Simplicial Homotopy Theory, Completion, and Functor Calculus

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Preface

CHAPTER 1

Basic constructions: Mapping properties

In homotopy theory, we often work with—and construct for this purpose—new objects, built from simpler pieces, according to a characterization of their mapping properties. We will eventually be working in several contexts, but for starters, let's explore some constructions in the familiar setting of the category of sets and their maps (i.e., functions), denoted Set, and the category of topological spaces and their maps (i.e., continuous functions), denoted Top. We are intentionally overlapping with material the reader has already encountered, but with an emphasis on mapping properties, rather than focusing primarily on the underlying set of elements. This point of view will have significant payoffs later.

Let's start our constructions in Set, and then find our way to their analogs in Top. We will cycle back and forth, from Set to Top, for various examples. This will begin the development in the reader of some intuition for such constructions; very soon, working formally with these constructions (e.g., via mapping properties) will begin to feel quite natural. Once this happens, working in other settings (e.g., chain complexes over a ring, simplicial sets, simplicial modules over a ring, various flavors of spectra, algebras over operads in symmetric spectra, etc...) becomes more accessible; in other words, you've got to start somewhere, and Set and Top are useful places to begin building intuition and understanding.

1.1. Products

Let X, Y be sets. Classically, the product $X \times Y$ in Set is defined to be the set of ordered pairs

$$X \times Y := \{(x, y) |\, x \in X, y \in Y\}$$

There is a diagram of the form

$$X \xleftarrow{\mathrm{pr}_0} X \times Y \xrightarrow{\mathrm{pr}_1} Y \tag{1.1}$$

in Set, defined by $\mathrm{pr}_0(x, y) = x$ and $\mathrm{pr}_1(x, y) = y$ (i.e., pr_0 and pr_1 are the usual projection maps to X and Y, respectively), which satisfies the following mapping property; for further reading, see [**5**, p. 69] and [**7**, p. 77].

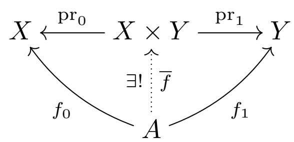
Remark 1.1.1. In homotopy theory, we often start our indexing at 0, rather than at 1; for instance, when we get to simplicial objects, non-negative chain complexes, cosimplicial objects, and so forth. So we might as well get used to it early on—hence the subscripts in (1.1) on our projection maps. On the other hand, the reader should not feel locked into this notation—when working in various situations, use whatever notation feels right to you. For instance, we could have denoted $(\mathrm{pr}_0, \mathrm{pr}_1)$ by $(\mathrm{pr}_X, \mathrm{pr}_Y)$, $(\mathrm{pr}_1, \mathrm{pr}_2)$, or $(\mathrm{pr}, \mathrm{pr}')$, or we simply could have called it (p, p'), (a, b), or (p', p''); as long as your colleagues (or audience) know what you are talking about.

PROPOSITION 1.1.2 (Universal property of products in Set). *If X, Y are sets, then diagram* (1.1) *is terminal with respect to all such diagrams into X, Y in Set; i.e., for any set A and diagram of the form*

$$X \xleftarrow{f_0} A \xrightarrow{f_1} Y$$

in Set, there exists a unique map $\overline{f}$ in Set which makes the diagram

(1.2)



commute; i.e., such that $\mathrm{pr}_0 \overline{f} = f_0$ and $\mathrm{pr}_1 \overline{f} = f_1$.

PROOF. Uniqueness is forced on us, since the diagram commutes implies that $\overline{f}(a) = (f_0(a), f_1(a))$ for each $a \in A$. Existence follows since this is a well-defined map in Set. $\square$

REMARK 1.1.3. The reader should not feel obliged to use the symbol $\overline{f}$ for the map induced by f_0, f_1 in (1.2); for instance, we could have used the symbol f, f', ξ, a_1, α, or $\overline{\alpha}$, in place of $\overline{f}$.

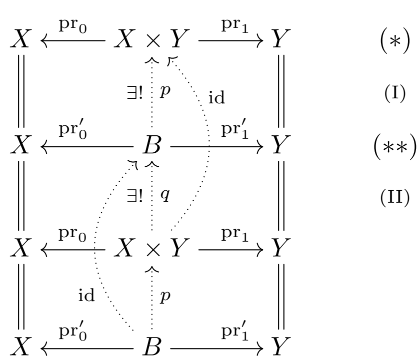
REMARK 1.1.4. The upshot is: giving a map $\overline{f}\colon A \to X \times Y$ in Set is the same as giving a pair of maps $X \xleftarrow{f_0} A \xrightarrow{f_1} Y$ in Set. For this reason, sometimes $\overline{f}$ is written as $\overline{f} = (f_0, f_1)$.

REMARK 1.1.5. The mapping properties in (1.2) characterize the product $X \times Y$ in Set, up to isomorphism. Let's verify this. Suppose there is a set B together with a diagram of the form

$$X \xleftarrow{\mathrm{pr}'_0} B \xrightarrow{\mathrm{pr}'_1} Y \tag{1.3}$$

in Set, which satisfies the universal property in (1.2); i.e., such that diagram (1.3) is terminal with respect to all such diagrams into X, Y in Set. Let's verify that $B \cong X \times Y$ in Set: we want to compare the sets $X \times Y$ and B, and the only game in town is the universal property; so let's use it. Consider the solid diagram of the form

(1.4)



in Set. Diagram $(*)$ is terminal with respect to all such diagrams into X, Y in Set, hence there exists a unique map p in Set which makes diagram (I) commute. Diagram $(**)$ is terminal with respect to all such diagrams into X, Y in Set, hence there exists a unique map q in Set which makes diagram (II) commute. The identity map id makes diagram (I) + (II) commute; but pq also makes (I) + (II) commute. Hence, by uniqueness, $pq = \mathrm{id}$. Similarly, by uniqueness, $qp = \mathrm{id}$. Hence we have verified (using four applications of the universal property) that $B \cong X \times Y$ in Set. Conversely, if there is a set B together with an isomorphism $p\colon B \xrightarrow{\cong} X \times Y$ in Set, then it is easy to verify that the diagram of the form (1.3), with $\mathrm{pr}'_0 := \mathrm{pr}_0\, p$ and $\mathrm{pr}'_1 := \mathrm{pr}_1\, p$ satisfies the mapping properties in (1.2); i.e., it is terminal with respect to all such diagrams into X, Y in Set.

A *product diagram* in Set is a diagram of the form (1.1) that satisfies the universal property in (1.2). For instance, if X is a set, then the diagram

$$X = X \longrightarrow *$$

is a product diagram in Set and hence $X \cong X \times *$ in Set.

Remark 1.1.6. A diagram that is naturally isomorphic to a product diagram in Set, is a product diagram in Set.

What happens if we replace Set with Top in our above discussion? The reader already knows from their background how to work with products of topological spaces, but let's pretend we forgot; this is a good idea since the intuition we develop here, in this familiar situation, will carry over and guide us in more complicated ones.

Let X, Y be topological spaces. Classically, the product $X \times Y$ in Top is defined to be $X \times Y$ in Set, equipped with an appropriate topology. Let's follow our noses to rediscover this topology, called the product topology. The idea is: we want the same mapping properties in (1.2) to be true, but for Set replaced by Top. In particular, this means that we need a topology on the set $X \times Y$ such that the maps in (1.1) are continuous; i.e., inverse images of open subsets are open. There are two extremes. If we give the set $X \times Y$ the largest (or finest) topology possible (i.e., the most open sets possible, where every subset is open, and hence is the discrete topology) then the maps $\mathrm{pr}_0, \mathrm{pr}_1$ are certainly continuous, but the set $X \times Y$ (with this discrete topology) then becomes difficult to map into in Top. The other extreme is to give the set $X \times Y$ the smallest (or coarsest) topology such that $\mathrm{pr}_0, \mathrm{pr}_1$ are continuous—this is called the *topology induced on* the set $X \times Y$ by the functions $\mathrm{pr}_0, \mathrm{pr}_1$; for further reading, see [**1**, pp. 65–66], [**4**, p. 30], and [**8**, p. 4].

Proposition 1.1.7. *The topology induced on the set $X \times Y$ by the functions $X \xleftarrow{\mathrm{pr}_0} X \times Y \xrightarrow{\mathrm{pr}_1} Y$ is characterized by the property that if A is a topological space, a function $\overline{f}\colon A \to X \times Y$ is continuous if and only if the composites $\mathrm{pr}_0\,\overline{f}$, $\mathrm{pr}_1\,\overline{f}$ are continuous.*

Proof. Here is the basic idea. The smallest topology on the set $X \times Y$ such that pr_0 is continuous, called the topology induced by pr_0, is given by the collection of inverse images

$$\{\mathrm{pr}_0^{-1}(U) \,|\, U \subset X \text{ is open}\} \tag{1.5}$$

Similarly, the smallest topology on the set $X \times Y$ such that pr_1 is continuous, called the topology induced by pr_1, is given by the collection of inverse images

$$\{\mathrm{pr}_1^{-1}(V) \,|\, V \subset Y \text{ is open}\} \tag{1.6}$$

The topology induced by $\mathrm{pr}_0, \mathrm{pr}_1$ is the smallest topology on the set $X \times Y$ which contains each of these topologies: it is the topology generated by the union of these two collections of subsets—it consists of $\emptyset$, $X \times Y$, all finite intersections of the generating subsets, and all arbitrary unions of these finite intersections. In particular, a function of the form $\overline{f}$ is continuous if and only if inverse images of the generating subsets—also called subbasis elements—in (1.5) and (1.6) are open. $\square$

This means that if we give the set $X \times Y$ the topology induced by $\mathrm{pr}_0, \mathrm{pr}_1$, then the desired universal property will be satisfied—this is the *product topology* on the set $X \times Y$. This is the topology we equip the set $X \times Y$ with when we consider the diagram

$$X \xleftarrow{\mathrm{pr}_0} X \times Y \xrightarrow{\mathrm{pr}_1} Y \tag{1.7}$$

in Top, defined by $\mathrm{pr}_0(x, y) = x$ and $\mathrm{pr}_1(x, y) = y$ (i.e., pr_0 and pr_1 are the usual projection maps to X and Y, respectively). Notice how we have been naturally led to the product topology by considering desirable mapping properties. Hence we have verified that diagram (1.7) satisfies the following mapping property.

Proposition 1.1.8 (Universal property of products in Top). *If X, Y are topological spaces, then diagram* (1.7) *is terminal with respect to all such diagrams into X, Y in* Top*; i.e., for any topological space A and diagram of the form*

$$X \xleftarrow{f_0} A \xrightarrow{f_1} Y$$

in Top*, there exists a unique map $\overline{f}$ in* Top *which makes the diagram*

$$\begin{array}{ccccc} X & \xleftarrow{\mathrm{pr}_0} & X \times Y & \xrightarrow{\mathrm{pr}_1} & Y \\ & {\scriptstyle f_0}\nwarrow & \uparrow {\scriptstyle \exists!\ \overline{f}} & \nearrow{\scriptstyle f_1} & \\ & & A & & \end{array} \tag{1.8}$$

commute; i.e., such that $\mathrm{pr}_0 \overline{f} = f_0$ and $\mathrm{pr}_1 \overline{f} = f_1$.

Proof. This follows from Propositions 1.1.2 and 1.1.7. $\square$

Remark 1.1.9. The upshot is: giving a map $\overline{f} \colon A \to X \times Y$ in Top is the same as giving a pair of maps $X \xleftarrow{f_0} A \xrightarrow{f_1} Y$ in Top; for this reason, sometimes $\overline{f}$ is written as $\overline{f} = (f_0, f_1)$.

For instance, a left-hand diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle (p_0,p_1)} \\ B & \xrightarrow[(h_0,h_1)]{} & X \times Y \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p_0} \\ B & \xrightarrow[h_0]{} & X \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{g} & C \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p_1} \\ B & \xrightarrow[h_1]{} & Y \end{array}$$

in Top commutes if and only if the corresponding right-hand diagrams in Top commute. Similarly, a left-hand diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{(g_0,g_1)} & C \times D \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p_0 \times p_1} \\ B & \xrightarrow[(h_0,h_1)]{} & X \times Y \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{g_0} & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p_0} \\ B & \xrightarrow[h_0]{} & X \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{g_1} & D \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p_1} \\ B & \xrightarrow[h_1]{} & Y \end{array}$$

in Top commutes if and only if the corresponding right-hand diagrams in Top commute; here, $p_0 \times p_1$ denotes the map (of the indicated form) induced by p_0, p_1.

REMARK 1.1.10. In more detail, $p_0 \times p_1$ is the map induced

$$\begin{array}{ccccc} C & \xleftarrow{\mathrm{pr}_0} & C \times D & \xrightarrow{\mathrm{pr}_1} & D \\ {\scriptstyle p_0}\downarrow & & \exists!\ \vdots\ {\scriptstyle p_0 \times p_1} & & \downarrow{\scriptstyle p_1} \\ X & \xleftarrow{\mathrm{pr}_0} & X \times Y & \xrightarrow{\mathrm{pr}_1} & Y \end{array}$$

by the outer diagram; i.e., $p_0 \times p_1 = (p_0 \,\mathrm{pr}_0, p_1 \,\mathrm{pr}_1)$.

A left-hand diagram of the form

$$\begin{array}{ccc} & & Y \\ & \nearrow^{f_0} & \uparrow{\scriptstyle p_0} \\ X & \xrightarrow{H} & M \\ & \searrow_{f_1} & \downarrow{\scriptstyle p_1} \\ & & Y \end{array} \qquad\qquad X \xrightarrow{H} M \xrightarrow{(p_0,p_1)} Y \times Y, \quad X \xrightarrow{(f_0,f_1)} Y \times Y$$

in Top commutes if and only if the corresponding right-hand diagram in Top commutes; we will be going back and forth between such diagrams when we work with right homotopies below.

REMARK 1.1.11. Arguing as above (Remark 1.1.5), the mapping properties in (1.8) characterize the product $X \times Y$ in Top, up to isomorphism.

A *product diagram* in Top is a diagram of the form (1.7) that satisfies the universal property in (1.8). For instance, if X is a topological space, then the diagram

$$X = = = X \longrightarrow *$$

is a product diagram in Top and hence $X \cong X \times *$ in Top.

REMARK 1.1.12. A diagram that is naturally isomorphic to a product diagram in Top, is a product diagram in Top.

1.2. Coproducts

If we reverse all the arrows in a product diagram and its mapping properties, we are naturally led to the mapping properties of a coproduct diagram: let's develop this idea. Let X, Y be sets. Classically, the coproduct (or disjoint union) $X \amalg Y$ in Set is defined to be the union of disjoint copies of X and Y

$$X \amalg Y := X \times \{0\} \cup Y \times \{1\} \tag{1.9}$$

REMARK 1.2.1. Note that $X \cong X \times \{0\}$, $Y \cong Y \times \{1\}$ in Set and these isomorphisms are simply a formal way to ensure that the union on the right-hand side of (1.9) is a union of disjoint sets. For instance, $X \amalg X$ is supposed to be the disjoint union of two copies of X; i.e., if we start with a set X, then we can think of $X \times \{0\}$ and $X \times \{1\}$ as two disjoint copies of the set X, and hence their union $X \times \{0\} \cup X \times \{1\}$ is a union of two disjoint copies of X, which we write as $X \amalg X$.

There is a diagram of the form

$$X \xrightarrow{\mathrm{in}_0} X \amalg Y \xleftarrow{\mathrm{in}_1} Y \tag{1.10}$$

in Set, defined by $\mathrm{in}_0(x) = (x, 0)$ and $\mathrm{in}_1(y) = (y, 1)$ (i.e., in_0 and in_1 are the usual inclusion maps of X and Y, respectively).

REMARK 1.2.2. For notational convenience reasons, we usually identify X with its copy $X \times \{0\}$ and Y with its copy $Y \times \{1\}$; in this case, the maps in (1.10) are defined by $\mathrm{in}_0(x) = x$ and $\mathrm{in}_1(y) = y$. For instance, if we are working with $X \amalg X$ (where $Y = X$), then we simply have to keep track of which copy of X we are mapping into; but this is indicated by the subscript on the inclusion map itself. We do not want to think of the first copy of X as $X \times \{0\}$ and the second copy of X as $X \times \{1\}$ (this is too messy, notationally), but instead to simply think of having two disjoint copies of X. In other words, we prefer to keep the notation as simple as possible—this will not cause any confusion.

