HIGHER STABILIZATION AND HIGHER FREUDENTHAL SUSPENSION

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ABSTRACT. We prove that the stabilization (resp. iterated suspension) functor participates in a derived adjunction comparing pointed spaces with certain (highly homotopy coherent) homotopy coalgebras, in the sense of Blumberg-Riehl, that is a Dwyer-Kan equivalence after restriction to 1-connected spaces, with respect to the associated enrichments. A key ingredient of our proof, of independent interest, is a higher stabilization theorem (resp. higher Freudenthal suspension theorem) for pointed spaces that provides strong estimates for the uniform cartesian-ness of certain cubical diagrams associated to iterating the space level stabilization map (resp. Freudenthal suspension map)—these technical results provide, in particular, new proofs (with strong estimates) of the stabilization and iterated loop-suspension completion results of Carlsson and the subsequent work of Arone-Kankaanrinta, and Bousfield and Hopkins, respectively, for 1-connected spaces; this is the stabilization (resp. Freudenthal suspension) analog of Dundas' higher Hurewicz theorem.

1. INTRODUCTION

We have written this paper simplicially: in other words, "space" means "simplicial set" unless otherwise noted; see Dwyer-Henn [27] for a useful introduction to these ideas, together with Bousfield-Kan [14], Goerss-Jardine [36], and Hovey [43]. We work in the category of symmetric spectra (see Hovey-Shipley-Smith [44] and Schwede [60]), equipped with the injective stable model structure, so that "S-modules" means "symmetric spectra", which are the same as modules over the sphere spectrum S; in particular, fibrant S-modules enjoy the property of being Ω-spectra [44, 1.4] that are objectwise Kan complexes. Alternatively, the results here could be developed in the context of EKMM spectra [30], or even Bousfield-Friedlander [12] spectra.

Our main result is that the stabilization functor—the classical construction of associating a spectrum to a pointed space by tensoring with the sphere spectrum—participates in a Dwyer-Kan equivalence with certain (highly homotopy coherent) homotopy coalgebra spectra, in the sense of Blumberg-Riehl [10], where the stabilization construction naturally lands, after restriction to 1-connected spaces. In the statement of the following theorem, $\tilde{\mathsf{K}}$ is the homotopical comonad (Remark 1.3, Definition 3.4, and (23)–(24)) naturally acting (Remark 3.6) on the suspension spectrum $\Sigma^\infty X$ (see (23)) of a pointed space X, and $\mathfrak{C}(Y)$ is the cosimplicial cobar construction (Definition 3.10) associated to the $\tilde{\mathsf{K}}$-coalgebra Y (Definition 3.5).

Theorem 1.1. *The stabilization functor*

$$\Sigma^\infty\colon \mathsf{Ho}(\mathsf{S}_*)\to\mathsf{Ho}(\mathsf{coAlg}_{\tilde{K}})$$

restricts to an equivalence between the homotopy categories of 1*-connected spaces and* 1*-connected* $\tilde{K}$*-coalgebra spectra (Remark 1.3 and Definition 3.4); more precisely,* Σ^∞ *participates in a derived adjunction (Proposition 4.2) comparing pointed spaces to* $\tilde{K}$*-coalgebra spectra*

$$\mathrm{Map}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}}}(\Sigma^\infty X, Y) \simeq \mathrm{Map}_{\mathsf{S}_*}(X, \mathrm{holim}_\Delta \mathfrak{C}(Y)) \tag{1}$$

that is a Dwyer-Kan equivalence after restriction to the full subcategories of 1*-connected spaces and* 1*-connected* $\tilde{\mathsf{K}}$*-coalgebra spectra, with respect to the associated enrichments (Definition 3.18).*

This can be thought of as a stabilization analog of the Quillen [57] (resp. Sullivan [63]) main result that the rational chains (resp. cochains) functor participates in a derived equivalence with certain coalgebra (resp. algebra) complexes, after restriction to 1-connected spaces up to rational equivalence.

Our second main result is that an analogous statement is true when we replace the stabilization functor Σ^∞ with the iterated suspension functor Σ^r. In the statement of the following theorem, $\tilde{\mathsf{K}}_r$ is the homotopical comonad (Remark 1.3, Definition 3.4 and (26)–(27)) naturally acting (Remark 3.6) on the iterated suspension $\Sigma^r X$ (see (26)) of a pointed space X, and $\mathfrak{C}_r(Z)$ is the cosimplicial cobar construction (Definition 3.10) associated to the $\tilde{\mathsf{K}}_r$-coalgebra Z (Definition 3.5).

Theorem 1.2. *Let* $r \geq 1$*. The iterated suspension functor*

$$\Sigma^r \colon \mathsf{Ho}(\mathsf{S}_*) \rightarrow \mathsf{Ho}(\mathsf{coAlg}_{\tilde{\mathsf{K}}_r})$$

restricts to an equivalence between the homotopy categories of 1*-connected spaces and* $(1+r)$*-connected* $\tilde{\mathsf{K}}_r$*-coalgebras (Remark 1.3 and Definition 3.4); more precisely,* Σ^r *participates in a derived adjunction (Proposition 4.2) comparing pointed spaces to* $\tilde{\mathsf{K}}_r$*-coalgebras*

$$\mathrm{Map}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}_r}}(\Sigma^r X, Z) \simeq \mathrm{Map}_{\mathsf{S}_*}(X, \mathrm{holim}_\Delta \mathfrak{C}_r(Z)) \tag{2}$$

that is a Dwyer-Kan equivalence after restriction to the full subcategories of 1*-connected spaces and* $(1+r)$*-connected* $\tilde{\mathsf{K}}_r$*-coalgebras, with respect to the associated enrichments (Definition 3.18).*

Remark 1.3*.* Here, $\tilde{\mathsf{K}}$ and $\tilde{\mathsf{K}}_r$ are point-set level (highly homotopy coherent) homotopical comonads in the sense of Blumberg-Riehl [10]; in particular, they are homotopy invariant, they satisfy highly homotopy coherent analogs of the usual counit and coassociativity diagrams for comonads, and their (highly homotopy coherent) homotopy coalgebras play a key role in the formulation and proofs of our main results.

The following are immediate corollaries of our main results.

Corollary 1.4. *A pair of* 1*-connected pointed spaces* X *and* X' *are weakly equivalent if and only if the suspension spectra* $\Sigma^\infty X$ *and* $\Sigma^\infty X'$ *(resp. iterated suspension spaces* $\Sigma^r X$ *and* $\Sigma^r X'$*) are weakly equivalent as derived* $\tilde{\mathsf{K}}$*-coalgebra spectra (resp.* $\tilde{\mathsf{K}}_r$*-coalgebras).*

Corollary 1.5. *Let* X, X' *be pointed spaces. Assume that* X' *is* 1*-connected and fibrant.*

(a) (Existence) *Given any map* ϕ *in* $[\Sigma^\infty X, \Sigma^\infty X']_{\tilde{\mathsf{K}}}$ *(resp.* $[\Sigma^r X, \Sigma^r X']_{\tilde{\mathsf{K}}_r}$*), there exists a map* f *in* $[X, X']$ *such that* $\phi = \Sigma^\infty(f)$ *(resp.* $\phi = \Sigma^r(f)$*).*

(b) (Uniqueness) *For each pair of maps* f, g *in* $[X, X']$, $f = g$ *if and only if* $\Sigma^\infty(f) = \Sigma^\infty(g)$ *(resp.* $\Sigma^r(f) = \Sigma^r(g)$*) in the homotopy category of* $\tilde{\mathsf{K}}$*-coalgebra spectra (resp.* $\tilde{\mathsf{K}}_r$*-coalgebras).*

Corollary 1.6. *A* $\tilde{\mathsf{K}}$*-coalgebra spectrum* Y *(resp.* $\tilde{\mathsf{K}}_r$*-coalgebra space* Z*) is weakly equivalent to the suspension spectrum* $\Sigma^\infty X$ *(resp. iterated suspension* $\Sigma^r X$*) of some* 1*-connected space* X*, via derived* $\tilde{\mathsf{K}}$*-coalgebra maps (resp.* $\tilde{\mathsf{K}}_r$*-coalgebra maps), if and only if* Y *is* 1*-connected (resp.* Z *is* $(1 + r)$*-connected).*

The following two theorems, of independent interest, are our main technical results (see Section 5.9 for the proofs). The higher stabilization (resp. higher Freudenthal suspension) theorem provides strong estimates for the uniform cartesian-ness of certain cubical diagrams associated to n-fold iterations of the space level stabilization (resp. Freudenthal suspension) map; it can be thought of as the stabilization (resp. Freudenthal suspension) analog of Dundas' higher Hurewicz theorem [24, 2.6]; see also the elaboration in Dundas-Goodwillie-McCarthy [25, A.8.3]. In the statement of the following theorems, $\tilde{\Omega}^\infty$ is the derived version of the 0-th space functor Ω^∞ (Definition 3.4 and (23)) and $\tilde{\Omega}^r$ is the derived version of the iterated loop space functor Ω^r (Definition 3.4 and (26)).

Theorem 1.7 (Higher stabilization)**.** *Let* $k \geq 1$*,* W *a finite set, and* $\mathcal{X}$ *a* W*-cube of pointed spaces. If* $\mathcal{X}$ *is* $(k(\mathrm{id}+1)+1)$*-cartesian, then so is* $\mathcal{X} \to \tilde{\Omega}^\infty\Sigma^\infty\mathcal{X}$*.*

Theorem 1.8 (Higher Freudenthal suspension)**.** *Let* $k \geq 1$*,* W *a finite set, and* $\mathcal{X}$ *a* W*-cube of pointed spaces. If* $\mathcal{X}$ *is* $(k(\mathrm{id}+1)+1)$*-cartesian, then so is* $\mathcal{X} \to \tilde{\Omega}^r\Sigma^r\mathcal{X}$*.*

It is worth pointing out that Theorems 1.7 and 1.8 provide new proofs (with strong estimates) of the $\tilde{\Omega}^\infty\Sigma^\infty$-completion results in Carlsson [16] and the subsequent work of Arone-Kankaanrinta [2], and the $\tilde{\Omega}^r\Sigma^r$-completion results of Bousfield [11] and Hopkins (see [11]), respectively, for 1-connected spaces; this is elaborated in Remarks 5.28 and 5.29. These uniform cartesian-ness estimates imply certain uniform cocartesian-ness estimates, and vice-versa (Proposition 5.14); this uniformity phenomenon is the stabilization (resp. Freudenthal suspension) analog of a closely related uniformity correspondence appearing in [24, 2.4] and [25, A.8.3.2] that naturally arises from a homotopical analysis of iterations of the Hurewicz map for integral homology.

Remark 1.9. We can state our main result in the context of Lurie's theory of ∞-categories in the following way [20, 1.3]. Associated to the simplicial Quillen adjunction $(\Sigma^\infty, \Omega^\infty)$ in (23) (resp. (Σ^r, Ω^r) in (26)) is a corresponding adjunction of ∞-categories [49, 5.2.4.6]. Riehl-Verity [59] show that an adjunction of ∞-categories of the form

$$\mathsf{F} : \mathcal{A} \rightleftarrows \mathcal{B} : \mathsf{G}$$

determines a homotopy coherent comonad K whose underlying functor is FG. If the ∞-category $\mathcal{A}$ admits suitable limits, the adjunction (F, G) lifts to an adjunction of ∞-categories of the form

$$\bar{\mathsf{F}} : \mathcal{A} \rightleftarrows \mathcal{C}_{\mathsf{K}} : \bar{\mathsf{C}}$$

where $\mathcal{C}_{\mathsf{K}}$ is an ∞-category of coalgebras over the comonad K and $\bar{\mathsf{C}}$ is a suitable cobar construction applied to K-coalgebras. This is dual to [59, 7.2.4]. Our Theorem 1.1 (resp. 1.2) then implies that, when applied to the adjunction $(\Sigma^\infty, \Omega^\infty)$ (resp. (Σ^r, Ω^r)), the resulting adjunction $(\bar{F}, \bar{C})$ restricts to an equivalence between the

∞-categories of 1-connected objects on each side (resp. 1-connected objects on the left side and $(1+r)$-connected objects on the right side).

In terms of (the opposite of) Lurie's version of the Barr-Beck theorem [50, 4.7.0.3]: Our main result reduces to proving that the left derived stabilization functor Σ^∞ (resp. iterated suspension functor Σ^r) commutes,

$$\Sigma^\infty \operatorname{holim}_\Delta \mathfrak{C}(Y) \simeq \operatorname{holim}_\Delta \Sigma^\infty \mathfrak{C}(Y) \tag{3}$$

$$\text{resp.} \quad \Sigma^r \operatorname{holim}_\Delta \mathfrak{C}_r(Z) \simeq \operatorname{holim}_\Delta \Sigma^r \mathfrak{C}_r(Z) \tag{4}$$

up to weak equivalence, with the right derived limit functor $\operatorname{holim}_\Delta$, when composed with the cosimplicial cobar construction $\mathfrak{C}$ (resp. $\mathfrak{C}_r$) associated to the homotopical comonad $\tilde{\mathsf{K}}$ (resp. $\tilde{\mathsf{K}}_r$) and evaluated on 1-connected $\tilde{\mathsf{K}}$-coalgebras (resp. $(1+r)$-connected $\tilde{\mathsf{K}}_r$-coalgebras)—see Theorem 2.4. This condition is precisely the crux of verifying (the opposite of) [50, 4.7.0.3].

1.10. **Strategy of attack and related work.** We are leveraging a line of attack developed in [20] for resolving the 0-connected case of a conjecture in Francis-Gaitsgory [31], together with a modification of that strategy developed in [9] for integral chains. We exploit Cohn's work [21] showing that this extends to (highly homotopy coherent) homotopy coalgebras over the associated homotopical comonad (Blumberg-Riehl [10]). A key ingredient underlying our homotopical estimates are certain uniform cartesian-ness estimates (Theorems 1.7 and 1.8) related to the $\tilde{\Omega}^\infty\Sigma^\infty$-completion map studied in Carlsson [16], and subsequently in the work of Arone-Kankaanrinta [2], and the $\tilde{\Omega}^r\Sigma^r$-completion map studied in Bousfield [11] and Hopkins (see [11]).

We were motivated by the results of Hopkins [42] on iterated suspension, the subsequent work of Goerss [34] on desuspension and Klein-Schwanzl-Vogt [46] on comultiplication and suspension, and the work of Klein [47] on moduli of suspension spectra. We benefited from a careful study of the density argument in Dundas-Goodwillie-McCarthy [25] and the higher Hurewicz theorem in Dundas [24]. Our results, from a technical point of view, are enabled by Goodwillie's higher (dual) Blakers-Massey theorems [37], the homotopical comonads and their associated homotopy coalgebras studied in Blumberg-Riehl [10] and exploited in Cohn [21], together with the enrichments and framework developed in Arone-Ching [1].

We were encouraged (via rough analogy) by the earlier work of Quillen [57] and Sullivan [63] (see also Bousfield-Gugenheim [13]), in light of the results in [19] for structured ring spectra, together with the work of Smirnov [62] on a coalgebraic study of homology, and the work of Dwyer-Hopkins (see [51, C]), Goerss [35], Karoubi [45], Kriz [48], and Mandell [51, 52].

Another way to think about Theorems 1.1 and 1.2 is that the functors Σ^∞ and Σ^r, as they appear in (1) and (2), satisfy homotopical descent on objects and morphisms; see Arone-Ching [1] and Hess [40], for a discussion of related ideas, Carlsson-Milgram [18] and May [53] for background on related topics, Edwards-Hastings [29] for a concise discussion of stabilization and abelianization, and Behrens-Rezk [6] for an interesting survey of closely related ideas.

1.11. **Organization of the paper.** In Section 2 we outline the argument of our main result. In Section 3 we review the completion constructions, their associated cosimplicial cobar constructions, together with the (highly homotopy coherent) homotopy coalgebras over the associated homotopical comonads that naturally arise when making sense of these completion constructions in our context. Furthermore, in Section 3 we set up the framework for the homotopy theory of these homotopy coalgebras. In Section 4 we describe the derived unit and derived counit maps associated to (1) and (2). In Section 5 we develop the homotopical estimates that underlie our main results, and in the short Section 6 we remind the reader about notation for various hom-objects used throughout the paper. For the experts that are already familiar with the enrichments and framework in Arone-Ching [1], it will suffice to read Sections 2 and 5 for a complete proof of the main results.

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2. Outline of the argument

In this section we will outline the proof of our main results. Since the derived unit map (Definition 4.3) associated to (1) is tautologically the $\tilde{\Omega}^\infty\Sigma^\infty$-completion map $X'{\rightarrow}X'^{\wedge}_{\tilde{\Omega}^\infty\Sigma^\infty}$, which is proved to be a weak equivalence on 1-connected spaces in Carlsson [16], and subsequently in Arone-Kankaanrinta [2], proving the main result in the stabilization case reduces to verifying that the derived counit map (Definition 4.5) associated to (1) is a weak equivalence. Similarly, since the derived unit map (Definition 4.3) associated to (2) is tautologically the $\tilde{\Omega}^r\Sigma^r$-completion map $X'{\rightarrow}X'^{\wedge}_{\tilde{\Omega}^r\Sigma^r}$, which is proved to be a weak equivalence on 1-connected spaces in the work of Bousfield [11] and Hopkins (see [11]), proving the main result in the iterated suspension case reduces to verifying that the derived counit map (Definition 4.5) associated to (2) is a weak equivalence.

The following theorem is proved in Section 5, just after Proposition 5.44.

Theorem 2.1. *Let $r\geq 1$. If Y is a 1-connected $\tilde{\mathsf{K}}$-coalgebra spectrum (resp. Z is a $(1+r)$-connected $\tilde{\mathsf{K}}_r$-coalgebra) and $n\geq 1$, then the natural map*

$$\operatorname{holim}_{\Delta^{\leq n}}\mathfrak{C}(Y)\longrightarrow\operatorname{holim}_{\Delta^{\leq n-1}}\mathfrak{C}(Y) \tag{5}$$

$$\textit{resp.}\quad \operatorname{holim}_{\Delta^{\leq n}}\mathfrak{C}_r(Z)\longrightarrow\operatorname{holim}_{\Delta^{\leq n-1}}\mathfrak{C}_r(Z) \tag{6}$$

is an $(n+2)$-connected map between 1-connected objects.

Theorem 2.2. *Let $r\geq 1$. If Y is a 1-connected $\tilde{\mathsf{K}}$-coalgebra spectrum and $n\geq 0$, then the natural maps*

$$\operatorname{holim}_{\Delta}\mathfrak{C}(Y)\longrightarrow\operatorname{holim}_{\Delta^{\leq n}}\mathfrak{C}(Y) \tag{7}$$

$$\Sigma^\infty\operatorname{holim}_{\Delta}\mathfrak{C}(Y)\longrightarrow\Sigma^\infty\operatorname{holim}_{\Delta^{\leq n}}\mathfrak{C}(Y) \tag{8}$$

are $(n+3)$-connected maps between 1-connected objects. Similarly, if Z is a $(1+r)$-connected $\tilde{\mathsf{K}}_r$-coalgebra and $n \geq 0$, then

$$\operatorname{holim}_{\Delta} \mathfrak{C}_r(Z) \longrightarrow \operatorname{holim}_{\Delta^{\leq n}} \mathfrak{C}_r(Z) \tag{9}$$

$$\Sigma^r \operatorname{holim}_{\Delta} \mathfrak{C}_r(Z) \longrightarrow \Sigma^r \operatorname{holim}_{\Delta^{\leq n}} \mathfrak{C}_r(Z) \tag{10}$$

the natural map (9) *(resp.* (10)*) is an $(n+3)$-connected map between 1-connected objects (resp. $(n+3+r)$-connected map between $(1+r)$-connected objects).*

Proof. Consider the first part. By Theorem 2.1 the maps in the holim tower $\{\operatorname{holim}_{\Delta^{\leq n}} \mathfrak{C}(Y)\}_n$ have connectivity strictly increasing with n; furthermore, the map above level n is $(n+3)$-connected, the map above level $n+1$ is $(n+4)$-connected, the map above level $n+2$ is $(n+5)$-connected, and so forth. It follows that the map (7) is $(n+3)$-connected. The second part follows from the first part. The other case is similar. □

We prove the following theorem in Section 5, just after Theorem 5.41. It provides estimates sufficient for verifying that stabilization (resp. iterated suspension) commutes past the desired homotopy limits.

Theorem 2.3. *Let $r \geq 1$. If Y is a 1-connected $\tilde{\mathsf{K}}$-coalgebra spectrum (resp. Z is a $(1+r)$-connected $\tilde{\mathsf{K}}_r$-coalgebra) and $n \geq 1$, then the natural map*

$$\Sigma^\infty \operatorname{holim}_{\Delta^{\leq n}} \mathfrak{C}(Y) \longrightarrow \operatorname{holim}_{\Delta^{\leq n}} \Sigma^\infty \mathfrak{C}(Y) \tag{11}$$

$$\textit{resp.} \quad \Sigma^r \operatorname{holim}_{\Delta^{\leq n}} \mathfrak{C}_r(Z) \longrightarrow \operatorname{holim}_{\Delta^{\leq n}} \Sigma^r \mathfrak{C}_r(Z) \tag{12}$$

is $(n+5)$-connected (resp. $(n+5+r)$-connected); the map is a weak equivalence for $n=0$.

Theorem 2.4. *Let $r \geq 1$. If Y is a 1-connected $\tilde{\mathsf{K}}$-coalgebra spectrum (resp. Z is a $(1+r)$-connected $\tilde{\mathsf{K}}_r$-coalgebra), then the natural maps*

$$\Sigma^\infty \operatorname{holim}_{\Delta} \mathfrak{C}(Y) \xrightarrow{\simeq} \operatorname{holim}_{\Delta} \Sigma^\infty \mathfrak{C}(Y) \xrightarrow{\simeq} Y \tag{13}$$

$$\textit{resp.} \quad \Sigma^r \operatorname{holim}_{\Delta} \mathfrak{C}_r(Z) \xrightarrow{\simeq} \operatorname{holim}_{\Delta} \Sigma^r \mathfrak{C}_r(Z) \xrightarrow{\simeq} Z \tag{14}$$

are weak equivalences.

Proof. For the case of the left-hand map in (13), it is enough to verify that the connectivities of the natural maps (8) and (11) are strictly increasing with n, and Theorems 2.2 and 2.3 complete the proof. Consider the right-hand map. Since $\Sigma^\infty \mathfrak{C}(Y) \simeq F\Sigma^\infty \mathfrak{C}(Y)$ and the latter is isomorphic to the cosimplicial cobar construction $\operatorname{Cobar}(F\tilde{\mathsf{K}}, \tilde{\mathsf{K}}, Y)$, which has extra codegeneracy maps s^{-1} (Dwyer-Miller-Neisendorfer [28, 6.2]), it follows from the cofinality argument in Dror-Dwyer [22, 3.16] that the right-hand map in (13) is a weak equivalence. The other case is similar. □

Proof of Theorems 1.1 and 1.2. We want to verify that the natural map

$$\Sigma^\infty \operatorname{holim}_{\Delta} \mathfrak{C}(Y) \rightarrow Y$$

is a weak equivalence; since this is the composite (13), Theorem 2.4 completes the proof. The other case is similar. □

The following is an immediate corollary of the connectivity estimates in Theorem 2.1. These types of homotopy spectral sequences have been studied, for instance, in [7, 8, 14, 15].

Corollary 2.5. *If Y is a 1-connected $\tilde{\mathsf{K}}$-coalgebra spectrum (resp. Z is a $(1+r)$-connected $\tilde{\mathsf{K}}_r$-coalgebra), then the homotopy spectral sequence*

$$E^2_{-s,t} = \pi^s\pi_t\mathfrak{C}(Y) \Longrightarrow \pi_{t-s}\operatorname{holim}_\Delta \mathfrak{C}(Y)$$

$$\big(\textit{resp.}\quad E^2_{-s,t} = \pi^s\pi_t\mathfrak{C}_r(Z) \Longrightarrow \pi_{t-s}\operatorname{holim}_\Delta \mathfrak{C}_r(Z)\big)$$

converges strongly. By strong convergence *of $\{E^r\}$ to $\pi_*\operatorname{holim}_\Delta \mathfrak{C}(Y)$ we mean that (i) for each $(-s,t)$, there exists an r such that $E^r_{-s,t} = E^\infty_{-s,t}$ and (ii) for each i, $E^\infty_{-s,s+i} = 0$ except for finitely many s; see, for instance,* [14, IV.5.6, IX.5.3, IX.5.4] *and* [26, p. 255]*; similarly for the other case.*

3. Homotopical comonads and their homotopy coalgebras

If X is a pointed space, the stabilization map has the form

$$\pi_*(X) \to \pi^{\mathrm{s}}_*(X) = \operatorname{colim}_r \pi_{*+r}(\Sigma^r X) \tag{15}$$

This comparison map between homotopy groups and stable homotopy groups comes from a space level stabilization map of the form

$$X \to \tilde{\Omega}^\infty\Sigma^\infty(X) \tag{16}$$

and applying π_* to (16) recovers the map (15); here, $\tilde{\Omega}^\infty = \Omega^\infty F$ (Definition 3.4) denotes the right-derived functor of the underlying 0-th space functor $\Omega^\infty = \mathrm{Ev}_0$, $\Sigma^\infty = S\otimes -$ denotes the stabilization functor given by tensoring with the sphere spectrum S, and F denotes a simplicial fibrant replacement monad.

With a space level stabilization map in hand, it is natural to form a cosimplicial resolution of X with respect to $\tilde{\Omega}^\infty\Sigma^\infty$ of the form

$$X \longrightarrow \tilde{\Omega}^\infty\Sigma^\infty(X) \rightrightarrows (\tilde{\Omega}^\infty\Sigma^\infty)^2(X) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} (\tilde{\Omega}^\infty\Sigma^\infty)^3(X)\cdots \tag{17}$$

showing only the coface maps. The homotopical comonad $\tilde{K} = \Sigma^\infty\tilde{\Omega}^\infty$, which is the derived functor of the comonad $K = \Sigma^\infty\Omega^\infty$ associated to the $(\Sigma^\infty, \Omega^\infty)$ adjunction (see (23)), can be thought of as encoding the spectrum level co-operations on the suspension spectra; compare with [9] for integral chains.

