabla_\alpha (u - u), \alpha \leq n - 1$, immediately gives a bound for the mixed normal-tangential derivatives:
\begin{equation}
|\nabla_{\alpha n} u(x_0)| \leq C, \quad 1 \leq \alpha \leq n - 1.
\end{equation}
For the pure normal second derivative $\nabla_{nn} u$, since $\text{tr}(W[u]) \geq 0$, that is,
\begin{equation}
\Delta u + \frac{2s - nt}{2} |\nabla u|^2 + \text{tr} A \geq 0,
\end{equation}
$\Delta u$ is bounded from below and we need only derive an upper bound
\begin{equation}
\nabla_{nn} u(x_0) \leq C.
\end{equation}
This is equivalent to

\[(2.15) \quad W_{nn}[u](x_0) \leq C.\]

Let \(\tilde{W}[u]\) be the restriction of \(W[u]\) on \(\partial M\) and \(\lambda'(\tilde{W}[u])\) denote the eigenvalues of \((g|_{\partial M})^{-1}\tilde{W}[u]\). Note that under a principal local frame

\[
\lambda'(\tilde{W}[u]) = \lambda'(\{W_{\alpha\beta}[u]\}_{\alpha,\beta<n}) \quad \text{on} \quad \partial M,
\]

and

\[(2.16) \quad \sigma_k(\lambda(W[u])) = \sigma_{k-1}(\lambda'(W_{\alpha\beta}[u]))W_{nn}[u] + \sigma_k(\lambda'(W_{\alpha\beta}[u]))
- \sum_{\gamma=1}^{n-1} (W_{\gamma\alpha}[u])^2 \sigma_{k-2}(\lambda'(W_{\alpha\beta\gamma}[u]));\]

where

\[
W_{\alpha\beta;\gamma} = \begin{cases} W_{\alpha\beta} & \text{if } \alpha \neq \gamma, \beta \neq \gamma, \\ 0 & \text{if } \alpha = \gamma, \text{ or } \beta = \gamma. \end{cases}
\]

We see from (2.10) and (2.12) that the proof of Theorem 2.1 will be complete once we have proved the following Lemma.

\[\square\]

**Lemma 2.4.** There exists a uniform constant \(c_0 > 0\) such that

\[(2.17) \quad \sigma_{k-1}(\lambda'(\tilde{W}[u])) \geq c_0 \quad \text{on} \quad \partial M.\]

**Proof.** We will assume \(k \geq 2\). (By convention \(\sigma_0 = 1\) and \(\sigma_{-1} = 0\).) Since \(\{\Gamma^j\}\) is positive definite, \(\lambda'(\tilde{W}[u]) \in \Gamma' = \Gamma_{k-1}(n-1)\) on \(\partial M\). Let \(d_{\Gamma'}(\lambda')\) denote the distance between \(\lambda'\) and \(\partial \Gamma'\) for \(\lambda' \in \Gamma'\). From (2.10) we see that (2.17) is equivalent to

\[(2.18) \quad d_{\Gamma'}(\lambda'(\tilde{W}[u])) \geq c'_0 \quad \text{on} \quad \partial M\]

for some uniform constant \(c'_0 > 0\).

Consider a minimum point \(x_0 \in \partial M\) of the function

\[
d(x) := d_{\Gamma'}(\lambda'(\tilde{W}[u])(x)), \quad x \in \partial M.
\]

It suffices to show \(d(x_0) \geq c'_0 > 0\). For simplicity we may assume \(\sigma_k(W[u](x_0)) = 1\). Choose a principal local frame \(e_1, \ldots, e_n\) about \(x_0\) such that \(\{W_{\alpha\beta}[u](x_0)\}_{\alpha,\beta<n}\) is diagonal and

\[
W_{11}[u](x_0) \leq \cdots \leq W_{n-1,n-1}[u](x_0).
\]
As in [4] one can find \( \mu' = (\mu_1, \ldots, \mu_{n-1}) \in \mathbb{R}^{n-1} \) with

\[
\mu_1 \geq \cdots \geq \mu_{n-1} \geq 0, \quad \sum_{\alpha<n} \mu_\alpha^2 = 1,
\]

and

\[
\sum_{\alpha<n} \mu_\alpha W_{\alpha\alpha}[u](x_0) = d(x_0)
\]

such that

\[
(2.19) \quad \Gamma' \text{ lies in } \left\{ \lambda' \in \mathbb{R}^{n-1}: \mu' \cdot \lambda' > 0 \right\}.
\]

Note that \( \mu' \) is the unit normal to a supporting plane of \( \Gamma' \). We have

\[
(2.20) \quad \sum_{\alpha<n} \mu_\alpha W_{\alpha\alpha}[u](x) \geq d(x) \geq d(x_0) \quad \text{for all } x \in \partial M \text{ near } x_0.
\]

Since \( u - \underline{u} = 0 \) on \( \partial M \), \( \nabla_\alpha(u - \underline{u}) = 0 \) for \( \alpha<n \) and by (2.9),

\[
W_{\alpha\beta}[u] = W_{\alpha\beta}[\underline{u}] - \nabla_\alpha(u - \underline{u}) \left( B_{\alpha\beta} + \frac{t}{2} \nabla_\alpha(u + \underline{u}) \delta_{\alpha\beta} \right), \quad 1 \leq \alpha, \beta \leq n - 1
\]

on \( \partial M \) near \( x_0 \), where

\[
B_{\alpha\beta} = \langle \nabla_\alpha e_\beta, e_n \rangle, \quad 1 \leq \alpha, \beta \leq n - 1
\]

which, when considered on \( \partial M \), gives the coefficients of the second fundamental form of \( \partial M \). Therefore, by (2.20),

\[
(2.21) \quad h(x) := \nabla_\alpha(u - \underline{u})(x) \left[ \sum_{\alpha<n} \mu_\alpha B_{\alpha\alpha}(x) + \frac{t}{2} \nabla_\alpha(u + \underline{u}) \right] - \sum_{\alpha<n} \mu_\alpha W_{\alpha\alpha}[u](x) + d(x_0) \leq 0,
\]

for \( x \in \partial M \) near \( x_0 \).