Following the notational convention in Remark 1.2.2, we rewrite diagram (1.10) as a diagram of the form

$$X \xrightarrow{\mathrm{in}_0} X \amalg Y \xleftarrow{\mathrm{in}_1} Y \tag{1.11}$$

in Set, defined by $\mathrm{in}_0(x) = x$ and $\mathrm{in}_1(y) = y$ (i.e., in_0 and in_1 are the usual inclusion maps of X and Y, respectively), which satisfies the following mapping property; for further reading, see [**5**, p. 63] and [**7**, p. 81].

PROPOSITION 1.2.3 (Universal property of coproducts in Set). *If X, Y are sets, then diagram (1.11) is initial with respect to all such diagrams out of X, Y in Set; i.e., for any set A and diagram of the form*

$$X \xrightarrow{f_0} A \xleftarrow{f_1} Y$$

in Set, there exists a unique map $\overline{f}$ in Set which makes the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\mathrm{in}_0} & X \amalg Y & \xleftarrow{\mathrm{in}_1} & Y \\ & {\scriptstyle f_0}\searrow & \vdots\, {\scriptstyle \exists!\ \overline{f}} & \swarrow{\scriptstyle f_1} & \\ & & A & & \end{array} \tag{1.12}$$

commute; i.e., such that $\overline{f}\,\mathrm{in}_0 = f_0$ and $\overline{f}\,\mathrm{in}_1 = f_1$.

PROOF. Uniqueness is forced on us, since the diagram commutes implies that $\overline{f}(x) = f_0(x)$ for each $x \in X$ and $\overline{f}(y) = f_1(y)$ for each $y \in Y$. Existence follows since this is a well-defined map in Set. $\square$

REMARK 1.2.4. Similar to Remark 1.1.3, the reader should not feel obliged to use the symbol $\overline{f}$ for the map induced by f_0, f_1 in (1.12); for instance, we could have used the symbol f, f', ξ, a_1, α, or $\overline{\alpha}$, in place of $\overline{f}$.

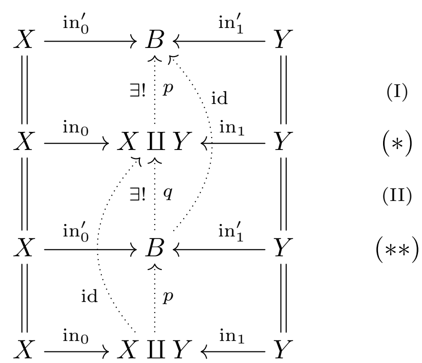
Remark 1.2.5. The upshot is: giving a map $\overline{f}\colon X \amalg Y \to A$ in Set is the same as giving a pair of maps $X \xrightarrow{f_0} A \xleftarrow{f_1} Y$ in Set. For this reason, sometimes $\overline{f}$ is written as $\overline{f} = (f_0, f_1)$.

Remark 1.2.6. The mapping properties in (1.12) characterize the coproduct $X \amalg Y$ in Set, up to isomorphism. Let's verify this. Suppose there is a set B together with a diagram of the form

$$X \xrightarrow{\mathrm{in}_0'} B \xleftarrow{\mathrm{in}_1'} Y \tag{1.13}$$

in Set, which satisfies the universal property in (1.12); i.e., such that diagram (1.13) is initial with respect to all such diagrams out of X, Y in Set. Let's verify that $B \cong X \amalg Y$ in Set: we want to compare the sets $X \amalg Y$ and B, and the only game in town is the universal property; so let's use it. Consider the solid diagram of the form

(1.14)



in Set. Diagram $(*)$ is initial with respect to all such diagrams out of X, Y in Set, hence there exists a unique map p in Set which makes diagram (I) commute. Diagram $(**)$ is initial with respect to all such diagrams out of X, Y in Set, hence there exists a unique map q in Set which makes diagram (II) commute. The identity map id makes diagram (I) + (II) commute; but pq also makes (I) + (II) commute. Hence, by uniqueness, $pq = \mathrm{id}$. Similarly, by uniqueness, $qp = \mathrm{id}$. Hence we have verified (using four applications of the universal property) that $B \cong X \amalg Y$ in Set. Conversely, if there is a set B together with an isomorphism $p\colon X \amalg Y \xrightarrow{\cong} B$ in Set, then it is easy to verify that the diagram of the form (1.13), with $\mathrm{in}_0' := p\,\mathrm{in}_0$ and $\mathrm{in}_1' := p\,\mathrm{in}_1$ satisfies the mapping properties in (1.12); i.e., it is initial with respect to all such diagrams out of X, Y in Set.

A *coproduct diagram* in Set is a diagram of the form (1.11) that satisfies the universal property in (1.12). For instance, if X is a set, then the diagram

$$X = X \longleftarrow \emptyset$$

is a coproduct diagram in Set and hence $X \cong X \amalg \emptyset$ in Set.

Remark 1.2.7. A diagram that is naturally isomorphic to a coproduct diagram in Set, is a coproduct diagram in Set.

What happens if we replace Set with Top in our above discussion? The reader already knows from their background how to work with coproducts (or disjoint unions) of topological spaces, but let's pretend we forgot; this is a good idea as the intuition we develop here, in this familiar situation, will carry over and guide us in more complicated ones.

Let X, Y be topological spaces. Classically, the coproduct (or disjoint union) $X \amalg Y$ in Top is defined to be $X \amalg Y$ in Set, equipped with an appropriate topology. Let's follow our noses to rediscover this topology, called the coproduct topology. The idea is: we want the same mapping properties in (1.12) to be true, but for Set replaced by Top. In particular, this means that we need a topology on the set $X \amalg Y$ such that the maps in (1.11) are continuous; i.e., inverse images of open subsets are open. There are two extremes. If we give the set $X \amalg Y$ the smallest topology possible (i.e., the fewest open sets possible, where only $\emptyset$ and $X \amalg Y$ are open, and hence is the trivial (or indiscrete) topology) then the maps $\mathrm{in}_0, \mathrm{in}_1$ are certainly continuous, but the set $X \amalg Y$ (with this trivial topology) then becomes difficult to map out of in Top. The other extreme is to give the set $X \amalg Y$ the largest topology such that $\mathrm{in}_0, \mathrm{in}_1$ are continuous—this is called the *topology coinduced on* the set $X \amalg Y$ by the functions $\mathrm{in}_0, \mathrm{in}_1$; for further reading, see [**1**, pp. 131–132], [**4**, p. 29], and [**8**, pp. 4–5].

Proposition 1.2.8. *The topology coinduced on the set $X \amalg Y$ by the functions $X \xrightarrow{\mathrm{in}_0} X \amalg Y \xleftarrow{\mathrm{in}_1} Y$ is characterized by the property that if A is a topological space, a function $\overline{f}\colon X \amalg Y \to A$ is continuous if and only if the composites $\overline{f}\,\mathrm{in}_0$, $\overline{f}\,\mathrm{in}_1$ are continuous.*

Proof. Here is the basic idea. The largest topology on the set $X \amalg Y$ such that in_0 is continuous, called the topology coinduced by in_0, is given by the collection

$$\{W \subset X \amalg Y \,|\, \mathrm{in}_0^{-1}(W) \subset X \text{ is open}\} \tag{1.15}$$

Similarly, the largest topology on the set $X \amalg Y$ such that in_1 is continuous, called the topology coinduced by in_1, is given by the collection

$$\{W \subset X \amalg Y \,|\, \mathrm{in}_1^{-1}(W) \subset Y \text{ is open}\} \tag{1.16}$$

The topology coinduced by $\mathrm{in}_0, \mathrm{in}_1$ is the largest topology on the set $X \amalg Y$ which is contained in each of these topologies: it is the intersection of these two collections of subsets. □

This means that if we give the set $X \amalg Y$ the topology coinduced by $\mathrm{in}_0, \mathrm{in}_1$, then the desired universal property will be satisfied—this is the *coproduct topology* on the set $X \amalg Y$. This is the topology that we equip the set $X \amalg Y$ with when we consider the diagram

$$X \xrightarrow{\mathrm{in}_0} X \amalg Y \xleftarrow{\mathrm{in}_1} Y \tag{1.17}$$

in Top, defined by $\mathrm{in}_0(x) = x$ and $\mathrm{in}_1(y) = y$ (i.e., in_0 and in_1 are the usual inclusion maps of X and Y, respectively). Notice how we have been naturally led to the coproduct topology by considering desirable mapping properties. Hence we have verified that diagram (1.17) satisfies the following mapping property.

Proposition 1.2.9 (Universal property of coproducts in Top). *If X, Y are topological spaces, then diagram* (1.17) *is initial with respect to all such diagrams out of X, Y in Top; i.e., for any topological space A and diagram of the form*

$$X \xrightarrow{f_0} A \xleftarrow{f_1} Y$$

in Top*, there exists a unique map* $\overline{f}$ *in* Top *which makes the diagram*

(1.18)
$$\begin{array}{ccccc} X & \xrightarrow{\mathrm{in}_0} & X \amalg Y & \xleftarrow{\mathrm{in}_1} & Y \\ & {\scriptstyle f_0}\searrow & \Big\downarrow {\scriptstyle \exists!\ \overline{f}} & \swarrow{\scriptstyle f_1} & \\ & & A & & \end{array}$$

commute; i.e., such that $\overline{f}\,\mathrm{in}_0 = f_0$ *and* $\overline{f}\,\mathrm{in}_1 = f_1$*.*

Proof. This follows from Propositions 1.2.3 and 1.2.8. □

Remark 1.2.10. The upshot is: giving a map $\overline{f}\colon X \amalg Y \to A$ in Top is the same as giving a pair of maps $X \xrightarrow{f_0} A \xleftarrow{f_1} Y$ in Top. For this reason, sometimes $\overline{f}$ is written as $\overline{f} = (f_0, f_1)$.

For instance, a left-hand diagram of the form

$$\begin{array}{ccc} X \amalg Y & \xrightarrow{(g_0,g_1)} & C \\ {\scriptstyle (i_0,i_1)}\Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ Z & \xrightarrow[h]{} & D \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{g_0} & C \\ {\scriptstyle i_0}\Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ Z & \xrightarrow[h]{} & D \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{g_1} & C \\ {\scriptstyle i_1}\Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ Z & \xrightarrow[h]{} & D \end{array}$$

in Top commutes if and only if the corresponding right-hand diagrams in Top commute. Similarly, a left-hand diagram of the form

$$\begin{array}{ccc} X \amalg Y & \xrightarrow{(g_0,g_1)} & C \\ {\scriptstyle i_0 \amalg i_1}\Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ W \amalg Z & \xrightarrow[(h_0,h_1)]{} & D \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{g_0} & C \\ {\scriptstyle i_0}\Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ W & \xrightarrow[h_0]{} & D \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{g_1} & C \\ {\scriptstyle i_1}\Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ Z & \xrightarrow[h_1]{} & D \end{array}$$

in Top commutes if and only if the corresponding right-hand diagrams in Top commute; here, $i_0 \amalg i_1$ denotes the map (of the indicated form) induced by i_0, i_1.

Remark 1.2.11. In more detail, $i_0 \amalg i_1$ is the map induced

$$\begin{array}{ccccc} X & \xrightarrow{\mathrm{in}_0} & X \amalg Y & \xleftarrow{\mathrm{in}_1} & Y \\ {\scriptstyle i_0}\Big\downarrow & & \Big\downarrow{\scriptstyle \exists!\ i_0 \amalg i_1} & & \Big\downarrow{\scriptstyle i_1} \\ W & \xrightarrow{\mathrm{in}_0} & W \amalg Z & \xleftarrow{\mathrm{in}_1} & Z \end{array}$$

by the outer diagram; i.e., $i_0 \amalg i_1 = (\mathrm{in}_0\, i_0, \mathrm{in}_1\, i_1)$.

A left-hand diagram of the form

$$\begin{array}{ccc} X & & \\ {\scriptstyle i_0}\Big\downarrow & \searrow{\scriptstyle f_0} & \\ C & \xrightarrow{H} & Y \\ {\scriptstyle i_1}\Big\uparrow & \nearrow{\scriptstyle f_1} & \\ X & & \end{array} \qquad X \amalg X \xrightarrow{(i_0,i_1)} C \xrightarrow{H} Y, \quad (f_0,f_1)\colon X \amalg X \to Y$$

in Top commutes if and only if the corresponding right-hand diagram in Top commutes; we will be going back and forth between such diagrams when we work with left homotopies below.

Remark 1.2.12. Arguing as above (Remark 1.2.6), the mapping properties in (1.18) characterize the coproduct $X \amalg Y$ in Top, up to isomorphism.

A *coproduct diagram* in Top is a diagram of the form (1.17) that satisfies the universal property in (1.18). For instance, if X is a topological space, then the diagram

$$X = X \longleftarrow \emptyset$$

is a coproduct diagram in Top and hence $X \cong X \amalg \emptyset$ in Top.

Remark 1.2.13. A diagram that is naturally isomorphic to a coproduct diagram in Top, is a coproduct diagram in Top.

1.3. Subsets and subspaces

Let Y be a set and $X \subset Y$ a subset. There is a map of the form $i\colon X \to Y$ in Set, defined by $i(x) = x$ (i.e., i is the usual inclusion map to Y), which satisfies the following mapping property.

Proposition 1.3.1 (Universal property of subsets in Set). *If Y is a set and $X \subset Y$ a subset, then the map i is terminal with respect to all maps into Y with image contained in X in Set; i.e., for any set A and map of the form $\alpha\colon A \to Y$ in Set with $\alpha(A) \subset X$, there exists a unique map $\overline{\alpha}$ in Set which makes the diagram*

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ {\scriptstyle \exists!}\uparrow{\scriptstyle \overline{\alpha}} & \nearrow_{\alpha} & \\ A & & \end{array} \tag{1.19}$$

commute; i.e., such that $i\,\overline{\alpha} = \alpha$.

Proof. Uniqueness is forced on us, since the diagram commutes implies that $\overline{\alpha}(a) = \alpha(a)$ for each $a \in A$. Existence follows since this is a well-defined map in Set. ☐

Remark 1.3.2. The upshot is: giving a map $\overline{\alpha}\colon A \to X$ in Set is the same as giving a map $\alpha\colon A \to Y$ in Set with $\alpha(A) \subset X$.

Let's turn this into a notion that makes sense up to isomorphism. Let $i\colon X \to Y$ be an injective map in Set. Then i can be written as the composite $X \cong i(X) \subset Y$ in Set, and hence satisfies the following universal property.

Proposition 1.3.3 (Universal property of injections in Set). *If $i\colon X \to Y$ is an injective map in Set, then i is terminal with respect to all maps into Y with image contained in $i(X)$ in Set; i.e., for any set A and map of the form $\alpha\colon A \to Y$ in Set with $\alpha(A) \subset i(X)$, there exists a unique map $\overline{\alpha}$ in Set which makes the diagram*

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ {\scriptstyle \exists!}\uparrow{\scriptstyle \overline{\alpha}} & \nearrow_{\alpha} & \\ A & & \end{array} \tag{1.20}$$

commute; i.e., such that $i\,\overline{\alpha} = \alpha$.

REMARK 1.3.4. The upshot is: giving a map $\overline{\alpha}\colon A \to X$ in Set is the same as giving a map $\alpha\colon A \to Y$ in Set with $\alpha(A) \subset i(X)$.

What happens if we replace Set with Top in our above discussion? The reader already knows from their background how to work with subspaces, but let's pretend we forgot. This will help the reader develop their intuition.

Let Y be a topological space and $X \subset Y$ a subset. Classically, the set X is called a subspace of Y when X is equipped with an appropriate topology. Let's follow our noses to rediscover this topology, called the subspace topology on X with respect to Y. The idea is: we want the same mapping properties in (1.19) to be true, but for Set replaced by Top. In particular, this means that we need a topology on the set X such that the inclusion function i is continuous; i.e., inverse images of open subsets are open. There are two extremes. If we give the set X the largest topology possible (i.e., the most open sets possible, where every subset is open, and hence is the discrete topology) then the map i is certainly continuous, but the set X (with this discrete topology) then becomes difficult to map into in Top. The other extreme is to give the set X the smallest topology such that i is continuous—this is called the *topology induced on* the set X by the function i; for further reading, see [**3**, p. 49], [**4**, p. 30], [**6**, p. 88], and [**8**, p. 4].