By analogy with the techniques in Bousfield-Kan [14], by iterating the space level stabilization map (16) Carlsson [16], and subsequently Arone-Kankaanrinta [2], study the cosimplicial resolution of X with respect to $\tilde{\Omega}^\infty\Sigma^\infty$, and taking the homotopy limit of the resolution (17) produce the $\tilde{\Omega}^\infty\Sigma^\infty$-completion map

$$X \to X^\wedge_{\tilde{\Omega}^\infty\Sigma^\infty} \tag{18}$$

This map arises as the derived unit map (Definition 4.3) associated to (1).

Similarly, if $r \geq 1$, the Freudenthal suspension map has the form

$$\pi_*(X) \to \pi_{*+r}(\Sigma^r X) \tag{19}$$

This map comes from a space level Freudenthal suspension map of the form

$$X \to \tilde{\Omega}^r\Sigma^r(X) \tag{20}$$

Applying π_* to the map (20) recovers the map (19); here, $\tilde{\Omega}^r = \Omega^r\Phi$ (Definition 3.4) denotes the right-derived functor of the iterated loop space functor $\Omega^r = \mathbf{hom}_*(S^r, -)$, $\Sigma^r = S^r \wedge -$ denotes the iterated suspension functor given by smashing with the r-sphere $S^r = (S^1)^{\wedge r}$, and Φ denotes a simplicial fibrant replacement monad; see Section 6 for a reminder on the $\mathbf{hom}_*$ functor.

Once one has such a Freudenthal suspension map on the level of spaces, it is natural to form a cosimplicial resolution of X with respect to $\tilde{\Omega}^r\Sigma^r$ of the form

$$X \longrightarrow \tilde{\Omega}^r\Sigma^r(X) \rightrightarrows (\tilde{\Omega}^r\Sigma^r)^2(X) \substack{\rightarrow\\\rightarrow\\\rightarrow} (\tilde{\Omega}^r\Sigma^r)^3(X)\cdots \tag{21}$$

showing only the coface maps. The homotopical comonad $\tilde{\mathsf{K}}_r = \Sigma^r\tilde{\Omega}^r$, which is the derived functor of the comonad $\mathsf{K}_r = \Sigma^r\Omega^r$ associated to the (Σ^r, Ω^r) adjunction (see (26)), can be thought of as encoding the space level co-operations on the iterated suspension spaces. Bousfield [11] and Hopkins (see [11]) study the cosimplicial resolution of X with respect to $\tilde{\Omega}^r\Sigma^r$, and taking the homotopy limit of the resolution (21) produces the $\tilde{\Omega}^r\Sigma^r$-completion map

$$X \rightarrow X^{\wedge}_{\tilde{\Omega}^r\Sigma^r} \tag{22}$$

This map arises as the derived unit map (Definition 4.3) associated to (2).

3.1. **Cosimplical cobar constructions associated to homotopy coalgebras.** Consider any pointed space X and S-module Y, and recall that the suspension spectrum $\Sigma^\infty(X) = S\otimes X$ and 0-th space $\Omega^\infty(Y) = \mathrm{Ev}_0(Y) = Y_0$ functors fit into an adjunction

$$\mathsf{S}_* \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathsf{Mod}_S \tag{23}$$

with left adjoint on top. Associated to the adjunction in (23) is the monad $\Omega^\infty\Sigma^\infty$ on pointed spaces S_* and the comonad $\mathsf{K} := \Sigma^\infty\Omega^\infty$ on S-modules Mod_S of the form

$$\begin{aligned} &\mathrm{id} \xrightarrow{\eta} \Omega^\infty\Sigma^\infty \quad \text{(unit)}, && \mathrm{id} \xleftarrow{\varepsilon} \mathsf{K} \quad \text{(counit)},\\ &\Omega^\infty\Sigma^\infty\Omega^\infty\Sigma^\infty \rightarrow \Omega^\infty\Sigma^\infty \quad \text{(multiplication)}, && \mathsf{K}\mathsf{K} \xleftarrow{m} \mathsf{K} \quad \text{(comultiplication)}. \end{aligned} \tag{24}$$

The suspension spectrum $\Sigma^\infty X$ is naturally equipped with a K-coalgebra structure. Denote by $m\colon \Omega^\infty \rightarrow \Omega^\infty\mathsf{K} = \Omega^\infty\Sigma^\infty\Omega^\infty$ the right K-coaction map on Ω^∞ (defined by $m := \eta\,\mathrm{id}$).

Definition 3.2. Let Y be a K-coalgebra. The *cosimplicial cobar construction* $C(Y) := \mathrm{Cobar}(\Omega^\infty, \mathsf{K}, Y)$ in $(\mathsf{S}_*)^\Delta$ looks like

$$C(Y): \qquad \Omega^\infty Y \underset{d^1}{\overset{d^0}{\rightrightarrows}} \Omega^\infty\mathsf{K}Y \substack{\rightarrow\\\rightarrow\\\rightarrow} \Omega^\infty\mathsf{K}\mathsf{K}Y\cdots \tag{25}$$

(showing only the coface maps) and is defined objectwise by $C(Y)^n := \Omega^\infty\mathsf{K}^nY$ with the obvious coface and codegeneracy maps; see, for instance, the face and degeneracy maps in the simplicial bar constructions described in Gugenheim-May [39, A.1] or May [54, Section 7], and dualize. For instance, the indicated coface maps in (25) are defined by $d^0 := m\,\mathrm{id}$ and $d^1 := \mathrm{id}\,m$. Here, we denote by $m\colon Y \rightarrow \mathsf{K}Y$ the K-coaction map on Y. Compare with [20, 3.15].

Similarly, consider any pointed spaces X, X' and recall that the iterated suspension space $\Sigma^r(X) := S^r \wedge X$ and iterated loop space $\Omega^r(X') := \mathbf{hom}_*(S^r, X')$ functors fit into the left-hand adjunction

$$\mathsf{S}_* \underset{\Omega^r}{\overset{\Sigma^r}{\rightleftarrows}} \mathsf{S}_* \qquad \mathsf{S}_* \underset{\Omega}{\overset{\Sigma}{\rightleftarrows}} \mathsf{S}_* \;\cdots\; \mathsf{S}_* \underset{\Omega}{\overset{\Sigma}{\rightleftarrows}} \mathsf{S}_* \qquad (r \text{ copies}) \tag{26}$$

with left adjoint on top; here, $S^r := (S^1)^{\wedge r}$ for $r \geq 1$ where $S^1 := \Delta[1]/\partial\Delta[1]$. Note that the left-hand adjunction is naturally isomorphic to the right-hand r-fold iteration of the suspension-loop adjunctions, by uniqueness of adjoints up to natural isomorphism. Associated to the adjunction in (26) is the monad $\Omega^r\Sigma^r$ on pointed spaces S_* and the comonad $\mathsf{K}_r := \Sigma^r\Omega^r$ on pointed spaces S_* of the form

$$\begin{aligned} &\mathrm{id} \xrightarrow{\eta} \Omega^r\Sigma^r \quad \text{(unit)}, && \mathrm{id} \xleftarrow{\varepsilon} \mathsf{K}_r \quad \text{(counit)}, \\ &\Omega^r\Sigma^r\Omega^r\Sigma^r \rightarrow \Omega^r\Sigma^r \quad \text{(multiplication)}, && \mathsf{K}_r\mathsf{K}_r \xleftarrow{m} \mathsf{K}_r \quad \text{(comultiplication)}. \end{aligned} \tag{27}$$

The iterated suspension space $\Sigma^r X$ is naturally equipped with a K_r-coalgebra structure. Denote by $m\colon \Omega^r \rightarrow \Omega^r\mathsf{K}_r = \Omega^r\Sigma^r\Omega^r$ the right K_r-coaction map on Ω^r (defined by $m := \eta\,\mathrm{id}$).

Definition 3.3. Let Z be a K_r-coalgebra. The *cosimplicial cobar construction* $C_r(Z) := \mathrm{Cobar}(\Omega^r, \mathsf{K}_r, Z)$ in $(\mathsf{S}_*)^\Delta$ looks like

$$C_r(Z): \qquad \Omega^r Z \underset{d^1}{\overset{d^0}{\rightrightarrows}} \Omega^r\mathsf{K}_r Z \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \Omega^r\mathsf{K}_r\mathsf{K}_r Z \cdots \tag{28}$$

(showing only the coface maps) and is defined objectwise by $C_r(Z)^n := \Omega^r(\mathsf{K}_r)^n Z$ with the obvious coface and codegeneracy maps; see, for instance, the face and degeneracy maps in the simplicial bar constructions described in Gugenheim-May [39, A.1] or May [54, Section 7], and dualize. For instance, in (28) the indicated coface maps are defined by $d^0 := m\,\mathrm{id}$ and $d^1 := \mathrm{id}\,m$. Here, we denote by $m\colon Z \rightarrow \mathsf{K}_r Z$ the K_r-coaction map on Z. Compare with [20, 3.15].

It will be useful to interpret the cosimplicial $\tilde{\Omega}^\infty\Sigma^\infty$-resolution of X in terms of a cosimplicial cobar construction that naturally arises as a "fattened" version of (25); this leads to the notion of a homotopy $\tilde{\mathsf{K}}$-coalgebra appearing in Blumberg-Riehl [10] and exploited in Cohn [21].

Definition 3.4. Denote by $\eta\colon \mathrm{id} \rightarrow F$ and $m\colon FF \rightarrow F$ the unit and multiplication maps of the simplicial fibrant replacement monad F on Mod_S (see [10, 6.1], and also [33] and [58]). It follows that $\tilde{\Omega}^\infty := \Omega^\infty F$ and $\tilde{\mathsf{K}} := \mathsf{K}F$ are the derived functors of Ω^∞ and K, respectively. The comultiplication $m\colon \tilde{\mathsf{K}} \rightarrow \tilde{\mathsf{K}}\tilde{\mathsf{K}}$ and counit $\varepsilon\colon \tilde{\mathsf{K}} \rightarrow F$ maps are defined by the composites

$$\mathsf{K}F \xrightarrow{m\,\mathrm{id}} \mathsf{K}\mathsf{K}F = \mathsf{K}\mathrm{id}\mathsf{K}F \xrightarrow{\mathrm{id}\,\eta\,\mathrm{id}\,\mathrm{id}} \mathsf{K}F\mathsf{K}F \tag{29}$$

$$\mathsf{K}F \xrightarrow{\varepsilon\,\mathrm{id}} \mathrm{id}F = F \tag{30}$$

respectively.

Similarly, denote by $\eta\colon \mathrm{id} \rightarrow \Phi$ and $m\colon \Phi\Phi \rightarrow \Phi$ the unit and multiplication maps of the simplicial fibrant replacement monad Φ on pointed spaces S_* (see [10, 6.1], and also [33] and [58]). It follows that $\tilde{\Omega}^r := \Omega^r\Phi$ and $\tilde{\mathsf{K}}_r := \mathsf{K}_r\Phi$ are the derived functors of Ω^r and K_r, respectively. The comultiplication $m\colon \tilde{\mathsf{K}}_r \rightarrow \tilde{\mathsf{K}}_r\tilde{\mathsf{K}}_r$ and counit $\varepsilon\colon \tilde{\mathsf{K}}_r \rightarrow \Phi$ maps are defined by the composites

$$\mathsf{K}_r\Phi \xrightarrow{m\,\mathrm{id}} \mathsf{K}_r\mathsf{K}_r\Phi = \mathsf{K}_r\mathrm{id}\mathsf{K}_r\Phi \xrightarrow{\mathrm{id}\,\eta\,\mathrm{id}\,\mathrm{id}} \mathsf{K}_r\Phi\mathsf{K}_r\Phi \tag{31}$$

$$\mathsf{K}_r\Phi \xrightarrow{\varepsilon\,\mathrm{id}} \mathrm{id}\Phi = \Phi \tag{32}$$

respectively.

It is shown in Blumberg-Riehl [10, 4.2, 4.4], and subsequently exploited in Cohn [21], that the derived functor $\tilde{\mathsf{K}} := \mathsf{K}F$ of the comonad K is very nearly a comonad itself with the structure maps $m\colon \tilde{\mathsf{K}}\rightarrow\tilde{\mathsf{K}}\tilde{\mathsf{K}}$ and $\varepsilon\colon \tilde{\mathsf{K}}\rightarrow F$ above. For instance, it is proved in [10] that $\tilde{\mathsf{K}}$ defines a comonad on the homotopy category of Mod_S, which is a reflection of the the fact that $\tilde{\mathsf{K}}$ has the structure of a highly homotopy coherent comonad (see [10]); in particular, $\tilde{\mathsf{K}}$ has a strictly coassociative comultiplication $m\colon \tilde{\mathsf{K}}\rightarrow\tilde{\mathsf{K}}\tilde{\mathsf{K}}$ and satisfies left and right counit identities up to factors of $F\simeq \mathrm{id}$.

In more detail, the homotopical comonad $\tilde{\mathsf{K}}$ makes the following diagrams

$$
\begin{tikzcd}
\tilde{\mathsf{K}} \arrow[r, "m"] \arrow[d, "m"'] & \tilde{\mathsf{K}}\tilde{\mathsf{K}} \arrow[d, "m\,\mathrm{id}"] \\
\tilde{\mathsf{K}}\tilde{\mathsf{K}} \arrow[r, "\mathrm{id}\,m"'] & \tilde{\mathsf{K}}\tilde{\mathsf{K}}\tilde{\mathsf{K}}
\end{tikzcd}
\qquad
\begin{tikzcd}
F\tilde{\mathsf{K}} \arrow[r, "\mathrm{id}\,m"] \arrow[d, equal] & F\tilde{\mathsf{K}}\tilde{\mathsf{K}} \arrow[d, "(*)"] \\
F\tilde{\mathsf{K}} \arrow[r, equal] & F\tilde{\mathsf{K}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{\mathsf{K}} \arrow[r, "m"] \arrow[d, equal] & \tilde{\mathsf{K}}\tilde{\mathsf{K}} \arrow[d, "(**)"] \\
\tilde{\mathsf{K}} \arrow[r, equal] & \tilde{\mathsf{K}}
\end{tikzcd}
$$

commute. Here, the map $(*)$ is the composite $F\tilde{\mathsf{K}}\tilde{\mathsf{K}} \xrightarrow{\mathrm{id}\,\varepsilon\,\mathrm{id}} FF\tilde{\mathsf{K}} \xrightarrow{m\,\mathrm{id}} F\tilde{\mathsf{K}}$ and the map $(**)$ is the composite $\mathsf{K}F\tilde{\mathsf{K}} \xrightarrow{\mathrm{id}\,\mathrm{id}\,\varepsilon} \mathsf{K}FF \xrightarrow{\mathrm{id}\,m} \mathsf{K}F$; in other words, the map $(*)$ (resp. $(**)$) simply inserts $\varepsilon\colon \mathsf{K}\rightarrow\mathrm{id}$ on the left (resp. right) and multiplies down the resulting FF term to F.

Similarly, the homotopical comonad $\tilde{\mathsf{K}}_r$ makes the following diagrams

$$
\begin{tikzcd}
\tilde{\mathsf{K}}_r \arrow[r, "m"] \arrow[d, "m"'] & \tilde{\mathsf{K}}_r\tilde{\mathsf{K}}_r \arrow[d, "m\,\mathrm{id}"] \\
\tilde{\mathsf{K}}_r\tilde{\mathsf{K}}_r \arrow[r, "\mathrm{id}\,m"'] & \tilde{\mathsf{K}}_r\tilde{\mathsf{K}}_r\tilde{\mathsf{K}}_r
\end{tikzcd}
\qquad
\begin{tikzcd}
\Phi\tilde{\mathsf{K}}_r \arrow[r, "\mathrm{id}\,m"] \arrow[d, equal] & \Phi\tilde{\mathsf{K}}_r\tilde{\mathsf{K}}_r \arrow[d, "(*)"] \\
\Phi\tilde{\mathsf{K}}_r \arrow[r, equal] & \Phi\tilde{\mathsf{K}}_r
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{\mathsf{K}}_r \arrow[r, "m"] \arrow[d, equal] & \tilde{\mathsf{K}}_r\tilde{\mathsf{K}}_r \arrow[d, "(**)"] \\
\tilde{\mathsf{K}}_r \arrow[r, equal] & \tilde{\mathsf{K}}_r
\end{tikzcd}
$$

commute. Here, the map $(*)$ is the composite $\Phi\tilde{\mathsf{K}}_r\tilde{\mathsf{K}}_r \xrightarrow{\mathrm{id}\,\varepsilon\,\mathrm{id}} \Phi\Phi\tilde{\mathsf{K}}_r \xrightarrow{m\,\mathrm{id}} \Phi\tilde{\mathsf{K}}_r$ and the map $(**)$ is the composite $\mathsf{K}_r\Phi\tilde{\mathsf{K}}_r \xrightarrow{\mathrm{id}\,\mathrm{id}\,\varepsilon} \mathsf{K}_r\Phi\Phi \xrightarrow{\mathrm{id}\,m} \mathsf{K}_r\Phi$; in other words, the map $(*)$ (resp. $(**)$) simply inserts $\varepsilon\colon \mathsf{K}_r\rightarrow\mathrm{id}$ on the left (resp. right) and multiplies down the resulting $\Phi\Phi$ term to Φ.

The following notion of a homotopy $\tilde{\mathsf{K}}$-coalgebra appearing in Blumberg-Riehl [10] and exploited in Cohn [21], captures exactly the left $\tilde{\mathsf{K}}$-coaction structure that stabilization $\Sigma^\infty X$ of a pointed space X satisfies; this is precisely the structure being encoded by the cosimplicial $\tilde{\Omega}^\infty\Sigma^\infty$ resolution (17).

Definition 3.5. A *homotopy* $\tilde{\mathsf{K}}$*-coalgebra* (or $\tilde{\mathsf{K}}$-coalgebra, for short) is a $Y\in\mathsf{Mod}_S$ together with a map $m\colon Y\rightarrow\tilde{\mathsf{K}}Y$ in Mod_S such that the following diagrams

$$
\begin{tikzcd}
Y \arrow[r, "m"] \arrow[d, "m"'] & \tilde{\mathsf{K}}Y \arrow[d, "m\,\mathrm{id}"] \\
\tilde{\mathsf{K}}Y \arrow[r, "\mathrm{id}\,m"'] & \tilde{\mathsf{K}}\tilde{\mathsf{K}}Y
\end{tikzcd}
\qquad
\begin{tikzcd}
FY \arrow[r, "\mathrm{id}\,m"] \arrow[d, equal] & F\tilde{\mathsf{K}}Y \arrow[d, "(*)"] \\
FY \arrow[r, equal] & FY
\end{tikzcd}
\tag{33}
$$

commute. Here, the map $(*)$ is the composite $F\tilde{\mathsf{K}}Y \xrightarrow{\mathrm{id}\,\varepsilon\,\mathrm{id}} FFY \xrightarrow{m\,\mathrm{id}} FY$; in other words, the map $(*)$ simply inserts $\varepsilon\colon \mathsf{K}\rightarrow\mathrm{id}$ on the left and multiplies down the resulting FF term to F. We will sometimes refer to a $\tilde{\mathsf{K}}$-coalgebra structure on Y as a *left* $\tilde{\mathsf{K}}$*-coaction* on Y.

Similarly, a *homotopy* $\tilde{\mathsf{K}}_r$*-coalgebra* (or $\tilde{\mathsf{K}}_r$-coalgebra, for short) is a $Z \in \mathsf{S}_*$ together with a map $m\colon Z \to \tilde{\mathsf{K}}_r Z$ in S_* such that the following diagrams

$$\begin{tikzcd} Z \arrow[r, "m"] \arrow[d, "m"'] & \tilde{\mathsf{K}}_r Z \arrow[d, "m\,\mathrm{id}"] \\ \tilde{\mathsf{K}}_r Z \arrow[r, "\mathrm{id}\,m"'] & \tilde{\mathsf{K}}_r\tilde{\mathsf{K}}_r Z \end{tikzcd} \qquad \begin{tikzcd} \Phi Z \arrow[r, "\mathrm{id}\,m"] \arrow[d, equal] & \Phi\tilde{\mathsf{K}}_r Z \arrow[d, "(*)"] \\ \Phi Z \arrow[r, equal] & \Phi Z \end{tikzcd} \tag{34}$$

commute. Here, the map $(*)$ is the composite $\Phi\tilde{\mathsf{K}}_r Z \xrightarrow{\mathrm{id}\,\varepsilon\,\mathrm{id}} \Phi\Phi Z \xrightarrow{m\,\mathrm{id}} \Phi Z$; in other words, the map $(*)$ simply inserts $\varepsilon\colon \mathsf{K}_r \to \mathrm{id}$ on the left and multiplies down the resulting $\Phi\Phi$ term to Φ. We will sometimes refer to a $\tilde{\mathsf{K}}_r$-coalgebra structure on Z as a *left* $\tilde{\mathsf{K}}_r$*-coaction* on Z.

Remark 3.6. Associated to the adjunction $(\Sigma^\infty, \Omega^\infty)$ is a left K-coaction (or K-coalgebra structure) $m\colon \Sigma^\infty X \to \mathsf{K}\Sigma^\infty X$ on $\Sigma^\infty X$, defined by $m = \mathrm{id}\,\eta\,\mathrm{id}$, for any $X \in \mathsf{S}_*$. This map induces a corresponding left $\tilde{\mathsf{K}}$-coaction $m\colon \Sigma^\infty X \to \tilde{\mathsf{K}}\Sigma^\infty X$ that is the composite

$$\Sigma^\infty X \xrightarrow{m} \mathsf{K}\Sigma^\infty X = \mathsf{K}\mathrm{id}\Sigma^\infty X \xrightarrow{\mathrm{id}\,\eta\,\mathrm{id}\,\mathrm{id}} \mathsf{K}F\Sigma^\infty X$$

Similarly, associated to the adjunction (Σ^r, Ω^r) is a left K_r-coaction (or K_r-coalgebra structure) $m\colon \Sigma^r X \to \mathsf{K}_r\Sigma^r X$ on $\Sigma^r X$, defined by $m = \mathrm{id}\,\eta\,\mathrm{id}$, for any $X \in \mathsf{S}_*$. This map induces a corresponding left $\tilde{\mathsf{K}}_r$-coaction $m\colon \Sigma^r X \to \tilde{\mathsf{K}}_r\Sigma^r X$ that is the composite

$$\Sigma^r X \xrightarrow{m} \mathsf{K}_r\Sigma^r X = \mathsf{K}_r\mathrm{id}\Sigma^r X \xrightarrow{\mathrm{id}\,\eta\,\mathrm{id}\,\mathrm{id}} \mathsf{K}_r\Phi\Sigma^r X$$

Remark 3.7. More generally, every K-coalgebra structure $m\colon Y \to \mathsf{K}Y$ on $Y \in \mathsf{Mod}_S$ induces a $\tilde{\mathsf{K}}$-coalgebra structure $m\colon Y \to \tilde{\mathsf{K}}Y$ that is the composite $Y \xrightarrow{m} \mathsf{K}Y = \mathsf{K}\mathrm{id}Y \xrightarrow{\mathrm{id}\,\eta\,\mathrm{id}} \mathsf{K}FY$. Similarly, every K_r-coalgebra structure $m\colon Z \to \mathsf{K}_r Z$ on $Z \in \mathsf{S}_*$ induces a $\tilde{\mathsf{K}}_r$-coalgebra structure $m\colon Z \to \tilde{\mathsf{K}}_r Z$ that is the composite $Z \xrightarrow{m} \mathsf{K}_r Z = \mathsf{K}_r\mathrm{id}Z \xrightarrow{\mathrm{id}\,\eta\,\mathrm{id}} \mathsf{K}_r\Phi Z$.

Remark 3.8. The derived functor $\tilde{\Omega}^\infty$ has a naturally occurring right homotopy $\tilde{\mathsf{K}}$-coaction map (or right $\tilde{\mathsf{K}}$-coaction map, for short) $m\colon \tilde{\Omega}^\infty \to \tilde{\Omega}^\infty\tilde{\mathsf{K}}$, defined by the composite

$$\Omega^\infty F \xrightarrow{m\,\mathrm{id}} \Omega^\infty\mathsf{K}F = \Omega^\infty\mathrm{id}\mathsf{K}F \xrightarrow{\mathrm{id}\,\eta\,\mathrm{id}\,\mathrm{id}} \Omega^\infty F\mathsf{K}F$$

that makes the following diagrams

$$\begin{tikzcd} \tilde{\Omega}^\infty \arrow[r, "m"] \arrow[d, "m"'] & \tilde{\Omega}^\infty\tilde{\mathsf{K}} \arrow[d, "m\,\mathrm{id}"] \\ \tilde{\Omega}^\infty\tilde{\mathsf{K}} \arrow[r, "\mathrm{id}\,m"'] & \tilde{\Omega}^\infty\tilde{\mathsf{K}}\tilde{\mathsf{K}} \end{tikzcd} \qquad \begin{tikzcd} \tilde{\Omega}^\infty \arrow[r, "m"] \arrow[d, equal] & \tilde{\Omega}^\infty\tilde{\mathsf{K}} \arrow[d, "(**)"] \\ \tilde{\Omega}^\infty \arrow[r, equal] & \tilde{\Omega}^\infty \end{tikzcd}$$

commute. Here, the map $(**)$ is the composite $\Omega^\infty F\tilde{\mathsf{K}} \xrightarrow{\mathrm{id}\,\mathrm{id}\,\varepsilon} \Omega^\infty FF \xrightarrow{\mathrm{id}\,m} \Omega^\infty F$; in other words, the map $(**)$ simply inserts $\varepsilon\colon \mathsf{K} \to \mathrm{id}$ on the right and multiplies down the resulting FF term to F.