Next, since

\[
\lambda'(W_{\alpha\beta}[u](x_0)) \equiv \lambda'(\{W_{\alpha\beta}[u](x_0)\}_{\alpha,\beta<n}) \in \Gamma'
\]

we may assume

\[
d(x_0) < \frac{1}{2} d_{\Gamma'}(\lambda'(W_{\alpha\beta}[u](x_0))),
\]
for otherwise we are done. By Lemma 6.2 of [4] we then have

\[ \nabla_n(u - u)(x_0) \left[ \sum_{\alpha < \eta} \mu_{\alpha} B_{\alpha \alpha}(x_0) + \frac{t}{2} \nabla_n(u + u)(x_0) \right] \]

\[ = \sum_{\alpha < \eta} \mu_{\alpha} (W_{\alpha \alpha}[u](x_0) - W_{\alpha \alpha}[u](x_0)) \geq d_{\Gamma}(\lambda(W_{\alpha \beta}[u](x_0))) - d(x_0) \]

\[ \geq \frac{1}{2} d_{\Gamma}(\lambda(W_{\alpha \beta}[u](x_0))) > 0. \]

It follows that

\[ \sum_{\alpha < \eta} \mu_{\alpha} B_{\alpha \alpha}(x_0) + \frac{t}{2} \nabla_n(u + u)(x_0) \geq c_1 \]

for some uniform constant \( c_1 > 0 \) depending on \( |\nabla u| \).

We now first assume \( t > 0 \). Rewrite the function \( h \) in the form

\[ h = \frac{t(\nabla_n u)^2}{2} + a(x) \nabla_n u + b(x). \]

We see that

\[-\mathcal{L}h = -(t \nabla_n u + a) \mathcal{L} \nabla_n u - \nabla_n u \mathcal{L} a - \mathcal{L} b - F^{ij} \nabla_{ni} u (t \nabla_{nj} u + 2 \nabla j a) \leq C \left( 1 + \sum F^{ij} \right) \]

by (2.11) and the inequality

\[-2F^{ij} \nabla_{ni} u \nabla j a \leq tF^{ij} \nabla_{ni} u \nabla_{nj} u + t^{-1} F^{ij} \nabla i a \nabla j a. \]

By Lemma 2.3,

\[ \nabla_n h(x_0) \leq C \]

and, by (2.23),

\[ (c_1 + t \nabla_n (u - u))(x_0)) \nabla_{nn} u(x_0) \leq C. \]

Since \( \nabla_n (u - u)(x_0) \geq 0 \), we obtain

\[ \nabla_{nn} u(x_0) \leq C. \]
The case \( t = 0 \) is simpler: we may replace \( h \) by the function

\[
\tilde{h} := h \left( \sum_{\alpha < \nu} \mu_{\alpha} A_{\alpha} \right)^{-1} = \nabla_n (u - u) + \Phi
\]

where

\[
\Phi = \left( d(x_0) - \sum_{\alpha < \nu} \mu_{\alpha} W_{\alpha} [u](x) \right) \left( \sum_{\alpha < \nu} \mu_{\alpha} B_{\alpha} \right)^{-1}
\]

which by (2.23) is smooth in a neighborhood of \( x_0 \). From (2.21) we have \( \tilde{h} \leq 0 \) on \( \partial M \) near \( x_0 \) and

\[
-\mathcal{L} \tilde{h} = -\mathcal{L} \nabla_n (u - u) + \mathcal{L} \Phi \leq C \left( 1 + \sum F^u \right)
\]

by (2.11). Applying Lemma 2.3 to \( \tilde{h} \) we obtain (2.25) which, combining with (2.10) and (2.12), gives an a priori bound for \( \lambda(W[u](x_0)) \).

To finish the proof we note that \( \sigma_k = 0 \) on \( \partial \Gamma_k^+ \). On the other hand, since \( \psi \) is positive and continuous, there exist constants \( \underline{\psi}, \overline{\psi} > 0 \) depending on the \( C^1 \) norm of \( u \) such that

\[
\underline{\psi} \leq \psi(x, u, \nabla u) \leq \overline{\psi} \quad \text{on } \bar{M}.
\]

Since \( \lambda(W[u](x_0)) \) lies in a bounded subset of \( \Gamma_k^+ \) and \( F(W[u](x_0)) \geq \underline{\psi} > 0 \), we must have

\[
\text{dist} (\lambda(W[u](x_0)), \partial \Gamma_k^+) \geq c_2
\]

for some uniform constant \( c_2 > 0 \). Finally, since \( \{ F^{ij} \} \) is positive definite in \( \Gamma_k^+ \), we have \( \sigma_{k-1}(W[u](x_0)) = k F^{mn}(W[u](x_0)) \geq c_3 > 0 \) and consequently, \( d(x_0) \geq c'_0 \).

The proof of Theorem 2.1 is complete.

3. Global estimates. Our primary goal of this section is to derive the a priori global \( C^2 \) estimate

\[
\| u \|_{C^2(\bar{M})} \leq C
\]

for admissible solutions of (1.7) under suitable assumptions. Once this is established (1.7) becomes uniformly elliptic. One may therefore obtain global \( C^{2,\alpha} \) and higher order estimates using the Evans-Krylov Theorem ([8], [25]) and classical Schauder theory. These estimates will ultimately lead to our main existence result, Theorem 4.1, via the continuity method and degree theory arguments.
For simplicity we shall assume $\psi \in C^\infty(\bar{M} \times \mathbb{R})$, i.e. $\psi$ does not depend on $\nabla u$. We divide our proof of (3.1) into several steps.