PROPOSITION 1.3.5. *The topology induced on the set X by the inclusion function $i\colon X \to Y$ is characterized by the property that if A is a topological space, a function $\overline{\alpha}\colon A \to X$ is continuous if and only if the composite $i\,\overline{\alpha}$ is continuous.*

PROOF. Here is the basic idea. The smallest topology on the set X such that i is continuous, called the topology induced by i, is given by the collection

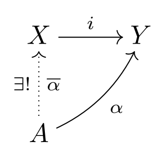
$$\{i^{-1}(U) \,|\, U \subset Y \text{ is open}\} \tag{1.21}$$

of inverse images. □

This means that if we give the set X the topology induced by the inclusion function $i\colon X \to Y$, then the desired universal property will be satisfied—this is the *subspace topology* on the set X with respect to Y. Notice how we have been naturally led to the subspace topology by considering desirable mapping properties. This is the topology that we equip the set X with when we regard $X \subset Y$ as a subspace of Y; in this case, we call the inclusion map $i\colon X \to Y$, defined by $i(x) = x$, a *subspace inclusion*. Hence we have verified that $i\colon X \to Y$ satisfies the following mapping property.

PROPOSITION 1.3.6 (Universal property of subspaces in Top). *If Y is a topological space and $X \subset Y$ is a subspace, then the subspace inclusion $i\colon X \to Y$ is terminal with respect to all maps into Y with image contained in X in Top; i.e., for any topological space A and map of the form $\alpha\colon A \to Y$ in Top with $\alpha(A) \subset X$, there exists a unique map $\overline{\alpha}$ in Top which makes the diagram*

(1.22)



commute; i.e., such that $i\,\overline{\alpha} = \alpha$.

REMARK 1.3.7. The upshot is: giving a map $\overline{\alpha}\colon A \to X$ in Top is the same as giving a map $\alpha\colon A \to Y$ in Top with $\alpha(A) \subset X$.

Let's turn this into a notion that makes sense up to isomorphism. Let Y be a topological space and $i\colon X \to Y$ an injective map in Set. The idea is: we want the same mapping properties in (1.20) to be true, but for Set replaced by Top. Reasoning as above, we can give the set X the smallest topology such that i is continuous—this is called the *topology induced on* the set X by the function i; for further reading, see [**3**, p. 49], [**4**, p. 30], and [**8**, p. 4].

PROPOSITION 1.3.8. *The topology induced on the set X by the injective map $i\colon X \to Y$ is characterized by the property that if A is a topological space, a function $\overline{\alpha}\colon A \to X$ is continuous if and only if the composite $i\,\overline{\alpha}$ is continuous.*

PROOF. Here is the basic idea. The smallest topology on the set X such that i is continuous, called the topology induced by i, is given by the collection

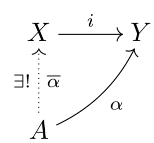
$$\{i^{-1}(U) \,|\, U \subset Y \text{ is open}\} \tag{1.23}$$

of inverse images. □

This means that if we give the set X the topology induced by the injective function $i\colon X \to Y$, then the desired universal property will be satisfied; in this case, we call the injection $i\colon X \to Y$ a *subspace injection* (or simply, a subspace inclusion, since that's what it is up to isomorphism). Notice how we have been naturally led to this topology by considering desirable mapping properties. Hence we have verified that $i\colon X \to Y$ satisfies the following mapping property.

PROPOSITION 1.3.9 (Universal property of subspace injections in Top). *If Y is a topological space and $i\colon X \to Y$ is a subspace injection, then i is terminal with respect to all maps into Y with image contained in $i(X)$ in Top; i.e., for any topological space A and map of the form $\alpha\colon A \to Y$ in Top with $\alpha(A) \subset i(X)$, there exists a unique map $\overline{\alpha}$ in Top which makes the diagram*

(1.24)



commute; i.e., such that $i\,\overline{\alpha} = \alpha$.

REMARK 1.3.10. The upshot is: giving a map $\overline{\alpha}\colon A \to X$ in Top is the same as giving a map $\alpha\colon A \to Y$ in Top with $\alpha(A) \subset i(X)$.

REMARK 1.3.11. It's worth pointing out that i in (1.24) can be written as the composite $X \cong i(X) \subset Y$ in Top, where $i(X)$ has the subspace topology with respect to Y.

1.4. Quotient sets and quotient spaces

If we reverse all the arrows in a subset inclusion (resp. subspace inclusion) and its mapping properties, we are naturally led to the mapping properties of a quotient set projection (resp. quotient space projection): let's develop this idea. Let X be a set and $\sim$ an equivalence relation on X. Recall that $\sim$ partitions the set X into equivalence classes and denote by $X/{\sim}$ the corresponding set of equivalence classes, called the quotient set of X over the equivalence relation $\sim$. If $x \in X$, denote by $[x] \in X/{\sim}$ the equivalence class containing x; in particular, $[x] \subset X$ and $x' \in [x]$ if and only if $x \sim x'$. There is a map of the form $p\colon X \to X/{\sim}$ in Set, defined by $p(x) = [x]$ (i.e., p is the usual projection function of X), which satisfies the following mapping property.

Proposition 1.4.1 (Universal property of quotient sets in Set). *If X is a set and $\sim$ is an equivalence relation on X, then the map p is initial with respect to all maps out of X that identify equivalent elements of X in* Set*; i.e., for any set A and map of the form $\alpha\colon X \to A$ in* Set *satisfying $\alpha(x) = \alpha(x')$ if $x \sim x'$, there exists a unique map $\overline{\alpha}$ in* Set *which makes the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & A \\ {\scriptstyle p}\downarrow & \exists! & \nearrow{\scriptstyle \overline{\alpha}} \\ X/{\sim} & & \end{array} \tag{1.25}$$

commute; i.e., such that $\overline{\alpha}\, p = \alpha$.

Proof. Uniqueness is forced on us, since the diagram commutes implies that $\overline{\alpha}([x]) = \alpha(x)$ for each $x \in X$. Existence follows since this is a well-defined map in Set. □

Remark 1.4.2. The upshot is: giving a map $\overline{\alpha}\colon X/{\sim} \to A$ in Set is the same as giving a map $\alpha\colon X \to A$ in Set satisfying $\alpha(x) = \alpha(x')$ if $x \sim x'$.

Let's turn this into a notion that makes sense up to isomorphism. Let $p\colon X \to Y$ be a surjective map in Set. Denote by $\sim$ the equivalence relation on X associated to p, defined by $x \sim x'$ if and only if $p(x) = p(x')$. Then p can be written as the composite $X \to X/{\sim} \cong Y$ in Set, and hence satisfies the following universal property.

Proposition 1.4.3 (Universal property of surjections in Set). *If $p\colon X \to Y$ is a surjective map in* Set *and $\sim$ denotes the equivalence relation on X associated to p, then the map p is initial with respect to all maps out of X that identify equivalent elements of X in* Set*; i.e., for any set A and map of the form $\alpha\colon X \to A$ in* Set *satisfying $\alpha(x) = \alpha(x')$ if $x \sim x'$, there exists a unique map $\overline{\alpha}$ in* Set *which makes the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & A \\ {\scriptstyle p}\downarrow & \exists! & \nearrow{\scriptstyle \overline{\alpha}} \\ Y & & \end{array} \tag{1.26}$$

commute; i.e., such that $\overline{\alpha}\, p = \alpha$.

Remark 1.4.4. The upshot is: giving a map $\overline{\alpha}\colon Y \to A$ in Set is the same as giving a map $\alpha\colon X \to A$ in Set satisfying $\alpha(x) = \alpha(x')$ if $x \sim x'$.

What happens if we replace Set with Top in our above discussion? The reader already knows from their background how to work with quotient spaces, but let's pretend we forgot. This will help the reader develop their own intuition.

Let X be a topological space and $\sim$ an equivalence relation on X. Classically, the quotient space $X/\sim$ in Top is defined to be the quotient set $X/\sim$ in Set, equipped with an appropriate topology. Let's follow our noses to rediscover this topology, called the quotient space topology (or identification topology). The idea is: we want the same mapping properties in (1.25) to be true, but for Set replaced by Top. In particular, this means that we need a topology on the set $X/{\sim}$ such that the projection function p is continuous; i.e., inverse images of open subsets are open. There are two extremes. If we give the set $X/{\sim}$ the smallest topology possible (i.e., the fewest open sets possible, where only $\emptyset$ and $X/{\sim}$ are open, and hence is the trivial (or indiscrete) topology) then the map p is certainly continuous, but the set $X/{\sim}$ (with this trivial topology) then becomes difficult to map out of in Top. The other extreme is to give the set $X/{\sim}$ the largest topology such that p is continuous—this is called the *topology coinduced on* the set $X/{\sim}$ by the function p; for further reading, see [**4**, p. 29], [**6**, p. 138] and [**8**, pp. 4–5].

PROPOSITION 1.4.5. *The topology coinduced on the set $X/{\sim}$ by the projection function $p\colon X \to X/{\sim}$ is characterized by the property that if A is a topological space, a function $\overline{\alpha}\colon X/{\sim} \to A$ is continuous if and only if the composite $\overline{\alpha}\, p$ is continuous.*

PROOF. Here is the basic idea. The largest topology on the set $X/{\sim}$ such that p is continuous, called the topology coinduced by p, is given by the collection

$$\{W \subset X/{\sim} \mid p^{-1}(W) \subset X \text{ is open}\} \tag{1.27}$$

of inverse images. □

This means that if we give the set $X/{\sim}$ the topology coinduced by the projection function $p\colon X \to X/{\sim}$, then the desired universal property will be satisfied—this is the *quotient space topology* (or identification topology) on the set $X/{\sim}$. Notice how we have been naturally led to the quotient space topology by considering desirable mapping properties. This is the topology that we equip the set $X/{\sim}$ with when we regard $X/{\sim}$ as a quotient space of X; in this case, we call the projection map $p\colon X \to X/{\sim}$, defined by $p(x) = [x]$, a *quotient space projection* (or identification projection). Hence we have verified that $p\colon X \to X/{\sim}$ satisfies the following mapping property.

PROPOSITION 1.4.6 (Universal property of quotient spaces in Top). *If X is a topological space and $\sim$ is an equivalence relation on X, then the quotient space projection $p\colon X \to X/{\sim}$ is initial with respect to all maps out of X that identify equivalent elements of X in Top; i.e., for any topological space A and map of the form $\alpha\colon X \to A$ in Top satisfying $\alpha(x) = \alpha(x')$ if $x \sim x'$, there exists a unique map $\overline{\alpha}$ in Top which makes the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & A \\ {\scriptstyle p}\downarrow & \exists! & \nearrow {\scriptstyle \overline{\alpha}} \\ X/{\sim} & & \end{array} \tag{1.28}$$

commute; i.e., such that $\overline{\alpha}\, p = \alpha$.

REMARK 1.4.7. The upshot is: giving a map $\overline{\alpha}\colon X/{\sim} \to A$ in Top is the same as giving a map $\alpha\colon X \to A$ in Top satisfying $\alpha(x) = \alpha(x')$ if $x \sim x'$.

Let's turn this into a notion that makes sense up to isomorphism. Let X be a topological space and $p\colon X \to Y$ a surjective map in Set. The idea is: we want the same mapping properties in (1.26) to be true, but for Set replaced by Top. Reasoning as above, we can give the set Y the largest topology such that p is continuous—this is called the *topology coinduced on* the set Y by the function p; for further reading, see [**4**, p. 29] and [**8**, pp. 4–5].

Proposition 1.4.8. *The topology coinduced on the set Y by the surjective function $p\colon X \to Y$ is characterized by the property that if A is a topological space, a function $\overline{\alpha}\colon Y \to A$ is continuous if and only if the composite $\overline{\alpha}\,p$ is continuous.*

Proof. Here is the basic idea. The largest topology on the set Y such that p is continuous, called the topology coinduced by p, is given by the collection

$$\{W \subset Y \,|\, p^{-1}(W) \subset X \text{ is open}\} \tag{1.29}$$

of inverse images. □

This means that if we give the set Y the topology coinduced by the surjective function $p\colon X \to Y$, then the desired universal property will be satisfied; in this case, we call the surjection $p\colon X \to Y$ a *quotient space surjection* (or identification surjection, or simply, a quotient space projection or identification projection, since that's what it is up to isomorphism). Notice how we have been naturally led to this topology by considering desirable mapping properties. Hence we have verified that $p\colon X \to Y$ satisfies the following mapping property.

Proposition 1.4.9 (Universal property of quotient space surjections in Top). *If $p\colon X \to Y$ is a quotient space surjection in* Top *and $\sim$ denotes the equivalence relation on X associated to p, then the map p is initial with respect to all maps out of X that identify equivalent elements of X in* Top*; i.e., for any topological space A and map of the form $\alpha\colon X \to A$ in* Top *satisfying $\alpha(x) = \alpha(x')$ if $x \sim x'$, there exists a unique map $\overline{\alpha}$ in* Top *which makes the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & A \\ {\scriptstyle p}\downarrow & \exists! & \nearrow{\scriptstyle \overline{\alpha}} \\ Y & & \end{array} \tag{1.30}$$

commute; i.e., such that $\overline{\alpha}\,p = \alpha$.

Remark 1.4.10. The upshot is: giving a map $\overline{\alpha}\colon Y \to A$ in Top is the same as giving a map $\alpha\colon X \to A$ in Top satisfying $\alpha(x) = \alpha(x')$ if $x \sim x'$.

Remark 1.4.11. It's worth pointing out that p in (1.30) can be written as the composite $X \to X/{\sim} \cong Y$ in Top, where $X/{\sim}$ has the quotient space topology.

1.5. Equalizers

Consider any pair of maps in Set of the form

$$Y \underset{f'}{\overset{f}{\rightrightarrows}} Z \tag{1.31}$$

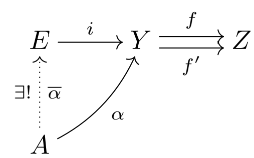
Classically, the equalizer of the pair of maps (1.31) in Set is defined to be the subset $E \subset Y$ given by $E := \{y \in Y \,|\, f(y) = f'(y)\}$. There is a map i of the form

$$E \xrightarrow{i} Y \underset{f'}{\overset{f}{\rightrightarrows}} Z \qquad fi = f'i \tag{1.32}$$

in Set, defined by $i(y) = y$ (i.e., i is the usual inclusion map to Y), which satisfies the following universal property; for further reading, see [**5**, p. 70] and [**7**, p. 78].

Proposition 1.5.1 (Universal property of equalizers in Set). *If f, f' is a pair of maps in* Set *of the form* (1.31)*, then the map i in* (1.32) *is terminal with respect to all maps into Y that equalize f, f' in* Set*; i.e., for any set A and map of the form $\alpha\colon A \to Y$ in* Set *with $f\alpha = f'\alpha$, there exists a unique map $\overline{\alpha}$ in* Set *which makes the diagram*

(1.33)



$$fi = f'i \qquad f\alpha = f'\alpha$$

commute; i.e., such that $i\,\overline{\alpha} = \alpha$.

Proof. Uniqueness is forced on us, since the diagram commutes implies that $\overline{\alpha}(a) = \alpha(a)$ for each $a \in A$. Existence follows since this is a well-defined map in Set. $\square$

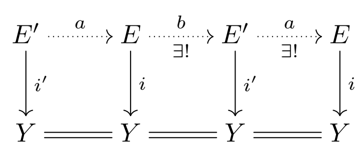
Remark 1.5.2. The upshot is: giving a map $\overline{\alpha}\colon A \to E$ in Set is the same as giving a map $\alpha\colon A \to Y$ in Set with $f\alpha = f'\alpha$.

Remark 1.5.3. The mapping properties in (1.33) characterize the equalizer E of f, f' in Set, up to isomorphism. Let's verify this. Suppose there is a set E' together with a map i' of the form

$$E' \xrightarrow{i'} Y \underset{f'}{\overset{f}{\rightrightarrows}} Z \qquad fi' = f'i' \tag{1.34}$$

in Set, which satisfies the universal property in (1.33); i.e., such that the map i' is terminal with respect to all maps into Y that equalize f, f' in Set. Let's verify that $E' \cong E$ in Set: we want to compare the sets E and E', and the only game in town is the universal property; so let's use it. Consider the solid diagram of the form

(1.35)



in Set. By the universal property of the map i, there exists a unique map a in Set which makes the right-hand square commute (i.e., such that $ia = i'$). By the universal property of the map i', there exists a unique map b in Set which makes the middle square commute (i.e., such that $i'b = i$). The identity map id on E satisfies $i\,\mathrm{id} = i$; but ab also satisfies $iab = i$. Hence, by uniqueness, $ab = \mathrm{id}$. Similarly, by uniqueness, $ba = \mathrm{id}$. Hence we have verified (using four applications of the universal property) that $E' \cong E$ in Set. Conversely, if there is a set E' together with an isomorphism $a\colon E' \xrightarrow{\cong} E$ in Set, then it is easy to verify that the map $i' := ia$ satisfies the mapping properties in (1.33); i.e., it is terminal with respect to all maps into Y that equalize f, f' in Set.