Similarly, the derived functor $\tilde{\Omega}^r$ has a naturally occurring right homotopy $\tilde{\mathsf{K}}_r$-coaction map (or right $\tilde{\mathsf{K}}_r$-coaction map, for short) $m\colon \tilde{\Omega}^r{\to}\tilde{\Omega}^r\tilde{\mathsf{K}}_r$, defined by the composite

$$\Omega^r\Phi \xrightarrow{m\,\mathrm{id}} \Omega^r\mathsf{K}_r\Phi = \Omega^r\mathrm{id}\mathsf{K}_r\Phi \xrightarrow{\mathrm{id}\,\eta\,\mathrm{id}\,\mathrm{id}} \Omega^r\Phi\mathsf{K}_r\Phi$$

that makes the following diagrams

$$\begin{tikzcd}
\tilde{\Omega}^r \arrow[r, "m"] \arrow[d, "m"'] & \tilde{\Omega}^r\tilde{\mathsf{K}}_r \arrow[d, "m\,\mathrm{id}"] \\
\tilde{\Omega}^r\tilde{\mathsf{K}}_r \arrow[r, "\mathrm{id}\,m"'] & \tilde{\Omega}^r\tilde{\mathsf{K}}_r\tilde{\mathsf{K}}_r
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{\Omega}^r \arrow[r, "m"] \arrow[d, equal] & \tilde{\Omega}^r\tilde{\mathsf{K}}_r \arrow[d, "(**)"] \\
\tilde{\Omega}^r \arrow[r, equal] & \tilde{\Omega}^r
\end{tikzcd}$$

commute. Here, the map $(**)$ is the composite $\Omega^r\Phi\tilde{\mathsf{K}}_r \xrightarrow{\mathrm{id}\,\mathrm{id}\,\varepsilon} \Omega^r\Phi\Phi \xrightarrow{\mathrm{id}\,m} \Omega^r\Phi$; in other words, the map $(**)$ simply inserts $\varepsilon\colon \mathsf{K}_r{\to}\mathrm{id}$ on the right and multiplies down the resulting $\Phi\Phi$ term to Φ.

The following cosimplicial cobar constructions provide a generalization of the resolutions (17) and (21): this is because the resolution $X \to \mathfrak{C}(\Sigma^\infty X)$ is identical to (17) and the resolution $X \to \mathfrak{C}_r(\Sigma^r X)$ is identical to (21).

Remark 3.9. In the following definition, it is important to note that the cobar constructions $\mathfrak{C}(Y), \mathfrak{C}_r(Z)$ are genuinely cosimplicial diagrams (equipped with both coface and codegeneracy maps satisfying the cosimplicial identities); we have simply suppressed showing the codegeneracy maps to keep the diagrams (35) and (36) reasonably sized for typesetting purposes.

Definition 3.10. Let Y be a $\tilde{\mathsf{K}}$-coalgebra spectrum. The *cosimplicial cobar construction* $\mathfrak{C}(Y) := \mathrm{Cobar}(\tilde{\Omega}^\infty, \tilde{\mathsf{K}}, Y)$ in $(\mathsf{S}_*)^\Delta$ looks like

$$\mathfrak{C}(Y): \qquad \tilde{\Omega}^\infty Y \underset{d^1}{\overset{d^0}{\rightrightarrows}} \tilde{\Omega}^\infty\tilde{\mathsf{K}}Y \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \tilde{\Omega}^\infty\tilde{\mathsf{K}}\tilde{\mathsf{K}}Y \cdots \tag{35}$$

(showing only the coface maps) and is defined objectwise by $\mathfrak{C}(Y)^n := \tilde{\Omega}^\infty\tilde{\mathsf{K}}^n Y = \Omega^\infty F(\mathsf{K}F)^n Y$ with the obvious coface and codegeneracy maps; for instance, in (35) the indicated coface maps are defined by $d^0 := m\,\mathrm{id}$ and $d^1 := \mathrm{id}\,m$. Compare with [20, 3.15].

Similarly, Let Z be a $\tilde{\mathsf{K}}_r$-coalgebra. The *cosimplicial cobar construction* $\mathfrak{C}_r(Z) := \mathrm{Cobar}(\tilde{\Omega}^r, \tilde{\mathsf{K}}_r, Z)$ in $(\mathsf{S}_*)^\Delta$ looks like

$$\mathfrak{C}_r(Z): \qquad \tilde{\Omega}^r Z \underset{d^1}{\overset{d^0}{\rightrightarrows}} \tilde{\Omega}^r\tilde{\mathsf{K}}_r Z \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \tilde{\Omega}^r\tilde{\mathsf{K}}_r\tilde{\mathsf{K}}_r Z \cdots \tag{36}$$

(showing only the coface maps) and is defined objectwise by $\mathfrak{C}_r(Z)^n := \tilde{\Omega}^r\tilde{\mathsf{K}}_r^n Z = \Omega^r\Phi(\mathsf{K}_r\Phi)^n Z$ with the obvious coface and codegeneracy maps; for instance, in (36) the indicated coface maps are defined by $d^0 := m\,\mathrm{id}$ and $d^1 := \mathrm{id}\,m$. Compare with [20, 3.15].

Remark 3.11. It may be helpful to note, when comparing with [10], that the counit map (30) is identical to the composite

$$\mathsf{K}F = \mathrm{id}\mathsf{K}F \xrightarrow{\eta\,\mathrm{id}\,\mathrm{id}} F\mathsf{K}F \xrightarrow{\mathrm{id}\,\varepsilon\,\mathrm{id}} F\mathrm{id}F = FF \xrightarrow{m} F$$

Similary, the counit map (32) is identical to the composite

$$\mathsf{K}_r\Phi = \mathrm{id}\mathsf{K}_r\Phi \xrightarrow{\eta\,\mathrm{id}\,\mathrm{id}} \Phi\mathsf{K}_r\Phi \xrightarrow{\mathrm{id}\,\varepsilon\,\mathrm{id}} \Phi\mathrm{id}\Phi = \Phi\Phi \xrightarrow{m} \Phi$$

3.12. **Enrichments, box products, and composition maps.** In this section we setup the homotopy theory of $\tilde{\mathsf{K}}$-coalgebras (resp. $\tilde{\mathsf{K}}_r$-coalgebras) using a tiny modification of the framework developed in Arone-Ching [1]; it is closely related to [20], together with the observation in Cohn [21] that this framework extends to the homotopy coalgebras in Blumberg-Riehl [10].

Definition 3.13. Let Y, Y' be $\tilde{\mathsf{K}}$-coalgebra spectra (resp. Z, Z' be $\tilde{\mathsf{K}}_r$-coalgebras). A *morphism* of $\tilde{\mathsf{K}}$-coalgebra spectra from Y to Y' is a map $f\colon Y \to Y'$ in Mod_S that makes the left-hand diagram

$$\begin{tikzcd} Y \arrow[r, "m"] \arrow[d, "f"'] & \tilde{\mathsf{K}}Y \arrow[d, "\mathrm{id}\,f"] \\ Y' \arrow[r, "m"'] & \tilde{\mathsf{K}}Y' \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r, "m"] \arrow[d, "g"'] & \tilde{\mathsf{K}}_r Z \arrow[d, "\mathrm{id}\,g"] \\ Z' \arrow[r, "m"'] & \tilde{\mathsf{K}}_r Z' \end{tikzcd} \tag{37}$$

in Mod_S commute. Similarly, a *morphism* of $\tilde{\mathsf{K}}_r$-coalgebras from Z to Z' is a map $g\colon Z \to Z'$ in S_* that makes the right-hand diagram in S_* commute.

This motivates the following homotopically meaningful cosimplicial resolution of $\tilde{\mathsf{K}}$-coalgebra (resp. $\tilde{\mathsf{K}}_r$-coalgebra) maps; for a reminder on the **Hom** functors, see Section 6.

Definition 3.14. Let Y, Y' be $\tilde{\mathsf{K}}$-coalgebra spectra (resp. Z, Z' be $\tilde{\mathsf{K}}_r$-coalgebras). The cosimplicial object $\mathbf{Hom}_{\mathsf{Mod}_S}\big(Y, F\tilde{\mathsf{K}}^\bullet Y'\big)$ (resp. $\mathbf{Hom}_{\mathsf{S}_*}\big(Z, \Phi\tilde{\mathsf{K}}_r^\bullet Z'\big)$) in sSet looks like (showing only the coface maps)

$$\mathbf{Hom}_{\mathsf{Mod}_S}(Y, FY') \overset{d^0}{\underset{d^1}{\rightrightarrows}} \mathbf{Hom}_{\mathsf{Mod}_S}\big(Y, F\tilde{\mathsf{K}}Y'\big) \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathbf{Hom}_{\mathsf{Mod}_S}\big(Y, F\tilde{\mathsf{K}}\tilde{\mathsf{K}}Y'\big) \cdots$$

$$\text{resp.}\quad \mathbf{Hom}_{\mathsf{S}_*}(Z, \Phi Z') \overset{d^0}{\underset{d^1}{\rightrightarrows}} \mathbf{Hom}_{\mathsf{S}_*}\big(Z, \Phi\tilde{\mathsf{K}}_r Z'\big) \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathbf{Hom}_{\mathsf{S}_*}\big(Z, \Phi\tilde{\mathsf{K}}_r\tilde{\mathsf{K}}_r Z'\big) \cdots$$

and is defined objectwise by

$$\mathbf{Hom}_{\mathsf{Mod}_S}\big(Y, F\tilde{\mathsf{K}}^\bullet Y'\big)^n := \mathbf{Hom}_{\mathsf{Mod}_S}\big(Y, F\tilde{\mathsf{K}}^n Y'\big)$$
$$\text{resp.}\quad \mathbf{Hom}_{\mathsf{S}_*}\big(Z, \Phi\tilde{\mathsf{K}}_r^\bullet Z'\big)^n := \mathbf{Hom}_{\mathsf{S}_*}\big(Z, \Phi\tilde{\mathsf{K}}_r^n Z'\big)$$

with the obvious coface and codegeneracy maps (see [1, 1.3]).

Remark 3.15. This is simply the resolution in Arone-Ching [1], but "fattened-up" by F (resp. Φ). For instance, on the level of hom-sets (simplicial degree 0), let's verify that $s^0 d^1 = \mathrm{id}$ on $\mathbf{Hom}_{\mathsf{Mod}_S}(Y, FY')$. Start with $f\colon Y \to FY'$ and consider the commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[dd, equal] & FY' \arrow[r, "\mathrm{id}\, m"] \arrow[dd, equal] & F\mathsf{K}FY' \arrow[d, "\mathrm{id}\,\varepsilon\,\mathrm{id}^2"] \arrow[dd, bend right=60, "(*)"'] \\
 & & FFY' \arrow[d, "m\,\mathrm{id}"] \\
Y \arrow[r, "f"] & FY' \arrow[r, equal] & FY'
\end{tikzcd}$$

The composite along the upper horizontal and right-hand vertical maps is s^0d^1f and the composite along the bottom horizontal maps is f; the diagram commutes verifies that $s^0d^1 = \mathrm{id}$. Similarly, on the level of hom-sets (simplicial degree 0), let's verify that $s^0d^0 = \mathrm{id}$ on $\mathbf{Hom}_{\mathsf{Mod}_S}(Y, FY')$. Start with $f\colon Y {\rightarrow} FY'$ and consider the commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "m"] \arrow[d, "\eta\,\mathrm{id}"] & \mathsf{K}FY \arrow[r, "\mathrm{id}^2 f"] \arrow[d, "\eta\,\mathrm{id}^3"] & \mathsf{K}FFY' \arrow[r, "\mathrm{id}\, m\,\mathrm{id}"] \arrow[d, "\eta\,\mathrm{id}^4"] & \mathsf{K}FY' \arrow[r, "\eta\,\mathrm{id}^3"] \arrow[d, "\eta\,\mathrm{id}^3"] & F\mathsf{K}FY' \arrow[d, equal] \\
FY \arrow[r, "\mathrm{id}\, m"] \arrow[dd, equal] & F\mathsf{K}FY \arrow[r, "\mathrm{id}^3 f"] \arrow[d, "\mathrm{id}\,\varepsilon\,\mathrm{id}^2"] \arrow[dd, bend right=60, "(*)"'] & F\mathsf{K}FFY' \arrow[r, "\mathrm{id}^2\, m\,\mathrm{id}"] \arrow[d, "\mathrm{id}\,\varepsilon\,\mathrm{id}^3"] & F\mathsf{K}FY' \arrow[r, equal] \arrow[d, "\mathrm{id}\,\varepsilon\,\mathrm{id}^2"] & F\mathsf{K}FY' \arrow[d, "\mathrm{id}\,\varepsilon\,\mathrm{id}^2"] \\
 & FFY \arrow[r, "\mathrm{id}^2 f"] \arrow[d, "m\,\mathrm{id}"] & FFFY' \arrow[r, "\mathrm{id}\, m\,\mathrm{id}"] \arrow[d, "m\,\mathrm{id}^2"] & FFY' \arrow[r, equal] \arrow[d, "m\,\mathrm{id}"] & FFY' \arrow[d, "m\,\mathrm{id}"] \\
FY \arrow[r, equal] & FY \arrow[r, "\mathrm{id} f"] & FFY' \arrow[r, "m\,\mathrm{id}"] & FY' \arrow[r, equal] & FY' \\
Y \arrow[u, "\eta\,\mathrm{id}"] \arrow[rr, "f"] & & FY' \arrow[u, "\eta\,\mathrm{id}^2"] \arrow[r, equal] & FY' \arrow[u, equal] &
\end{tikzcd}$$

The composite along the upper horizontal and right-hand vertical maps is s^0d^0f and the composite along the bottom horizontal maps is f; the diagram commutes verifies that $s^0d^0 = \mathrm{id}$.

Definition 3.16. The *realization* functor $|-|\colon \mathsf{sSet}{\rightarrow}\mathsf{CGHaus}$ for simplicial sets is defined objectwise by the coend $X \mapsto X \times_{\Delta} \Delta^{(-)}$; here, Δ^n in CGHaus denotes the topological standard n-simplex for each $n \geq 0$ (see [36, I.1.1]).

Remark 3.17. Recall that if $Y, Y' \in \mathsf{Mod}_S$ and $Z, Z' \in \mathsf{S}_*$, then the mapping spaces $\mathrm{Map}_{\mathsf{Mod}_S}(Y, Y')$ and $\mathrm{Map}_{\mathsf{S}_*}(Z, Z')$ in CGHaus are defined by realization

$$\mathrm{Map}_{\mathsf{Mod}_S}(Y, Y') := |\,\mathbf{Hom}_{\mathsf{Mod}_S}(Y, Y')| \qquad \mathrm{Map}_{\mathsf{S}_*}(Z, Z') := |\,\mathbf{Hom}_{\mathsf{S}_*}(Z, Z')|$$

of the indicated simplicial sets.

Definition 3.18. Let Y, Y' be $\tilde{\mathsf{K}}$-coalgebra spectra. The *mapping spaces* of derived $\tilde{\mathsf{K}}$-coalgebra maps $\mathbf{Hom}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}}}(Y, Y')$ in sSet and $\mathrm{Map}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}}}(Y, Y')$ in CGHaus are defined by the restricted totalizations

$$\mathbf{Hom}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}}}(Y, Y') := \mathrm{Tot}^{\mathsf{res}}\, \mathbf{Hom}_{\mathsf{Mod}_S}\big(Y, F\tilde{\mathsf{K}}^{\bullet} Y'\big)$$
$$\mathrm{Map}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}}}(Y, Y') := \mathrm{Tot}^{\mathsf{res}}\, \mathrm{Map}_{\mathsf{Mod}_S}\big(Y, F\tilde{\mathsf{K}}^{\bullet} Y'\big)$$

of the indicated cosimplicial objects.

Similarly, let Z, Z' be $\tilde{\mathsf{K}}_r$-coalgebras. The *mapping spaces* of derived $\tilde{\mathsf{K}}_r$-coalgebra maps $\mathbf{Hom}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}_r}}(Z, Z')$ in sSet and $\mathrm{Map}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}_r}}(Z, Z')$ in CGHaus are defined by the restricted totalizations

$$\mathbf{Hom}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}_r}}(Z, Z') := \mathrm{Tot}^{\mathsf{res}}\,\mathbf{Hom}_{\mathsf{Mod}_S}\big(Z, \Phi\tilde{\mathsf{K}}_r^{\bullet} Z'\big)$$
$$\mathrm{Map}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}_r}}(Z, Z') := \mathrm{Tot}^{\mathsf{res}}\,\mathrm{Map}_{\mathsf{Mod}_S}\big(Z, \Phi\tilde{\mathsf{K}}_r^{\bullet} Z'\big)$$

of the indicated cosimplicial objects.

Remark 3.19. Note that there are natural zigzags of weak equivalences

$$\mathbf{Hom}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}}}(Y, Y') \simeq \operatorname*{holim}_{\Delta} \mathbf{Hom}_{\mathsf{Mod}_S}\big(Y, F\tilde{\mathsf{K}}^{\bullet} Y'\big)$$
$$\mathbf{Hom}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}_r}}(Z, Z') \simeq \operatorname*{holim}_{\Delta} \mathbf{Hom}_{\mathsf{S}_*}\big(Z, \Phi\tilde{\mathsf{K}}_r^{\bullet} Z'\big)$$

The following proposition is proved in [20, 4.9].

Proposition 3.20. *If $A \in (\mathsf{sSet})^{\Delta_{\mathsf{res}}}$ is objectwise fibrant, then the natural map*

$$|\,\mathrm{Tot}^{\mathsf{res}} A| \xrightarrow{\simeq} \mathrm{Tot}^{\mathsf{res}} |A|$$

in CGHaus *is a weak equivalence.*

The following proposition allows us to describe certain maps simplicially and then pass to the topological mapping space via realization.

Proposition 3.21. *Let Y, Y' be $\tilde{\mathsf{K}}$-coalgebra spectra (resp. Z, Z' be $\tilde{\mathsf{K}}_r$-coalgebras). Then the natural map*

$$|\,\mathbf{Hom}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}}}(Y, Y')| \xrightarrow{\simeq} \mathrm{Map}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}}}(Y, Y')$$
$$\textit{resp.}\quad |\,\mathbf{Hom}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}_r}}(Z, Z')| \xrightarrow{\simeq} \mathrm{Map}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}_r}}(Z, Z')$$

in CGHaus *is a weak equivalence.*

Proof. This follows from Proposition 3.20. □

Definition 3.22. Let Y, Y' be $\tilde{\mathsf{K}}$-coalgebra spectra (resp. Z, Z' be $\tilde{\mathsf{K}}_r$-coalgebras). A *derived $\tilde{\mathsf{K}}$-coalgebra map* f of the form $Y \to Y'$ (resp. *$\tilde{\mathsf{K}}_r$-coalgebra map* g of the form $Z \to Z'$) is any map in $(\mathsf{sSet})^{\Delta_{\mathsf{res}}}$ of the form

$$f\colon\ \Delta[-] \longrightarrow \mathbf{Hom}_{\mathsf{Mod}_S}\big(Y, F\tilde{\mathsf{K}}^{\bullet} Y'\big)$$
$$\text{resp.}\quad g\colon\ \Delta[-] \longrightarrow \mathbf{Hom}_{\mathsf{S}_*}\big(Z, \Phi\tilde{\mathsf{K}}_r^{\bullet} Z'\big)$$

A *topological derived $\tilde{\mathsf{K}}$-coalgebra map* h of the form $Y \to Y'$ (resp. *$\tilde{\mathsf{K}}_r$-coalgebra map* l of the form $Z \to Z'$) is any map in $(\mathsf{CGHaus})^{\Delta_{\mathsf{res}}}$ of the form

$$h\colon\ \Delta^{\bullet} \longrightarrow \mathrm{Map}_{\mathsf{Mod}_S}\big(Y, F\tilde{\mathsf{K}}^{\bullet} Y'\big)$$
$$\text{resp.}\quad l\colon\ \Delta^{\bullet} \longrightarrow \mathrm{Map}_{\mathsf{S}_*}\big(Z, \Phi\tilde{\mathsf{K}}_r^{\bullet} Z'\big)$$

The *underlying map* of a derived $\tilde{\mathsf{K}}$-coalgebra map f is the map $f_0\colon Y \to FY'$ that corresponds to the map $f_0\colon \Delta[0] \to \mathbf{Hom}_{\mathsf{Mod}_S}(Y, FY')$. Similarly, the *underlying map* of a derived $\tilde{\mathsf{K}}_r$-coalgebra map g is the map $g_0\colon Z \to \Phi Z'$ that corresponds to the map $g_0\colon \Delta[0] \to \mathbf{Hom}_{\mathsf{S}_*}(Z, \Phi Z')$.

Remark 3.23. Note that every derived $\tilde{\mathsf{K}}$-coalgebra map f (resp. $\tilde{\mathsf{K}}_r$-coalgebra map g) determines a topological derived $\tilde{\mathsf{K}}$-coalgebra map $|f|$ (resp. $\tilde{\mathsf{K}}_r$-coalgebra map $|g|$) by realization.

Box product pairings on cosimplicial objects can be thought of as encoding, on the cosimplicial level, pairings that formally look like the cup product pairings on the singular cochains of a space. These pairings can also encode the composition-of-loops pairings on based loop spaces. In our context, we use box product pairings to encode composition of derived $\tilde{\mathsf{K}}$-coalgebra (resp. $\tilde{\mathsf{K}}_r$-coalgebra) maps; see, for instance, Arone-Ching [1] and Batanin [5]. The following definition of the box product (and its associated coface and codegeneracy maps) appears in McClure-Smith [55, 2.1]; it goes back to Batanin [4, 3.2], and earlier to Artin-Mazur [3, III] in a dual version built for bisimplicial sets. The box product construction (and associated formulas for the coface and codegeneracy maps) below naturally arises by studying the cup product structure on the singular cochains of a space [55, (2.1)–(2.3)]; see Remark 3.26 for a more conceptual interpretation of the box product in terms of a left Kan extension along concatenation.

Definition 3.24. If $X, Y \in (\mathsf{sSet})^{\Delta}$ their *box product* $X \square Y \in (\mathsf{sSet})^{\Delta}$ is defined objectwise by a coequalizer of the form

$$(X\square Y)^n \cong \operatorname{colim}\Big(\coprod_{p+q=n} X^p \times Y^q \leftleftarrows \coprod_{r+s=n-1} X^r \times Y^s \Big)$$

where the top (resp. bottom) map is induced by $\mathrm{id}\times d^0$ (resp. $d^{r+1}\times \mathrm{id}$) on each (r,s) term of the indicated coproduct; note that $(X\square Y)^0 \cong X^0\times Y^0$. The coface maps $d^i\colon (X\square Y)^n \to (X\square Y)^{n+1}$ are induced by

$$\begin{cases} X^p\times Y^q \xrightarrow{d^i\times \mathrm{id}} X^{p+1}\times Y^q, & \text{if } i\leq p,\\ X^p\times Y^q \xrightarrow{\mathrm{id}\times d^{i-p}} X^{p}\times Y^{q+1}, & \text{if } i> p,\end{cases}$$

and the codegeneracy maps $s^j\colon (X\square Y)^n \to (X\square Y)^{n-1}$ are induced by

$$\begin{cases} X^p\times Y^q \xrightarrow{s^j\times \mathrm{id}} X^{p-1}\times Y^q, & \text{if } j< p,\\ X^p\times Y^q \xrightarrow{\mathrm{id}\times s^{j-p}} X^{p}\times Y^{q-1}, & \text{if } j\geq p.\end{cases}$$

If $X, Y \in \mathsf{CGHaus}^{\Delta}$, then their box product $X\square Y \in \mathsf{CGHaus}^{\Delta}$ is defined similarly by replacing $(\mathsf{sSet}, \times)$ with $(\mathsf{CGHaus}, \times)$; the box product is defined similarly for cosimplicial objects in any closed symmetric monoidal category $(\mathsf{M}, \otimes)$.

Remark 3.25. For instance, $(X\square Y)^1$ and $(X\square Y)^2$ are naturally isomorphic to the colimits of the left-hand and right-hand diagrams, respectively,

$$\begin{tikzcd} X^0\times Y^1 \\ X^0\times Y^0 \arrow[u, "\mathrm{id}\times d^0"] \arrow[r, "d^1\times \mathrm{id}"] & X^1\times Y^0 \end{tikzcd} \qquad \begin{tikzcd} X^0\times Y^2 \\ X^0\times Y^1 \arrow[u, "\mathrm{id}\times d^0"] \arrow[r, "d^1\times \mathrm{id}"] & X^1\times Y^1 \\ & X^1\times Y^0 \arrow[u, "\mathrm{id}\times d^0"] \arrow[r, "d^2\times \mathrm{id}"] & X^2\times Y^0 \end{tikzcd}$$

and so forth. These zigzag diagrams appear in Batanin [4]; we like them because they provide a useful way of working effectively with box products. Note that the box product construction is glued together by identifying the first coface maps of the second object Y with the last coface maps of the first object X, roughly speaking; this is what one would expect from a concatenation process (Remark 3.26).

Remark 3.26. From a conceptual point of view, the box product of cosimplicial objects can be understood as a left Kan extension that is built to model concatenation processes. In more detail: if $X, Y \in (\mathsf{sSet})^{\Delta}$, their box product $X \square Y \in (\mathsf{sSet})^{\Delta}$ is the left Kan extension of objectwise product

$$\begin{tikzcd}
\Delta \times \Delta \arrow[r, "X \times Y"] \arrow[d, "\amalg"] & \mathsf{sSet} \times \mathsf{sSet} \arrow[r, "\times"] & \mathsf{sSet} \\
\Delta \arrow[rr, "X \square Y", "\text{left Kan extension}"'] & & \mathsf{sSet}
\end{tikzcd}$$

along ordinal sum (or concatenation). This is proved in McClure-Smith [55, 2.3]; for an explicit description of the resulting adjunction see [20, 4.16] and [55, 2.4].