**Lemma 3.1.** If

$$\lim_{r \to +\infty} \psi(x, r) = +\infty, \quad \forall x \in \bar{M},$$

or

$$2s - nt < 2\lambda_1$$

where $\lambda_1$ is the first eigenvalue of the problem

$$\begin{cases}
\Delta u + \lambda (\text{tr } A)^+ u = 0 & \text{on } \bar{M} \\
u = 0 & \text{on } \partial M
\end{cases}$$

($\lambda_1 = +\infty$ if $\text{tr } A \leq 0$), then any admissible solution $u \in C^2(\bar{M})$ of (1.7) admits the a priori bound

$$\sup_M u \leq c_0 = c_0(\psi, \sup_{\partial M} u, \sup_{\bar{M}} F(A)).$$

**Proof.** Suppose $u(x_0) = \max_{\bar{M}} u$ for an interior point $x_0 \in M$. Then $\nabla u(x_0) = 0$ and $\nabla^2 u(x_0) \leq 0$. Thus

$$\psi(x_0, u(x_0)) = F(W[u](x_0)) \leq F(A(x_0)).$$

This immediately yields (3.5) if (3.2) holds.

Next, if $a := (2s - nt)/2 \leq 0$ then from (2.13) we have by the maximum principle $u \leq h$ on $\bar{M}$ where $h \in C^\infty(\bar{M})$ solves

$$\Delta h + \text{tr } A = 0 \quad \text{on } \bar{M}, \quad h = u \quad \text{on } \partial M.$$

Suppose now $0 < a < \lambda_1$ and let $v = e^{au}$. By (2.13) we calculate

$$\Delta v + a(\text{tr } A)^+ v \geq \Delta v + a(\text{tr } A)v \geq 0.$$

Since $a < \lambda_1$ there exists a unique solution $w \in C^\infty(\bar{M})$ of the problem

$$\Delta w + a(\text{tr } A)^+ w = 0 \quad \text{on } \bar{M}, \quad w = v \quad \text{on } \partial M,$$

and the maximum principle implies $v \leq w$. This proves (3.5) if (3.3) holds. \qed
**Lemma 3.2.** If \( \text{tr} W[u] \geq 0 \) and \( |u| \leq \mu \) on \( \bar{M} \), then

\[
\sup_{\partial M} \nabla \nu u \leq c_1 = c_1(\mu). \tag{3.9}
\]

Here \( \nu \) is the interior unit normal to \( \partial M \).

**Proof.** This follows immediately from \( u \leq h \) in \( \bar{M} \) if \( a := (2s - nt)/2 \leq 0 \) where \( h \) is the solution of (3.6). Suppose \( a > 0 \) let \( \tilde{w} \) be the solution of

\[
\Delta \tilde{w} + a(\text{tr} A)^+ e^{\alpha u} = 0 \text{ on } \bar{M}, \quad \tilde{w} = e^{\alpha u} \text{ on } \partial M.
\]

From (3.7) we see that

\[
\Delta v + a(\text{tr} A)^+ e^{\alpha u} \geq \Delta v + a(\text{tr} A)^+ v \geq 0
\]

where \( v = e^{\alpha u} \). By the maximum principle \( v \leq \tilde{w} \) on \( \bar{M} \) which implies (3.9). \( \square \)

**Theorem 3.3.** If \( u \) is an admissible solution of (1.7) then

\[
\max_{\bar{M}} |\nabla u| \leq C_1 \tag{3.10}
\]

where \( C_1 \) depends on \( \max_{\bar{M}} |u|, \max_{\partial M} |\nabla u| \) and other known data.

**Proof.** Set

\[
m_0 = \max_{\bar{M}} \eta(u), \quad w = \frac{1}{2} |\nabla u|^2
\]

where \( \eta(u) \) is a positive function to be chosen later. We wish to derive a bound for \( m_0 \) when it is attained at an interior point \( x_0 \in M \). Choose a local orthonormal frame \( e_1, \ldots, e_n \) about \( x_0 \) and differentiate the function \( \log w + \eta(u) \) which achieves a local maximum at \( x_0 \). We obtain at \( x_0 \),

\[
\frac{\nabla_i w}{w} + \eta' \nabla_i u = 0, \text{ for every } i = 1, \ldots, n \tag{3.11}
\]

and

\[
\frac{1}{w} F^{ij} \nabla_i w - \frac{1}{w^2} F^{ij} \nabla_i w \nabla_j w + \eta'' F^{ij} \nabla_i u \nabla_j u + \eta' F^{ij} \nabla_i u \nabla_j u \leq 0. \tag{3.12}
\]

We calculate

\[
\nabla_i w = \nabla_k u \nabla_i u, \tag{3.13}
\]

and

\[
\nabla_j w = \nabla_{ik} u \nabla_j u + \nabla_k u \nabla_{ij} u. \tag{3.14}
\]
Plug these into (3.11) and (3.12),

\[
\nabla_k u \nabla_{ik} u = -\eta' w \nabla_{ij} u
\]

and

\[
\frac{1}{w} F^{ij} \left( \delta_{kl} - \frac{\nabla_k u \nabla_{il} u}{2w} \right) \nabla_{ik} u \nabla_{jl} u + \frac{1}{w} F^{ij} \nabla_k u \nabla_{ij} u \\
+ \left( \eta'' - \frac{1}{2} \eta'^2 \right) F^{ij} \nabla_{ij} u + \eta' F^{ij} \nabla_{ij} u \leq 0.
\]