An *equalizer diagram* in Set is a diagram of the form (1.32) that satisfies the universal property in (1.33). For instance, if Y is a set, then the diagram

$$Y = Y \rightrightarrows *$$

is an equalizer diagram in Set.

REMARK 1.5.4. A diagram that is naturally isomorphic to an equalizer diagram in Set, is an equalizer diagram in Set.

What happens if we replace Set with Top in our above discussion? The basic idea is to calculate the equalizer E in Set, and then equip it with an appropriate topology. We have already worked out what we need—the subspace topology. Consider any pair of maps in Top of the form

$$Y \underset{f'}{\overset{f}{\rightrightarrows}} Z \tag{1.36}$$

Classically, the equalizer of the pair of maps (1.36) in Top is defined to be the subspace $E \subset Y$ given by $E := \{y \in Y \mid f(y) = f'(y)\}$. There is a map i of the form

$$E \xrightarrow{i} Y \underset{f'}{\overset{f}{\rightrightarrows}} Z \qquad fi = f'i \tag{1.37}$$

in Top, defined by $i(y) = y$ (i.e., i is the usual inclusion map to Y), which satisfies the following universal property.

PROPOSITION 1.5.5 (Universal property of equalizers in Top). *If f, f' is a pair of maps in Top of the form (1.36), then the map i in (1.37) is terminal with respect to all maps into Y that equalize f, f' in Top; i.e., for any topological space A and map of the form $\alpha\colon A \to Y$ in Top with $f\alpha = f'\alpha$, there exists a unique map $\overline{\alpha}$ in Top which makes the diagram*

$$\begin{array}{ccccc} E & \xrightarrow{i} & Y & \underset{f'}{\overset{f}{\rightrightarrows}} & Z \qquad fi = f'i \\ {\scriptstyle \exists!\ \overline{\alpha}}\uparrow & \nearrow_{\alpha} & & & \\ A & & & & \qquad f\alpha = f'\alpha \end{array} \tag{1.38}$$

commute; i.e., such that $i\,\overline{\alpha} = \alpha$.

PROOF. This follows from easily from above. $\square$

REMARK 1.5.6. The upshot is: giving a map $\overline{\alpha}\colon A \to E$ in Top is the same as giving a map $\alpha\colon A \to Y$ in Top with $f\alpha = f'\alpha$.

REMARK 1.5.7. Arguing as above (Remark 1.5.3), the mapping properties in (1.38) characterize the equalizer of f, f' in Top, up to isomorphism.

An *equalizer diagram* in Top is a diagram of the form (1.37) that satisfies the universal property in (1.38). For instance, if Y is a topological space, then the diagram $Y = Y \rightrightarrows *$ is an equalizer diagram in Top.

REMARK 1.5.8. A diagram that is naturally isomorphic to an equalizer diagram in Top, is an equalizer diagram in Top.

1.6. Coequalizers

If we reverse all the arrows in an equalizer diagram and its mapping properties, we are naturally led to the mapping properties of a coequalizer diagram: let's develop this idea. Consider any pair of maps in Set of the form

$$W \underset{g'}{\overset{g}{\rightrightarrows}} X \tag{1.39}$$

Classically, the coequalizer of the pair of maps (1.39) in Set is defined to be the quotient set $Q := X/\!\sim$, where $\sim$ is the equivalence relation on X generated by $g(w) \sim g'(w)$ for each $w \in W$. There is a map p of the form

$$W \underset{g'}{\overset{g}{\rightrightarrows}} X \xrightarrow{p} Q \qquad pg = pg' \tag{1.40}$$

in Set, defined by $p(x) = [x]$ (i.e., p is the usual projection map on X), which satisfies the following universal property; for further reading, see [**5**, p. 64] and [**7**, p. 81].

Proposition 1.6.1 (Universal property of coequalizers in Set). *If g, g' is a pair of maps in Set of the form* (1.39)*, then the map p in* (1.40) *is initial with respect to all maps out of X that coequalize g, g' in Set; i.e., for any set A and map of the form $\alpha\colon X \to A$ in Set satisfying $\alpha g = \alpha g'$, there exists a unique map $\overline{\alpha}$ in Set which makes the diagram*

$$\begin{array}{ccccc} W \underset{g'}{\overset{g}{\rightrightarrows}} & X & \xrightarrow{p} & Q & \qquad pg = pg' \\ & & \searrow^{\alpha} & \vdots\ \exists!\ \overline{\alpha} & \\ & & & A & \qquad \alpha g = \alpha g' \end{array} \tag{1.41}$$

commute; i.e., such that $\overline{\alpha}\, p = \alpha$.

Proof. Uniqueness is forced on us, since the diagram commutes implies that $\overline{\alpha}([x]) = \alpha(x)$ for each $x \in X$. Existence follows since this is a well-defined map in Set. □

Remark 1.6.2. The upshot is: giving a map $\overline{\alpha}\colon Q \to A$ in Set is the same as giving a map $\alpha\colon X \to A$ in Set with $\alpha g = \alpha g'$.

Remark 1.6.3. The mapping properties in (1.41) characterize the coequalizer of g, g' in Set, up to isomorphism. Let's verify this. Suppose there is a set Q' together with a map p' of the form

$$W \underset{g'}{\overset{g}{\rightrightarrows}} X \xrightarrow{p'} Q' \qquad p'g = p'g' \tag{1.42}$$

in Set, which satisfies the universal property in (1.41); i.e., such that the map p' is initial with respect to all maps out of X that coequalize g, g' in Set. Let's verify that $Q' \cong Q$ in Set: we want to compare the sets Q and Q', and the only game in town is the universal property; so let's use it. Consider the solid diagram of the form

$$(1.43)\qquad \begin{array}{ccccccc} X & = & X & = & X & = & X \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ Q & \xrightarrow[a]{} & Q' & \xrightarrow[b]{\exists!} & Q & \xrightarrow[a]{\exists!} & Q' \end{array}$$

in Set. By the universal property of the map p, there exists a unique map a in Set which makes the right-hand square commute (i.e., such that $ap = p'$). By the universal property of the map p', there exists a unique map b in Set which makes the middle square commute (i.e., such that $bp' = p$). The identity map id on Q' satisfies $\mathrm{id}\, p' = p'$; but ab also satisfies $abp' = p'$. Hence, by uniqueness, $ab = \mathrm{id}$. Similarly, by uniqueness, $ba = \mathrm{id}$. Hence we have verified (using four applications of the universal property) that $Q \cong Q'$ in Set. Conversely, if there is a set Q' together with an isomorphism $a\colon Q \xrightarrow{\cong} Q'$ in Set, then it is easy to verify that the map $p' := ap$ satisfies the mapping properties in (1.41); i.e., it is initial with respect to all maps out of X that coequalize g, g' in Set.

A *coequalizer diagram* in Set is a diagram of the form (1.40) that satisfies the universal property in (1.41). For instance, if X is a set, then the diagram

$$\emptyset \rightrightarrows X = X$$

is a coequalizer diagram in Set.

REMARK 1.6.4. A diagram that is naturally isomorphic to a coequalizer diagram in Set, is a coequalizer diagram in Set.

What happens if we replace Set with Top in our above discussion? The basic idea is to calculate the coequalizer Q in Set, and then equip it with an appropriate topology. We have already worked out what we need—the quotient space topology. Consider any pair of maps in Top of the form

$$(1.44)\qquad W \underset{g'}{\overset{g}{\rightrightarrows}} X$$

Classically, the coequalizer of the pair of maps (1.44) in Top is defined to be the quotient space $Q := X/{\sim}$, where $\sim$ is the equivalence relation on X generated by $g(w) \sim g'(w)$ for each $w \in W$. There is a map p of the form

$$(1.45)\qquad W \underset{g'}{\overset{g}{\rightrightarrows}} X \xrightarrow{p} Q \qquad pg = pg'$$

in Top, defined by $p(x) = [x]$ (i.e., p is the usual projection map on X), which satisfies the following universal property.

PROPOSITION 1.6.5 (Universal property of coequalizers in Top). *If g, g' is a pair of maps in* Top *of the form* (1.44)*, then the map p in* (1.45) *is initial with respect to all maps out of X that coequalize g, g' in* Top*; i.e., for any topological space A and map of the form $\alpha\colon X \to A$ in* Top *satisfying $\alpha g = \alpha g'$, there exists a* *unique map $\overline{\alpha}$ in* Top *which makes the diagram*

(1.46)
$$\begin{array}{ccccc} W & \overset{g}{\underset{g'}{\rightrightarrows}} & X & \xrightarrow{p} & Q \\ & & & \underset{\alpha}{\searrow} & \Big\downarrow{\scriptstyle \exists!\, \overline{\alpha}} \\ & & & & A \end{array} \qquad \begin{array}{c} pg = pg' \\ \\ \alpha g = \alpha g' \end{array}$$

commute; i.e., such that $\overline{\alpha}\, p = \alpha$.

Proof. This follows from easily from above. □

Remark 1.6.6. The upshot is: giving a map $\overline{\alpha}\colon Q \to A$ in Top is the same as giving a map $\alpha\colon X \to A$ in Top with $\alpha g = \alpha g'$.

Remark 1.6.7. Arguing as above (Remark 1.6.3), the mapping properties in (1.46) characterize the coequalizer of g, g' in Top, up to isomorphism.

A *coequalizer diagram* in Top is a diagram of the form (1.45) that satisfies the universal property in (1.46). For instance, if X is a topological space, then the diagram

$$\emptyset \rightrightarrows X = X$$

is a coequalizer diagram in Top.

Remark 1.6.8. A diagram that is naturally isomorphic to a coequalizer diagram in Top, is a coequalizer diagram in Top.

1.7. Pullbacks

Consider any diagram in Set of the form

(1.47)
$$\begin{array}{ccc} & & Y \\ & & \Big\downarrow{\scriptstyle f_1} \\ X & \xrightarrow[f_0]{} & Z \end{array}$$

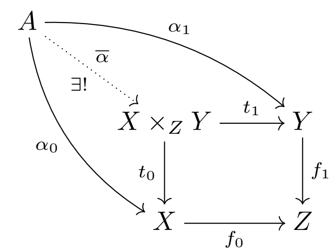
Classically, the pullback of the pair of maps (1.47) in Set is defined to be the subset of $X \times Y$ given by $X \times_Z Y := \{(x, y) \in X \times Y \,|\, f_0(x) = f_1(y)\}$. There is a diagram of the left-hand form

(1.48)
$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{t_1} & Y \\ {\scriptstyle t_0}\Big\downarrow & & \\ X & & \end{array} \qquad \begin{array}{ccc} X \times_Z Y & \xrightarrow{t_1} & Y \\ {\scriptstyle t_0}\Big\downarrow & & \Big\downarrow{\scriptstyle f_1} \\ X & \xrightarrow[f_0]{} & Z \end{array}$$

in Set, defined by $t_0(x, y) = x$ and $t_1(x, y) = y$, which makes the right-hand diagram commute (i.e., such that $f_0 t_0 = f_1 t_1$) and satisfies the following universal property; for further reading, see [**5**, p. 71] and [**7**, pp. 78–80].

Proposition 1.7.1 (Universal property of pullbacks in Set). *If f_0, f_1 is a pair of maps in* Set *of the form* (1.47)*, then the left-hand diagram in* (1.48) *is terminal* *with respect to all such diagrams in* Set*; i.e., for any set A and maps α_0, α_1 in* Set *which make the outer diagram*

(1.49)



commute (i.e., such that $f_0\alpha_0 = f_1\alpha_1$), there exists a unique map $\overline{\alpha}$ in Set *which makes the diagram commute; i.e, such that $t_0\,\overline{\alpha} = \alpha_0$ and $t_1\,\overline{\alpha} = \alpha_1$.*

Proof. Uniqueness is forced on us, since the diagram commutes implies that $\overline{\alpha}(a) = (\alpha_0(a), \alpha_1(a))$ for each $a \in A$. Existence follows since this is a well-defined map in Set. □

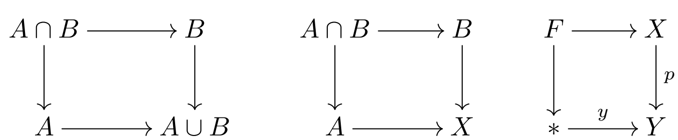
Remark 1.7.2. The upshot is: giving a map $\overline{\alpha}\colon\ A \to X \times_Z Y$ in Set is the same as giving maps α_0, α_1 in Set which make the outer diagram in (1.49) commute. For this reason, sometimes $\overline{\alpha}$ is written as $\overline{\alpha} = (\alpha_0, \alpha_1)$.

Remark 1.7.3. Arguing as above (e.g., Remark 1.5.3), the mapping properties in (1.49) characterize the pullback of f_0, f_1 in Set, up to isomorphism.

A *pullback diagram* (or cartesian diagram) in Set is a commutative diagram of the right-hand form in (1.48) that satisfies the universal property in (1.49). In this case, t_0 (resp. t_1) is called a *pullback* of f_1 along f_0 (resp. f_0 along f_1).

Remark 1.7.4. A diagram that is naturally isomorphic to a pullback diagram in Set, is a pullback diagram in Set. The pullback of an isomorphism in Set, is an isomorphism in Set; i.e., if f_0 (resp. f_1) in (1.49) is an isomorphism in Set, then so is t_1 (resp. t_0).

For instance, if X is a set and $A, B \subset X$ are subsets, then the left-hand and middle diagrams



of inclusion maps are pullback diagrams in Set. If we consider their universal properties only, the middle diagram encodes the same information as the left-hand diagram, but is less efficient at the task (at least when $X \neq A \cup B$); i.e., the lower right-hand set in the middle diagram is larger than it needs to be. If $p\colon\ X \to Y$ is a map in Set and $y \in Y$, then the right-hand diagram is a pullback diagram in Set; here, $F := p^{-1}(\{y\}) \subset X$, the upper horizontal map is an inclusion, and the bottom horizontal map picks out the point $y \in Y$. The resulting sequence of maps $F \to X \to Y$ is called a *fiber sequence* in Set and F is called the *fiber* of p over the point y.

Let's look at a few more examples. Consider any sets X, Y and map $g\colon X \to Y$ in Set, then the commutative diagrams

$$\begin{array}{ccc} X \times Y & \xrightarrow{\mathrm{pr}_1} & Y \\ {\scriptstyle \mathrm{pr}_0}\downarrow & & \downarrow \\ X & \longrightarrow & * \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{g} & Y \\ \| & & \| \\ X & \xrightarrow[g]{} & Y \end{array}$$

are pullback diagrams in Set. Consider any pair of left-hand pullback diagrams of the form

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{array} \qquad \begin{array}{ccc} A' & \longrightarrow & C' \\ \downarrow & & \downarrow \\ B' & \longrightarrow & D' \end{array} \qquad \begin{array}{ccc} A \times A' & \longrightarrow & C \times C' \\ \downarrow & & \downarrow \\ B \times B' & \longrightarrow & D \times D' \end{array} \tag{1.50}$$

in Set. Then the corresponding right-hand diagram is a pullback diagram in Set.

REMARK 1.7.5. Notice how we have not labeled the arrows above. This is a good idea, as it is less distracting, and furthermore, will get the reader in the habit of coming up with their own notation for the arrows when working with various diagrams. In other words, we will want to add our own labels for the arrows when verifying the above assertions—this is easily done in each case by verifying the universal property.