Remark 3.27. Let's illustrate how to use these zigzag diagrams to work effectively with the box product; this will help the reader to develop a feel for the construction. For instance, consider the coface and codegeneracy maps for the truncated box product $X \square Y$ diagram of the form

$$\begin{tikzcd}
(X \square Y)^0 \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] & (X \square Y)^1 \arrow[r, shift left=2, "d^0"] \arrow[r] \arrow[r, shift right=2, "d^2"'] \arrow[d, "s^0"] & (X \square Y)^2 \arrow[d, shift right, "s^0"'] \arrow[d, shift left, "s^1"] \\
& (X \square Y)^0 & (X \square Y)^1
\end{tikzcd} \tag{38}$$

Let's unwind from Definition 3.24 exactly how the indicated coface d^i and codegeneracy s^j maps are defined. We will then use the resulting descriptions to verify several of the cosimplicial identities involving these particular maps.

The map $d^0\colon (X \square Y)^0 \to (X \square Y)^1$ is induced by the map (note that $i = 0$)

$$X^0 \times Y^0 \xrightarrow{d^0 \times \mathrm{id}} X^1 \times Y^0$$

and hence, in terms of zigzag diagrams, is induced by the diagram

$$\begin{tikzcd}
& X^0 \times Y^1 & \\
X^0 \times Y^0 \arrow[rr, bend right, "d^0 \times \mathrm{id}"'] & X^0 \times Y^0 \arrow[u, "\mathrm{id} \times d^0"] \arrow[r, "d^1 \times \mathrm{id}"] & X^1 \times Y^0
\end{tikzcd}$$

The map $d^1\colon (X \square Y)^0 \to (X \square Y)^1$ is induced by the map (note that $i = 1$)

$$X^0 \times Y^0 \xrightarrow{\mathrm{id} \times d^1} X^0 \times Y^1$$

and hence, in terms of zigzag diagrams, is induced by the diagram

$$\begin{tikzcd}
& X^0 \times Y^1 & \\
X^0 \times Y^0 \arrow[ur, bend left, "\mathrm{id} \times d^1"] & X^0 \times Y^0 \arrow[u, "\mathrm{id} \times d^0"] \arrow[r, "d^1 \times \mathrm{id}"] & X^1 \times Y^0
\end{tikzcd}$$

The map $d^0\colon (X\square Y)^1 \rightarrow (X\square Y)^2$ is induced by the maps (note that $i=0$)

$$X^0\times Y^1 \xrightarrow{d^0\times\mathrm{id}} X^1\times Y^1$$
$$X^1\times Y^0 \xrightarrow{d^0\times\mathrm{id}} X^2\times Y^0$$

and hence, in terms of zigzag diagrams, is induced by the diagram

$$\begin{tikzcd}
 & & X^0\times Y^2 & & \\
X^0\times Y^1 \arrow[rrr, bend right=10, "d^0\times\mathrm{id}"] & & X^0\times Y^1 \arrow[u, "\mathrm{id}\times d^0"] \arrow[r, "d^1\times\mathrm{id}"] & X^1\times Y^1 & \\
X^0\times Y^0 \arrow[u, "\mathrm{id}\times d^0"] \arrow[r, "d^1\times\mathrm{id}"'] \arrow[rrr, bend left=10, "d^0\times\mathrm{id}"] & X^1\times Y^0 \arrow[rrr, bend right=10, "d^0\times\mathrm{id}"'] & & X^1\times Y^0 \arrow[u, "\mathrm{id}\times d^0"] \arrow[r, "d^2\times\mathrm{id}"] & X^2\times Y^0
\end{tikzcd}$$

This diagram commutes (note that $d^2d^0=d^0d^1$).

The map $d^1\colon (X\square Y)^1 \rightarrow (X\square Y)^2$ is induced by the maps (note that $i=1$)

$$X^0\times Y^1 \xrightarrow{\mathrm{id}\times d^1} X^0\times Y^2$$
$$X^1\times Y^0 \xrightarrow{d^1\times\mathrm{id}} X^2\times Y^0$$

and hence, in terms of zigzag diagrams, is induced by the diagram

$$\begin{tikzcd}
 & & X^0\times Y^2 & & \\
X^0\times Y^1 \arrow[rru, bend left=15, "\mathrm{id}\times d^1"] & & X^0\times Y^1 \arrow[u, "\mathrm{id}\times d^0"] \arrow[r, "d^1\times\mathrm{id}"] & X^1\times Y^1 & \\
X^0\times Y^0 \arrow[u, "\mathrm{id}\times d^0"] \arrow[r, "d^1\times\mathrm{id}"'] \arrow[rru, bend left=15, "\mathrm{id}\times d^0"] \arrow[rrr, bend left=10, "d^1\times\mathrm{id}"] & X^1\times Y^0 \arrow[rrr, bend right=10, "d^1\times\mathrm{id}"'] & & X^1\times Y^0 \arrow[u, "\mathrm{id}\times d^0"] \arrow[r, "d^2\times\mathrm{id}"] & X^2\times Y^0
\end{tikzcd}$$

This diagram commutes (note that $d^1d^0=d^0d^0$ and $d^2d^1=d^1d^1$).

The map $d^2\colon (X\square Y)^1 \rightarrow (X\square Y)^2$ is induced by the maps (note that $i=2$)

$$X^0\times Y^1 \xrightarrow{\mathrm{id}\times d^2} X^0\times Y^2$$
$$X^1\times Y^0 \xrightarrow{\mathrm{id}\times d^1} X^1\times Y^1$$

and hence, in terms of zigzag diagrams, is induced by the diagram

$$\begin{tikzcd}
 & & X^0\times Y^2 & & \\
X^0\times Y^1 \arrow[urr, bend left, "\mathrm{id}\times d^2"] & & X^0\times Y^1 \arrow[u, "\mathrm{id}\times d^0"] \arrow[r, "d^1\times\mathrm{id}"] & X^1\times Y^1 & \\
X^0\times Y^0 \arrow[u, "\mathrm{id}\times d^0"] \arrow[urr, bend left, "\mathrm{id}\times d^1"] \arrow[r, "d^1\times\mathrm{id}"] & X^1\times Y^0 \arrow[urr, bend right, "\mathrm{id}\times d^1"] & & X^1\times Y^0 \arrow[u, "\mathrm{id}\times d^0"] \arrow[r, "d^2\times\mathrm{id}"] & X^2\times Y^0
\end{tikzcd}$$

This diagram commutes (note that $d^2d^0 = d^0d^1$).

The map $s^0\colon\ (X\Box Y)^1\to(X\Box Y)^0$ is induced by the maps (note that $j = 0$)

$$X^0\times Y^1 \xrightarrow{\mathrm{id}\times s^0} X^0\times Y^0$$

$$X^1\times Y^0 \xrightarrow{s^0\times\mathrm{id}} X^0\times Y^0$$

and hence, in terms of zigzag diagrams, is induced by the diagram

$$\begin{tikzcd}
X^0\times Y^1 \arrow[rr, bend left, "\mathrm{id}\times s^0"'] & & X^0\times Y^0 \\
X^0\times Y^0 \arrow[u, "\mathrm{id}\times d^0"] \arrow[r, "d^1\times\mathrm{id}"] & X^1\times Y^0 \arrow[ur, bend right, "s^0\times\mathrm{id}"'] &
\end{tikzcd}$$

This diagram commutes (note that $s^0d^0 = \mathrm{id}$ and $s^0d^1 = \mathrm{id}$).

The map $s^0\colon\ (X\Box Y)^2\to(X\Box Y)^1$ is induced by the maps (note that $j = 0$)

$$X^0\times Y^2 \xrightarrow{\mathrm{id}\times s^0} X^0\times Y^1$$

$$X^1\times Y^1 \xrightarrow{s^0\times\mathrm{id}} X^0\times Y^1$$

$$X^2\times Y^0 \xrightarrow{s^0\times\mathrm{id}} X^1\times Y^0$$

and hence, in terms of zigzag diagrams, is induced by the diagram

$$\begin{tikzcd}
X^0\times Y^2 \arrow[rrr, bend left, "\mathrm{id}\times s^0"'] & & & X^0\times Y^1 & \\
X^0\times Y^1 \arrow[u, "\mathrm{id}\times d^0"] \arrow[r, "d^1\times\mathrm{id}"] & X^1\times Y^1 \arrow[urr, bend left, "s^0\times\mathrm{id}"'] & & X^0\times Y^0 \arrow[u, "\mathrm{id}\times d^0"] \arrow[r, "d^1\times\mathrm{id}"] & X^1\times Y^0 \\
 & X^1\times Y^0 \arrow[u, "\mathrm{id}\times d^0"] \arrow[urr, bend left, "s^0\times\mathrm{id}"] \arrow[r, "d^2\times\mathrm{id}"'] & X^2\times Y^0 \arrow[urr, bend right, "s^0\times\mathrm{id}"'] & &
\end{tikzcd}$$

This diagram commutes (note that $s^0d^0 = \mathrm{id}$, $s^0d^1 = \mathrm{id}$, and $s^0d^2 = d^1s^0$).

The map $s^1\colon\ (X\Box Y)^2\to(X\Box Y)^1$ is induced by the maps (note that $j = 1$)

$$X^0\times Y^2 \xrightarrow{\mathrm{id}\times s^1} X^0\times Y^1$$

$$X^1\times Y^1 \xrightarrow{\mathrm{id}\times s^0} X^1\times Y^0$$

$$X^2\times Y^0 \xrightarrow{s^1\times\mathrm{id}} X^1\times Y^0$$

and hence, in terms of zigzag diagrams, is induced by the diagram

$$
\begin{tikzcd}
X^0 \times Y^2 \arrow[rrr, bend left=5, "\mathrm{id}\times s^1"'] & & & X^0 \times Y^1 & \\
X^0 \times Y^1 \arrow[u, "\mathrm{id}\times d^0"] \arrow[r, "d^1\times\mathrm{id}"'] \arrow[rrr, bend left=10, "\mathrm{id}\times s^0"] & X^1 \times Y^1 \arrow[rrrr, bend right=10, "\mathrm{id}\times s^0"'] & & X^0 \times Y^0 \arrow[u, "\mathrm{id}\times d^0"] \arrow[r, "d^1\times\mathrm{id}"] & X^1 \times Y^0 \\
& X^1 \times Y^0 \arrow[u, "\mathrm{id}\times d^0"] \arrow[r, "d^2\times\mathrm{id}"] & X^2 \times Y^0 \arrow[rru, bend right=10, "s^1\times\mathrm{id}"'] & &
\end{tikzcd}
$$

This diagram commutes (note that $s^0d^0 = \mathrm{id}$, $s^1d^2 = \mathrm{id}$, and $s^1d^0 = d^0s^0$).

Consider diagram (38). Let's use the descriptions above to verify several of the cosimplicial identities.

Let's verify that $s^0d^0 = \mathrm{id}$ in (38). This is because the diagram

$$
\begin{tikzcd}
X^0 \times Y^0 \arrow[r, "d^0\times\mathrm{id}"] \arrow[d, equal] & X^1 \times Y^0 \arrow[r, "s^0\times\mathrm{id}"] & X^0 \times Y^0 \arrow[d, equal] \\
X^0 \times Y^0 \arrow[rr, "\mathrm{id}\times\mathrm{id}"] & & X^0 \times Y^0
\end{tikzcd}
$$

commutes (note that $s^0d^0 = \mathrm{id}$).

Let's verify that $s^0d^1 = \mathrm{id}$ in (38). This is because the diagram

$$
\begin{tikzcd}
X^0 \times Y^0 \arrow[r, "\mathrm{id}\times d^1"] \arrow[d, equal] & X^0 \times Y^1 \arrow[r, "\mathrm{id}\times s^0"] & X^0 \times Y^0 \arrow[d, equal] \\
X^0 \times Y^0 \arrow[rr, "\mathrm{id}\times\mathrm{id}"] & & X^0 \times Y^0
\end{tikzcd}
$$

commutes (note that $s^0d^1 = \mathrm{id}$).

Let's verify that $d^1d^0 = d^0d^0$ in (38). This is because the diagram

$$
\begin{tikzcd}
X^0 \times Y^0 \arrow[r, "d^0\times\mathrm{id}"] \arrow[d, equal] & X^1 \times Y^0 \arrow[r, "d^1\times\mathrm{id}"] & X^2 \times Y^0 \arrow[d, equal] \\
X^0 \times Y^0 \arrow[r, "d^0\times\mathrm{id}"] & X^1 \times Y^0 \arrow[r, "d^0\times\mathrm{id}"] & X^2 \times Y^0
\end{tikzcd}
$$

commutes (note that $d^1d^0 = d^0d^0$).

Let's verify that $d^2d^0 = d^0d^1$ in (38). This is because the diagram

$$
\begin{tikzcd}
X^0 \times Y^0 \arrow[r, "d^0\times\mathrm{id}"] \arrow[d, equal] & X^1 \times Y^0 \arrow[r, "\mathrm{id}\times d^1"] & X^1 \times Y^1 \arrow[d, equal] \\
X^0 \times Y^0 \arrow[r, "\mathrm{id}\times d^1"] & X^0 \times Y^1 \arrow[r, "d^0\times\mathrm{id}"] & X^1 \times Y^1
\end{tikzcd}
$$

commutes.

Let's verify that $d^2d^1 = d^1d^1$ in (38). This is because the diagram

$$\begin{tikzcd}
X^0 \times Y^0 \arrow[r, "\mathrm{id}\times d^1"] \arrow[d, equal] & X^0 \times Y^1 \arrow[r, "\mathrm{id}\times d^2"] & X^0 \times Y^2 \arrow[d, equal] \\
X^0 \times Y^0 \arrow[r, "\mathrm{id}\times d^1"] & X^0 \times Y^1 \arrow[r, "\mathrm{id}\times d^1"] & X^0 \times Y^2
\end{tikzcd}$$

commutes (note that $d^2d^1 = d^1d^1$).

Let's verify that $s^0d^2 = d^1s^0$ in (38). This is because the two diagrams

$$\begin{tikzcd}
X^0 \times Y^1 \arrow[r, "\mathrm{id}\times d^2"] \arrow[d, equal] & X^0 \times Y^2 \arrow[r, "\mathrm{id}\times s^0"] & X^0 \times Y^1 \arrow[d, equal] \\
X^0 \times Y^1 \arrow[r, "\mathrm{id}\times s^0"] & X^0 \times Y^0 \arrow[r, "\mathrm{id}\times d^1"] & X^0 \times Y^1
\end{tikzcd}
\qquad
\begin{tikzcd}
X^1 \times Y^0 \arrow[r, "\mathrm{id}\times d^1"] \arrow[d, equal] & X^1 \times Y^1 \arrow[r, "s^0\times \mathrm{id}"] & X^0 \times Y^1 \arrow[d, equal] \\
X^1 \times Y^0 \arrow[r, "s^0\times \mathrm{id}"] & X^0 \times Y^0 \arrow[r, "\mathrm{id}\times d^1"] & X^0 \times Y^1
\end{tikzcd}$$

commute (note that $s^0d^2 = d^1s^0$).

Let's verify that $s^1d^0 = d^0s^0$ in (38). This is because the two diagrams

$$\begin{tikzcd}
X^0 \times Y^1 \arrow[r, "d^0\times \mathrm{id}"] \arrow[d, equal] & X^1 \times Y^1 \arrow[r, "\mathrm{id}\times s^0"] & X^1 \times Y^0 \arrow[d, equal] \\
X^0 \times Y^1 \arrow[r, "\mathrm{id}\times s^0"] & X^0 \times Y^0 \arrow[r, "d^0\times \mathrm{id}"] & X^1 \times Y^0
\end{tikzcd}
\qquad
\begin{tikzcd}
X^1 \times Y^0 \arrow[r, "d^0\times \mathrm{id}"] \arrow[d, equal] & X^2 \times Y^0 \arrow[r, "s^1\times \mathrm{id}"] & X^1 \times Y^0 \arrow[d, equal] \\
X^1 \times Y^0 \arrow[r, "s^0\times \mathrm{id}"] & X^0 \times Y^0 \arrow[r, "d^0\times \mathrm{id}"] & X^1 \times Y^0
\end{tikzcd}$$

commute (note that $s^1d^0 = d^0s^0$).

Let's verify that $s^0s^0 = s^0s^1$ in (38). This is because the three diagrams

$$\begin{tikzcd}
X^0 \times Y^2 \arrow[r, "\mathrm{id}\times s^0"] \arrow[d, equal] & X^0 \times Y^1 \arrow[r, "\mathrm{id}\times s^0"] & X^0 \times Y^0 \arrow[d, equal] \\
X^0 \times Y^2 \arrow[r, "\mathrm{id}\times s^1"] & X^0 \times Y^1 \arrow[r, "\mathrm{id}\times s^0"] & X^0 \times Y^0
\end{tikzcd}
\qquad
\begin{tikzcd}
X^1 \times Y^1 \arrow[r, "s^0\times \mathrm{id}"] \arrow[d, equal] & X^0 \times Y^1 \arrow[r, "\mathrm{id}\times s^0"] & X^0 \times Y^0 \arrow[d, equal] \\
X^1 \times Y^1 \arrow[r, "\mathrm{id}\times s^0"] & X^1 \times Y^0 \arrow[r, "s^0\times \mathrm{id}"] & X^0 \times Y^0
\end{tikzcd}$$

$$\begin{tikzcd}
X^2 \times Y^0 \arrow[r, "s^0\times \mathrm{id}"] \arrow[d, equal] & X^1 \times Y^0 \arrow[r, "s^0\times \mathrm{id}"] & X^0 \times Y^0 \arrow[d, equal] \\
X^2 \times Y^0 \arrow[r, "s^1\times \mathrm{id}"] & X^1 \times Y^0 \arrow[r, "s^0\times \mathrm{id}"] & X^0 \times Y^0
\end{tikzcd}$$

commute (note that $s^0s^0 = s^0s^1$).

The following pairings are exactly what you would expect them to be. The basic idea is that in cosimplicial degree 0, the composition map μ (below) should send $\alpha\colon Y \to FY'$ and $\alpha'\colon Y' \to FY''$ to the composite

$$Y \xrightarrow{\alpha} FY' \xrightarrow{\mathrm{id}\alpha'} FFY'' \xrightarrow{m\mathrm{id}} FY''$$

where $m\colon FF \to F$ is the multiplication on F.

Proposition 3.28. *Let Y, Y', Y'' be $\tilde{\mathsf{K}}$-coalgebra spectra (resp. Z, Z', Z'' be $\tilde{\mathsf{K}}_r$-coalgebras). There is a natural map of the form*

$$\mathbf{Hom}_{\mathsf{Mod}_S}\big(Y, F\tilde{\mathsf{K}}^\bullet Y'\big) \square \mathbf{Hom}_{\mathsf{Mod}_S}\big(Y', F\tilde{\mathsf{K}}^\bullet Y''\big) \xrightarrow{\mu} \mathbf{Hom}_{\mathsf{Mod}_S}\big(Y, F\tilde{\mathsf{K}}^\bullet Y''\big)$$

$$\textit{resp.}\quad \mathbf{Hom}_{\mathsf{S}_*}\big(Z, \Phi\tilde{\mathsf{K}}_r^\bullet Z'\big) \square \mathbf{Hom}_{\mathsf{S}_*}\big(Z', \Phi\tilde{\mathsf{K}}_r^\bullet Z''\big) \xrightarrow{\mu} \mathbf{Hom}_{\mathsf{S}_*}\big(Z, \Phi\tilde{\mathsf{K}}_r^\bullet Z''\big)$$

in $(\mathsf{sSet})^{\Delta}$*. We sometimes refer to* μ *as the* composition *map.*

Proof. This is proved exactly as in [1, 1.6]; μ is the map induced by the collection of composites

$$\begin{aligned}
&\mathbf{Hom}_{\mathsf{Mod}_S}\big(Y, F\tilde{\mathsf{K}}^p Y'\big) \times \mathbf{Hom}_{\mathsf{Mod}_S}\big(Y', F\tilde{\mathsf{K}}^q Y''\big) \xrightarrow{\mathrm{id}\times F\tilde{\mathsf{K}}^p} \\
&\mathbf{Hom}_{\mathsf{Mod}_S}\big(Y, F\tilde{\mathsf{K}}^p Y'\big) \times \mathbf{Hom}_{\mathsf{Mod}_S}\big(F\tilde{\mathsf{K}}^p Y', F\tilde{\mathsf{K}}^p F\tilde{\mathsf{K}}^q Y''\big) \xrightarrow{\mathrm{comp}} \\
&\mathbf{Hom}_{\mathsf{Mod}_S}\big(Y, F\tilde{\mathsf{K}}^p F\tilde{\mathsf{K}}^q Y''\big) \xrightarrow{\simeq} \mathbf{Hom}_{\mathsf{Mod}_S}\big(Y, F\tilde{\mathsf{K}}^{p+q} Y''\big)
\end{aligned}$$

where $p, q \geq 0$; here, the indicated weak equivalence is the map induced by multiplication $FF \rightarrow F$ of the simplicial fibrant replacement monad. The other case is similar. $\square$

Proposition 3.29. *Let* $A, B \in (\mathsf{sSet})^{\Delta}$*. There is a natural isomorphism of the form* $|A \square B| \cong |A| \square |B|$ *in* $(\mathsf{CGHaus})^{\Delta}$*.*

Proof. This follows from the fact that realization commutes with finite products and all small colimits [32, 36]. $\square$

Proposition 3.30. *Let* Y, Y', Y'' *be* $\tilde{\mathsf{K}}$*-coalgebra spectra (resp.* Z, Z', Z'' *be* $\tilde{\mathsf{K}}_r$*-coalgebras). There is a natural map of the form*

$$\begin{aligned}
&\mathrm{Map}_{\mathsf{Mod}_S}\big(Y, F\tilde{\mathsf{K}}^{\bullet} Y'\big) \square \mathrm{Map}_{\mathsf{Mod}_S}\big(Y', F\tilde{\mathsf{K}}^{\bullet} Y''\big) \xrightarrow{\mu} \mathrm{Map}_{\mathsf{Mod}_S}\big(Y, F\tilde{\mathsf{K}}^{\bullet} Y''\big) \\
&\textit{resp.}\quad \mathrm{Map}_{\mathsf{S}_*}\big(Z, \Phi\tilde{\mathsf{K}}^{\bullet}_r Z'\big) \square \mathrm{Map}_{\mathsf{S}_*}\big(Z', \Phi\tilde{\mathsf{K}}^{\bullet}_r Z''\big) \xrightarrow{\mu} \mathrm{Map}_{\mathsf{S}_*}\big(Z, \Phi\tilde{\mathsf{K}}^{\bullet}_r Z''\big)
\end{aligned}$$

in $(\mathsf{CGHaus})^{\Delta}$*. We sometimes refer to* μ *as the* composition *map.*

Proof. This follows from Proposition 3.28 by applying realization, together with Proposition 3.29. $\square$

Definition 3.31. The non-Σ operad A in CGHaus is the coendomorphism operad of $\Delta^{\bullet}$ with respect to the box product $\square$ and is defined objectwise by the end construction ([1, 1.12])

$$\mathsf{A}(n) := \mathrm{Map}_{\Delta_{\mathsf{res}}}\big(\Delta^{\bullet}, (\Delta^{\bullet})^{\square n}\big) := \mathrm{Map}\big(\Delta^{\bullet}, (\Delta^{\bullet})^{\square n}\big)^{\Delta_{\mathsf{res}}}$$

In other words, $\mathsf{A}(n)$ is the space of restricted cosimplicial maps from $\Delta^{\bullet}$ to $(\Delta^{\bullet})^{\square n}$; in particular, note that $\mathsf{A}(0) = *$.

Consider the natural collection of maps ([1, 1.13])

$$\begin{aligned}
\mathsf{A}(n) \times \mathrm{Map}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}}}(Y_0, Y_1) \times \cdots \times \mathrm{Map}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}}}(Y_{n-1}, Y_n)& \\
\longrightarrow \mathrm{Map}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}}}(Y_0, Y_n),& \qquad n \geq 0,
\end{aligned} \tag{39}$$

$$\begin{aligned}
\text{resp.}\quad \mathsf{A}(n) \times \mathrm{Map}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}_r}}(Z_0, Z_1) \times \cdots \times \mathrm{Map}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}_r}}(Z_{n-1}, Z_n)& \\
\longrightarrow \mathrm{Map}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}_r}}(Z_0, Z_n),& \qquad n \geq 0,
\end{aligned} \tag{40}$$

induced by (iterations of) the composition map μ; in particular, in the case $n = 0$, note that (39) (resp. (40)) denotes the unit map

$$\begin{aligned}
* = \mathsf{A}(0) &\longrightarrow \mathrm{Map}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}}}(Y_0, Y_0) \\
\text{resp.}\quad * = \mathsf{A}(0) &\longrightarrow \mathrm{Map}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}_r}}(Z_0, Z_0)
\end{aligned}$$

Remark 3.32. The notion of an A_∞ composition, and the corresponding notion of an A_∞ category, is studied, for instance, in Batanin [5].

Proposition 3.33. *The collection of maps* (39) *(resp.* (40)*) determine a topological A_∞ category with objects the $\tilde{\mathsf{K}}$-coalgebra spectra (resp. $\tilde{\mathsf{K}}_r$-coalgebras) and morphism spaces the mapping spaces* $\mathrm{Map}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}}}(Y,Y')$ *(resp.* $\mathrm{Map}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}_r}}(Z,Z')$*).*

Proof. This is proved exactly as in [1, 1.14]. □

Definition 3.34. The *homotopy category* of $\tilde{\mathsf{K}}$-coalgebras (resp. $\tilde{\mathsf{K}}_r$-coalgebras), denoted $\mathsf{Ho}(\mathsf{coAlg}_{\tilde{\mathsf{K}}})$ (resp. $\mathsf{Ho}(\mathsf{coAlg}_{\tilde{\mathsf{K}}_r})$), is the category with objects the $\tilde{\mathsf{K}}$-coalgebra spectra (resp. $\tilde{\mathsf{K}}_r$-coalgebras) and morphism sets $[Y,Y']_{\tilde{\mathsf{K}}}$ from Y to Y' (resp. $[Z,Z']_{\tilde{\mathsf{K}}_r}$ from Z to Z') the path components

$$
\begin{aligned}
&[Y,Y']_{\tilde{\mathsf{K}}} := \pi_0 \,\mathrm{Map}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}}}(Y,Y') \\
\text{resp.}\quad &[Z,Z']_{\tilde{\mathsf{K}}_r} := \pi_0 \,\mathrm{Map}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}_r}}(Z,Z')
\end{aligned}
$$

of the indicated mapping spaces; compare with [1, 1.15].