The term \( F^{ij} \nabla_{ik} u \nabla_{jk} u \) is crucial the local gradient estimates of Guan-Wang [16]. For our purpose here, however, we will only need the fact that the first term in (3.16) is nonnegative; and only when \( t = 0 \) and \( s > 0 \). The rest of terms are treated as follows. First, from (1.8) we see that

\[
\eta' F^{ij} \nabla_{ij} u = \eta' F^{ij} (W_{ij}[u] - s \nabla_i u \nabla_j u + tw \delta_{ij} - A_{ij}) \\
\geq -|\eta'| \psi - s|\eta' F^{ij} \nabla_i u \nabla_j u + (tw \eta' - C|\eta'|) \sum F^{ii}.
\]

Next, differentiate equation (1.7),

\[
F^{ij} \nabla_k W_{ij}[u] = \nabla_k \psi + \psi_u \nabla_k u.
\]

By (1.8), (3.11) and (3.15),

\[
\nabla_k u \nabla_k W_{ij}[u] = \nabla_k u (\nabla_{kij} u + s \nabla_{kij} u \nabla_j u \\
+ s \nabla_i u \nabla_{kij} u - t \nabla_k w \delta_{ij} + \nabla_k A_{ij}) \\
= \nabla_k u \nabla_{kij} u - 2sw \eta' \nabla_i u \nabla_j u \\
+ 2tw^2 \eta' \delta_{ij} + \nabla_k u \nabla_k A_{ij}.
\]

Since

\[
\nabla_{ij} u = \nabla_{kij} u + R^l_{ij} \nabla_l u
\]

by (3.18) we have

\[
F^{ij} \nabla_{ij} u \nabla_{ij} u \geq F^{ij} \nabla_k u \nabla_{kij} u - C|\nabla u|^2 \sum F^{ii} \\
\geq F^{ij} \nabla_k u \nabla_k W_{ij}[u] + 2sw \eta' F^{ij} \nabla_i u \nabla_j u \\
- (2tw^2 \eta' + C \sum F^{ii}) \sum F^{ii} \\
\geq 2sw \eta' F^{ij} \nabla_i u \nabla_j u - 2tw^2 \eta' \sum F^{ii} \\
- Cw \left( 1 + \sum F^{ii} \right).
\]
Finally, combining (3.16), (3.17) and (3.20) we derive

\[
\left( \eta'' - \frac{1}{2} \eta'^2 + s \eta' \right) F^{ij} \nabla_i u \nabla_j u - t\eta' \sum F^{ii} \leq C(1 + |\eta'|) \left( 1 + \sum F^{ii} \right).
\]

If \( t \neq 0 \) let \( \eta(u) = av^N \) with \( a > 0 \), \( |N| \geq 2 \), \( tN < 0 \) and

\[
v = 1 + u - \inf_M u.
\]

We have

\[
0 < -t Na \left( 1 + \sup_M u - \inf_M u \right)^{-(|N|+1)} \leq -t \eta'
\]

\[
= -t N a v^{N-1} \leq -t N a \left( 1 + \sup_M u - \inf_M u \right)^{|N|-1}
\]

and

\[
\eta'' - \frac{1}{2} \eta'^2 + s \eta' = \eta' \left( v^{-1}(N-1) - \frac{1}{2} \eta' + s \right) \geq \frac{N \eta'}{2} > 0
\]

if we first choose \( |N| \) sufficiently large and \( a \) sufficiently small. A bound for \( |\nabla u(x_0)| \) thus follows immediately from (3.21) and (1.10).

If \( t = 0 \) we need some additional properties of \( F \). Let us assume \( W_{ij}[u](x_0) \) to be diagonal and, without loss of generality, that

\[
\nabla_1 u(x_0) \geq \frac{1}{n} |\nabla u(x_0)| > 0.
\]

From (1.8) and (3.15) we have

\[
\nabla_{ij} u = -s \nabla_i u \nabla_j u - A_{ij}, \quad i \neq j
\]

and

\[
-\eta' w \nabla_1 u = \nabla_k u \nabla_{1k} u = \nabla_1 u \nabla_{11} u + \sum_{k=2}^n \nabla_k u(-s \nabla_1 u \nabla_k u - A_{1k})
\]

\[
= \nabla_1 u \nabla_{11} u - s \nabla_1 u(|\nabla u|^2 - |\nabla_{11} u|^2) + \nabla_1 u A_{11} - \sum_{k=1}^n \nabla_k u A_{1k}.
\]

It follows that (recall \( t = 0 \))

\[
W_{11}[u] = -\eta' w + s|\nabla u|^2 + \frac{1}{\nabla_1 u} \sum \nabla_k u A_{1k}
\]
\[
\begin{align*}
& \leq -\eta' w + s |\nabla u|^2 + n \sum_{j,k} |A_{jk}| \\
& \leq - (\eta' - 2s)w + C.
\end{align*}
\]

We next choose \( \eta \) so that

\begin{align}
0 < \alpha &\leq \eta' \leq C, \quad \eta' - 2s \geq \alpha \\
\eta'' - \frac{1}{2} \eta'^2 + s\eta' &\geq \alpha \eta' > 0
\end{align}

for some \( \alpha > 0 \). If \( s \leq 0 \) \((t = 0)\) we can simply take \( \eta = av^N \) with \( N \geq 2 \) large and \( a > 0 \) small as before. If \( s > 0 \) \((t = 0)\) let

\[
\eta(u) = 1 + (2s - \alpha)v - \ln (1 - ae^{(2s - \alpha)v})
\]

where \( v \) as in (3.22). By a straightforward calculation,

\[
\eta'(u) = \frac{2s - \alpha}{1 - ae^{(2s - \alpha)v}}
\]

and

\[
\eta'' - \eta'^2 + (2s - \alpha)\eta' = 0.
\]