For instance, consider any set X and pullback diagram of the left-hand form

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{array} \qquad \begin{array}{ccc} A \times X & \longrightarrow & C \\ \downarrow & & \downarrow \\ B \times X & \longrightarrow & D \end{array} \qquad \begin{array}{ccc} X & \longrightarrow & * \\ \| & & \| \\ X & \longrightarrow & * \end{array} \tag{1.51}$$

in Set. Then the corresponding middle diagram is a pullback diagram in Set; this is because the right-hand diagram is a pullback diagram in Set, together with (1.50). Similarly, consider any pullback diagram of the left-hand form in (1.51). If $B \to X$ is a map in Set, then the corresponding diagram of the form

$$\begin{array}{ccc} A & \longrightarrow & C \times X \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \times X \end{array}$$

is a pullback diagram in Set; this is because verifying the universal property of pullbacks reduces to verifying it for the left-hand diagram in (1.51).

Any commutative diagram of the form

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\ X' & \longrightarrow & Y' \end{array}$$

in Set is a pullback diagram in Set. The following is a useful observation.

Proposition 1.7.6. *Consider any commutative diagram of the form*

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & E \\ \downarrow & (I) & \downarrow & (II) & \downarrow \\ B & \longrightarrow & D & \longrightarrow & F \end{array}$$

in Set *and denote by* $(I) + (II)$ *the outer diagram.*

(a) *If* (I) *and* (II) *are pullback diagrams in* Set*, then so is* $(I) + (II)$*.*
(b) *If* (II) *and* $(I) + (II)$ *are pullback diagrams in* Set*, then so is* (I)*.*

Proof. Each part follows by checking the universal property of the desired pullback diagram, by using the universal property of the two known pullback diagrams; for instance, first verify existence of the desired map, and then check its uniqueness. □

What happens if we replace Set with Top in our above discussion? The basic idea is to calculate the pullback $X \times_Z Y$ in Set, and then equip it with an appropriate topology. We have already worked out what we need—the product topology and the subspace topology. Consider any diagram in Top of the form

(1.52)
$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle f_1} \\ X & \xrightarrow[f_0]{} & Z \end{array}$$

Classically, the pullback of the pair of maps (1.52) in Top is defined to be the subspace of $X \times Y$ given by $X \times_Z Y := \{(x, y) \in X \times Y \,|\, f_0(x) = f_1(y)\}$. There is a diagram of the left-hand form

(1.53)
$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{t_1} & Y \\ {\scriptstyle t_0}\downarrow & & \\ X & & \end{array} \qquad \begin{array}{ccc} X \times_Z Y & \xrightarrow{t_1} & Y \\ {\scriptstyle t_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ X & \xrightarrow[f_0]{} & Z \end{array}$$

in Top, defined by $t_0(x, y) = x$ and $t_1(x, y) = y$, which makes the right-hand diagram commute (i.e., such that $f_0 t_0 = f_1 t_1$); it satisfies the following universal property.

Proposition 1.7.7 (Universal property of pullbacks in Top). *If* f_0, f_1 *is a pair of maps in* Top *of the form* (1.52)*, then the left-hand diagram in* (1.53) *is terminal with respect to all such diagrams in* Top*; i.e., for any topological space* A *and maps* α_0, α_1 *in* Top *which make the outer diagram*

(1.54)
$$\begin{array}{cccc} A & & \xrightarrow{\alpha_1} & \\ & \overset{\overline{\alpha}}{\underset{\exists!}{\searrow}} & & \\ & X \times_Z Y & \xrightarrow{t_1} & Y \\ {\scriptstyle \alpha_0}\searrow & {\scriptstyle t_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ & X & \xrightarrow[f_0]{} & Z \end{array}$$

commute (i.e., such that $f_0\alpha_0 = f_1\alpha_1$), there exists a unique map $\overline{\alpha}$ in Top *which makes the diagram commute; i.e, such that $t_0\,\overline{\alpha} = \alpha_0$ and $t_1\,\overline{\alpha} = \alpha_1$.*

Proof. This follows from easily from above. $\square$

Remark 1.7.8. The upshot is: giving a map $\overline{\alpha}\colon A \to X \times_Z Y$ in Top is the same as giving maps α_0, α_1 in Top which make the outer diagram in (1.54) commute. For this reason, sometimes $\overline{\alpha}$ is written as $\overline{\alpha} = (\alpha_0, \alpha_1)$.

For instance, consider any maps (p_0, p_1) and (h_0, h_1) of the indicated left-hand form in (1.55); i.e., such that $f_0p_0 = f_1p_1$ and $f_0h_0 = f_1h_1$. Then the left-hand diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle (p_0,p_1)} \\ B & \xrightarrow[(h_0,h_1)]{} & X \times_Z Y \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p_0} \\ B & \xrightarrow[h_0]{} & X \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{g} & C \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p_1} \\ B & \xrightarrow[h_1]{} & Y \end{array} \tag{1.55}$$

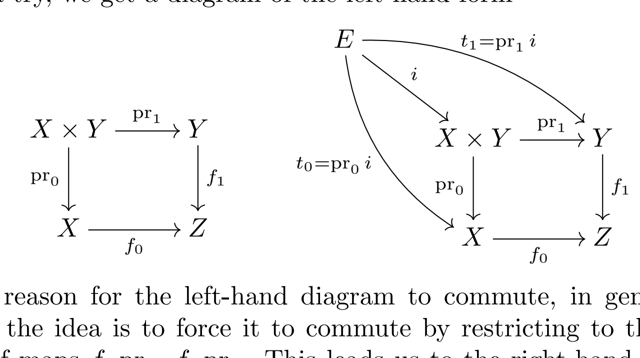
in Top commutes if and only if the corresponding right-hand diagrams in Top commute.

Remark 1.7.9. Arguing as above (e.g., Remark 1.5.3), the mapping properties in (1.54) characterize the pullback of f_0, f_1 in Top, up to isomorphism.

A *pullback diagram* (or cartesian diagram) in Top is a commutative diagram of the right-hand form in (1.53) that satisfies the universal property in (1.54). In this case, t_0 (resp. t_1) is called a *pullback* of f_1 along f_0 (resp. f_0 along f_1).

Remark 1.7.10. A diagram that is naturally isomorphic to a pullback diagram in Top, is a pullback diagram in Top. The pullback of an isomorphism in Top, is an isomorphism in Top.

Remark 1.7.11. Let's reformulate our approach to showing the existence of pullback diagrams in Top (resp. Set), in terms of products and equalizers. Suppose we start off with the diagram in (1.52) (resp. in (1.47)). We want to build its pullback diagram, but perhaps we don't remember how it works. Here is a useful approach: as a first step, look for something that has naturally occurring maps into both X and Y; we already know of such a construction, it is the product $X \times Y$. Giving this a try, we get a diagram of the left-hand form



There is no reason for the left-hand diagram to commute, in general. So as a second step, the idea is to force it to commute by restricting to the equalizer E of the pair of maps $f_0\,\mathrm{pr}_0$, $f_1\,\mathrm{pr}_1$. This leads us to the right-hand outer diagram which commutes (we just forced it to); this right-hand outer diagram is a pullback diagram. It is easy to check that $E \cong X \times_Z Y$: we could either use the classical constructions of product and equalizer to recover the description in (1.53) (resp. (1.48)), or we could work directly with the universal properties of product diagrams and equalizer diagrams. Conceptually, this means that once we know products and equalizers exist in Top (resp. Set), we know that pullbacks exist in Top (resp. Set).

1.8. Pushouts

If we reverse all the arrows in a pullback diagram and its mapping properties, we are naturally led to the mapping properties of a pushout diagram: let's develop this idea. Consider any diagram in Set of the form

$$\begin{array}{ccc} W & \xrightarrow{g_1} & Y \\ {\scriptstyle g_0}\downarrow & & \\ X & & \end{array} \tag{1.56}$$

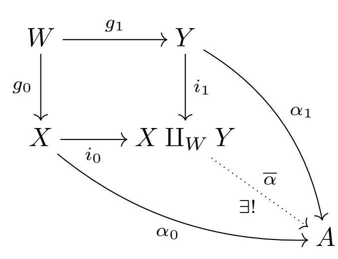
Classically, the pushout of the pair of maps (1.56) in Set is defined to be the quotient set given by $X \amalg_W Y := (X \amalg Y)/{\sim}$, where $\sim$ is the equivalence relation on $X \amalg Y$ generated by $g_0(w) \sim g_1(w)$ for each $w \in W$. There is a diagram of the left-hand form

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle i_1} \\ X & \xrightarrow[i_0]{} & X \amalg_W Y \end{array} \qquad \begin{array}{ccc} W & \xrightarrow{g_1} & Y \\ {\scriptstyle g_0}\downarrow & & \downarrow{\scriptstyle i_1} \\ X & \xrightarrow[i_0]{} & X \amalg_W Y \end{array} \tag{1.57}$$

in Set, defined by $i_0(x) = [x]$ and $i_1(y) = [y]$, which makes the right-hand diagram commute (i.e., such that $i_0 g_0 = i_1 g_1$) and satisfies the following universal property; for further reading, see [**5**, pp. 65–66] and [**7**, p. 81].

Proposition 1.8.1 (Universal property of pushouts in Set). *If g_0, g_1 is a pair of maps in* Set *of the form* (1.56), *then the left-hand diagram in* (1.57) *is initial with respect to all such diagrams in* Set*; i.e., for any set A and maps α_0, α_1 in* Set *which make the outer diagram*

(1.58)



commute (i.e., such that $\alpha_0 g_0 = \alpha_1 g_1$), there exists a unique map $\overline{\alpha}$ in Set *which makes the diagram commute; i.e, such that $\overline{\alpha}\, i_0 = \alpha_0$ and $\overline{\alpha}\, i_1 = \alpha_1$.*

Proof. Uniqueness is forced on us, since the diagram commutes implies that $\overline{\alpha}([x]) = \alpha_0(x)$ for each $x \in X$ and $\overline{\alpha}([y]) = \alpha_1(y)$ for each $y \in Y$. Existence follows since this is a well-defined map in Set. □

Remark 1.8.2. The upshot is: giving a map $\overline{\alpha}\colon X \amalg_W Y \to A$ in Set is the same as giving maps α_0, α_1 in Set which make the outer diagram in (1.58) commute. For this reason, sometimes $\overline{\alpha}$ is written as $\overline{\alpha} = (\alpha_0, \alpha_1)$.

REMARK 1.8.3. Arguing as above, the mapping properties in (1.58) characterize the pushout of g_0, g_1 in Set, up to isomorphism.

A *pushout diagram* (or cocartesian diagram) in Set is a commutative diagram of the right-hand form in (1.57) that satisfies the universal property in (1.58). In this case, i_0 (resp. i_1) is called a *pushout* of g_1 along g_0 (resp. g_0 along g_1).

REMARK 1.8.4. A diagram that is naturally isomorphic to a pushout diagram in Set, is a pushout diagram in Set. The pushout of an isomorphism in Set is an isomorphism in Set; i.e., if g_0 (resp. g_1) in (1.58) is an isomorphism in Set, then so is i_1 (resp. i_0).

For instance, if X is a set and $A, B \subset X$ are subsets, then the left-hand diagram

$$\begin{CD} A \cap B @>>> B \\ @VVV @VVV \\ A @>>> A \cup B \end{CD} \qquad \begin{CD} (A \cap B) \amalg (A \cap B) @>>> B \\ @VVV @VVV \\ A @>>> A \cup B \end{CD} \qquad \begin{CD} W @>g>> X \\ @VVV @VVV \\ * @>>> X/g(W) \end{CD}$$

of inclusion maps is a pushout diagram in Set. Similarly, the middle diagram of inclusion maps and maps induced by inclusions, is a pushout diagram in Set. If we consider their universal properties only, the middle diagram encodes the same information as the left-hand diagram, but is less efficient at the task (at least when $A \cap B \neq \emptyset$); i.e., the upper left-hand set in the middle diagram is larger than it needs to be. If $g\colon W \to X$ is a map in Set and $a \in W$, then the right-hand diagram is a pushout diagram in Set; here, $X/g(W) := X/{\sim}$, where $\sim$ is the equivalence relation on X generated by $g(a) \sim g(w)$ for each $w \in W$. The resulting sequence of maps $W \to X \to X/g(W)$ is called a *cofiber sequence* in Set and $X/g(W)$ is called the *cofiber* of g. The bottom horizontal map picks out the point $[g(a)] \in X/g(W)$ that the subset $g(W)$ was collapsed to.

Let's look at a few more examples. Consider any sets X, Y and map $g\colon X \to Y$ in Set, then the commutative diagrams

$$\begin{CD} \emptyset @>>> Y \\ @VVV @VV{\mathrm{in}_1}V \\ X @>>{\mathrm{in}_0}> X \amalg Y \end{CD} \qquad \begin{CD} X @>g>> Y \\ @| @| \\ X @>>g> Y \end{CD}$$

are pushout diagrams in Set. Consider any pair of left-hand pushout diagrams of the form

$$\begin{CD} A @>>> C \\ @VVV @VVV \\ B @>>> D \end{CD} \qquad \begin{CD} A' @>>> C' \\ @VVV @VVV \\ B' @>>> D' \end{CD} \qquad \begin{CD} A \amalg A' @>>> C \amalg C' \\ @VVV @VVV \\ B \amalg B' @>>> D \amalg D' \end{CD} \tag{1.59}$$

in Set. Then the corresponding right-hand diagram is a pushout diagram in Set.

REMARK 1.8.5. Notice how we have not labeled the arrows above. This is a good idea, as it is less distracting, and furthermore, will get the reader in the habit of coming up with their own notation for the arrows when working with various diagrams. In other words, we will want to add our own labels for the arrows when verifying the above assertions—this is easily done in each case by verifying the universal property.

For instance, consider any set X and pushout diagram of the left-hand form

$$\begin{CD} A @>>> C \\ @VVV @VVV \\ B @>>> D \end{CD} \qquad \begin{CD} A @>>> C \amalg X \\ @VVV @VVV \\ B @>>> D \amalg X \end{CD} \qquad \begin{CD} \emptyset @>>> X \\ @| @| \\ \emptyset @>>> X \end{CD} \tag{1.60}$$

in Set. Then the corresponding middle diagram is a pushout diagram in Set; this is because the right-hand diagram is a pushout diagram in Set, together with (1.59). Similarly, consider any pushout diagram of the left-hand form in (1.60). If $X \to C$ is a map in Set, then the corresponding diagram of the form

$$\begin{CD} A \amalg X @>>> C \\ @VVV @VVV \\ B \amalg X @>>> D \end{CD}$$

is a pushout diagram in Set; this is because verifying the universal property of pushouts reduces to verifying it for the left-hand diagram in (1.60).

Any commutative diagram of the form

$$\begin{CD} X @>>> Y \\ @V{\cong}VV @VV{\cong}V \\ X' @>>> Y' \end{CD}$$

in Set is a pushout diagram in Set. The following is a useful observation.

Proposition 1.8.6. *Consider any commutative diagram of the form*

$$\begin{CD} A @>>> C @>>> E \\ @VVV (I) @VVV (II) @VVV \\ B @>>> D @>>> F \end{CD}$$

in Set *and denote by* $(I) + (II)$ *the outer diagram.*

(a) *If* (I) *and* (II) *are pushout diagrams in* Set*, then so is* $(I) + (II)$.
(b) *If* (I) *and* $(I) + (II)$ *are pushout diagrams in* Set*, then so is* (II).

Proof. Each part follows by checking the universal property of the desired pushout diagram, by using the universal property of the two known pushout diagrams; for instance, first verify existence of the desired map, and then check its uniqueness. □

What happens if we replace Set with Top in our above discussion? We have already worked out what we need. Consider any diagram in Top of the form

$$\begin{CD} W @>{g_1}>> Y \\ @V{g_0}VV \\ X \end{CD} \tag{1.61}$$

Classically, the pushout of the pair of maps in (1.61) (in Top) is defined to be the quotient space given by $X \amalg_W Y := (X \amalg Y)/{\sim}$, where $\sim$ is the equivalence relation on $X \amalg Y$ generated by $g_0(w) \sim g_1(w)$ for each $w \in W$. There is a diagram of the left-hand form

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle i_1} \\ X & \xrightarrow[i_0]{} & X \amalg_W Y \end{array} \qquad\qquad \begin{array}{ccc} W & \xrightarrow{g_1} & Y \\ {\scriptstyle g_0}\downarrow & & \downarrow{\scriptstyle i_1} \\ X & \xrightarrow[i_0]{} & X \amalg_W Y \end{array} \tag{1.62}$$

in Top, defined by $i_0(x) = [x]$ and $i_1(y) = [y]$, which makes the right-hand diagram commute (i.e., such that $i_0 g_0 = i_1 g_1$); it satisfies the following universal property.