Definition 3.35. A derived $\tilde{\mathsf{K}}$-coalgebra map f of the form $Y \to Y'$ is a *weak equivalence* if the underlying map $f_0\colon Y \to FY'$ is a weak equivalence. Similarly, a derived $\tilde{\mathsf{K}}_r$-coalgebra map g of the form $Z \to Z'$ is a *weak equivalence* if the underlying map $g_0\colon Z \to \Phi Z'$ is a weak equivalence.

Proposition 3.36. *Let Y, Y' be $\tilde{\mathsf{K}}$-coalgebra spectra (resp. Z, Z' be $\tilde{\mathsf{K}}_r$-coalgebras). A derived $\tilde{\mathsf{K}}$-coalgebra map f of the form $Y \to Y'$ (resp. $\tilde{\mathsf{K}}_r$-coalgebra map g of the form $Z \to Z'$) is a weak equivalence if and only if it represents an isomorphism in the homotopy category of $\tilde{\mathsf{K}}$-coalgebras (resp. $\tilde{\mathsf{K}}_r$-coalgebras).*

Proof. This is proved exactly as in [1, 1.16]. □

4. The derived unit and counit maps

The purpose of this section is to describe the natural weak equivalences (1) and (2) appearing in the statement of our main results, together with the derived unit and counit maps associated with them; for a reminder on the **Hom** functors, see Section 6.

Proposition 4.1. *Let X be a pointed space and Y a $\tilde{\mathsf{K}}$-coalgebra spectrum (resp. Z a $\tilde{\mathsf{K}}_r$-coalgebra). The natural isomorphisms associated to the* $(\Sigma^\infty, \Omega^\infty)$ *(resp.* (Σ^r, Ω^r)*) adjunction induce well-defined isomorphisms*

$$
\begin{aligned}
&\mathbf{Hom}_{\mathsf{Mod}_S}\big(\Sigma^\infty X, F\tilde{\mathsf{K}}^\bullet Y\big) \xrightarrow{\cong} \mathbf{Hom}_{\mathsf{S}_*}\big(X, \tilde{\Omega}^\infty \tilde{\mathsf{K}}^\bullet Y\big) \\
\textit{resp.}\quad &\mathbf{Hom}_{\mathsf{S}_*}\big(\Sigma^r X, \Phi\tilde{\mathsf{K}}_r^\bullet Z\big) \xrightarrow{\cong} \mathbf{Hom}_{\mathsf{S}_*}\big(X, \tilde{\Omega}^r \tilde{\mathsf{K}}_r^\bullet Z\big)
\end{aligned}
$$

of cosimplicial objects in sSet*, natural in X, Y (resp. X, Z).*

Proof. The is because the collection ($n \geq 0$) of composite maps

$$
\begin{aligned}
\hom(\Sigma^\infty(X)\dot{\otimes}\Delta[n], F\tilde{\mathsf{K}}^\bullet Y) &\cong \hom(\Sigma^\infty(X\dot{\otimes}\Delta[n]), F\tilde{\mathsf{K}}^\bullet Y) \\
&\cong \hom(X\dot{\otimes}\Delta[n], \tilde{\Omega}^\infty\tilde{\mathsf{K}}^\bullet Y)
\end{aligned}
$$

is a well-defined map of cosimplicial objects in Set, natural in X, Y. The other case is similar. □

The following proposition establishes the natural weak equivalences (1) and (2).

Proposition 4.2. *Let X be a pointed space and Y a $\tilde{\mathsf{K}}$-coalgebra spectrum (resp. Z a $\tilde{\mathsf{K}}_r$-coalgebra). There are natural zigzags of weak equivalences*

$$\mathrm{Map}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}}}(\Sigma^\infty X, Y) \simeq \mathrm{Map}_{\mathsf{S}_*}(X, \mathrm{holim}_\Delta \mathfrak{C}(Y)) \tag{41}$$

$$\textit{resp.}\quad \mathrm{Map}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}_r}}(\Sigma^r X, Z) \simeq \mathrm{Map}_{\mathsf{S}_*}(X, \mathrm{holim}_\Delta \mathfrak{C}_r(Z)) \tag{42}$$

in CGHaus*; applying π_0 gives the natural isomorphism $[\Sigma^\infty X, Y]_{\tilde{\mathsf{K}}} \cong [X, \mathrm{holim}_\Delta \mathfrak{C}(Y)]$ (resp $[\Sigma^r X, Z]_{\tilde{\mathsf{K}}_r} \cong [X, \mathrm{holim}_\Delta \mathfrak{C}_r(Z)]$).*

Proof. There are natural zigzags of weak equivalences of the form

$$\begin{aligned}
\mathbf{Hom}_{\mathsf{S}_*}(X, \mathrm{holim}_\Delta \mathfrak{C}(Y)) &\simeq \mathbf{Hom}_{\mathsf{S}_*}\big(X, \mathrm{Tot}^{\mathsf{res}} \mathfrak{C}(Y)\big) \\
&\cong \mathrm{Tot}^{\mathsf{res}} \mathbf{Hom}_{\mathsf{S}_*}\big(X, \tilde{\Omega}^\infty \tilde{\mathsf{K}}^\bullet Y\big) \\
&\cong \mathrm{Tot}^{\mathsf{res}} \mathbf{Hom}_{\mathsf{Mod}_S}\big(\Sigma^\infty X, F\tilde{\mathsf{K}}^\bullet Y\big) \\
&\cong \mathbf{Hom}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}}}(\Sigma^\infty X, Y)
\end{aligned}$$

in sSet; applying realization finishes the proof. The other case is similar. $\square$

Definition 4.3. The *derived unit map* associated to the natural zigzag of weak equivalences in (41) (resp. (42)) is the map of the form $X \to \mathrm{holim}_\Delta \mathfrak{C}(\Sigma^\infty X)$ (resp. $X \to \mathrm{holim}_\Delta \mathfrak{C}_r(\Sigma^r X)$) in pointed spaces with representing map

$$X \to \mathrm{Tot}^{\mathsf{res}} \mathfrak{C}(\Sigma^\infty X) \tag{43}$$

$$\text{resp.}\quad X \to \mathrm{Tot}^{\mathsf{res}} \mathfrak{C}_r(\Sigma^r X) \tag{44}$$

corresponding to the identity map $\mathrm{id}\colon \Sigma^\infty X {\to} \Sigma^\infty X$ (resp. $\mathrm{id}\colon \Sigma^r X {\to} \Sigma^r X$) in $\mathsf{coAlg}_{\tilde{\mathsf{K}}}$ (resp. $\mathsf{coAlg}_{\tilde{\mathsf{K}}_r}$).

Remark 4.4. If X is a pointed space, then there is a zigzag of weak equivalences

$$\begin{aligned}
X^\wedge_{\tilde{\Omega}^\infty\Sigma^\infty} &\simeq \mathrm{holim}_\Delta \mathfrak{C}(\Sigma^\infty X) \simeq \mathrm{Tot}^{\mathsf{res}} \mathfrak{C}(\Sigma^\infty X) \\
\text{resp.}\quad X^\wedge_{\tilde{\Omega}^r\Sigma^r} &\simeq \mathrm{holim}_\Delta \mathfrak{C}_r(\Sigma^r X) \simeq \mathrm{Tot}^{\mathsf{res}} \mathfrak{C}_r(\Sigma^r X)
\end{aligned}$$

in S_*, natural with respect to all such X; this is because $\mathfrak{C}(\Sigma^\infty X)$ (resp. $\mathfrak{C}_r(\Sigma^r X)$) is objectwise fibrant. In particular, the derived unit map (43) is tautologically the $\tilde{\Omega}^\infty\Sigma^\infty$-completion map $X {\to} X^\wedge_{\tilde{\Omega}^\infty\Sigma^\infty}$ studied in Carlsson [16] and subsequently in Arone-Kankaanrinta [2]. Similarly, the derived unit map (44) is tautologically the $\tilde{\Omega}^r\Sigma^r$-completion map $X {\to} X^\wedge_{\tilde{\Omega}^r\Sigma^r}$ studied by Bousfield [11] and Hopkins (see [11]).

Definition 4.5. The *derived counit map* associated to the natural zigzag of weak equivalences in (41) (resp. (42)) is the derived $\tilde{\mathsf{K}}$-coalgebra (resp. $\tilde{\mathsf{K}}_r$-coalgebra) map of the form $\Sigma^\infty \mathrm{holim}_\Delta \mathfrak{C}(Y) \to Y$ (resp. $\Sigma^r \mathrm{holim}_\Delta \mathfrak{C}_r(Z) \to Z$) with underlying map

$$\Sigma^\infty \mathrm{Tot}^{\mathsf{res}} \mathfrak{C}(Y) \longrightarrow FY \tag{45}$$

$$\text{resp.}\quad \Sigma^r \mathrm{Tot}^{\mathsf{res}} \mathfrak{C}_r(Z) \longrightarrow \Phi Z \tag{46}$$

corresponding to the identity map

$$\mathrm{id}\colon \mathrm{Tot}^{\mathsf{res}} \mathfrak{C}(Y) {\to} \mathrm{Tot}^{\mathsf{res}} \mathfrak{C}(Y) \tag{47}$$

$$\text{resp.}\quad \mathrm{id}\colon \mathrm{Tot}^{\mathsf{res}} \mathfrak{C}_r(Z) {\to} \mathrm{Tot}^{\mathsf{res}} \mathfrak{C}_r(Z) \tag{48}$$

in S_*, via the adjunctions [9, 5.4] and $(\Sigma^\infty, \Omega^\infty)$ (resp. (Σ^r, Ω^r)). In more detail, the derived counit map is the derived $\tilde{\mathsf{K}}$-coalgebra (resp. $\tilde{\mathsf{K}}_r$-coalgebra) map defined by the composite

$$\begin{aligned} \Delta[-] &\xrightarrow{(*)} \mathbf{Hom}_{\mathsf{S}_*}\big(\mathrm{Tot}^{\mathsf{res}}\,\mathfrak{C}(Y), \mathfrak{C}(Y)\big) \\ &\cong\ \mathbf{Hom}_{\mathsf{Mod}_S}\big(\Sigma^\infty\,\mathrm{Tot}^{\mathsf{res}}\,\mathfrak{C}(Y), F\tilde{\mathsf{K}}^\bullet Y\big) \end{aligned} \tag{49}$$

$$\begin{aligned} \text{resp.}\quad \Delta[-] &\xrightarrow{(*)} \mathbf{Hom}_{\mathsf{S}_*}\big(\mathrm{Tot}^{\mathsf{res}}\,\mathfrak{C}_r(Z), \mathfrak{C}_r(Z)\big) \\ &\cong\ \mathbf{Hom}_{\mathsf{S}_*}\big(\Sigma^r\,\mathrm{Tot}^{\mathsf{res}}\,\mathfrak{C}_r(Z), \Phi\tilde{\mathsf{K}}_r^\bullet Z\big) \end{aligned} \tag{50}$$

in $(\mathsf{sSet})^{\Delta_{\mathsf{res}}}$, where $(*)$ corresponds to the map (47) (resp. (48)).

Remark 4.6. Let X, X' be pointed spaces. If X' is fibrant and the natural coaugmentation $X' \simeq X'^{\wedge}_{\tilde{\Omega}^\infty\Sigma^\infty}$ (resp. $X' \simeq X'^{\wedge}_{\tilde{\Omega}^r\Sigma^r}$) is a weak equivalence, then there is a natural zigzag

$$\begin{aligned} \Sigma^\infty\colon\ \mathrm{Map}_{\mathsf{S}_*}(X, X') &\xrightarrow{\simeq} \mathrm{Map}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}}}(\Sigma^\infty X, \Sigma^\infty X') \\ \text{resp.}\quad \Sigma^r\colon\ \mathrm{Map}_{\mathsf{S}_*}(X, X') &\xrightarrow{\simeq} \mathrm{Map}_{\mathsf{coAlg}_{\tilde{\mathsf{K}}_r}}(\Sigma^r X, \Sigma^r X') \end{aligned}$$

of weak equivalences; applying π_0 gives the map $[f] \mapsto [\Sigma^\infty f]$ (resp. $[f] \mapsto [\Sigma^r f]$). This follows from the natural zigzags

$$\begin{aligned} \mathrm{Map}_{\mathsf{S}_*}(X, X') &\simeq\ \mathrm{Map}_{\mathsf{S}_*}(X, X'^{\wedge}_{\tilde{\Omega}^\infty\Sigma^\infty}) \\ &\simeq\ \mathrm{Map}_{\mathsf{S}_*}\big(X, \mathrm{holim}_\Delta\,\mathfrak{C}(\Sigma^\infty X')\big) \simeq\ \mathrm{Map}_{\mathsf{coAlg}_{\mathsf{K}}}(\Sigma^\infty X, \Sigma^\infty X') \end{aligned}$$

of weak equivalences; see [1, 2.15] and [40, 5.5]. The other case is similar.

5. Homotopical analysis

The purpose of this section is to prove Theorems 1.7 and 1.8; we will then use these to prove Theorems 2.3 and 2.1 (Section 5.30). The following definitions appear in [37, Section 1, 1.12] in the context of spaces.

Definition 5.1. Let W be a finite set and M a category.

- Denote by $\mathcal{P}(W)$ the poset of all subsets of W, ordered by inclusion $\subset$ of sets. We will often regard $\mathcal{P}(W)$ as the category associated to this partial order in the usual way; the objects are the elements of $\mathcal{P}(W)$, and there is a morphism $U \to V$ if and only if $U \subset V$.
- Denote by $\mathcal{P}_0(W) \subset \mathcal{P}(W)$ the poset of all nonempty subsets of W; it is the full subcategory of $\mathcal{P}(W)$ containing all objects except the initial object $\emptyset$.
- Denote by $\mathcal{P}_1(W) \subset \mathcal{P}(W)$ the poset of all subsets of W not equal to W; it is the full subcategory of $\mathcal{P}(W)$ containing all objects except the terminal object W.
- A *W-cube* $\mathcal{X}$ in M is a $\mathcal{P}(W)$-shaped diagram $\mathcal{X}$ in M; in other words, a functor $\mathcal{X}\colon \mathcal{P}(W) \to \mathsf{M}$.

Remark 5.2. If $\mathcal{X}$ is a W-cube in M where $|W| = n$, we will sometimes refer to $\mathcal{X}$ simply as an *n-cube* in M. In particular, a 0-cube is an object in M and a 1-cube is a morphism in M.

Definition 5.3. Let W be a finite set and M a category. Let $\mathcal{X}$ be a W-cube in M and consider any subsets $U \subset V \subset W$. Denote by $\partial_U^V \mathcal{X}$ the $(V-U)$-cube defined objectwise by

$$T \mapsto (\partial_U^V \mathcal{X})_T := \mathcal{X}_{T \cup U}, \qquad T \subset V - U.$$

In other words, $\partial_U^V \mathcal{X}$ is the $(V-U)$-cube formed by all maps in $\mathcal{X}$ between $\mathcal{X}_U$ and $\mathcal{X}_V$. We say that $\partial_U^V \mathcal{X}$ is a *face* of $\mathcal{X}$ of *dimension* $|V-U|$.

Definition 5.4. Let W be a finite set. Let $\mathcal{X}$ be a W-cube in Mod_S (resp. S_*) and $k \in \mathbb{Z}$.

- $\mathcal{X}$ is a *cofibration cube* if the map $\operatorname{colim}_{\mathcal{P}_1(V)} \mathcal{X} \rightarrow \operatorname{colim}_{\mathcal{P}(V)} \mathcal{X} \cong \mathcal{X}_V$ is a cofibration for each $V \subset W$; in particular, each $\mathcal{X}_V$ is cofibrant.
- $\mathcal{X}$ is *k-cocartesian* if the map $\operatorname{hocolim}_{\mathcal{P}_1(W)} \mathcal{X} \rightarrow \operatorname{hocolim}_{\mathcal{P}(W)} \mathcal{X} \simeq \mathcal{X}_W$ is k-connected.
- $\mathcal{X}$ is *∞-cocartesian* if the map $\operatorname{hocolim}_{\mathcal{P}_1(W)} \mathcal{X} \rightarrow \operatorname{hocolim}_{\mathcal{P}(W)} \mathcal{X} \simeq \mathcal{X}_W$ is a weak equivalence.
- $\mathcal{X}$ is a *fibration cube* if the map $\mathcal{X}_V \cong \lim_{\mathcal{P}(W-V)} \partial_V^W \mathcal{X} \rightarrow \lim_{\mathcal{P}_0(W-V)} \partial_V^W \mathcal{X}$ is a fibration for each $V \subset W$; in particular, each $\mathcal{X}_V$ is fibrant.
- $\mathcal{X}$ is *k-cartesian* if the map $\mathcal{X}_\emptyset \simeq \operatorname{holim}_{\mathcal{P}(W)} \mathcal{X} \rightarrow \operatorname{holim}_{\mathcal{P}_0(W)} \mathcal{X}$ is k-connected.
- $\mathcal{X}$ is *∞-cartesian* if the map $\mathcal{X}_\emptyset \simeq \operatorname{holim}_{\mathcal{P}(W)} \mathcal{X} \rightarrow \operatorname{holim}_{\mathcal{P}_0(W)} \mathcal{X}$ is a weak equivalence.

Remark 5.5. In particular, a 1-cube $\mathcal{X}$ (e.g., a W-cube with $W = \{1\}$) is k-cartesian if the map $\mathcal{X}_\emptyset \rightarrow \mathcal{X}_{\{1\}}$ is k-connected and a 0-cube $\mathcal{Y}$ is k-cartesian if the map $\mathcal{Y}_\emptyset \rightarrow *$ is k-connected. Similarly, a 1-cube $\mathcal{X}$ is k-cocartesian if the map $\mathcal{X}_\emptyset \rightarrow \mathcal{X}_{\{1\}}$ is k-connected and a 0-cube $\mathcal{Y}$ is k-cocartesian if the map $* \rightarrow \mathcal{Y}_\emptyset$ is k-connected.

The following definitions appear in [24, Section 2], [25, A.8.0.1, A.8.3.1].

Definition 5.6. Let T, W be finite sets such that $|T| \leq |W|$ and M a category. Let $\mathcal{X}$ be a W-cube in M. A *T-subcube of* $\mathcal{X}$ is a T-cube resulting from the precomposite of $\mathcal{X}$ along an injection $\xi\colon \mathcal{P}(T) \rightarrow \mathcal{P}(W)$ satisfying that if $U, V \subset T$, then $\xi(U \cap V) = \xi(U) \cap \xi(V)$ and $\xi(U \cup V) = \xi(U) \cup \xi(V)$. If $|T| = d$, we will often refer to a T-subcube of $\mathcal{X}$ simply as a *d-subcube* of $\mathcal{X}$.

Remark 5.7. In general, not all subcubes of $\mathcal{X}$ are faces of $\mathcal{X}$. For instance, consider a 2-cube $\mathcal{X}$ of the form (e.g, $\mathcal{X}$ is a W-cube with $W = \{1, 2\}$)

$$\begin{tikzcd} \mathcal{X}_\emptyset \arrow[r] \arrow[d] & \mathcal{X}_{\{1\}} \arrow[d] \\ \mathcal{X}_{\{2\}} \arrow[r] & \mathcal{X}_{\{1,2\}} \end{tikzcd} \tag{51}$$

Then the composite $\mathcal{X}_\emptyset \rightarrow \mathcal{X}_{\{1,2\}}$ is a 1-subcube of $\mathcal{X}$, but is not a 1-dimensional face of $\mathcal{X}$. There are exactly four 1-dimensional faces of $\mathcal{X}$, the maps indicated in (51), and exactly five 1-subcubes of $\mathcal{X}$.

Definition 5.8. Let $f\colon \mathbb{N} \rightarrow \mathbb{N}$ be a function and W a finite set. A W-cube $\mathcal{X}$ is *f-cartesian* (resp. *f-cocartesian*) if each d-subcube of $\mathcal{X}$ is $f(d)$-cartesian (resp. $f(d)$-cocartesian); here, $\mathbb{N}$ denotes the non-negative integers.

5.9. **Higher Freudenthal suspension and higher stabilization.** The purpose of this section is to prove Theorems 1.7 and 1.8, that play a key role in the proofs of our main results (Section 5.30). The strategy of attack developed in this section is motivated by Dundas [24, 2.6]; see also Dundas-Goodwillie-McCarthy [25, A.8.3].

The cartesian-ness estimates in higher Freudenthal suspension (Theorem 1.8) arise by homotopically analyzing what happens when we iteratively apply the Freudenthal suspension map $\mathrm{id} \rightarrow \tilde{\Omega}\Sigma$ to go from a pointed space X (i.e., a 0-cube), to a 1-cube, to a 2-cube, to a 3-cube, and so forth.

Remark 5.10 (Freudenthal suspension and stabilization). Let $k \geq 1$. Suppose $\mathcal{X}$ is a 0-cube in pointed spaces and $\mathcal{X}_\emptyset$ is k-connected. We know by Freudenthal suspension, which can be understood as a consequence of the higher Blakers-Massey theorem (see, for instance, [25, A.8.2]), that the map $\mathcal{X}_\emptyset \rightarrow \tilde{\Omega}\Sigma\mathcal{X}_\emptyset$ is $(2k+1)$-connected. More generally, it follows by repeated application of Freudenthal suspension that the map $\mathcal{X}_\emptyset \rightarrow \tilde{\Omega}^r\Sigma^r\mathcal{X}_\emptyset$ (resp. $\mathcal{X}_\emptyset \rightarrow \tilde{\Omega}^\infty\Sigma^\infty\mathcal{X}_\emptyset$) is a $(2k+1)$-connected map between k-connected spaces.

Suppose X is 1-connected and let $\mathcal{X}$ be the 1-cube $X \rightarrow \tilde{\Omega}\Sigma X$. Then we know by Remark 5.10 that each 0-subcube of $\mathcal{X}$ is 2-cartesian and the 1-subcube of $\mathcal{X}$ is 3-cartesian. In other words, the 1-cube $\mathcal{X}$ is $(\mathrm{id}+2)$-cartesian. Now let's study what happens when we apply the Freudenthal suspension map to $\mathcal{X}$ itself.

Suppose $\mathcal{X}$ is a 1-cube of pointed spaces (e.g., $\mathcal{X}$ is a W-cube with $W = \{1\}$) and $\mathcal{X}$ is $(\mathrm{id}+2)$-cartesian. Let's verify that the 2-cube $\mathcal{X} \rightarrow \tilde{\Omega}\Sigma\mathcal{X}$ is also $(\mathrm{id}+2)$-cartesian. It suffices to assume that $\mathcal{X}$ is a cofibration W-cube. Consider the 2-cube of the form

$$\begin{tikzcd} \mathcal{X}: \arrow[d] & \mathcal{X}_\emptyset \arrow[r] \arrow[d] & \mathcal{X}_{\{1\}} \arrow[d] \\ \tilde{\Omega}\Sigma\mathcal{X}: & \tilde{\Omega}\Sigma\mathcal{X}_\emptyset \arrow[r] & \tilde{\Omega}\Sigma\mathcal{X}_{\{1\}} \end{tikzcd}$$

We already know by Remark 5.10 that each 0-subcube is 2-cartesian and each 1-subcube is 3-cartesian. Hence it suffices to analyze the cartesian-ness of the 2-subcube $\mathcal{X} \rightarrow \tilde{\Omega}\Sigma\mathcal{X}$. It is difficult to analyze the cartesian-ness directly, so we take an indirect attack as follows. Let C be the homotopy cofiber of $\mathcal{X}$ and $\mathcal{C}$ the 1-cube $* \rightarrow C$. Then the associated 2-cube $\mathcal{X} \rightarrow \mathcal{C}$ is ∞-cocartesian and has the form

$$\begin{tikzcd} \mathcal{X}: \arrow[d] & \mathcal{X}_\emptyset \arrow[r] \arrow[d] & \mathcal{X}_{\{1\}} \arrow[d] \\ \mathcal{C}: & * \arrow[r] & C \end{tikzcd}$$

Since the 0-subcubes of $\mathcal{X}$ are 2-cartesian, we know that the vertical maps are 2-connected. Since the upper horizontal map is 3-connected, we know the bottom horizontal map is 3-connected.

Since it is difficult to analyze the cartesian-ness of $\mathcal{X} \rightarrow \tilde{\Omega}\Sigma\mathcal{X}$ directly, we consider the following commutative diagram of 2-cubes

$$\begin{tikzcd} \mathcal{X} \arrow[r, "(\#)"] \arrow[d, "(*)"'] & \mathcal{C} \arrow[d, "(\#)''"] \\ \tilde{\Omega}\Sigma\mathcal{X} \arrow[r, "(\#)'"] & \tilde{\Omega}\Sigma\mathcal{C} \end{tikzcd} \tag{52}$$

We take an indirect attack and analyze first the cartesian-ness of (#), (#)$'$, (#)$''$. Consider the case of (#). We want to estimate the cartesian-ness of $\mathcal{X} \to \mathcal{C}$. To carry this out, the idea is to use Goodwillie's higher Blakers-Massey theorem [37, 2.5], which we recall here for the convenience of the reader.