We now fix \( a > 0 \) and \( \alpha > 0 \) sufficiently small such that

\[
\frac{\alpha}{s} \leq ae^{(2s - \alpha)} \leq ae^{(2s - \alpha) \sup M} \leq 1 - \alpha.
\]

It follows that

\[
2s + \alpha \leq \eta'(u) \leq \frac{2s}{\alpha}
\]

and

\[
\eta'' - \frac{1}{2} \eta'^2 + s\eta' = \left( \frac{1}{2} \eta' - s + \alpha \right) \eta' \geq \frac{3\alpha}{2} \eta',
\]

verifying (3.28) and (3.29).

If \( W_{11}[u](x_0) \geq 0 \), which always holds if \( k = n \), we then obtain a bound

\begin{align}
(3.30) \quad w(x_0) &\leq C
\end{align}
from (3.27) and (3.28). So we assume \( k < n \) and

\[(3.31) \quad W_{11}[u] < 0.\]

Since \( \psi \) does not depend on \( \nabla u \), (2.26) holds for some positive \( \bar{\psi} \) and \( \bar{\psi} \). Therefore, by (1.14)

\[(3.32) \quad F^{11} \geq \mu_0 \left( 1 + \sum F^{ii} \right)\]

for some \( \mu_0 = \mu_0(\bar{\psi}, \bar{\psi}) > 0 \). From (3.21) and (3.29) we obtain

\[(3.33) \quad \eta \nabla^i \leq C(1 + \eta^i)\]

and hence (3.30).

**Theorem 3.4.** Let \( u \) be an admissible solution of (1.7). If \( t > 0 \), or \( t = 0 \) and \( A \in \Gamma_k^+ \), then

\[(3.34) \quad \max_M |\nabla^2 u| \leq C_2\]

where \( C_2 \) depends on \( \|u\|_{C^1(\bar{M})} \), \( \max_{\partial M} |\nabla^2 u| \) and other known data.

**Proof.** Set

\[\Psi = \max_{x \in \bar{M}} \max_{|\xi| = 1, \xi \in T_x \bar{M}} e^\nu \{ \nabla_\xi^2 u + s \nabla_\xi u \} \]

where \( \nu \) is a function of \( w \) and \( u \) to be chosen later. We may assume \( \Psi \) to be attained at an interior point \( x_0 \in M \) and for some unit vector \( \xi \in T_{x_0} \bar{M} \). Choose a smooth orthonormal local frame \( e_1, \ldots, e_n \) about \( x_0 \) such that \( e_1(x_0) = \xi \) and \( \{ W_{ij}[u](x_0) \} \) is diagonal. Let \( G = \nabla_{11} u + s |\nabla_1 u|^2 \) and assume \( G \geq 1 \). At \( x_0 \) where the function \( e^\nu G \), which is locally defined near \( x_0 \), attains its maximum,

\[(3.35) \quad \nabla_i G + G \nabla_i u = \nabla_{11} u + 2s \nabla_1 u \nabla_{11} u + G \nabla_i u = 0, \quad \text{for every } i\]

and

\[(3.36) \quad F^{ii} \left\{ \nabla_{11} G - \frac{|\nabla G|^2}{G} + G \nabla_{11} u \right\} \leq 0.\]

Next,

\[(3.37) \quad F^{ii} \nabla_i G = F^{ii} \left\{ \nabla_{ii1} u + 2s(\nabla_1 u \nabla_1 u + |\nabla_1 u|^2) \right\} \]

\[\geq F^{ii} \left\{ \nabla_{11} u + 2s(\nabla_1 u \nabla_1 u + |\nabla_1 u|^2) \right\} \]

\[- C \sum_{j,l} (1 + |\nabla_j u|) \sum F^{ii} A_{ii}.\]
From (1.8) and (3.18),

\[ F^ii \nabla_{1ii} = F^ii \nabla_1 \left\{ W^ii[u] - s |\nabla_{i}u|^2 + \frac{t}{2} |\nabla u|^2 - A^ii \right\} \]

\[ = \nabla_1 F + t \nabla_{k} u \nabla_{kk} u \sum_i F^{ii} \]

\[ - 2s^2 F^{ii} \nabla_{i} u \nabla_{1ii} - \sum_i F^{ii} \nabla_{1} A^ii \]

and

\[ F^{ii} \nabla_{1} u \nabla_{1ii} = F^{ii} \nabla_{1} \left\{ W^{ii}[u] - s |\nabla_{i}u|^2 + \frac{t}{2} |\nabla u|^2 - A^ii \right\} \]

\[ \geq F^{ii} \nabla_{1} W^{ii}[u] - 2s^{2} F^{ii} \{ \nabla_{i} u \nabla_{1ii} u + |\nabla_{1ii} u|^2 \} \]

\[ + t \sum_k \{ \nabla_k u \nabla_{1kk} u + |\nabla_{1kk} u|^2 \} \sum_i F^{ii} - C \sum_i F^{ii} \].