Proposition 1.8.7 (Universal property of pushouts in Top). *If g_0, g_1 is a pair of maps in* Top *of the form* (1.61)*, then the left-hand diagram in* (1.62) *is initial with respect to all such diagrams in* Top*; i.e., for any topological space A and maps α_0, α_1 in* Top *which make the outer diagram*

$$\begin{array}{cccc} W & \xrightarrow{g_1} & Y & \\ {\scriptstyle g_0}\downarrow & & \downarrow{\scriptstyle i_1} & \searrow{\scriptstyle \alpha_1} \\ X & \xrightarrow[i_0]{} & X \amalg_W Y & \\ & \searrow{\scriptstyle \alpha_0} & & \searrow{\scriptstyle \exists!\, \overline{\alpha}} \\ & & & A \end{array} \tag{1.63}$$

commute (i.e., such that $\alpha_0 g_0 = \alpha_1 g_1$), there exists a unique map $\overline{\alpha}$ in Top *which makes the diagram commute; i.e, such that $\overline{\alpha}\, i_0 = \alpha_0$ and $\overline{\alpha}\, i_1 = \alpha_1$.*

Proof. This follows from easily from above. □

Remark 1.8.8. The upshot is: giving a map $\overline{\alpha}\colon X \amalg_W Y \to A$ in Top is the same as giving maps α_0, α_1 in Top which make the outer diagram in (1.63) commute. For this reason, sometimes $\overline{\alpha}$ is written as $\overline{\alpha} = (\alpha_0, \alpha_1)$.

For instance, consider any maps (k_0, k_1) and (n_0, n_1) of the indicated left-hand form in (1.64); i.e., such that $k_0 g_0 = k_1 g_1$ and $n_0 g_0 = n_1 g_1$. Then the left-hand diagram of the form

$$\begin{array}{ccc} X \amalg_W Y & \xrightarrow{(n_0,n_1)} & C \\ {\scriptstyle (k_0,k_1)}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[h]{} & D \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{n_0} & C \\ {\scriptstyle k_0}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[h]{} & D \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{n_1} & C \\ {\scriptstyle k_1}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[h]{} & D \end{array} \tag{1.64}$$

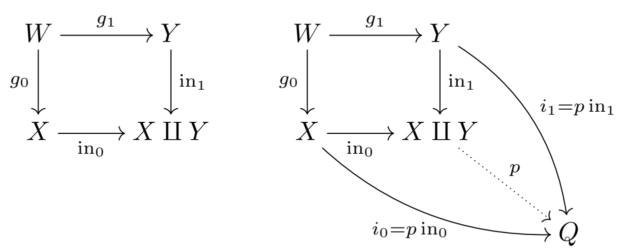
in Top commutes if and only if the corresponding right-hand diagrams in Top commute.

Remark 1.8.9. Arguing as above, the mapping properties in (1.63) characterize the pushout of g_0, g_1 in Top, up to isomorphism.

A *pushout diagram* (or cocartesian diagram) in Top is a commutative diagram of the right-hand form in (1.62) that satisfies the universal property in (1.63). In this case, i_0 (resp. i_1) is called a *pushout* of g_1 along g_0 (resp. g_0 along g_1).

REMARK 1.8.10. A diagram that is naturally isomorphic to a pushout diagram in Top, is a pushout diagram in Top. The pushout of an isomorphism in Top, is an isomorphism in Top.

REMARK 1.8.11. Let's reformulate our approach to showing the existence of pushout diagrams in Top (resp. Set)—in terms of coproducts and coequalizers. Suppose we start off with the diagram in (1.61) (resp. in (1.56)). We want to build its pushout diagram, but perhaps we don't remember how it works. Here is a useful approach: as a first step, look for something that receives naturally occurring maps out of both X and Y; we already know of such a construction, it is the coproduct $X \amalg Y$. Giving this a try, we get a diagram of the left-hand form



There is no reason for the left-hand diagram to commute, in general. So as a second step, the idea is to force it to commute by mapping to the coequalizer Q of the pair of maps $\mathrm{in}_0\, g_0$, $\mathrm{in}_1\, g_1$. This leads us to the right-hand outer diagram which commutes (we just forced it to); this right-hand outer diagram is a pushout diagram. It is easy to check that $Q \cong X \amalg_W Y$: we could either use the classical constructions of coproduct and coequalizer to recover the description in (1.62) (resp. (1.57)), or we could work directly with the universal properties of coproduct diagrams and coequalizer diagrams. Conceptually, this means that once we know coproducts and coequalizers exist in Top (resp. Set), we know that pushouts exist in Top (resp. Set).

1.9. Small products: Products indexed over a set

Let $\{B_0, \dots, B_n\}$ be a collection of sets, indexed on the set $\{0, \dots, n\}$, for some $n \geq 0$, and denote by B the union of the sets B_i for each $0 \leq i \leq n$. Recall that an $(n+1)$-tuple $(b_0, \dots, b_n)$ of elements of B is a function of the form $b\colon \{0, \dots, n\} \to B$; we usually denote the value of b at i by b_i, instead of $b(i)$, and we usually denote the $(n+1)$-tuple b by $(b_0, \dots, b_n)$. In particular, a 2-tuple (b_0, b_1) is the same as an ordered pair (b_0, b_1). Classically, the product $B_0 \times \cdots \times B_n$ in Set is defined to be the set of $(n+1)$-tuples

$$B_0 \times \cdots \times B_n := \{(b_0, \dots, b_n) | \, b_0 \in B_0, \dots, b_n \in B_n\} \tag{1.65}$$

There are maps of the form

$$B_i \xleftarrow{\mathrm{pr}_i} B_0 \times \cdots \times B_n \qquad 0 \leq i \leq n \tag{1.66}$$

in Set, defined by $\mathrm{pr}_i(b_0, \dots, b_n) = b_i$, $0 \leq i \leq n$ (i.e., each map pr_i is the usual projection map to B_i), which satisfies the following mapping property; for further reading, see [**5**, p. 69], [**6**, p. 113] and [**7**, p. 77].

Proposition 1.9.1 (Universal property of finite products in Set). *Let $n \geq 0$. If $B_0, \ldots, B_n$ are sets, then diagram (1.66) is terminal with respect to all such diagrams into B_i, $0 \leq i \leq n$, in* Set*; i.e., for any set A and diagram of the form*

$$B_i \xleftarrow{f_i} A \qquad 0 \leq i \leq n$$

in Set*, there exists a unique map $\overline{f}$ in* Set *which makes the diagram*

$$\begin{array}{ccc} B_i & \xleftarrow{\mathrm{pr}_i} & B_0 \times \cdots \times B_n \\ & {\scriptstyle f_i}\nwarrow & \uparrow {\scriptstyle \exists!\ \overline{f}} \\ & & A \end{array} \qquad\qquad 0 \leq i \leq n \tag{1.67}$$

commute; i.e., such that $\mathrm{pr}_i \overline{f} = f_i$ for each $0 \leq i \leq n$.

Proof. Uniqueness is forced on us, since the diagram commutes implies that $\overline{f}(a) = (f_0(a), \ldots, f_n(a))$ for each $a \in A$. Existence follows since this is a well-defined map in Set. $\square$

Remark 1.9.2. The upshot is: giving a map $\overline{f}\colon A \to B_0 \times \cdots \times B_n$ in Set is the same as giving maps $f_i\colon A \to B_i$, $0 \leq i \leq n$, in Set. For this reason, sometimes $\overline{f}$ is written as $\overline{f} = (f_0, \ldots, f_n)$.

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More generally, let $\{B_\alpha\}_{\alpha \in I}$ be a collection of sets, indexed on a set I, and denote by B the union of the sets B_α for each $\alpha \in I$. Recall that an I-tuple $(b_\alpha)_{\alpha \in I}$ of elements of B is a function of the form $b\colon I \to B$; we usually denote the value of b at α by b_α, instead of $b(\alpha)$, and we usually denote the I-tuple b by $(b_\alpha)_{\alpha \in I}$, or simply (b_α), when the index set I is clear. Classically, the product $\prod_{\alpha \in I} B_\alpha$ in Set is defined to be the set of I-tuples

$$\prod_{\alpha \in I} B_\alpha := \{(b_\alpha)_{\alpha \in I} \,|\, b_\alpha \in B_\alpha \text{ for each } \alpha \in I\} \tag{1.68}$$

There are maps of the form

$$B_\alpha \xleftarrow{\mathrm{pr}_\alpha} \textstyle\prod_{\alpha \in I} B_\alpha \qquad \alpha \in I \tag{1.69}$$

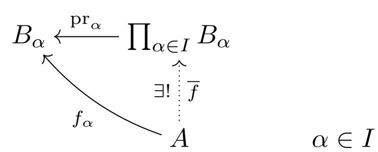
in Set, defined by $\mathrm{pr}_{\alpha'}\big((b_\alpha)_{\alpha \in I}\big) = b_{\alpha'}$ for each $\alpha' \in I$ (i.e., each map pr_α is the usual projection map to B_α), which satisfies the following mapping property; for further reading, see [**5**, p. 69], [**6**, p. 113] and [**7**, p. 77].

Proposition 1.9.3 (Universal property of products in Set). *Let $\{B_\alpha\}_{\alpha \in I}$ be a collection of sets, indexed on a set I. Then diagram (1.69) is terminal with respect to all such diagrams into B_α, $\alpha \in I$, in* Set*; i.e., for any set A and diagram of the form*

$$B_\alpha \xleftarrow{f_\alpha} A \qquad \alpha \in I$$

in Set*, there exists a unique map* $\overline{f}$ *in* Set *which makes the diagram*

(1.70)



commute; i.e., such that $\mathrm{pr}_\alpha \overline{f} = f_\alpha$ *for each* $\alpha \in I$*.*

Proof. Uniqueness is forced on us, since the diagram commutes implies that $\overline{f}(a) = (f_\alpha(a))_{\alpha\in I}$ for each $a \in A$. Existence follows since this is a well-defined map in Set. □

Remark 1.9.4. The upshot is: giving a map $\overline{f}\colon A \to \prod_{\alpha\in I} B_\alpha$ in Set is the same as giving maps $f_\alpha\colon A \to B_\alpha$, $\alpha \in I$, in Set. For this reason, sometimes $\overline{f}$ is written as $\overline{f} = (f_\alpha)_{\alpha\in I}$ or simply $\overline{f} = (f_\alpha)$

Remark 1.9.5. Arguing as above, the mapping properties in (1.70) characterize the product $\prod_{\alpha\in I} B_\alpha$ in Set, up to isomorphism.

A *product diagram* in Set is a diagram of the form (1.69) that satisfies the universal property in (1.70). For instance, if $I = \{0, \dots, n\}$ for some $n \geq 0$, then diagram (1.66) is a product diagram in Set and $B_0 \times \cdots \times B_n = \prod_{\alpha\in I} B_\alpha$ in Set.

What happens if we replace Set with Top in our above discussion? The reader already knows from their background how to work with products of topological spaces, but let's pretend we forgot; this is a good idea as the intuition we develop here, in this familiar situation, will carry over and guide us in more complicated ones.

Let $\{B_\alpha\}_{\alpha\in I}$ be a collection of topological spaces, indexed on a set I. Classically, the product $\prod_{\alpha\in I} B_\alpha$ in Top is defined to be $\prod_{\alpha\in I} B_\alpha$ in Set, equipped with an appropriate topology. Let's follow our noses to rediscover this topology, called the product topology. The idea is: we want the same mapping properties in (1.70) to be true, but for Set replaced by Top. In particular, this means that we need a topology on the set $\prod_{\alpha\in I} B_\alpha$ such that the maps in (1.69) are continuous; i.e., inverse images of open subsets are open. There are two extremes. If we give the set $\prod_{\alpha\in I} B_\alpha$ the largest topology possible (i.e., the most open sets possible, where every subset is open, and hence is the discrete topology) then the maps in (1.69) are certainly continuous, but the set $\prod_{\alpha\in I} B_\alpha$ (with this discrete topology) then becomes difficult to map into in Top. The other extreme is to give the set $\prod_{\alpha\in I} B_\alpha$ the smallest topology such that the maps in (1.69) are continuous—this is called the *topology induced on* the set $\prod_{\alpha\in I} B_\alpha$ by the functions pr_α, $\alpha \in I$.

Proposition 1.9.6. *The topology induced on the set* $\prod_{\alpha\in I} B_\alpha$ *by the functions* pr_α*,* $\alpha \in I$*, is characterized by the property that if* A *is a topological space, a function* $\overline{f}\colon A \to \prod_{\alpha\in I} B_\alpha$ *is continuous if and only if the composites* $\mathrm{pr}_\alpha \overline{f}$*,* $\alpha \in I$*, are continuous.*

Proof. Here is the basic idea. Consider any $\alpha' \in I$. The smallest topology on the set $\prod_{\alpha\in I} B_\alpha$ such that $\mathrm{pr}_{\alpha'}$ is continuous, called the topology induced by $\mathrm{pr}_{\alpha'}$, is given by the collection of inverse images

$$\{\mathrm{pr}_{\alpha'}^{-1}(U) \mid U \subset X_{\alpha'} \text{ is open}\} \tag{1.71}$$

The topology induced by the collection of functions pr_α, $\alpha \in I$, is the smallest topology on the set $\prod_{\alpha\in I} B_\alpha$ which contains the topology (1.71) for each $\alpha' \in I$: it is the topology generated by the union of these I-indexed collections of subsets—it consists of $\emptyset$, $\prod_{\alpha\in I} B_\alpha$, all finite intersections of the generating subsets, and all arbitrary unions of these finite intersections. In particular, a function of the form $\overline{f}$ is continuous if and only if inverse images of the generating subsets—also called subbasis elements—in (1.71) are open for each $\alpha' \in I$. $\square$

This means that if we give the set $\prod_{\alpha\in I} B_\alpha$ the topology induced by the functions pr_α, $\alpha \in I$, then the desired universal property will be satisfied—this is the *product topology* on the set $\prod_{\alpha\in I} B_\alpha$. This is the topology we equip the set $\prod_{\alpha\in I} B_\alpha$ with when we consider the diagram

$$B_\alpha \xleftarrow{\mathrm{pr}_\alpha} \prod_{\alpha\in I} B_\alpha \qquad \alpha \in I \tag{1.72}$$

in Top, defined by $\mathrm{pr}_{\alpha'}\big((b_\alpha)_{\alpha\in I}\big) = b_{\alpha'}$ for each $\alpha' \in I$ (i.e., each map pr_α is the usual projection map to B_α). Notice how we have been naturally led to the product topology by considering desirable mapping properties. Hence we have verified that diagram (1.72) satisfies the following mapping property.

PROPOSITION 1.9.7 (Universal property of products in Top). *Let $\{B_\alpha\}_{\alpha\in I}$ be a collection of sets, indexed on a set I. Then diagram* (1.72) *is terminal with respect to all such diagrams into B_α, $\alpha \in I$, in* Top*; i.e., for any topological space A and diagram of the form*

$$B_\alpha \xleftarrow{f_\alpha} A \qquad \alpha \in I$$

in Top*, there exists a unique map $\overline{f}$ in* Top *which makes the diagram*

$$\begin{array}{ccc} B_\alpha & \xleftarrow{\mathrm{pr}_\alpha} & \prod_{\alpha\in I} B_\alpha \\ & \nwarrow^{f_\alpha} & \uparrow{\scriptstyle \exists!\, \overline{f}} \\ & & A \end{array} \qquad \alpha \in I \tag{1.73}$$

commute; i.e., such that $\mathrm{pr}_\alpha \overline{f} = f_\alpha$ for each $\alpha \in I$.

PROOF. This follows from easily from above. $\square$

REMARK 1.9.8. The upshot is: giving a map $\overline{f}\colon A \to \prod_{\alpha\in I} B_\alpha$ in Top is the same as giving maps $f_\alpha\colon A \to B_\alpha$, $\alpha \in I$, in Top. For this reason, sometimes $\overline{f}$ is written as $\overline{f} = (f_\alpha)_{\alpha\in I}$ or simply $\overline{f} = (f_\alpha)$.

REMARK 1.9.9. Arguing as above, the mapping properties in (1.73) characterize the product $\prod_{\alpha\in I} B_\alpha$ in Top, up to isomorphism.