Proposition 5.11 (Higher Blakers-Massey theorem)**.** *Let W be a nonempty finite set. Let $\mathcal{X}$ be a W-cube of pointed spaces. Suppose that*

(i) *for each nonempty subset $V \subset W$, the V-cube $\partial_\emptyset^V \mathcal{X}$ (formed by all maps in $\mathcal{X}$ between $\mathcal{X}_\emptyset$ and $\mathcal{X}_V$) is k_V-cocartesian,*

(ii) *$k_U \leq k_V$ for each $U \subset V$.*

Then $\mathcal{X}$ is k-cartesian, where k is the minimum of $1 - |W| + \sum_{V \in \lambda} k_V$ over all partitions λ of W by nonempty sets.

By higher Blakers-Massey (Proposition 5.11), the 2-cube $\mathcal{X} \to \mathcal{C}$ is k-cartesian, where $k+1$ is the minimum of

$$\begin{gathered} k_{\{1\}} + k_{\{2\}} = 3 + 2 \\ k_{\{1,2\}} = \infty \end{gathered}$$

Hence $k = 4$ and we have calculated that $\mathcal{X} \to \mathcal{C}$ is 4-cartesian. Consider the case of (#)$'$. The 2-cube $\Sigma\mathcal{X} \to \Sigma\mathcal{C}$ is ∞-cocartesian and has the form

$$\begin{tikzcd} \Sigma\mathcal{X}: \arrow[d] & \Sigma\mathcal{X}_\emptyset \arrow[r] \arrow[d] & \Sigma\mathcal{X}_{\{1\}} \arrow[d] \\ \Sigma\mathcal{C}: & \Sigma * \arrow[r] & \Sigma C \end{tikzcd}$$

We know the vertical maps are 3-connected and the horizontal maps are 4-connected. By higher Blakers-Massey (Proposition 5.11), the 2-cube $\Sigma\mathcal{X} \to \Sigma\mathcal{C}$ is k-cartesian, where $k+1$ is the minimum of

$$\begin{gathered} k_{\{1\}} + k_{\{2\}} = 4 + 3 \\ k_{\{1,2\}} = \infty \end{gathered}$$

Hence $k = 6$ and we have calculated that $\Sigma\mathcal{X} \to \Sigma\mathcal{C}$ is 6-cartesian; therefore $\tilde{\Omega}\Sigma\mathcal{X} \to \tilde{\Omega}\Sigma\mathcal{C}$ is 5-cartesian. Consider the case of (#)$''$. The 2-cube $\mathcal{C} \to \tilde{\Omega}\Sigma\mathcal{C}$ has the form

$$\begin{tikzcd} \mathcal{C}: \arrow[d] & * \arrow[r] \arrow[d] & C \arrow[d] \\ \tilde{\Omega}\Sigma\mathcal{C}: & \tilde{\Omega}\Sigma * \arrow[r] & \tilde{\Omega}\Sigma C \end{tikzcd}$$

We know from above that C is 3-connected, hence by Remark 5.10 the right-hand vertical map is 7-connected. Taking homotopy fibers horizontally gives the map $\tilde{\Omega}C \to \tilde{\Omega}\tilde{\Omega}\Sigma C$ which is $\tilde{\Omega}$ applied to the right-hand vertical map. Hence we have calculated that $\mathcal{C} \to \tilde{\Omega}\Sigma\mathcal{C}$ is 6-cartesian ([37, 1.18]). Putting it all together, it follows from diagram (52) and [37, 1.8] that $\mathcal{X} \to \tilde{\Omega}\Sigma\mathcal{X}$ is 4-cartesian. Hence we have shown that the 2-cube $\mathcal{X} \to \tilde{\Omega}\Sigma\mathcal{X}$ satisfies: each 0-subcube is 2-cartesian, each 1-subcube is 3-cartesian, each 2-subcube is 4-cartesian. Hence we have verified that the 2-cube $\mathcal{X} \to \tilde{\Omega}\Sigma\mathcal{X}$ is $(\mathrm{id}+2)$-cartesian.

Suppose $\mathcal{X}$ is a 2-cube of pointed spaces (e.g., $\mathcal{X}$ is a W-cube with $W = \{1,2\}$) and $\mathcal{X}$ is $(\mathrm{id}+2)$-cartesian. Let's verify that the 3-cube $\mathcal{X} \to \tilde{\Omega}\Sigma\mathcal{X}$ is also $(\mathrm{id}+2)$-cartesian. It suffices to assume that $\mathcal{X}$ is a cofibration W-cube. Consider the 3-cube of the form $\mathcal{X} \to \tilde{\Omega}\Sigma\mathcal{X}$. It is difficult to analyze the cartesian-ness directly, so we take an indirect attack as follows. Let C be the iterated homotopy cofiber of $\mathcal{X}$ and $\mathcal{C}$ the 2-cube of form

$$\begin{tikzcd} * \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & C \end{tikzcd} \tag{53}$$

Then the associated 3-cube $\mathcal{X} \to \mathcal{C}$ is ∞-cocartesian and has the form

$$\begin{tikzcd} X_\emptyset \arrow[rr] \arrow[dr] \arrow[dd] & & X_{\{1\}} \arrow[dr] \arrow[dd] & \\ & X_{\{2\}} \arrow[rr] \arrow[dd] & & X_{\{1,2\}} \arrow[dd] \\ * \arrow[rr] \arrow[dr] & & * \arrow[dr] & \\ & * \arrow[rr] & & C \end{tikzcd} \tag{54}$$

Let's analyze the connectivity of C. It suffices to estimate the cocartesian-ness of $\mathcal{X}$. To carry this out, the idea is to use Goodwillie's higher dual Blakers-Massey theorem [37, 2.6], which we recall here for the convenience of the reader.

Proposition 5.12 (Higher dual Blakers-Massey theorem)**.** *Let W be a nonempty finite set. Let $\mathcal{X}$ be a W-cube of pointed spaces. Suppose that*

(i) *for each nonempty subset $V \subset W$, the V-cube $\partial^W_{W-V}\mathcal{X}$ (formed by all maps in $\mathcal{X}$ between $\mathcal{X}_{W-V}$ and $\mathcal{X}_W$) is k_V-cartesian,*
(ii) *$k_U \leq k_V$ for each $U \subset V$.*

Then $\mathcal{X}$ is k-cocartesian, where k is the minimum of $|W| - 1 + \sum_{V\in\lambda} k_V$ over all partitions λ of W by nonempty sets.

By higher dual Blakers-Massey (Proposition 5.12), the 2-cube $\mathcal{X}$ is k-cocartesian, where $k-1$ is the minimum of

$$\begin{gathered} k_{\{1\}} + k_{\{2\}} = 3 + 3 \\ k_{\{1,2\}} = 4 \end{gathered}$$

Hence $k = 5$ and we have calculated that $\mathcal{X}$ is 5-cocartesian. Therefore C is 5-connected, and hence the two maps of the form $* \to C$ in (54) are 5-connected. Furthermore, since the 0-subcubes of $\mathcal{X}$ are 2-cartesian, we know that the vertical maps in (54) are 2-connected.

Remark 5.13. It is worth pointing out that the 2-cube $\mathcal{X}$, which was assumed to be $(\mathrm{id}+2)$-cartesian, satisfies: each 0-subcube is 1-cocartesian, each 1-subcube is 3-cocartesian, and each 2-subcube is 5-cocartesian. Hence the 2-cube $\mathcal{X}$ is $(2\mathrm{id}+1)$-cocartesian. Conversely, by higher Blakers-Massey (Proposition 5.11), if a 2-cube $\mathcal{X}$ is $(2\mathrm{id}+1)$-cocartesian, then $\mathcal{X}$ is also $(\mathrm{id}+2)$-cartesian. A more general version of this uniformity correspondence is described below (Proposition 5.14).

Since it is difficult to analyze the cartesian-ness of $\mathcal{X} \to \tilde{\Omega}\Sigma\mathcal{X}$ directly, we consider the following commutative diagram of 3-cubes

(55)
$$\begin{tikzcd}
\mathcal{X} \arrow[r, "(\#)"] \arrow[d, "(*)"'] & \mathcal{C} \arrow[d, "(\#)''"] \\
\tilde{\Omega}\Sigma\mathcal{X} \arrow[r, "(\#)'"] & \tilde{\Omega}\Sigma\mathcal{C}
\end{tikzcd}$$

We take an indirect attack and analyze first the cartesian-ness of $(\#)$, $(\#)'$, $(\#)''$. Consider the case of $(\#)$. By higher Blakers-Massey (Proposition 5.11), the 3-cube $\mathcal{X} \to \mathcal{C}$ is k-cartesian, where $k+2$ is the minimum of

$$\begin{aligned}
k_{\{1\}} + k_{\{2\}} + k_{\{3\}} &= 3+3+2 \\
k_{\{1\}} + k_{\{2,3\}} &= 3+4 \\
k_{\{2\}} + k_{\{1,3\}} &= 3+4 \\
k_{\{3\}} + k_{\{1,2\}} &= 2+5 \\
k_{\{1,2,3\}} &= \infty
\end{aligned}$$

Hence $k = 5$ and we have calculated that $\mathcal{X} \to \mathcal{C}$ is 5-cartesian. Consider the case of $(\#)'$. The 3-cube $\Sigma\mathcal{X} \to \Sigma\mathcal{C}$ is ∞-cocartesian. Noting that Σ shifts the cocartesian-ness estimates by $+1$: By higher Blakers-Massey (Proposition 5.11), the 3-cube $\Sigma\mathcal{X} \to \Sigma\mathcal{C}$ is k-cartesian, where $k+2$ is the minimum of

$$\begin{aligned}
k_{\{1\}} + k_{\{2\}} + k_{\{3\}} &= 4+4+3 \\
k_{\{1\}} + k_{\{2,3\}} &= 4+5 \\
k_{\{2\}} + k_{\{1,3\}} &= 4+5 \\
k_{\{3\}} + k_{\{1,2\}} &= 3+6 \\
k_{\{1,2,3\}} &= \infty
\end{aligned}$$

Hence $k = 7$ and we have calculated that $\Sigma\mathcal{X} \to \Sigma\mathcal{C}$ is 7-cartesian; therefore $\tilde{\Omega}\Sigma\mathcal{X} \to \tilde{\Omega}\Sigma\mathcal{C}$ is 6-cartesian. Consider the case of $(\#)''$. The 3-cube $\mathcal{C} \to \tilde{\Omega}\Sigma\mathcal{C}$ has the form

$$\begin{tikzcd}
* \arrow[rr] \arrow[dr] \arrow[dd] & & * \arrow[dr] \arrow[dd] & \\
& * \arrow[rr] \arrow[dd] & & C \arrow[dd] \\
\tilde{\Omega}\Sigma * \arrow[rr, dashed] \arrow[dr] & & \tilde{\Omega}\Sigma * \arrow[dr] & \\
& \tilde{\Omega}\Sigma * \arrow[rr] & & \tilde{\Omega}\Sigma C
\end{tikzcd}$$

We know from above that C is 5-connected, hence by Remark 5.10 the right-hand vertical map of the form $C \to \tilde{\Omega}\Sigma C$ is 11-connected. Taking homotopy fibers horizontally and then "into the page" gives the map $(\tilde{\Omega})^2 C \to (\tilde{\Omega})^2\tilde{\Omega}\Sigma C$ which is $(\tilde{\Omega})^2$ applied to the right-hand vertical map $C \to \tilde{\Omega}\Sigma C$. Hence we have calculated that $\mathcal{C} \to \tilde{\Omega}\Sigma\mathcal{C}$ is 9-cartesian ([37, 1.18]). Putting it all together, it follows from diagram (55) and [37, 1.8] that $\mathcal{X} \to \tilde{\Omega}\Sigma\mathcal{X}$ is 5-cartesian. What about the subcubes of $\mathcal{X} \to \tilde{\Omega}\Sigma\mathcal{X}$? We have nearly all the cartesian-ness estimates we need, except for the 2-subcube $\tilde{\Omega}\Sigma\mathcal{X}$. We know from Remark 5.13 that $\mathcal{X}$ is $(2\mathrm{id}+1)$-cocartesian, and hence $\Sigma\mathcal{X}$ is $(2\mathrm{id}+2)$-cocartesian. By higher Blakers-Massey (Proposition 5.11), the 2-cube $\Sigma\mathcal{X}$ is k-cartesian, where $k+1$ is the minimum of

$$\begin{aligned} k_{\{1\}} + k_{\{2\}} &= 4+4 \\ k_{\{1,2\}} &= 6 \end{aligned}$$

Hence $k = 5$ and we have calculated that $\Sigma\mathcal{X}$ is 5-cartesian; therefore $\tilde{\Omega}\Sigma\mathcal{X}$ is 4-cartesian. Hence we have shown that the 3-cube $\mathcal{X} \rightarrow \tilde{\Omega}\Sigma\mathcal{X}$ satisfies: each 0-subcube is 2-cartesian, each 1-subcube is 3-cartesian, each 2-subcube is 4-cartesian, each 3-subcube is 5-cartesian. Hence we have verified that the 3-cube $\mathcal{X} \rightarrow \tilde{\Omega}\Sigma\mathcal{X}$ is $(\mathrm{id}+2)$-cartesian.

A more general version of these cartesian-ness estimates are worked out below in Theorem 5.16. First it will be useful to observe the following uniformity correspondence; compare with [25, A.8.3]. A special case of this correspondence was worked out in Remark 5.13.

Proposition 5.14 (Uniformity correspondence). *Let $k \geq 1$ and W a finite set. A W-cube of pointed spaces is $(k(\mathrm{id}+1)+1)$-cartesian if and only if it is $((k+1)(\mathrm{id}+1)-1)$-cocartesian.*

Remark 5.15. Note that a W-cube $\mathcal{X}$ is $(k(\mathrm{id}+1)+1)$-cartesian means that $\mathcal{X}$ satisfies: each 0-subcube is $(k+1)$-cartesian, each 1-subcube is $(2k+1)$-cartesian, each 2-subcube is $(3k+1)$-cartesian, each 3-subcube is $(4k+1)$-cartesian, and so forth.

Similarly, note that a W-cube $\mathcal{X}$ is $((k+1)(\mathrm{id}+1)-1)$-cocartesian means that $\mathcal{X}$ satisfies: each 0-subcube is k-cocartesian, each 1-subcube is $(2k+1)$-cocartesian, each 2-subcube is $(3k+2)$-cocartesian, each 3-subcube is $(4k+3)$-cocartesian, and so forth.

Proof of Proposition 5.14. This is tautologically true for $|W| = 0, 1$. Let $n \geq 2$. Assume the statement is true for all $|W| < n$; let's verify it is true for $|W| = n$. Let $W = \{1, \cdots, n\}$ and suppose $\mathcal{X}$ is a W-cube of pointed spaces. Assume that $\mathcal{X}$ is $(k(\mathrm{id}+1)+1)$-cartesian; let's verify $\mathcal{X}$ is $((k+1)(\mathrm{id}+1)-1)$-cocartesian. By the induction hypothesis, it suffices to verify that $\mathcal{X}$ is $(k(n+1)+n)$-cocartesian; this follows easily from higher dual Blakers-Massey (Proposition 5.12). Conversely, assume that $\mathcal{X}$ is $((k+1)(\mathrm{id}+1)-1)$-cocartesian; let's verify $\mathcal{X}$ is $(k(\mathrm{id}+1)+1)$-cartesian. By the induction hypothesis, it suffices to verify that $\mathcal{X}$ is $(k(n+1)+1)$-cartesian; this follows easily from higher Blakers-Massey (Proposition 5.11). $\square$

Theorem 5.16 (Higher Freudenthal suspension: Theorem 1.8 restated). *Let $k \geq 1$, W a finite set, and $\mathcal{X}$ a W-cube of pointed spaces. If $\mathcal{X}$ is $(k(\mathrm{id}+1)+1)$-cartesian, then so is $\mathcal{X} \rightarrow \tilde{\Omega}^r\Sigma^r\mathcal{X}$.*

Proof. Consider the case $|W| = 0$. This is recalled in Remark 5.10. Consider the case $|W| \geq 1$. Suppose $\mathcal{X}$ is a W-cube and $\mathcal{X}$ is $(k(\mathrm{id}+1)+1)$-cartesian. Let's verify that $\mathcal{X} \rightarrow \tilde{\Omega}^r\Sigma^r\mathcal{X}$ is a $(k(\mathrm{id}+1)+1)$-cartesian $(|W|+1)$-cube. It suffices to assume that $\mathcal{X}$ is a cofibration W-cube; see [37, 1.13]. Let C be the iterated homotopy cofiber of $\mathcal{X}$ and $\mathcal{C}$ the W-cube defined objectwise by $\mathcal{C}_V = *$ for $V \neq W$ and

$\mathcal{C}_W = C$. Then $\mathcal{X}\to\mathcal{C}$ is ∞-cocartesian. Consider the commutative diagram

(56)
$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d, "(*)"'] & \mathcal{C} \arrow[d] \\ \tilde{\Omega}^r\Sigma^r\mathcal{X} \arrow[r] & \tilde{\Omega}^r\Sigma^r\mathcal{C} \end{tikzcd}$$

of $|W|$-cubes.

Let's verify that $(*)$ is $(k(|W|+2)+1)$-cartesian as a $(|W|+1)$-cube of pointed spaces. We know that $\mathcal{X}$ is $((k+1)(\mathrm{id}+1)-1)$-cocartesian by the uniformity correspondence in Proposition 5.14, and in particular, C is $((k+1)(|W|+1)-1)$-connected. For $d<|W|$, any $(d{+}1)$ dimensional subcube of $\mathcal{X}$ is $((k{+}1)(d{+}2){-}1) = ((k{+}1)(d{+}1){+}k)$-cocartesian and any d dimensional subcube of $\mathcal{X}$ is $((k{+}1)(d{+}1){-}1)$-cocartesian. So if $\mathcal{X}|T$ is some d-subcube of $\mathcal{X}$ with T not containing the terminal set W, then $\mathcal{X}|T\to\mathcal{C}|T = *$ is $(k+1)(d+1)$-cocartesian by [37, 1.7]. Furthermore, even if T contains the terminal set W, we know that $\mathcal{X}|T\to\mathcal{C}|T$ is still $(k{+}1)(d{+}1)$-cocartesian by [37, 1.7]; this is because $(k+1)(d+1)<(k+1)(|W|+1)-1$ since $k\geq 1$ and $d<|W|$. Hence $\mathcal{X}|T\to\mathcal{C}|T$ is $(k+1)(d+1)$-cocartesian for any d-subcube $\mathcal{X}|T$ of $\mathcal{X}$. It follows easily from higher Blakers-Massey (Proposition 5.11) that $\mathcal{X}\to\mathcal{C}$ is $(k(|W|+2)+1)$-cartesian. Similarly, it follows that $\Sigma^r\mathcal{X}\to\Sigma^r\mathcal{C}$ is $(k(|W|+2)+1+2r)$-cartesian and hence $\tilde{\Omega}^r\Sigma^r\mathcal{X}\to\tilde{\Omega}^r\Sigma^r\mathcal{C}$ is $(k(|W|+2)+1+r)$-cartesian. Also, $\mathcal{C}\to\tilde{\Omega}^r\Sigma^r\mathcal{C}$ is at least $(k(|W|{+}2){+}1)$-cartesian since $C\to\tilde{\Omega}^r\Sigma^rC$ is $(2[(k{+}1)(|W|{+}1){-}1]{+}1)$-connected by Remark 5.10; this is because the cartesian-ness of $\mathcal{C}\to\tilde{\Omega}^r\Sigma^r\mathcal{C}$ is the same as the connectivity of the map $\tilde{\Omega}^{|W|}C\to\tilde{\Omega}^{|W|}\tilde{\Omega}^r\Sigma^rC$ (by considering iterated homotopy fibers, together with [37, 1.18]).

Putting it all together, it follows from diagram (56) and [37, 1.8] that the map $(*)$ is $(k(|W|{+}2){+}1)$-cartesian; this is because $k(|W|{+}2){+}1 < k(|W|{+}2){+}1{+}r$. Doing this also on all subcubes gives the result. $\square$

The proof of the following is similar.

Theorem 5.17 (Higher stabilization: Theorem 1.7 restated)**.** *Let $k\geq 1$, W a finite set, and $\mathcal{X}$ a W-cube of pointed spaces. If $\mathcal{X}$ is $(k(\mathrm{id}+1)+1)$-cartesian, then so is $\mathcal{X}\to\tilde{\Omega}^\infty\Sigma^\infty\mathcal{X}$.*

Proof. Consider the case $|W|=0$. This is recalled in Remark 5.10. Consider the case $|W|\geq 1$. Suppose $\mathcal{X}$ is a W-cube and $\mathcal{X}$ is $(k(\mathrm{id}+1)+1)$-cartesian. Let's verify that $\mathcal{X}\to\tilde{\Omega}^\infty\Sigma^\infty\mathcal{X}$ is a $(k(\mathrm{id}+1)+1)$-cartesian $(|W|+1)$-cube. It suffices to assume that $\mathcal{X}$ is a cofibration W-cube; see [37, 1.13]. Let C be the iterated homotopy cofiber of $\mathcal{X}$ and $\mathcal{C}$ the W-cube defined objectwise by $\mathcal{C}_V = *$ for $V\neq W$ and $\mathcal{C}_W = C$. Then $\mathcal{X}\to\mathcal{C}$ is ∞-cocartesian. Consider the commutative diagram

(57)
$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d, "(*)"'] & \mathcal{C} \arrow[d] \\ \tilde{\Omega}^\infty\Sigma^\infty\mathcal{X} \arrow[r] & \tilde{\Omega}^\infty\Sigma^\infty\mathcal{C} \end{tikzcd}$$

of $|W|$-cubes.

Let's verify that $(*)$ is $(k(|W|+2)+1)$-cartesian as a $(|W|+1)$-cube of pointed spaces. We know that $\mathcal{X}$ is $((k+1)(\mathrm{id}+1)-1)$-cocartesian by the uniformity correspondence in Proposition 5.14, and in particular, C is $((k+1)(|W|+1)-1)$-connected. For $d<|W|$, any $(d{+}1)$ dimensional subcube of $\mathcal{X}$ is $((k{+}1)(d{+}2){-}1) =$

$((k+1)(d+1)+k)$-cocartesian and any d dimensional subcube of $\mathcal{X}$ is $((k+1)(d+1)-1)$-cocartesian. So if $\mathcal{X}|T$ is some d-subcube of $\mathcal{X}$ with T not containing the terminal set W, then $\mathcal{X}|T \to \mathcal{C}|T = *$ is $(k+1)(d+1)$-cocartesian by [37, 1.7]. Furthermore, even if T contains the terminal set W, we know that $\mathcal{X}|T \to \mathcal{C}|T$ is still $(k+1)(d+1)$-cocartesian by [37, 1.7]; this is because $(k+1)(d+1) < (k+1)(|W|+1)-1$ since $k \geq 1$ and $d < |W|$. Hence $\mathcal{X}|T \to \mathcal{C}|T$ is $(k+1)(d+1)$-cocartesian for any d-subcube $\mathcal{X}|T$ of $\mathcal{X}$. It follows easily from higher Blakers-Massey (Proposition 5.11) that $\mathcal{X} \to \mathcal{C}$ is $(k(|W|+2)+1)$-cartesian.

We know that $\Sigma^\infty \mathcal{X} \to \Sigma^\infty \mathcal{C}$ is ∞-cocartesian and hence ∞-cartesian; therefore $\tilde{\Omega}^\infty \Sigma^\infty \mathcal{X} \to \tilde{\Omega}^\infty \Sigma^\infty \mathcal{C}$ is ∞-cartesian. Also, $\mathcal{C} \to \tilde{\Omega}^\infty \Sigma^\infty \mathcal{C}$ is at least $(k(|W|+2)+1)$-cartesian since $C \to \tilde{\Omega}^\infty \Sigma^\infty C$ is $(2[(k+1)(|W|+1)-1]+1)$-connected by Remark 5.10; this is because the cartesian-ness of $\mathcal{C} \to \tilde{\Omega}^\infty \Sigma^\infty \mathcal{C}$ is the same as the connectivity of the map $\tilde{\Omega}^{|W|} C \to \tilde{\Omega}^{|W|} \tilde{\Omega}^\infty \Sigma^\infty C$ (by considering iterated homotopy fibers, together with [37, 1.18]).

Putting it all together, it follows from diagram (57) and [37, 1.8] that the map $(*)$ is $(k(|W|+2)+1)$-cartesian. Doing this also on all subcubes gives the result. $\square$

5.18. **Completion with respect to $\tilde{\Omega}^\infty \Sigma^\infty$ and $\tilde{\Omega}^r \Sigma^r$.** In this section we explain how Theorems 1.7 and 1.8 provide new proofs (with strong estimates) of the stabilization and iterated loop-suspension completion results of Carlsson [16] (and the subsequent work of Arone-Kankaanrinta [2]), and Bousfield [11] and Hopkins (see [11]), respectively, for 1-connected spaces. Along the way, we recall the notion of coface n-cubes and certain cofinality statements needed in the proofs of Theorems 2.3 and 2.1 (Section 5.30).

Definition 5.19. A cosimplicial pointed space Z is *coaugmented* if it comes with a map $d^0\colon Z^{-1} \to Z^0$ of pointed spaces such that $d^0 d^0 = d^1 d^0\colon Z^{-1} \to Z^1$; in this case, it follows easily from the cosimplicial identities ([36, I.1]) that d^0 induces a map $Z^{-1} \to Z$ of Δ-shaped diagrams in S_*, where Z^{-1} denotes the constant cosimplicial object with value Z^{-1}; i.e., via the inclusion $Z^{-1} \in \mathsf{S}_* \subset \mathsf{S}_*^\Delta$ of constant diagrams.

Definition 5.20. Let $n \geq -1$ and suppose Z is a cosimplicial pointed space coaugmented by $d^0\colon Z^{-1} \to Z^0$. The *coface* $(n+1)$-cube, denoted $\mathcal{X}_{n+1}$, associated to the coaugmented cosimplicial pointed space $Z^{-1} \to Z$, is the canonical $(n+1)$-cube built from the coface relations [36, I.1] $d^j d^i = d^i d^{j-1}$, if $i < j$, associated to the coface maps of the n-truncation

$$\begin{tikzcd} Z^{-1} \arrow[r, "d^0"] & Z^0 \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] & Z^1 \ \cdots \ Z^n \end{tikzcd}$$

of $Z^{-1} \to Z$; in particular, $\mathcal{X}_0$ is the pointed space (or 0-cube) Z^{-1}.