From (3.35),

\[ 2s F^{ii} \nabla_{i} u \nabla_{1ii} \leq 2s F^{ii} \nabla_{i} u \nabla_{1ii} u + C \sum_i F^{ii} \]

\[ = -2s F^{ii} \nabla_{i} (2s \nabla_{1} u \nabla_{1ii} u + G \nabla_{i} v) + C \sum_i F^{ii} \]

and, similarly,

\[ t \nabla_{k} u \nabla_{1kk} u \geq -t \nabla_{k} u (2s \nabla_{1} u \nabla_{kk} u + G \nabla_{k} v) - C. \]

Since \( W^{ii}[u(x_0)] \) is diagonal and \( \operatorname{tr} W \geq 0 \), from (3.25) we have

\[ \sum_{k, l} |\nabla_{kl} u| \leq \sum_k |\nabla_{kk} u| + C \leq 2n \nabla_{1} u + C \leq 2nG + C. \]

Combining (3.37)-(3.42), we derive

\[ F^{ii} \nabla_{i} G \geq F^{ii} \nabla_{1} W^{ii}[u] + 2s G F^{ii} \nabla_{i} u \nabla_{i} v \]

\[ - tG \sum_k \nabla_{kk} u \nabla_{k} v \sum_i F^{ii} + (tG - C)G \sum_i F^{ii} \]

If \( t > 0 \) we simply take \( v \equiv 0 \). By the concavity of \( F \),

\[ F^{ii} \nabla_{1} W^{ii}[u] \geq \nabla_{1} F(W[u]) \geq -C(1 + \nabla_{1} u). \]
From (3.35), (3.36), (3.43) and (3.42) we obtain
\[(3.45) \quad G(x_0) \leq \nabla_{11} u(x_0) + C \leq C.\]

In the rest of this proof we consider the case \(t = 0\), and we will follow an idea of Urbas [39]. Let
\[v = \eta(w) - \beta \tilde{u}\]
where \(\beta\) is a positive constant to be chosen later,
\[\tilde{u} = \begin{cases} u, & \text{if } s = 0, \\ -e^{-su}/s, & \text{if } s \neq 0, \end{cases}\]
and
\[\eta(t) = \left(1 - \frac{t}{2N}\right)^{-\frac{1}{2}}, \quad 0 \leq t \leq N = \sup_M w.\]

We have
\[(3.46) \quad 1 \leq \eta \leq \sqrt{2}, \quad \eta' = \frac{\eta^3}{4N}, \quad \eta'' = \frac{3\eta^2}{\eta} \geq \left(\sqrt{2} + \frac{1}{\eta}\right) \eta^2.\]

Note that
\[(3.47) \quad \nabla_i \tilde{u} = e^{-su}\nabla_i u, \quad \nabla_{ij} \tilde{u} = e^{-su}(\nabla_{ij} u - s \nabla_i u \nabla_j u)\]
and
\[(3.48) \quad \nabla_i v = \eta' \nabla_i w + \eta'' |\nabla_i w|^2 - \beta \nabla_i \tilde{u}.\]

From (3.47),
\[(3.49) \quad F^{ii}(\nabla_i \tilde{u} + 2s \nabla_i u \nabla_i \tilde{u}) = e^{-su}(F - F^{ii} A_{ii}).\]

Next,
\[(3.50) \quad F^{ii}(\nabla_i w + 2s \nabla_i u \nabla_i w)\]
\[= F^{ii} \{ |\nabla_i u|^2 + \nabla_i \mu \nabla_i u + 2s \nabla_i u \nabla_i u \nabla_i u \} \]
\[\geq F^{ii} |\nabla_i u|^2 + F^{ii} \nabla_i \mu |\nabla_i u + 2s \nabla_i u \nabla_i u | - C \sum F^{ii} \]
\[= F^{ii} |\nabla_i u|^2 + F^{ii} \{ |\nabla_i \mu \nabla_i (\nabla_i u + s |\nabla_i u|^2)\} - C \sum F^{ii} \]
\[\geq \frac{1}{2} F^{ii} (W_{ii}(u))^2 + \nabla_i \mu (\nabla_i F - F^{ii} \nabla_i A_{ii}) - C \sum F^{ii} \]
\[\geq \frac{1}{2} F^{ii} (W_{ii}(u))^2 - C \sum F^{ii}.\]
Finally, from (3.36), (3.43), (3.48), (3.49) and (3.50),

\[
0 \geq F^{ii} \nabla_{11} W_{ii}[u] - G^{-1} F^{ii} | \nabla_i G |^2 + \eta'' G F^{ii} | \nabla_i w |^2 \\
+ \frac{\eta'}{2} G F^{ii} (W_{ii}[u])^2 - \beta e^{-u} G (F - F^{ii} A_{ii}) - CG \sum F^{ii}.
\]

By (3.35) and (3.46), for every \( i \)

\[
(G^{-1} \nabla_i G)^2 = | \nabla_i u |^2 = (\eta' \nabla_i w - \beta \nabla_i \tilde{u})^2 \\
\leq \left( \frac{1}{\eta} + \frac{1}{\eta} \right) \eta'^2 | \nabla_i w |^2 + (1 + \eta) \beta^2 | \nabla_i \tilde{u} |^2 \\
\leq \eta'' | \nabla_i w |^2 + C \beta^2.
\]

Let \( \lambda_i = W_{ii}[u](x_0) \) and assume

\[
\lambda_n = \min_i \lambda_i \leq -\theta G
\]

where \( \theta \) is any fixed number in \((0, 1/\sqrt{3})\). By (1.14),

\[
F^{ii} \geq \mu_0 \sum F^{ii}.
\]

From (3.44), (3.51) and (3.52) we therefore obtain

\[
\frac{\eta' \mu_0 \theta^2}{2} G^3 \sum F^{ii} - C(1 + \beta + \beta^2) G \sum F^{ii} \leq 0.
\]

Consequently,

\[
G \leq C \theta^{-1} (1 + \beta).
\]

It remains to treat the case

\[
\lambda_i > -\theta G, \quad \forall i.
\]