A *product diagram* in Top is a diagram of the form (1.72) that satisfies the universal property in (1.73). If $I = \{0, \dots, n\}$, for some $n \geq 0$, then we sometimes write diagram (1.72) in Top using the notation

$$B_i \xleftarrow{\mathrm{pr}_i} B_0 \times \cdots \times B_n \qquad 0 \leq i \leq n \tag{1.74}$$

defined by $\mathrm{pr}_i(b_0, \dots, b_n) = b_i$, $0 \leq i \leq n$ (i.e., each map pr_i is the usual projection map to B_i). In particular, (1.74) is a product diagram in Top and in this notation $B_0 \times \cdots \times B_n \cong \prod_{\alpha\in I} B_\alpha$ in Top.

REMARK 1.9.10. A diagram that is naturally isomorphic to a product diagram in Top (resp. Set), is a product diagram in Top (resp. Set).

1.10. Small coproducts: Coproducts indexed over a set

If we reverse all the arrows in a product diagram and its mapping properties, we are naturally led to the mapping properties of a coproduct diagram: let's develop this idea. Let $\{B_\alpha\}_{\alpha\in I}$ be a collection of sets, indexed on a set I. Classically, the coproduct (or disjoint union) $\coprod_{\alpha\in I} B_\alpha$ in Set is defined to be the union of disjoint copies of B_α, $\alpha \in I$,

$$\coprod_{\alpha\in I} B_\alpha := \bigcup_{\alpha\in I} B_\alpha \times \{\alpha\} \tag{1.75}$$

REMARK 1.10.1. Note that $B_\alpha \cong B_\alpha \times \{\alpha\}$, for each $\alpha \in I$, in Set and these isomorphisms are simply a formal way to ensure that the union on the right-hand side of (1.75) is a union of disjoint sets. For instance, if X is a set, then $\coprod_{\alpha\in I} X$ is supposed to be the disjoint union of copies of X, indexed on the set I; i.e., if we start with a set X, then we can think of $X \times \{\alpha\}$, $\alpha \in I$, as a collection of disjoint copies of the set X, and hence their union $\bigcup_{\alpha\in I} X \times \{\alpha\}$ is a union of disjoint copies of X, indexed on the set I, which we write as $\coprod_{\alpha\in I} X$.

There are maps of the form

$$B_\alpha \xrightarrow{\text{in}_\alpha} \textstyle\coprod_{\alpha\in I} B_\alpha \qquad \alpha \in I \tag{1.76}$$

in Set, defined by $\text{in}_\alpha(b) = (b, \alpha)$ for each $\alpha \in I$ (i.e., each map in_α is the usual inclusion map of B_α).

REMARK 1.10.2. For notational convenience reasons, we usually identify X with its copy $X \times \{\alpha\}$ for each $\alpha \in I$; in this case, the maps in (1.76) are defined by $\text{in}_\alpha(x) = x$. For instance, if we are working with $\coprod_{\alpha\in I} X$ (where $B_\alpha = X$ for each $\alpha \in I$), then we simply have to keep track of which copy of X we are mapping into; but this is indicated by the subscript on the inclusion map itself. We do not want to think of the α-indexed copy of X as $X \times \{\alpha\}$ for each $\alpha \in I$ (this is too messy, notationally), but instead to simply think of having an I-indexed collection of disjoint copies of X. In other words, we prefer to keep the notation as simple as possible—this will not cause any confusion.

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Following the notational convention in Remark 1.10.2, we rewrite the maps in (1.76) as maps of the form

$$B_\alpha \xrightarrow{\text{in}_\alpha} \textstyle\coprod_{\alpha\in I} B_\alpha \qquad \alpha \in I \tag{1.77}$$

in Set, defined by $\text{in}_\alpha(b) = b$ for each $\alpha \in I$ (i.e., each map in_α is the usual inclusion map of B_α), which satisfies the following mapping property; for further reading, see [**5**, p. 63] and [**7**, p. 81].

PROPOSITION 1.10.3 (Universal property of coproducts in Set). *Let $\{B_\alpha\}_{\alpha\in I}$ be a collection of sets, indexed on a set I. Then diagram* (1.77) *is initial with respect* *to all such diagrams out of* B_α, $\alpha \in I$, *in* Set*; i.e., for any set* A *and diagram of the form*

$$B_\alpha \xrightarrow{f_\alpha} A \qquad \alpha \in I$$

in Set*, there exists a unique map* $\overline{f}$ *in* Set *which makes the diagram*

$$\begin{array}{ccc} B_\alpha & \xrightarrow{\mathrm{in}_\alpha} & \coprod_{\alpha \in I} B_\alpha \\ & \searrow^{f_\alpha} & \Big\downarrow {\scriptstyle \exists!\ \overline{f}} \\ & & A \end{array} \qquad \alpha \in I \tag{1.78}$$

commute; i.e., such that $\overline{f}\,\mathrm{in}_\alpha = f_\alpha$ *for each* $\alpha \in I$*.*

PROOF. Uniqueness is forced on us, since the diagram commutes implies that $\overline{f}(b) = f_\alpha(b)$ for each $b \in B_\alpha$ and $\alpha \in I$. Existence follows since this is a well-defined map in Set. □

REMARK 1.10.4. The upshot is: giving a map $\overline{f}\colon \coprod_{\alpha \in I} B_\alpha \to A$ in Set is the same as giving maps $f_\alpha\colon B_\alpha \to A$, $\alpha \in I$, in Set. For this reason, sometimes $\overline{f}$ is written as $\overline{f} = (f_\alpha)_{\alpha \in I}$ or simply $\overline{f} = (f_\alpha)$.

REMARK 1.10.5. Arguing as above, the mapping properties in (1.78) characterize the coproduct $\coprod_{\alpha \in I} B_\alpha$ in Set, up to isomorphism.

A *coproduct diagram* in Set is a diagram of the form (1.77) that satisfies the universal property in (1.78). If $I = \{0, \dots, n\}$, for some $n \geq 0$, then we sometimes write diagram (1.77) in Set using the notation

$$B_i \xrightarrow{\mathrm{in}_i} B_0 \amalg \cdots \amalg B_n \qquad 0 \leq i \leq n \tag{1.79}$$

defined by $\mathrm{in}_i(b) = b$, $0 \leq i \leq n$ (i.e., each map in_i is the usual inclusion map of B_i). In particular, (1.79) is a coproduct diagram in Set and in this notation $B_0 \amalg \cdots \amalg B_n \cong \coprod_{\alpha \in I} B_\alpha$ in Set.

What happens if we replace Set with Top in our above discussion? The reader already knows from their background how to work with coproducts (or disjoint unions) of topological spaces, but let's pretend we forgot; this is a good idea as the intuition we develop here, in this familiar situation, will carry over and guide us in more complicated ones.

Let $\{B_\alpha\}_{\alpha \in I}$ be a collection of topological spaces, indexed on a set I. Classically, the coproduct $\coprod_{\alpha \in I} B_\alpha$ in Top is defined to be $\coprod_{\alpha \in I} B_\alpha$ in Set, equipped with an appropriate topology. Let's follow our noses to rediscover this topology, called the coproduct topology. The idea is: we want the same mapping properties in (1.78) to be true, but for Set replaced by Top. In particular, this means that we need a topology on the set $\coprod_{\alpha \in I} B_\alpha$ such that the maps in (1.77) are continuous; i.e., inverse images of open subsets are open. There are two extremes. If we give the set $\coprod_{\alpha \in I} B_\alpha$ the smallest topology possible (i.e., the fewest open sets possible, where only $\emptyset$ and $\coprod_{\alpha \in I} B_\alpha$ are open, and hence is the trivial topology) then the maps in (1.77) are certainly continuous, but the set $\coprod_{\alpha \in I} B_\alpha$ (with this trivial topology) then becomes difficult to map out of in Top. The other extreme is to give the set $\coprod_{\alpha \in I} B_\alpha$ the largest topology such that the maps in (1.77) are continuous—this is called the *topology coinduced on* the set $\coprod_{\alpha \in I} B_\alpha$ by the functions in_α, $\alpha \in I$.

PROPOSITION 1.10.6. *The topology coinduced on the set* $\coprod_{\alpha \in I} B_\alpha$ *by the functions* in_α, $\alpha \in I$, *is characterized by the property that if* A *is a topological space, a function* $\overline{f}\colon \coprod_{\alpha \in I} B_\alpha \to A$ *is continuous if and only if the composites* $\overline{f}\,\mathrm{in}_\alpha$, $\alpha \in I$, *are continuous.*

PROOF. Here is the basic idea. Consider any $\alpha' \in I$. The largest topology on the set $\coprod_{\alpha \in I} B_\alpha$ such that $\mathrm{in}_{\alpha'}$ is continuous, called the topology coinduced by $\mathrm{in}_{\alpha'}$, is given by the collection

$$\{W \subset \coprod\nolimits_{\alpha \in I} B_\alpha \,|\, \mathrm{in}_{\alpha'}^{-1}(W) \subset B_{\alpha'} \text{ is open}\} \tag{1.80}$$

The topology coinduced by the collection of functions in_α, $\alpha \in I$, is the largest topology on the set $\coprod_{\alpha \in I} B_\alpha$ which is contained in the topology (1.80), for each $\alpha' \in I$: it is the intersection of these I-indexed collections of subsets. □

This means that if we give the set $\coprod_{\alpha \in I} B_\alpha$ the topology coinduced by the functions in_α, $\alpha \in I$, then the desired universal property will be satisfied—this is the *coproduct topology* on the set $\coprod_{\alpha \in I} B_\alpha$. This is the topology we equip the set $\coprod_{\alpha \in I} B_\alpha$ with when we consider the diagram

$$B_\alpha \xrightarrow{\mathrm{in}_\alpha} \coprod\nolimits_{\alpha \in I} B_\alpha \qquad \alpha \in I \tag{1.81}$$

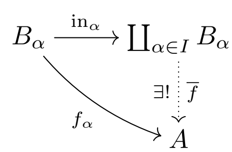
in Top, defined by $\mathrm{in}_\alpha(b) = b$ for each $\alpha \in I$ (i.e., each map in_α is the usual inclusion map of B_α). Notice how we have been naturally led to the coproduct topology by considering desirable mapping properties. Hence we have verified that diagram (1.81) satisfies the following mapping property.

PROPOSITION 1.10.7 (Universal property of coproducts in Top). *Let* $\{B_\alpha\}_{\alpha \in I}$ *be a collection of topological spaces, indexed on a set* I. *Then diagram* (1.81) *is initial with respect to all such diagrams out of* B_α, $\alpha \in I$, *in* Top*; i.e., for any topological space* A *and diagram of the form*

$$B_\alpha \xrightarrow{f_\alpha} A \qquad \alpha \in I$$

in Top*, there exists a unique map* $\overline{f}$ *in* Top *which makes the diagram*

(1.82)



$\alpha \in I$

commute; i.e., such that $\overline{f}\,\mathrm{in}_\alpha = f_\alpha$ *for each* $\alpha \in I$.

PROOF. This follows from easily from above. □

REMARK 1.10.8. The upshot is: giving a map $\overline{f}\colon \coprod_{\alpha \in I} B_\alpha \to A$ in Top is the same as giving maps $f_\alpha\colon B_\alpha \to A$, $\alpha \in I$, in Top. For this reason, sometimes $\overline{f}$ is written as $\overline{f} = (f_\alpha)_{\alpha \in I}$ or simply $\overline{f} = (f_\alpha)$.

REMARK 1.10.9. Arguing as above, the mapping properties in (1.82) characterize the coproduct $\coprod_{\alpha \in I} B_\alpha$ in Top, up to isomorphism.

A *coproduct diagram* in Top is a diagram of the form (1.81) that satisfies the universal property in (1.82). If $I = \{0, \ldots, n\}$, for some $n \geq 0$, then we sometimes write diagram (1.81) in Top using the notation

$$B_i \xrightarrow{\mathrm{in}_i} B_0 \amalg \cdots \amalg B_n \qquad 0 \leq i \leq n \tag{1.83}$$

defined by $\mathrm{in}_i(b) = b$, $0 \leq i \leq n$ (i.e., each map in_i is the usual inclusion map of B_i). In particular, (1.83) is a coproduct diagram in Top and in this notation $B_0 \amalg \cdots \amalg B_n \cong \coprod_{\alpha \in I} B_\alpha$ in Top.

Remark 1.10.10. A diagram that is naturally isomorphic to a coproduct diagram in Top (resp. Set), is a coproduct diagram in Top (resp. Set).

1.11. Limits and colimits

Products, equalizers, and pullbacks (resp. coproducts, coequalizers, and pushouts) are particular examples of limits (resp. colimits). Before we get to these, it will be useful to have in mind some language for diagrams, together with several examples of diagrams that we can easily draw.

A D-*shaped diagram* in Set is a functor $X\colon \mathsf{D} \to \mathsf{Set}$; here, D is a category which we sometimes call the indexing category for the diagram X. For instance, if D is the empty category (no objects and no arrows), then a diagram $X\colon \mathsf{D} \to \mathsf{Set}$ is the empty diagram in Set (no objects and no arrows). If D is the discrete category (no non-identity arrows) with objects the four integers

$$0 \qquad 1 \qquad 2 \qquad 3$$

then a diagram $X\colon \mathsf{D} \to \mathsf{Set}$ has the form

$$X(0) \qquad X(1) \qquad X(2) \qquad X(3) \tag{1.84}$$

in Set. It consists of sets (1.84) indexed on the set of integers $\{0, 1, 2, 3\}$.

Remark 1.11.1. For notational convenience reasons, we usually use the subscript notation when working with diagrams, instead of the parentheses notation, to denote a diagram X evaluated on an object i; i.e., we usually write X_i in place of $X(i)$; often we find that writing $X(i)$ takes up too much space, notationally. For instance, we usually write a diagram of the form (1.84), in subscript notation, as

$$X_0 \qquad X_1 \qquad X_2 \qquad X_3$$

which feels a little more compact and easy to look at.

If D is the category of the form

$$b \longrightarrow d \longleftarrow c \qquad \Big(\text{resp.} \quad b \longleftarrow a \longrightarrow c\Big)$$

(with exactly three objects and two non-identity arrows of the indicated form), then a diagram $X\colon \mathsf{D} \to \mathsf{Set}$ has the form

$$X_b \longrightarrow X_d \longleftarrow X_c \qquad \Big(\text{resp.} \quad X_b \longleftarrow X_a \longrightarrow X_c\Big)$$

in Set. If D is the category of the left-hand form

$$0 \rightrightarrows 1 \qquad X_0 \rightrightarrows X_1$$

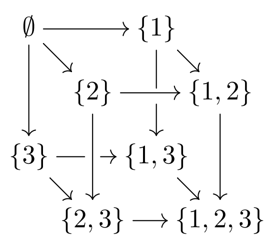
(with exactly two objects and two non-identity arrows of the indicated form), then a diagram $X\colon \mathsf{D} \to \mathsf{Set}$ has the right-hand form in Set. If D is the category of the form

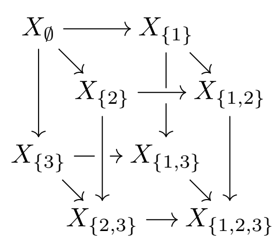
$$\begin{array}{c} 0 \longleftarrow 1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4 \longleftarrow \cdots \\ \Big(\text{resp.}\quad 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow \cdots \Big) \end{array}$$

with objects the non-negative integers and a single morphism $i \leftarrow j$ (resp. $i \to j$) for each $i \leq j$, then a diagram $X\colon \mathsf{D} \to \mathsf{Set}$ has the form

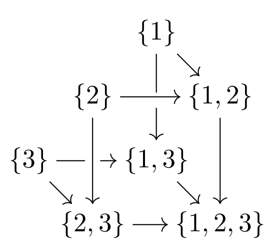
$$\begin{array}{c} X_0 \longleftarrow X_1 \longleftarrow X_2 \longleftarrow X_3 \longleftarrow X_4 \longleftarrow \cdots \\ \Big(\text{resp.}\quad X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow X_4 \longrightarrow \cdots \Big) \end{array}$$