Remark 5.21. For instance, the coface 1-cube $\mathcal{X}_1$ has the left-hand form

$$\begin{tikzcd} Z^{-1} \arrow[r, "d^0"] & Z^0 \end{tikzcd}$$

$$\begin{tikzcd} Z^{-1} \arrow[r, "d^0"] \arrow[d, "d^0"] & Z^0 \arrow[d, "d^0"] \\ Z^0 \arrow[r, "d^1"] & Z^1 \end{tikzcd}$$

$$\begin{tikzcd} Z^{-1} \arrow[rr, "d^0"] \arrow[dr, "d^0"] \arrow[dd, "d^0"'] & & Z^0 \arrow[dr, "d^0"] \arrow[dd, "d^1"' near start] & \\ & Z^0 \arrow[rr, "d^1" near start] \arrow[dd, "d^1" near start] & & Z^1 \arrow[dd, "d^2"] \\ Z^0 \arrow[rr, "d^0" near start] \arrow[dr, "d^0"'] & & Z^1 \arrow[dr, "d^0"] & \\ & Z^1 \arrow[rr, "d^1"] & & Z^2 \end{tikzcd}$$

the coface 2-cube $\mathcal{X}_2$ has the indicated middle form, and the coface 3-cube $\mathcal{X}_3$ has the indicated right-hand form.

Definition 5.22. Let $n \geq 0$. Denote by $\Delta^{\leq n} \subset \Delta$ the full subcategory of objects $[m]$ such that $m \leq n$.

The functor in the following definition, appearing in [61, 6.3], plays a key role in the homotopical analysis of this paper; see also [56, 9.4.1].

Definition 5.23. Define the totally ordered sets $[n] := \{0, 1, \ldots, n\}$ for each $n \geq 0$, and given their natural ordering. The functor $\mathcal{P}_0([n]) \rightarrow \Delta^{\leq n}$ is defined objectwise by $U \mapsto [|U| - 1]$, and which sends $U \subset V$ in $\mathcal{P}_0([n])$ to the composite

$$[|U| - 1] \cong U \subset V \cong [|V| - 1]$$

where the indicated isomorphisms are the unique isomorphisms of totally ordered sets.

Remark 5.24. For instance, the punctured 2-cube $\mathcal{P}_0([1]) \longrightarrow \Delta^{\leq 1}$ has the left-hand form

$$\begin{tikzcd}
& \{1\} \arrow[d, "d^0"] \\
\{0\} \arrow[r, "d^1"] & \{0,1\}
\end{tikzcd}
\qquad
\begin{tikzcd}
& & \{1\} \arrow[dr, "d^0"] \arrow[dd, "d^1"] & \\
& \{0\} \arrow[rr, "d^1"] \arrow[dd, "d^1"] & & \{0,1\} \arrow[dd, "d^2"] \\
\{2\} \arrow[rr, "d^0"] \arrow[dr, "d^0"'] & & \{1,2\} \arrow[dr, "d^0"] & \\
& \{0,2\} \arrow[rr, "d^1"] & & \{0,1,2\}
\end{tikzcd}$$

and the punctured 3-cube $\mathcal{P}_0([2]) \longrightarrow \Delta^{\leq 2}$ has the indicated right-hand form. It may be helpful to note that d^i is the inclusion that "misses the i-th element in its codomain", where position count starts from 0; e.g., the 0-th element in $\{1, 2\}$ is 1 and the 1-st element in $\{1, 2\}$ is 2.

The following proposition is proved in [14, XI.9.2].

Proposition 5.25. *Let* $\alpha\colon \mathsf{D}' \rightarrow \mathsf{D}$ *be a functor between small categories. If* Z *is a* D*-shaped diagram in pointed spaces and* α *is left cofinal* [14, XI.9.1]*, then the induced map* $\operatorname{holim}_{\mathsf{D}'} Z \xleftarrow{\simeq} \operatorname{holim}_{\mathsf{D}} Z$ *is a weak equivalence.*

The following proposition, proved in [61, 6.7], explains the homotopical significance of the punctured n-cube appearing in Definition 5.23; see also [17, 6.1–6.4] and [23, 18.7]. It was exploited early on in [42]; see also [25] and [56].

Proposition 5.26. *Let* $n \geq 0$*. The functor* $\mathcal{P}_0([n]) \rightarrow \Delta^{\leq n}$ *is left cofinal; hence, if* Z *is a cosimplicial pointed space, then the induced map*

$$\operatorname{holim}_{\mathcal{P}_0([n])} Z \xleftarrow{\simeq} \operatorname{holim}_{\Delta^{\leq n}} Z$$

is a weak equivalence.

Remark 5.27. We follow the conventions and definitions in [9, 5.11] for our model of homotopy limit; e.g., if the diagram is objectwise fibrant, it is the Bousfield-Kan homotopy limit formula defined as the totalization of the cosimplicial replacement.

Remark 5.28 (Higher stabilization implies $X \simeq X^{\wedge}_{\tilde{\Omega}^\infty\Sigma^\infty}$). Assume that X is a 1-connected pointed space. A result in Carlsson [16], and in the subsequent work of Arone-Kankaanrinta [2], is that the completion map $X \simeq X^{\wedge}_{\tilde{\Omega}^\infty\Sigma^\infty}$ is a weak equivalence. It is worth pointing out that higher stabilization (Theorem 1.7) provides a new proof, with strong estimates, of this result as follows.

To verify that the completion map $X \simeq X^{\wedge}_{\tilde{\Omega}^\infty\Sigma^\infty}$ is a weak equivalence, it suffices to verify that the map

$$X \rightarrow \operatorname{holim}_{\Delta^{\leq n}} \mathfrak{C}(\Sigma^\infty X) \tag{58}$$

into the n-th stage of the homotopy limit tower has connectivity strictly increasing with n. By Proposition 5.26, the map (58) can be built, up to weak equivalence, from the coface $(n+1)$-cube $\mathcal{X}_{n+1}$ (Defintion 5.20) associated to the cosimplicial resolution (17). In more detail: The map (58) can be described as the map $X \rightarrow \operatorname{holim}_{\mathcal{P}_0([n])} \mathcal{X}_{n+1}$; the connectivity of this map is the same as the cartesian-ness of the coface (n+1)-cube $\mathcal{X}_{n+1}$, but this is the same as the cartesian-ness of the (n+1)-cube $\mathcal{X}_n \rightarrow \tilde{\Omega}^\infty\Sigma^\infty\mathcal{X}_n$, for each $n \geq 0$.

Since X is a 1-connected pointed space, the map $X \rightarrow *$ is 2-connected, and hence the 0-cube $\mathcal{X}_0$ is $(\mathrm{id}+2)$-cartesian. Hence by higher stabilization (Theorem 1.7) we know that $\mathcal{X}_1$ is $(\mathrm{id}+2)$-cartesian, and therefore another application of Theorem 1.7 gives that $\mathcal{X}_2$ is $(\mathrm{id}+2)$-cartesian, and so forth. In a similar way, the coface $(n+1)$ cube $\mathcal{X}_{n+1}$ is $(\mathrm{id}+2)$-cartesian for each $n \geq 0$; hence Theorem 1.7 has provided us with strong estimates for the uniform cartesian-ness of cubes built by iterations of the stabilization map. In particular, we know that the $(n+1)$-cube $\mathcal{X}_{n+1}$ is $(n+1+2)$-cartesian for each $n \geq 0$, which means that the map (58) is $(n+3)$-connected for each $n \geq 0$. Therefore, these connectivity estimates imply that the map

$$X \rightarrow \operatorname{holim}_n \operatorname{holim}_{\Delta^{\leq n}} \mathfrak{C}(\Sigma^\infty X) \simeq \operatorname{holim}_{\Delta} \mathfrak{C}(\Sigma^\infty X) \simeq X^{\wedge}_{\tilde{\Omega}^\infty\Sigma^\infty}$$

is a weak equivalence; since this is the desired completion map, we have recovered the result in [2, 16] that $X \simeq X^{\wedge}_{\tilde{\Omega}^\infty\Sigma^\infty}$.

Remark 5.29 (Higher Freudenthal suspension implies $X \simeq X^{\wedge}_{\tilde{\Omega}^r\Sigma^r}$). Assume that X is a 1-connected pointed space. A result of Bousfield [11] and Hopkins (see [11]), is that the completion map $X \simeq X^{\wedge}_{\tilde{\Omega}^r\Sigma^r}$ is a weak equivalence. Similar to above, it is worth pointing out that higher Freudenthal suspension (Theorem 1.8) provides a new proof, with strong estimates, of this result as follows.

To verify that the completion map $X \simeq X^{\wedge}_{\tilde{\Omega}^r\Sigma^r}$ is a weak equivalence, it suffices to verify that the map

$$X \rightarrow \operatorname{holim}_{\Delta^{\leq n}} \mathfrak{C}_r(\Sigma^r X) \tag{59}$$

into the n-th stage of the homotopy limit tower has connectivity strictly increasing with n. By Proposition 5.26, the map (59) can be built, up to weak equivalence, from the coface $(n+1)$-cube $\mathcal{X}_{n+1}$ (Defintion 5.20) associated to the cosimplicial resolution (21). Arguing as in Remark 5.28, the coface $(n+1)$ cube $\mathcal{X}_{n+1}$ is $(\mathrm{id}+2)$-cartesian for each $n \geq 0$; hence Theorem 1.8 has provided us with strong estimates for the uniform cartesian-ness of cubes built by iterations of the higher Freudenthal suspension map. In particular, we know that the $(n+1)$-cube $\mathcal{X}_{n+1}$ is $(n+1+2)$-cartesian for each $n \geq 0$, which means that the map (59) is $(n+3)$-connected for each $n \geq 0$. Therefore, these connectivity estimates imply that the map

$$X \rightarrow \mathrm{holim}_n \, \mathrm{holim}_{\Delta^{\leq n}} \mathfrak{C}_r(\Sigma^r X) \simeq \mathrm{holim}_{\Delta} \mathfrak{C}_r(\Sigma^r X) \simeq X^{\wedge}_{\tilde{\Omega}^r \Sigma^r}$$

is a weak equivalence; since this is the desired completion map, we have recovered the result in [11] that $X \simeq X^{\wedge}_{\tilde{\Omega}^r \Sigma^r}$.

5.30. **Proofs of the main results.** Here we prove Theorems 2.1 and 2.3.

Definition 5.31. Let Z be a cosimplicial pointed space and $n \geq 0$. Assume that Z is objectwise fibrant. Denote by $Z\colon \mathcal{P}_0([n]) \rightarrow \mathsf{S}_*$ the corresponding composite diagram

$$\mathcal{P}_0([n]) \rightarrow \Delta^{\leq n} \subset \Delta \xrightarrow{Z} \mathsf{S}_*$$

The *associated* ∞*-cartesian* $(n+1)$*-cube built from* Z, denoted $\widetilde{Z}\colon \mathcal{P}([n]) \rightarrow \mathsf{S}_*$, is defined objectwise by

$$\widetilde{Z}_V := \begin{cases} \mathrm{holim}_{T \neq \emptyset} Z_T, & \text{for } V = \emptyset, \\ Z_V, & \text{for } V \neq \emptyset. \end{cases}$$

In other words, the $\widetilde{Z}$ construction is simply "filling in" the punctured $(n+1)$-cube $Z\colon \mathcal{P}_0([n]) \rightarrow \mathsf{S}_*$ with value $\widetilde{Z}_\emptyset = \mathrm{holim}_{\mathcal{P}_0([n])} Z \simeq \mathrm{holim}_{\Delta^{\leq n}} Z$ at the initial vertex to turn it into an $(n+1)$-cube that is ∞-cartesian.

Remark 5.32. For instance, in the case $n = 1$ the $\widetilde{Z}$ construction produces the ∞-cartesian 2-cube of the left-hand form

$$\begin{tikzcd} \mathrm{holim}_{\Delta^{\leq 1}} Z \arrow[r] \arrow[d] & Z^0 \arrow[d, "d^0"] \\ Z^0 \arrow[r, "d^1"] & Z^1 \end{tikzcd}
\qquad
\begin{tikzcd} \mathrm{holim}_{\Delta^{\leq 2}} Z \arrow[rr] \arrow[dr] \arrow[dd] & & Z^0 \arrow[dr, "d^0"] \arrow[dd, "d^1", near end] & \\ & Z^0 \arrow[rr, "d^1"] \arrow[dd, "d^1", near end] & & Z^1 \arrow[dd, "d^2"] \\ Z^0 \arrow[rr, "d^0", near start] \arrow[dr, "d^0"'] & & Z^1 \arrow[dr, "d^0"] & \\ & Z^1 \arrow[rr, "d^1"] & & Z^2 \end{tikzcd}$$

and in the case $n = 2$ the $\widetilde{Z}$ construction produces the ∞-cartesian 3-cube of the indicated right-hand form.

Definition 5.33. Let Z be a cosimplicial pointed space and $n \geq 0$. The *codegeneracy* n-cube, denoted $\mathcal{Y}_n$, associated to Z, is the canonical n-cube built from the codegeneracy relations [36, I.1] $s^j s^i = s^i s^{j+1}$, if $i \leq j$, associated to the codegeneracy maps of the n-truncation

$$\begin{tikzcd} Z^0 & Z^1 \arrow[l, "s^0"'] & Z^2 \arrow[l, shift right, "s^0"'] \arrow[l, shift left, "s^1"] & \cdots & Z^n \end{tikzcd}$$

of Z; in particular, $\mathcal{Y}_0$ is the pointed space (or 0-cube) Z^0.

Remark 5.34. For instance, the codegeneracy 1-cube $\mathcal{Y}_1$ has the left-hand form

$$\begin{tikzcd} Z^1 \arrow[r, "s^0"] & Z^0 \end{tikzcd} \qquad \begin{tikzcd} Z^2 \arrow[r, "s^1"] \arrow[d, "s^0"] & Z^1 \arrow[d, "s^0"] \\ Z^1 \arrow[r, "s^0"] & Z^0 \end{tikzcd} \qquad \begin{tikzcd} Z^3 \arrow[rr, "s^1"] \arrow[dr, "s^2"] \arrow[dd, "s^0"'] & & Z^2 \arrow[dr, "s^1"] \arrow[dd, "s^0"] & \\ & Z^2 \arrow[rr, "s^1"] \arrow[dd, "s^0"] & & Z^1 \arrow[dd, "s^0"] \\ Z^2 \arrow[rr, "s^0"] \arrow[dr, "s^1"'] & & Z^1 \arrow[dr, "s^0"] & \\ & Z^1 \arrow[rr, "s^0"] & & Z^0 \end{tikzcd}$$

the codegeneracy 2-cube $\mathcal{Y}_2$ has the indicated middle form, and the codegeneracy 3-cube $\mathcal{Y}_3$ has the indicated right-hand form.

Remark 5.35. It is important to note that the total homotopy fiber of an n-cube of pointed spaces is weakly equivalent to its iterated homotopy fiber [37, Section 1], and in this paper we use the terms interchangeably; we use the convention that the iterated homotopy fiber of a 0-cube $\mathcal{Y}$ (or object $\mathcal{Y}_\emptyset$) is the homotopy fiber of the unique map $\mathcal{Y}_\emptyset \to *$ and hence is weakly equivalent to $\mathcal{Y}_\emptyset$; see also [56, 5.5.4].

The following is proved in [56, 3.4.8].

Proposition 5.36. *Consider any* 2*-cube* $\mathcal{X}$ *of the form*

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[d, equal] & B \arrow[d, "s"] \\ A \arrow[r, equal] & A \end{tikzcd}$$

in S_**; in other words, suppose* s *is a retraction of* d*. There are natural weak equivalences* $\mathrm{hofib}(d) \simeq \tilde{\Omega}\,\mathrm{hofib}(s)$*.*

Remark 5.37. Let's develop some of the low dimensional versions of Theorem 2.3 for the special case $r = 1$; for intuition purposes, we will follow (as closely as possible) the analogous development for integral chains worked out in [9, after 3.25].

Let Z be a 2-connected $\tilde{\mathsf{K}}_1$-coalgebra. We want to estimate the connectivity of the map

$$\Sigma \operatorname{holim}_{\Delta^{\leq n}} \mathfrak{C}_1(Z) \to \operatorname{holim}_{\Delta^{\leq n}} \Sigma\mathfrak{C}_1(Z) \tag{60}$$

for each $n \geq 0$. In the case $n = 0$ this is the identity map $\Sigma\tilde{\Omega}Z \to \Sigma\tilde{\Omega}Z$ and hence a weak equivalence. Consider the case $n = 1$. Let's build $\widetilde{\mathfrak{C}_1(Z)}$, the ∞-cartesian 2-cube of the left-hand form

$$\begin{tikzcd} \operatorname{holim}_{\Delta^{\leq 1}} \mathfrak{C}_1(Z) \arrow[r] \arrow[d] & \mathfrak{C}_1(Z)^0 \arrow[d, "d^0"] \\ \mathfrak{C}_1(Z)^0 \arrow[r, "d^1"] & \mathfrak{C}_1(Z)^1 \end{tikzcd} \qquad \begin{tikzcd} \Sigma \operatorname{holim}_{\Delta^{\leq 1}} \mathfrak{C}_1(Z) \arrow[r] \arrow[d] & \Sigma\mathfrak{C}_1(Z)^0 \arrow[d, "\Sigma d^0"] \\ \Sigma\mathfrak{C}_1(Z)^0 \arrow[r, "\Sigma d^1"] & \Sigma\mathfrak{C}_1(Z)^1 \end{tikzcd}$$

Applying Σ gives the 2-cube $\Sigma\widetilde{\mathfrak{C}_1(Z)}$ of the indicated right-hand form. The connectivity of the map

$$\Sigma \operatorname{holim}_{\Delta^{\leq 1}} \mathfrak{C}_1(Z) \to \operatorname{holim}_{\Delta^{\leq 1}} \Sigma\mathfrak{C}_1(Z) \tag{61}$$

is the same as the cartesian-ness of the 2-cube $\Sigma\widetilde{\mathfrak{C}_1(Z)}$. The idea is to (i) estimate the cocartesian-ness of the 2-cube $\widetilde{\mathfrak{C}_1(Z)}$ (and its subcubes), (ii) applying Σ will increase cocartesian-ness estimates by 1, (iii) $\Sigma\widetilde{\mathfrak{C}_1(Z)}$ is a 2-cube in S_* and we will be able to use higher Blakers-Massey (Proposition 5.11) to obtain a cartesian-ness estimate of $\Sigma\widetilde{\mathfrak{C}_1(Z)}$. To carry this out, the idea is to use higher dual Blakers-Massey (Proposition 5.12) to estimate the cocartesian-ness of the 2-cube $\widetilde{\mathfrak{C}_1(Z)}$.

Taking $W = \{0,1\}$ since $\widetilde{\mathfrak{C}_1(Z)}$ is a 2-cube, the input to Proposition 5.12 requires that we estimate the cartesian-ness of each of the faces

$$\partial^W_{W-V}\widetilde{\mathfrak{C}_1(Z)}, \qquad \emptyset \neq V \subset W.$$

Hence we need to estimate the cartesian-ness of the two 1-faces indicated in the left-hand diagram

$$\begin{tikzcd} & \mathfrak{C}_1(Z)^0 \arrow[d, "d^0"] \\ \mathfrak{C}_1(Z)^0 \arrow[r, "d^1"] & \mathfrak{C}_1(Z)^1 \end{tikzcd} \qquad \begin{tikzcd} & \tilde{\Omega}Z \arrow[d, "d^0"] \\ \tilde{\Omega}Z \arrow[r, "d^1"] & \tilde{\Omega}\Sigma\tilde{\Omega}Z \end{tikzcd}$$

which have the form in the indicated right-hand diagram. We know that $d^0 = m\,\mathrm{id}$ is the Freudenthal suspension map on $\tilde{\Omega}Z$, and since $\tilde{\Omega}Z$ is 1-connected we know that d^0 is a 3-connected map and hence a 3-cartesian 1-cube. What about the map $d^1 = \mathrm{id}\,m$ involving the $\tilde{\mathsf{K}}_1$-coaction map on Z? The key observation is that the cosimplicial identities force a certain "uniformity of faces" behavior as follows. Consider the commutative diagrams (or 2-cubes) of the form

$$\begin{tikzcd} \tilde{\Omega}Z \arrow[r, "d^0"] \arrow[d, equal] & \tilde{\Omega}\Sigma\tilde{\Omega}Z \arrow[d, "s^0"] \\ \tilde{\Omega}Z \arrow[r, equal] & \tilde{\Omega}Z \end{tikzcd} \qquad \begin{tikzcd} \tilde{\Omega}Z \arrow[r, "d^1"] \arrow[d, equal] & \tilde{\Omega}\Sigma\tilde{\Omega}Z \arrow[d, "s^0"] \\ \tilde{\Omega}Z \arrow[r, equal] & \tilde{\Omega}Z \end{tikzcd} \tag{62}$$

coming from the cosimplicial identities [36, I.1]. Then by Proposition 5.36 we know

$$\mathrm{hofib}(d^0) \simeq \tilde{\Omega}\,\mathrm{hofib}(s^0), \qquad \mathrm{hofib}(d^1) \simeq \tilde{\Omega}\,\mathrm{hofib}(s^0),$$

and hence $\mathrm{hofib}(d^0) \simeq \mathrm{hofib}(d^1)$. Therefore, by this uniformity we know that d^1 is also a 3-connected map and hence a 3-cartesian 1-cube. Since we know that the 2-face of $\widetilde{\mathfrak{C}_1(Z)}$ is ∞-cartesian (by construction), it follows from Proposition 5.12 that $\widetilde{\mathfrak{C}_1(Z)}$ is k-cocartesian, where $k-1$ is the minimum of

$$k_{\{0,1\}} = \infty, \qquad k_{\{0\}} + k_{\{1\}} = 3 + 3 = 6.$$

Hence $k = 7$ and we have calculated that $\widetilde{\mathfrak{C}_1(Z)}$ is a 7-cocartesian 2-cube in S_*, hence $\Sigma\widetilde{\mathfrak{C}_1(Z)}$ is an 8-cocartesian 2-cube in S_*. Furthermore, since $\widetilde{\mathfrak{C}_1(Z)}$ is ∞-cartesian and d^0, d^1 are 3-connected, we know by [37, 1.6] that the other two 1-faces are also 3-connected. Putting it all together, we have shown that the 2-cube $\widetilde{\mathfrak{C}_1(Z)}$ in S_* is $(2\mathrm{id}+1)$-cocartesian and 7-cocartesian. Hence $\Sigma\widetilde{\mathfrak{C}_1(Z)}$ is $(2\mathrm{id}+2)$-cocartesian and 8-cocartesian. It follows from Proposition 5.11 that $\Sigma\widetilde{\mathfrak{C}_1(Z)}$ is k-cartesian, where $k+1$ is the minimum of

$$k_{\{0,1\}} = 8, \qquad k_{\{0\}} + k_{\{1\}} = 4 + 4 = 8.$$

Hence $k = 7$ and we have calculated that $\Sigma\widetilde{\mathfrak{C}_1(Z)}$ is a 7-cartesian 2-cube in S_*. The upshot is that $\Sigma\widetilde{\mathfrak{C}_1(Z)}$ is 7-cartesian and hence we have calculated that the map (61) is 7-connected.

Consider the case $n = 2$. Let's build the ∞-cartesian 3-cube $\widetilde{\mathfrak{C}_1(Z)}$. Applying Σ gives the 3-cube $\Sigma\widetilde{\mathfrak{C}_1(Z)}$ and the connectivity of the map

$$\Sigma \operatorname{holim}_{\Delta^{\leq 2}} \mathfrak{C}_1(Z) \to \operatorname{holim}_{\Delta^{\leq 2}} \Sigma\mathfrak{C}_1(Z) \tag{63}$$

is the same as the cartesian-ness of $\Sigma\widetilde{\mathfrak{C}_1(Z)}$. The idea is to (i) estimate the cocartesian-ness of the 3-cube $\widetilde{\mathfrak{C}_1(Z)}$ (and its subcubes), (ii) applying Σ will increase cocartesian-ness estimates by 1, (iii) $\Sigma\widetilde{\mathfrak{C}_1(Z)}$ is a 3-cube in S_* and we will be able to use higher Blakers-Massey (Proposition 5.11) to obtain a cartesian-ness estimate of $\Sigma\widetilde{\mathfrak{C}_1(Z)}$. To carry this out, the idea is to use higher dual Blakers-Massey (Proposition 5.12) to estimate the cocartesian-ness of the 3-cube $\widetilde{\mathfrak{C}_1(Z)}$.