We may further assume that

\[
G \geq \frac{1}{\theta} \sum_{k,l} | A_{kl} |
\]

which implies

\[
\frac{\lambda_i}{1 + \theta} \leq G \leq \frac{\lambda_i}{1 - \theta}, \quad \forall i.
\]
By (3.55) and (3.56),

\[ \lambda_1 - \lambda_i < \lambda_1 + \theta G \leq \frac{\lambda_1}{1 - \theta}, \quad \forall \, i. \quad (3.57) \]

Let

\[ I = \{ i: \ F^{ii} \leq 4F^{11} \}, \quad J = \{ i: \ F^{ii} > 4F^{11} \}. \]

From (3.52),

\[ \sum_{i \geq 1} F^{ii} \{ \eta'' G |\nabla w|^2 - G^{-1} |\nabla_1 G|^2 \} \]
\[ \geq -C\beta^2 \sum_{i \in I} F^{ii} - G^{-1} \sum_{i \in J} F^{ii} |\nabla_1 G|^2 \]
\[ \geq -C\beta^2 F^{11} - G^{-1} \sum_{i \in J} F^{ii} |\nabla_1 G|^2. \quad (3.58) \]

To control the last term in (3.58), in place of (3.44) we will make use of the following inequalities

\[ F^{ii} \nabla_{11} W_{ii}[u] \geq -CG - \sum_{i \in J} \frac{2}{\lambda_1 - \lambda_i} (F^{11} - F^{ii})(\nabla_1 W_{ii}[u])^2 \]
\[ \geq -CG + \frac{3(1 - \theta)}{2\lambda_1} \sum_{i \in J} F^{ii} (\nabla_1 W_{ii}[u])^2 \]
\[ \geq -CG + \frac{3(1 - \theta^2)}{2G} \sum_{i \in J} F^{ii} (\nabla_1 W_{ii}[u])^2. \quad (3.59) \]

Here the first inequality follows from differentiating equation (1.7) twice, combined with an identity from Andrews [1] and Gerhardt [11] (see Lemma 3.1 in [39]) and the concavity of \( \sigma_1^2 \); the second one follows from (3.57) and the fact that \( F^{ii} > 4F^{11} \) for \( i \in J \), while the third from (3.56). By a straightforward calculation, for any \( i \neq 1 \),

\[ \nabla_1 G = \nabla_{11} u + 2s\nabla_1 u \nabla_1 u \]
\[ = \nabla_{11} u + 2s\nabla_1 u \nabla_1 u + R_{11i} \nabla_1 u \]
\[ = \nabla_1 (W_{1i} - A_{1i}) + s\nabla_1 u \nabla_1 u - s\nabla_1 u \nabla_{11} u + R_{11i} \nabla_1 u \]
\[ = \nabla_1 (W_{1i} - A_{1i}) - sG\nabla_1 u - s\nabla_1 u A_{1i} + R_{11i} \nabla_1 u \]
by (3.25). It follows that

\begin{equation}
|\nabla i G|^2 \leq \frac{3(1 - \theta^2)}{2} (\nabla i W_{11})^2 + \mathcal{C}(1 + s^2 G^2), \quad \forall \, i \neq 1,
\end{equation}

and, by (3.58) and (3.59),

\begin{equation}
F^i [\nabla_{11} W_{ii} |u| - G^{-1} |\nabla i G|^2 + \eta'' G |\nabla i w|^2] \geq -C_\beta G \sum F^i.
\end{equation}

By our assumption that \(A \in \Gamma^+_k\) we can find some fixed \(\epsilon > 0\), depending only on \(A\) and \(k\), such that \(A - \epsilon g \in \Gamma^+_k\), and by the concavity of \(F\)

\begin{equation}
F^i A_{ii} \geq F(A - \epsilon g) + \epsilon \sum F^i > \epsilon \sum F^i.
\end{equation}

Plugging this and (3.61) into (3.51), we finally obtain

\begin{equation}
(\eta' \lambda^2 - C\beta^2) F^1 + \beta (\epsilon \sum F^i - F) + (\beta - C) \sum F^i \leq 0.
\end{equation}

Fixing \(\beta\) sufficiently large, this gives a bound \(\lambda_1 \leq C/\beta\) in view of (1.15). The proof is complete. \(\square\)

4. Existence. The main purpose of this section is to prove the following existence result based on the estimates derived in Sections 2 and 3.

**Theorem 4.1.** Let \(\psi \in C^\infty(\bar{M} \times \mathbb{R})\), \(\psi > 0\), and \(\varphi \in C^\infty(\partial M)\). Suppose (3.2) or (3.3) holds, and suppose either \(t > 0\) or \(t = 0\) and \(A \in \Gamma^+_k\). Then the Dirichlet problem (1.7)-(2.1) has an admissible solution \(u \in C^\infty(\bar{M})\), provided that there exists an admissible subsolution \(\underline{u} \in C^3(\bar{M})\) satisfying (2.2).

Moreover, for any integer \(l \geq 0\) there exists constant \(C > 0\) depending on \(k, n, l, M, f, \varphi, \|\underline{u}\|_{C^l(\bar{M})}\) and \(\lambda(g^{-1} W[\underline{u}])\) such that

\begin{equation}
\|u\|_{C^l(\bar{M})} \leq C \quad \text{on} \ \bar{M}.
\end{equation}

**Proof.** We will prove the existence of an admissible solution \(u \in C^\infty(\bar{M})\) satisfying \(u \geq \underline{u}\) in \(\bar{M}\). Obviously this implies

\[ \min_M u + \min_{\partial M} \nabla \nu u \geq \min_M \underline{u} + \min_{\partial M} \nabla \nu \underline{u}. \]

Combining this with Lemmas 3.1-3.2 we obtain the \(a\ priori\) estimate

\begin{equation}
\sup_M |u| + \sup_{\partial M} |\nabla \nu u| \leq C_0.
\end{equation}
We remark that $C_0$ is independent of the function $\psi$ on the right hand side of (1.7). By Theorems 3.3, 2.1 and 3.4 we then obtain the global $C^2$ estimate,

$$\|u\|_{C^2(\bar{M})} \leq C.$$  \hfill (4.3)

The higher order estimates in (4.1) follow from the Krylov-Evans Theorem ([8], [25]) and classical Schauder theory for elliptic linear equations.