Taking $W = \{0, 1, 2\}$ since $\widetilde{\mathfrak{C}_1(Z)}$ is a 3-cube, the input to Proposition 5.12 requires that we estimate the cartesian-ness of each of the faces $\partial^W_{W-V}\widetilde{\mathfrak{C}_1(Z)}$, $\emptyset \neq V \subset W$. Hence we need to estimate the cartesian-ness of three 2-faces and three 1-faces (or maps). The key observation is that exactly one of these 2-faces does not involve the $\tilde{\mathsf{K}}_1$-coaction map on Z; furthermore, this particular 2-face is precisely the coface 2-cube $\mathcal{X}_2$ in Remark 5.29 when taking $X = \tilde{\Omega} Z$. Since $\tilde{\Omega} Z$ is 1-connected, we know by higher Freudenthal suspension (Theorem 1.8) and Remark 5.29 that $\mathcal{X}_2$ is an $(\mathrm{id}+2)$-cartesian 2-cube; in particular, $\mathcal{X}_2$ is 4-cartesian. What about the other two 2-faces involving the $\tilde{\mathsf{K}}_1$-coaction map on Z? The key observation is that the cosimplicial identities force a certain "uniformity of faces" behavior as follows. For ease of notational purposes, let $A = \mathfrak{C}_1(Z)$ and consider the commutative diagrams of the form

(64)

$$\begin{tikzcd}
A^0 \arrow[r, "d^0"] \arrow[d, "d^0"'] \arrow[dr, phantom, "(F_1)"] & A^1 \arrow[d, "d^1"] & \\
A^1 \arrow[r, "d^0"] \arrow[d, "s^0"'] & A^2 \arrow[r, "s^0"] \arrow[d, "s^1"] & A^1 \arrow[d, "s^0"] \\
A^0 \arrow[r, "d^0"] & A^1 \arrow[r, "s^0"] & A^0
\end{tikzcd}
\qquad
\begin{tikzcd}
A^0 \arrow[r, "d^1"] \arrow[d, "d^1"'] \arrow[dr, phantom, "(F_2)"] & A^1 \arrow[r, "s^0"] \arrow[d, "d^2"] & A^0 \arrow[d, "d^1"] \\
A^1 \arrow[r, "d^1"] & A^2 \arrow[r, "s^0"] \arrow[d, "s^1"] & A^1 \arrow[d, "s^0"] \\
 & A^1 \arrow[r, "s^0"] & A^0
\end{tikzcd}
\qquad
\begin{tikzcd}
A^0 \arrow[r, "d^0"] \arrow[d, "d^1"'] \arrow[dr, phantom, "(F_3)"] & A^1 \arrow[r, "s^0"] \arrow[d, "d^2"] & A^0 \arrow[d, "d^1"] \\
A^1 \arrow[r, "d^0"] & A^2 \arrow[r, "s^0"] \arrow[d, "s^1"] & A^1 \arrow[d, "s^0"] \\
 & A^1 \arrow[r, "s^0"] & A^0
\end{tikzcd}$$

coming from the cosimplicial identities [36, I.1]. The upper left-hand square (F_1) is the coface 2-cube $\mathcal{X}_2$ which is $(\mathrm{id}+2)$-cartesian by above. The upper left-hand squares (F_2) and (F_3) are the remaining two 2-faces that we need cartesian-ness estimates for. The key observation is that the lower right-hand squares are each a copy of the codegeneracy 2-cube $\mathcal{Y}_2$ associated to A, and that furthermore, the indicated vertical and horizontal composites are the identity maps by the cosimplicial identities [36, I.1]; then by repeated application of Proposition 5.36 to these composites in (64), we know that

$$
\begin{aligned}
(\text{iterated hofib})(F_1) &\simeq \tilde{\Omega}^2(\text{iterated hofib})\mathcal{Y}_2\\
(\text{iterated hofib})(F_2) &\simeq \tilde{\Omega}^2(\text{iterated hofib})\mathcal{Y}_2\\
(\text{iterated hofib})(F_3) &\simeq \tilde{\Omega}^2(\text{iterated hofib})\mathcal{Y}_2
\end{aligned}
$$

and hence $(\text{iterated hofib})(F_1) \simeq (\text{iterated hofib})(F_2) \simeq (\text{iterated hofib})(F_3)$. Therefore, by this uniformity we know that (F_2) and (F_3) are also 4-cartesian 2-cubes. Similarly, we know that the three 1-faces (or maps) with codomain A^2 are 3-cartesian. It follows from [37, 1.6] that for $\widetilde{\mathfrak{C}_1(Z)}$ the 2-subcubes are 4-cartesian, the 1-subcubes are 3-cartesian, and the 0-subcubes are 2-cartesian. Putting it all together, we have shown that the 3-cube $\widetilde{\mathfrak{C}_1(Z)}$ in S_* is $(\mathrm{id}+2)$-cartesian. Since we know that the 3-face of $\widetilde{\mathfrak{C}_1(Z)}$ is ∞-cartesian (by construction), it follows from Proposition 5.12 that $\widetilde{\mathfrak{C}_1(Z)}$ is k-cocartesian, where $k-2$ is the minimum of

$$k_{\{0,1,2\}} = \infty, \quad k_{\{0\}} + k_{\{1,2\}} = 3+4 = 7, \quad k_{\{0\}} + k_{\{1\}} + k_{\{2\}} = 3+3+3 = 9.$$

Note that by the "uniformity of faces" behavior, we get nothing new from the other partitions of W; this is why we have not written them out here. Hence $k=9$ and we have calculated that $\widetilde{\mathfrak{C}_1(Z)}$ is a 9-cocartesian 3-cube in S_*. Furthermore, since each 2-subcube is $(\mathrm{id}+2)$-cartesian, then each 2-subcube is $(2\mathrm{id}+1)$-cocartesian (Remark 5.13 and Proposition 5.14). Hence $\Sigma\widetilde{\mathfrak{C}_1(Z)}$ is $(2\mathrm{id}+2)$-cocartesian and 10-cocartesian. It follows from Proposition 5.11 that $\Sigma\widetilde{\mathfrak{C}_1(Z)}$ is k-cartesian, where $k+2$ is the minimum of

$$k_{\{0,1,2\}} = 10, \quad k_{\{0\}} + k_{\{1,2\}} = 4+6 = 10, \quad k_{\{0\}} + k_{\{1\}} + k_{\{2\}} = 4+4+4 = 12.$$

Hence $k=8$ and we have calculated that $\Sigma\widetilde{\mathfrak{C}_1(Z)}$ is an 8-cartesian 3-cube in S_*. The upshot is that $\Sigma\widetilde{\mathfrak{C}_1(Z)}$ is 8-cartesian and hence we have calculated that the map (63) is 8-connected.

Note that there is more information in the argument above. Since the 2-face (F_1) is 4-cartesian, its total homotopy fiber is 3-connected, hence the indicated homotopy fiber (see [14, X.6.3]; compare with [56, 5.5.7])

$$\mathrm{hofib}(\mathrm{holim}_{\Delta^{\leq 2}} \mathfrak{C}_1(Z) \rightarrow \mathrm{holim}_{\Delta^{\leq 1}} \mathfrak{C}_1(Z)) \simeq \Omega^2(\text{iterated hofib})\mathcal{Y}_2$$

is 3-connected and therefore the map $\mathrm{holim}_{\Delta^{\leq 2}} \mathfrak{C}_1(Z) \rightarrow \mathrm{holim}_{\Delta^{\leq 1}} \mathfrak{C}_1(Z)$ is 4-connected. Also, since $\Omega^2(\text{iterated hofib})\mathcal{Y}_2$ is 3-connected, then $(\text{iterated hofib})\mathcal{Y}_2$ is 5-connected.

In a similar way, for each $n \geq 3$, the connectivity of the map

$$\Sigma\, \mathrm{holim}_{\Delta^{\leq n}} \mathfrak{C}_1(Z) \rightarrow \mathrm{holim}_{\Delta^{\leq n}} \Sigma\mathfrak{C}_1(Z)$$

is the same as the cartesian-ness of the (n+1)-cube $\Sigma\widetilde{\mathfrak{C}_1(Z)}$. We can organize our argument as follows, exactly as in the above cases for $n = 1, 2$.

The following calculation is proved in [14, X.6.3] for the Tot tower of a Reedy fibrant cosimplicial pointed space; compare with [56, 5.5.7]. The connectivity assumptions on Y (resp. Z) ensure that all spaces in sight (e.g., including the iterated homotopy fibers) are 1-connected; this is elaborated, for instance, in Remark 5.42.

Proposition 5.38. *Let Y (resp. Z) be a $\tilde{\mathsf{K}}$-coalgebra (resp. $\tilde{\mathsf{K}}_r$-coalgebra) and $n \geq 0$. If Y (resp. Z) is 1-connected (resp. $(1+r)$-connected), then there are natural zigzags of weak equivalences*

$$\mathrm{hofib}(\mathrm{holim}_{\Delta^{\leq n}} \mathfrak{C}(Y) \to \mathrm{holim}_{\Delta^{\leq n-1}} \mathfrak{C}(Y)) \simeq \tilde{\Omega}^n (\text{iterated hofib}) \mathcal{Y}_n$$

$$\textit{resp.} \qquad \mathrm{hofib}(\mathrm{holim}_{\Delta^{\leq n}} \mathfrak{C}_r(Z) \to \mathrm{holim}_{\Delta^{\leq n-1}} \mathfrak{C}_r(Z)) \simeq \tilde{\Omega}^n (\text{iterated hofib}) \mathcal{Y}_n$$

where $\mathcal{Y}_n$ denotes the codegeneracy n-cube associated to $\mathfrak{C}(Y)$ (resp. $\mathfrak{C}_r(Z)$).

First we recall the following closely related proposition, which appears in [20, 6.27]. It can be proved by arguing exactly as in [56, 5.5.7]; see also Remark 5.37.

Proposition 5.39 (Uniformity of faces)**.** *Let Y (resp. Z) be a $\tilde{\mathsf{K}}$-coalgebra (resp. $\tilde{\mathsf{K}}_r$-coalgebra) and $n \geq 0$. Let $\emptyset \neq T \subset [n]$ and $t \in T$. If Y (resp. Z) is 1-connected (resp. $(1+r)$-connected), then there is a weak equivalence*

$$(\text{iterated hofib}) \partial^T_{\{t\}} \widetilde{\mathfrak{C}(Y)} \simeq \tilde{\Omega}^{|T|-1} (\text{iterated hofib}) \mathcal{Y}_{|T|-1}$$

$$\textit{resp.} \qquad (\text{iterated hofib}) \partial^T_{\{t\}} \widetilde{\mathfrak{C}_r(Z)} \simeq \tilde{\Omega}^{|T|-1} (\text{iterated hofib}) \mathcal{Y}_{|T|-1}$$

where $\mathcal{Y}_{|T|-1}$ denotes the codegeneracy $(|T|-1)$-cube associated to $\mathfrak{C}(Y)$ (resp. $\mathfrak{C}_r(Z)$).

Proof. This is proved in [20] (compare [38, 3.4] and [61, 7.2]); the connectivity assumptions on Y (resp. Z) ensure that all spaces in sight (e.g., including the iterated homotopy fibers) are 1-connected. □

Theorem 5.40. *Let Y be a $\tilde{\mathsf{K}}$-coalgebra and $n \geq 1$. Consider the ∞-cartesian $(n+1)$-cube $\widetilde{\mathfrak{C}(Y)}$ (Definition 5.31) of pointed spaces built from $\mathfrak{C}(Y)$. If Y is 1-connected, then*

(a) *the cube $\widetilde{\mathfrak{C}(Y)}$ is $(2n+5)$-cocartesian,*
(b) *the cube $\Sigma^\infty \widetilde{\mathfrak{C}(Y)}$ is $(2n+5)$-cocartesian, and*
(c) *the cube $\Sigma^\infty \widetilde{\mathfrak{C}(Y)}$ is $(n+5)$-cartesian.*

Proof. Consider part (a). Taking $W = \{0, 1, \dots, n\}$ since $\widetilde{\mathfrak{C}(Y)}$ is an $(n+1)$-cube, our strategy is to use higher dual Blakers-Massey (Proposition 5.12) to estimate how close the W-cube $\widetilde{\mathfrak{C}(Y)}$ in S_* is to being cocartesian; the input to Proposition 5.12 requires that we estimate the cartesian-ness of each of the faces

$$\partial^W_{W-V} \widetilde{\mathfrak{C}(Y)}, \qquad \emptyset \neq V \subset W \tag{65}$$

Here is the key observation: for exactly one $w \in W$, the n-face $\partial^W_{\{w\}} \widetilde{\mathfrak{C}(Y)}$ does not involve the $\tilde{\mathsf{K}}$-coaction map on Y; furthermore, this particular n-face is precisely the coface n-cube $\mathcal{X}_n$ (Definition 5.20) associated to the cosimplicial resolution (17) when taking $X = \tilde{\Omega}^\infty Y$. Since $\tilde{\Omega}^\infty Y$ is 1-connected, we know by higher stabilization (Theorem 1.7) that $\mathcal{X}_n = \partial^W_{\{w\}} \widetilde{\mathfrak{C}(Y)}$ is an $(\mathrm{id}+2)$-cartesian n-cube. Furthermore, now that we have uniform cartesian-ness estimates for this particular n-face, we have the same uniform cartesian-ness estimates for the other desired faces in (65)—this is because of the "uniformity of faces" property enforced by the cosimplicial identities and summarized in Proposition 5.39. Hence we have verified that for each nonempty subset $V \subset W$, the V-cube $\partial^W_{W-V} \widetilde{\mathfrak{C}(Y)}$ is $(|V|+2)$-cartesian; since it is ∞-cartesian by construction when $V = W$, it follows immediately from higher dual

Blakers-Massey (Proposition 5.12) that $\widetilde{\mathfrak{C}(Y)}$ is $(2n+5)$-cocartesian in S_*, which finishes the proof of part (a). Part (b) follows from the fact that $\Sigma^\infty\colon \mathsf{S}_*{\rightarrow}\mathsf{Mod}_S$ is a left Quillen functor together with the property that Σ^∞ preserves connectivity of maps between 1-connected spaces. Part (c) follows easily from [19, 3.10]; in other words, that $\Sigma^\infty\widetilde{\mathfrak{C}(Y)}$ is k-cocartesian if and only if it is $(k-n)$-cartesian. Taking $k=(2n+5)$ from part (b) finishes the proof. □

The proof of the following is similar; however, in addition to estimating the cocartesian-ness of $\widetilde{\mathfrak{C}_r(Z)}$, we will also require estimates on the subcubes as well.

Theorem 5.41. *Let Z be a $\tilde{\mathsf{K}}_r$-coalgebra and $n\geq 1$. Consider the ∞-cartesian $(n+1)$-cube $\widetilde{\mathfrak{C}_r(Z)}$ (Definition 5.31) of pointed spaces built from $\mathfrak{C}_r(Z)$. If Z is $(1+r)$-connected, then*

(a) *the cube $\widetilde{\mathfrak{C}_r(Z)}$ is $(2n+5)$-cocartesian and $(2\cdot\mathrm{id}+1)$-cocartesian,*
(b) *the cube $\Sigma^r\widetilde{\mathfrak{C}_r(Z)}$ is $(2n+5+r)$-cocartesian and $(2\cdot\mathrm{id}+1+r)$-cocartesian, and*
(c) *the cube $\Sigma^r\widetilde{\mathfrak{C}_r(Z)}$ is $(n+5+r)$-cartesian.*

Proof. Consider part (a). Taking $W=\{0,1,\dots,n\}$ since $\widetilde{\mathfrak{C}_r(Z)}$ is an $(n+1)$-cube, our strategy is to use higher dual Blakers-Massey (Proposition 5.12) to estimate how close the W-cube $\widetilde{\mathfrak{C}_r(Z)}$ and its subcubes are to being cocartesian; the input to Proposition 5.12 requires that we estimate the cartesian-ness of each of the faces

$$\partial^W_{W-V}\widetilde{\mathfrak{C}_r(Z)},\qquad \emptyset\neq V\subset W.$$

We know from higher Freudenthal suspension (Theorem 1.8) on iterations of the Freudenthal suspension map applied to $\tilde{\Omega}^r Z$, together with the "uniformity of faces" property enforced by the cosimplicial identities and summarized in Proposition 5.39, that for each nonempty subset $V\subset W$, the V-cube $\partial^W_{W-V}\widetilde{\mathfrak{C}_r(Z)}$ is $(|V|+2)$-cartesian; since it is ∞-cartesian by construction when $V=W$, it follows immediately from higher dual Blakers-Massey (Proposition 5.12) that $\widetilde{\mathfrak{C}_r(Z)}$ is $(2n+5)$-cocartesian. Similarly, it follows that $\widetilde{\mathfrak{C}_r(Z)}$ is $(\mathrm{id}+2)$-cartesian, and hence by the uniformity correspondence in Proposition 5.14 we know that $\widetilde{\mathfrak{C}_r(Z)}$ is $(2\cdot\mathrm{id}+1)$-cocartesian which finishes the proof of part (a). Part (b) follows from the fact that $\Sigma^r\colon \mathsf{S}_*{\rightarrow}\mathsf{S}_*$ is a left Quillen functor together with the property that Σ^r increases the connectivity of maps between 1-connected spaces by r. Part (c) follows immediately from higher Blakers-Massey (Proposition 5.11) together with the cocartesian-ness estimates in part (b). □

Proof of Theorem 2.3. We want to estimate how connected the comparison map

$$\Sigma^\infty\,\mathrm{holim}_{\Delta^{\leq n}}\,\mathfrak{C}(Y)\longrightarrow \mathrm{holim}_{\Delta^{\leq n}}\,\Sigma^\infty\,\mathfrak{C}(Y)$$

is, which is equivalent to estimating how cartesian $\Sigma^\infty\widetilde{\mathfrak{C}(Y)}$ is, and Theorem 5.40(c) completes the proof. The other case is similar using Theorem 5.41(c). □

Remark 5.42. There is more information in the proof of Theorem 5.40 above. We know that for exactly one $w\in W$, the n-face $\partial^W_{\{w\}}\widetilde{\mathfrak{C}(Y)}$ (i.e., the unique n-face of this form not involving the $\tilde{\mathsf{K}}$-coaction map on Y) in the proof of Theorem 5.40

is precisely the coface n-cube $\mathcal{X}_n$ (Definition 5.20) associated to the cosimplicial resolution (17) when taking $X = \tilde{\Omega}^\infty Y$. Since $\tilde{\Omega}^\infty Y$ is 1-connected, we know that $\mathcal{X}_n$ is an $(\mathrm{id}+2)$-cartesian n-cube by higher stabilization (Theorem 1.7); in particular, this n-face $\partial^W_{\{w\}}\widetilde{\mathfrak{C}(Y)}$ is $(n+2)$-cartesian and hence its total homotopy fiber is $(n+1)$-connected. By Proposition 5.39, we know

$$(\text{iterated hofib})\partial^W_{\{w\}}\widetilde{\mathfrak{C}(Y)} \simeq \tilde{\Omega}^{(n+1)-1}(\text{iterated hofib})\mathcal{Y}_{(n+1)-1}$$

and hence by Proposition 5.38 we know that

$$\mathrm{hofib}(\mathrm{holim}_{\Delta^{\leq n}} \mathfrak{C}(Y) \rightarrow \mathrm{holim}_{\Delta^{\leq n-1}} \mathfrak{C}(Y)) \simeq \tilde{\Omega}^n(\text{iterated hofib})\mathcal{Y}_n$$

is $(n+1)$-connected; therefore the map $\mathrm{holim}_{\Delta^{\leq n}} \mathfrak{C}(Y) \rightarrow \mathrm{holim}_{\Delta^{\leq n-1}} \mathfrak{C}(Y)$ is $(n+2)$-connected. Also, since $\tilde{\Omega}^n(\text{iterated hofib})\mathcal{Y}_n$ is (n+1)-connected, then we know $(\text{iterated hofib})\mathcal{Y}_n$ is (2n+1)-connected. The upshot is that we have just proved Proposition 5.43 and the first part of Theorem 2.1.

Proposition 5.43. *Let Y be a $\tilde{\mathsf{K}}$-coalgebra and $n \geq 1$. Denote by $\mathcal{Y}_n$ the codegeneracy n-cube associated to the cosimplicial cobar construction $\mathfrak{C}(Y)$ of Y. If Y is 1-connected, then the total homotopy fiber of $\mathcal{Y}_n$ is $(2n+1)$-connected.*

Proof. This is proved in Remark 5.42. □

Proposition 5.44. *Let Z be a $\tilde{\mathsf{K}}_r$-coalgebra space and $n \geq 1$. Denote by $\mathcal{Y}_n$ the codegeneracy n-cube associated to the cosimplicial cobar construction $\mathfrak{C}_r(Z)$ of Z. If Z is $(1+r)$-connected, then the total homotopy fiber of $\mathcal{Y}_n$ is $(2n+1)$-connected.*

Proof. This is proved as in Remark 5.42 using $\widetilde{\mathfrak{C}_r(Z)}$ in place of $\widetilde{\mathfrak{C}(Y)}$. □

Proof of Theorem 2.1. The homotopy fiber of $\mathrm{holim}_{\Delta^{\leq n}} \mathfrak{C}(Y) \rightarrow \mathrm{holim}_{\Delta^{\leq n-1}} \mathfrak{C}(Y)$ is weakly equivalent to $\tilde{\Omega}^n$ of the total homotopy fiber of the codegeneracy n-cube $\mathcal{Y}_n$ associated to $\mathfrak{C}(Y)$ by Proposition 5.38, hence by Proposition 5.43 this map is $(n+2)$-connected. The other case is similar using Proposition 5.44. □

6. Appendix: Simplicial structures

In this background section, we recall the simplicial structures and adjunctions on pointed spaces and S-modules that are used throughout the paper; the expert may wish to skim through, or skip entirely, this background section. Our aim here is to be clear in our notation for various hom-objects and to point out Propositions 6.4 and 6.6 which are used throughout Section 4, for instance.

Definition 6.1. Let X, X' be pointed spaces and K a simplicial set. The *tensor product* $X\dot{\otimes}K$ in S_*, *mapping object* $\mathbf{hom}_{\mathsf{S}_*}(K, X)$ in S_*, and *mapping space* $\mathbf{Hom}_{\mathsf{S}_*}(X, X')$ in sSet are defined by

$$X\dot{\otimes}K := X \wedge K_+$$
$$\mathbf{hom}_{\mathsf{S}_*}(K, X') := \mathbf{hom}_*(K_+, X')$$
$$\mathbf{Hom}_{\mathsf{S}_*}(X, X')_n := \mathrm{hom}_{\mathsf{S}_*}(X\dot{\otimes}\Delta[n], X')$$

where the *pointed mapping space* $\mathbf{hom}_*(X, X')$ in S_* is $\mathbf{Hom}_{\mathsf{S}_*}(X, X')$ pointed by the constant map; see [36, II.3].

Definition 6.2. Let Y, Y' be S-modules and K a simplicial set. The *tensor product* $Y\dot{\otimes}K$ in Mod_S, *mapping object* $\mathbf{hom}_{\mathsf{Mod}_S}(K,Y)$ in Mod_S, and *mapping space* $\mathbf{Hom}_{\mathsf{Mod}_S}(Y,Y')$ in sSet are defined by

$$Y\dot{\otimes}K := Y \wedge K_+$$
$$\mathbf{hom}_{\mathsf{Mod}_S}(K,Y') := \mathrm{Map}(K_+,Y')$$
$$\mathbf{Hom}_{\mathsf{Mod}_S}(Y,Y')_n := \hom_{\mathsf{Mod}_S}(Y\dot{\otimes}\Delta[n],Y')$$

where $\mathrm{Map}(K_+,Y')$ denotes the function S-module; see [44, 2.2.9].

We sometimes drop the S_* and Mod_S decorations from the notation and simply write **Hom** and **hom**.

Proposition 6.3. *With the above definitions of mapping object, tensor product, and mapping space the categories of pointed spaces S_* and S-modules Mod_S are simplicial model categories.*

Proof. This is proved, for instance, in [36, II.3] and [44]. □

Recall that the adjunction $(\Sigma^\infty, \Omega^\infty)$ in (23) is a Quillen adjunction with left adjoint on top; in particular, for $X, Y \in \mathsf{S}_*$ there is an isomorphism

$$\hom(\Sigma^\infty X, Y) \cong \hom(X, \Omega^\infty Y) \tag{66}$$

in Set, natural in X, Y. The following proposition, which follows from [36, II.2.9], verifies that this adjunction meshes nicely with the simplicial structure.

Proposition 6.4. *Let X be a pointed space, Y an S-module, and K, L simplicial sets. Then*

(a) *there is a natural isomorphism $\sigma\colon \Sigma^\infty(X)\dot{\otimes}K \xrightarrow{\cong} \Sigma^\infty(X\dot{\otimes}K)$;*
(b) *there is an isomorphism*
$$\mathbf{Hom}(\Sigma^\infty X, Y) \cong \mathbf{Hom}(X, \Omega^\infty Y)$$
in sSet, natural in X, Y, that extends the adjunction isomorphism in (66);
(c) *there is an isomorphism*
$$\Omega^\infty \mathbf{hom}(K,Y) \cong \mathbf{hom}(K, \Omega^\infty Y)$$
in S_*, natural in K, Y.
(d) *there is a natural map $\sigma\colon \Omega^\infty(Y)\dot{\otimes}K \rightarrow \Omega^\infty(Y\dot{\otimes}K)$ induced by Ω^∞.*
(e) *the functors Σ^∞ and Ω^∞ are simplicial functors (Remark 6.5) with the structure maps σ of (a) and (d), respectively.*

Remark 6.5. For a useful reference on simplicial functors in the context of homotopy theory, see Hirschhorn [41, 9.8.5].

Similarly, recall that the adjunction (Σ^r, Ω^r) in (26) is a Quillen adjunction with left adjoint on top; in particular, for $X, X' \in \mathsf{S}_*$ there is an isomorphism

$$\hom(\Sigma^r X, X') \cong \hom(X, \Omega^r X') \tag{67}$$

in Set, natural in X, X'. The following proposition follows from [36, II.2.9].

Proposition 6.6. *Let X, Y be pointed spaces and K, L simplicial sets. Then*

(a) *there is a natural isomorphism $\sigma\colon \Sigma^r(X)\dot{\otimes}K \xrightarrow{\cong} \Sigma^r(X\dot{\otimes}K)$;*

(b) *there is an isomorphism*

$$\mathbf{Hom}(\Sigma^r X, Y) \cong \mathbf{Hom}(X, \Omega^r Y)$$

in sSet*, natural in* X, Y*, that extends the adjunction isomorphism in* (67)*;*

(c) *there is an isomorphism*

$$\Omega^r\,\mathbf{hom}(K, Y) \cong \mathbf{hom}(K, \Omega^r Y)$$

in S_**, natural in* K, Y*.*

(d) *there is a natural map* $\sigma\colon \Omega^r(Y)\dot{\otimes}K \rightarrow \Omega^r(Y\dot{\otimes}K)$ *induced by* Ω^r*.*

(e) *the functors* Σ^r *and* Ω^r *are simplicial functors (Remark 6.5) with the structure maps* σ *of (a) and (d), respectively.*

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