With the aid of (4.1) one may apply the continuity method and a degree argument to prove the existence of an admissible solution. The arguments are more or less standard; for completeness, however, we include an outline following the argument in [3] with some slight modifications; the reader is referred to [3] (also [13]) for details.

We assume $u \in C^\infty(M)$; the more general case $u \in C^3(M)$ follows by approximation. Set

$$A \equiv \{ v \in C^2(\bar{M}) : \text{v is admissible, } v \geq u \text{ in } \Omega \text{ and } v = u \text{ on } \partial \Omega \}.$$

We first consider a spatial case: $\psi(x, z) \geq 0$ for all $x \in \bar{M}$ and $|z| \leq C_0$ where $C_0$ as in (4.2).  For each fixed $t \in [0, 1]$ consider the Dirichlet problem

$$\begin{cases}
  F(W[u]) = t\psi(x, u) + (1 - t)F(W[u]) & \text{in } \bar{M}, \\
  u = \varphi & \text{on } \partial M.
\end{cases}$$  \hfill (4.4)

Note that $u$ is a subsolution of (4.4) for all $t \in [0, 1]$. By the maximum principle any admissible solution $u^t \in C^\infty(M)$ of (4.4) must belong to $A$. Thus $u^t$ satisfies the a priori estimates in (4.1). In particular, for any $0 < \alpha < 1$,

$$\|u^t\|_{C^{2,\alpha}(\bar{M})} \leq C \text{ independent of } t \in [0, 1].$$  \hfill (4.5)

By the continuity method, for each $t \in [0, 1]$ there exists a unique admissible solution to (4.4) in $C^\infty(M) \cap A$.

Turning to the general case, we assume that $u$ is not a solution since otherwise we are done. For $r > 0$ set

$$\mathcal{C}_r = \{ v \in C^{5,\alpha}(M) : v > 0 \text{ in } M, v|_{\partial M} = 0, \nabla v|_{\partial M} > 0 \text{ and } \|v\|_{C^{5,\alpha}(\bar{M})} < r \}$$

where $\alpha \in (0, 1)$ and $\nu$ is the unit interior normal to $\partial \Omega$. For $t \in [0, 1]$ and fixed $v \in \mathcal{C}_r$ consider the Dirichlet problem

$$\begin{cases}
  F(W[u]) = \psi^t(x, u) & \text{in } \bar{M}, \\
  u = \varphi & \text{on } \partial M,
\end{cases}$$  \hfill (4.6)
where
\[ \psi^t(x, z) \equiv (t \psi(x, z) + t' \psi(x, u)e^{\Lambda(z-u)}, 0 \leq t \leq 1. \]

Here \( t' = (1 - t)/2 \) and
\[ \Lambda = \max \left\{ 0, \max_{x \in \bar{M}, |z| \leq C_0} \frac{\psi(x, z)}{\psi(x, z)} \right\} \]

where \( C_0 \) is as in (4.2). We observe that \( u \) is a subsolution of (4.6) and
\[ \psi_{z}^t(x, z) \geq t(\psi_z(x, z) + \Lambda \psi(x, z)) e^{t\Lambda(z-u)} \geq 0, \quad \forall x \in \bar{M}, |z| \leq C_0, t \in [0, 1]. \]

Therefore there exists a unique admissible solution \( u^t \in C^\infty(\bar{M}) \) to (4.6) for each \( t \in [0, 1] \). By the maximum principle and Hopf lemma,
\[ (4.7) \quad u^t > u \text{ in } M \text{ and } \nabla_{\nu} u^t > \nabla_{\nu} u \text{ on } \partial M. \]

Thus the estimates in (4.1) hold for \( u^t \) uniformly in \( t \in [0, 1] \). In particular,
\[ (4.8) \quad \|u^t\|_{C^{3,\alpha}(\bar{M})} \leq C \text{ independent of } t \in [0, 1] \]

where \( C \) may depend on \( r \).

Consider the map \( T_t v \equiv u^t - u \). From (4.8) and (4.7) we see that \( T_t \) is a compact map in \( C^5 \), and that the equation
\[ (4.9) \quad v - T_t v = 0 \]

admits no solution on the boundary of \( C_r \) if \( r \) is fixed sufficiently large. (Note that the constant \( C \) in (4.8) does not depend on \( r \) for those \( u^t \) satisfying \( v - T_t v = 0 \), i.e. \( u^t = u + v \).) Thus the degree
\[ (4.10) \quad \deg (I - T_t, C_r, 0) = \gamma \]

is well defined and independent of \( t \) for \( r \) sufficiently large. When \( t = 0 \), (4.9) has a unique solution \( u^0 = u^0 - u \) which is a regular point of \( I - T^0 \). Consequently \( \gamma = \pm 1 \), and (4.9) has a solution \( u^t \in C_r \) for all \( 0 \leq t \leq 1 \). The function \( u = u + v^1 \)
is then a desired solution of (1.7)-(2.1). It follows from elliptic regularity theory that \( u \in C^\infty(\bar{M}) \).

Finally, we note that Theorem 4.1 implies Theorem 1.1 as (3.3) holds for (1.6).
REFERENCES