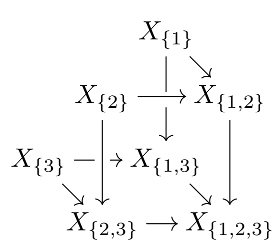
in Set. Define the sets $\mathbf{n} := \{1, 2, \ldots, n\}$ for each $n \geq 0$; we use the convention that $\mathbf{0} = \emptyset$ is the empty set. Denote by $\mathcal{P}(\mathbf{n})$ the poset (i.e., partially ordered set) of all subsets of $\mathbf{n}$, ordered by inclusion $\subset$ of sets. We will often regard $\mathcal{P}(\mathbf{n})$ as the category associated to this partial order in the usual way; the objects are the elements of $\mathcal{P}(\mathbf{n})$, and there is a morphism $U \to V$ if and only if $U \subset V$. For instance, if $\mathsf{D} = \mathcal{P}(\mathbf{3})$, then it is the category of the left-hand form



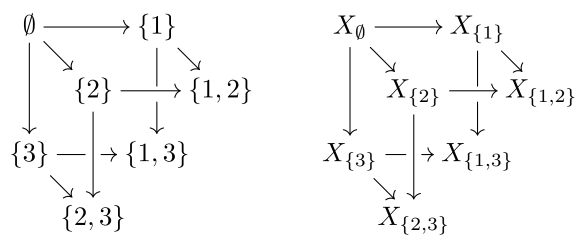


and a diagram $X\colon \mathsf{D} \to \mathsf{Set}$ has the indicated right-hand form in Set; i.e., X has the form of a 3-cube in Set. Denote by $\mathcal{P}_0(\mathbf{n}) \subset \mathcal{P}(\mathbf{n})$ the poset of all nonempty subsets of $\mathbf{n}$; it is the full subcategory of $\mathcal{P}(\mathbf{n})$ containing all objects except the initial object $\emptyset$. For instance, if $\mathsf{D} = \mathcal{P}_0(\mathbf{3})$, then it is the category of the left-hand form





and a diagram $X\colon \mathsf{D} \to \mathsf{Set}$ has the indicated right-hand form in Set. Denote by $\mathcal{P}_1(\mathbf{n}) \subset \mathcal{P}(\mathbf{n})$ the poset of all subsets of $\mathbf{n}$ not equal to $\mathbf{n}$; it is the full subcategory of $\mathcal{P}(\mathbf{n})$ containing all objects except the terminal object $\mathbf{n}$. For instance, if $\mathsf{D} = \mathcal{P}_1(\mathbf{3})$, then it is the category of the left-hand form

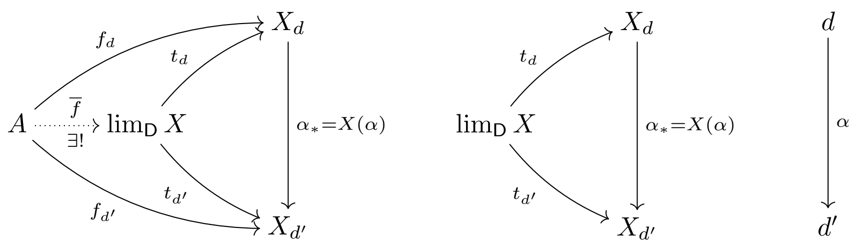


and a diagram $X\colon \mathsf{D} \to \mathsf{Set}$ has the indicated right-hand form in Set.

Let $X\colon \mathsf{D} \to \mathsf{Set}$ be a diagram. A *limit* of X, denoted $\lim_{\mathsf{D}} X$, is a set with the following mapping properties: (i) (Cone): there is a collection $\{t_d\}$ of maps

$$t_d\colon \lim_{\mathsf{D}} X \to X_d \qquad d \in \mathsf{D}$$

in Set, indexed on the objects $d \in \mathsf{D}$, which make the middle diagram

(1.85)


commute (i.e., such that $\alpha_* t_d = t_{d'}$) for each arrow α in D (such a collection $\{t_d\}$ is sometimes called a *cone* into X) and (ii) (Universal property): the cone $\{t_d\}$ is terminal with respect to all such cones into X; i.e., for any set A and collection $\{f_d\}$ of maps

$$f_d\colon A \to X_d \qquad d \in \mathsf{D}$$

in Set, indexed on the objects $d \in \mathsf{D}$, which make the left-hand outer diagram commute (i.e., such that $\alpha_* f_d = f_{d'}$) for each arrow α in D, there exists a unique map $\overline{f}$ in Set which makes the diagram commute; i.e., such that $t_d \overline{f} = f_d$ for each $d \in \mathsf{D}$. We call the cone $\{t_d\}$ the *limiting cone* of X (or the terminal cone into X).

REMARK 1.11.2. In other words, property (ii) states that every cone $\{f_d\}$ into X factors uniquely through the limiting cone $\{t_d\}$ of X.

REMARK 1.11.3. The upshot is: giving a map $\overline{f}\colon A \to \lim_{\mathsf{D}} X$ in Set is the same as giving a cone $\{f_d\}$ into X. For this reason, sometimes $\overline{f}$ is written as $\overline{f} = (f_d)_{d\in\mathsf{D}}$ or simply $\overline{f} = (f_d)$

To work effectively with the limit $\lim_{\mathsf{D}} X$ of the diagram X, we need to understand how to verify that a pair of maps into it are identical.

PROPOSITION 1.11.4. *Let $X\colon \mathsf{D} \to \mathsf{Set}$ be a diagram. Assume that its limit* $\lim_{\mathsf{D}} X$ (1.85) *exists. Consider any pair of maps of the form*

$$A \underset{g}{\overset{f}{\rightrightarrows}} \lim_{\mathsf{D}} X$$

in Set*. Then f and g are identical if and only if their corresponding cones into X are identical; i.e., in other words, $f = g$ if and only if $t_d f = t_d g$ for each $d \in \mathsf{D}$.*

PROOF. This follows from the universal property of limits. □

Here is an equivalent way to formulate this observation; note how the change in notation makes the statement even more obvious.

PROPOSITION 1.11.5. *Let* $X\colon \mathsf{D} \to \mathsf{Set}$ *be a diagram. Assume that its limit* $\lim_{\mathsf{D}} X$ (1.85) *exists. Consider any pair of maps of the form*

$$A \underset{(g_d)}{\overset{(f_d)}{\rightrightarrows}} \lim_{\mathsf{D}} X$$

in Set. *Then* (f_d) *and* (g_d) *are identical if and only if their corresponding cones into* X *are identical; i.e., in other words,* $(f_d) = (g_d)$ *if and only if* $f_d = g_d$ *for each* $d \in \mathsf{D}$.

For instance, consider any cones $\{p_d\}$ and $\{h_d\}$ into X of the indicated left-hand form in (1.85); i.e., such that $\alpha_* \, p_d = p_{d'}$ and $\alpha_* \, h_d = h_{d'}$ for each arrow α in D. Then the left-hand diagram of the form

(1.86)
$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle (p_d)} \\ B & \xrightarrow[(h_d)]{} & \lim_{\mathsf{D}} X \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p_d} \\ B & \xrightarrow[h_d]{} & X_d \end{array}$$

in Set commutes if and only if the corresponding right-hand diagram in Set commutes for each $d \in \mathsf{D}$.

REMARK 1.11.6. Arguing as above, the mapping properties in (1.85) characterize the limit $\lim_{\mathsf{D}} X$, up to isomorphism, provided that it exists.

A *limit diagram* in Set is a commutative diagram of the middle form (involving each arrow α in D) in (1.85) in Set, that satisfies the universal property of the left-hand form in (1.85).

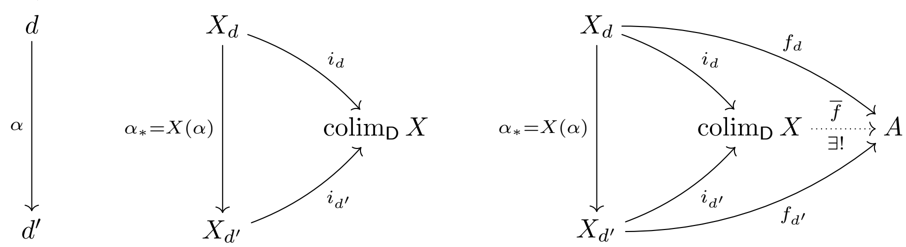
REMARK 1.11.7. A diagram that is naturally isomorphic to a limit diagram in Set, is a limit diagram in Set.

If we reverse all the arrows in a limit diagram and its mapping properties, we are naturally led to the mapping properties of a colimit diagram: let's develop this idea. Let $X\colon \mathsf{D} \to \mathsf{Set}$ be a diagram. A *colimit* of X, denoted $\operatorname{colim}_{\mathsf{D}} X$, is a set with the following mapping properties: (i) (Cone): there is a collection $\{i_d\}$ of maps

$$i_d\colon X_d \to \operatorname{colim}_{\mathsf{D}} X \qquad d \in \mathsf{D}$$

in Set, indexed on the objects $d \in \mathsf{D}$, which make the middle diagram

(1.87)



commute (i.e., such that $i_{d'}\,\alpha_* = i_d$) for each arrow α in D (such a collection $\{i_d\}$ is sometimes called a *cone* out of X) and (ii) (Universal property): the cone $\{i_d\}$ is initial with respect to all such cones out of X; i.e., for any set A and collection $\{f_d\}$ of maps

$$f_d\colon X_d \to A \qquad d \in \mathsf{D}$$

in Set, indexed on the objects $d \in \mathsf{D}$, which make the right-hand outer diagram commute (i.e., such that $f_{d'}\,\alpha_* = f_d$) for each arrow α in D, there exists a unique map $\overline{f}$ in Set which makes the diagram commute; i.e., such that $\overline{f}\,i_d = f_d$ for each $d \in \mathsf{D}$. We call the cone $\{i_d\}$ the *colimiting cone* of X (or the initial cone out of X).

Remark 1.11.8. In other words, property (ii) states that every cone $\{f_d\}$ out of X factors uniquely through the colimiting cone $\{i_d\}$ of X.

Remark 1.11.9. The upshot is: giving a map $\overline{f}\colon \operatorname{colim}_{\mathsf{D}} X \to A$ in Set is the same as giving a cone $\{f_d\}$ out of X. For this reason, sometimes $\overline{f}$ is written as $\overline{f} = (f_d)_{d\in\mathsf{D}}$ or simply $\overline{f} = (f_d)$

To work effectively with the colimit $\operatorname{colim}_{\mathsf{D}} X$ of the diagram X, we need to understand how to verify that a pair of maps out of it are identical.

Proposition 1.11.10. *Let $X\colon \mathsf{D} \to \mathsf{Set}$ be a diagram. Assume that its colimit $\operatorname{colim}_{\mathsf{D}} X$ (1.87) exists. Consider any pair of maps of the form*

$$\operatorname{colim}_{\mathsf{D}} X \underset{g}{\overset{f}{\rightrightarrows}} A$$

in Set. Then f and g are identical if and only if their corresponding cones out of X are identical; i.e., in other words, $f = g$ if and only if $f\,i_d = g\,i_d$ for each $d \in \mathsf{D}$.

Proof. This follows from the universal property of colimits. □

Here is an equivalent way to formulate this observation; note how the change in notation makes the statement even more obvious.

Proposition 1.11.11. *Let $X\colon \mathsf{D} \to \mathsf{Set}$ be a diagram. Assume that its colimit $\operatorname{colim}_{\mathsf{D}} X$ (1.87) exists. Consider any pair of maps of the form*

$$\operatorname{colim}_{\mathsf{D}} X \underset{(g_d)}{\overset{(f_d)}{\rightrightarrows}} A$$

in Set. Then (f_d) and (g_d) are identical if and only if their corresponding cones out of X are identical; i.e., in other words, $(f_d) = (g_d)$ if and only if $f_d = g_d$ for each $d \in \mathsf{D}$.

For instance, consider any cones (k_d) and (n_d) out of X of the indicated right-hand form in (1.87); i.e., such that $k_{d'}\,\alpha_* = k_d$ and $n_{d'}\,\alpha_* = n_d$ for each arrow α in D. Then the left-hand diagram of the form

$$(1.88)\qquad \begin{array}{ccc} \operatorname{colim}_{\mathsf{D}} X & \xrightarrow{(n_d)} & C \\ {\scriptstyle (k_d)}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[h]{} & D \end{array} \qquad\qquad \begin{array}{ccc} X_d & \xrightarrow{n_d} & C \\ {\scriptstyle k_d}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[h]{} & D \end{array}$$

in Set commutes if and only if the corresponding right-hand diagram in Set commutes for each $d \in \mathsf{D}$.

REMARK 1.11.12. Arguing as above, the mapping properties in (1.87) characterize the colimit $\operatorname{colim}_{\mathsf{D}} X$, up to isomorphism, provided that it exists.

A *colimit diagram* in Set is a commutative diagram of the middle form (involving each arrow α in D) in (1.87) in Set, that satisfies the universal property of the right-hand form in (1.87).

REMARK 1.11.13. A diagram that is naturally isomorphic to a colimit diagram in Set, is a colimit diagram in Set.

A category D is *small* if its collection of objects forms a set, and *finite* if (i) its collection of objects forms a finite set and (ii) D has only a finite number of morphisms between any pair of objects. A diagram $X \colon \mathsf{D} \to \mathsf{Set}$ is *small* (resp. *finite*) if the indexing category D is small (resp. finite). It turns out that every small diagram in Set has a limit and a colimit. Before we give a construction of these, let's make a few observations and also recognize several examples from above as particular instances of limits and colimits.

Before getting to some examples, it is worth pointing out, that the following is often the easiest way to verify that a particular set is isomorphic to the limit of a diagram in Set; it also verifies existence of the limit, if we didn't know that already.

PROPOSITION 1.11.14. *Let $X \colon \mathsf{D} \to \mathsf{Set}$ be a diagram. If Z is a set, then $Z \cong \lim_{\mathsf{D}} X$ in Set if there exists a collection $\{\alpha_d\}$ of maps*

$$\alpha_d \colon Z \to X_d \qquad d \in \mathsf{D}$$

in Set indexed on the objects $d \in \mathsf{D}$, such that $\{\alpha_d\}$ is a cone into X which is terminal with respect to all such cones into X.

PROOF. Since $\{\alpha_d\}$ is a terminal cone into X, we know from (1.85) that Z is a limit of X; furthermore, given any limit $\lim_{\mathsf{D}} X$ of X, it follows from the universal property of limits that $Z \cong \lim_{\mathsf{D}} X$ in Set. ☐

Similarly, the following is often the easiest way to verify that a particular set is isomorphic to the colimit of a diagram in Set; it also verifies existence of the colimit, if we didn't know that already.

PROPOSITION 1.11.15. *Let $X \colon \mathsf{D} \to \mathsf{Set}$ be a diagram. If Z is a set, then $Z \cong \operatorname{colim}_{\mathsf{D}} X$ in Set if there exists a collection $\{\alpha_d\}$ of maps*

$$\alpha_d \colon X_d \to Z \qquad d \in \mathsf{D}$$

in Set indexed on the objects $d \in \mathsf{D}$, such that $\{\alpha_d\}$ is a cone out of X which is initial with respect to all such cones out of X.

PROOF. Since $\{\alpha_d\}$ is an initial cone out of X, we know from (1.87) that Z is a colimit of X; furthermore, given any colimit $\operatorname{colim}_{\mathsf{D}} X$ of X, it follows from the universal property of limits that $Z \cong \operatorname{colim}_{\mathsf{D}} X$ in Set. ☐

If D is the empty category, then there exists a unique diagram $X \colon \mathsf{D} \to \mathsf{Set}$ (the empty diagram, with no sets and no maps). In this case, the limit $\lim_{\mathsf{D}} X \cong *$ is the terminal object in Set (i.e., a one-point set), which we usually refer to as *the* terminal object in Set (because it sounds better), even though there are many one-point sets; this will not cause any confusion. The colimit $\mathrm{colim}_{\mathsf{D}} X \cong \emptyset$ is the initial object in Set (i.e., the empty set).

If D is a discrete category with a set of objects (i.e., D is a small discrete category), then a diagram $X\colon \mathsf{D} \to \mathsf{Set}$ has the form of a collection of sets X_d, $d \in \mathsf{D}$. In this case, the limit $\lim_{\mathsf{D}} X \cong \prod_{d\in\mathsf{D}} X_d$ is the product of the collection of sets. The colimit $\mathrm{colim}_{\mathsf{D}} X \cong \coprod_{d\in\mathsf{D}} X_d$ is the coproduct (or disjoint union) of the collection of sets.

Let $X\colon \mathsf{D} \to \mathsf{Set}$ be a diagram of the left-hand form

$$\begin{array}{ccc} & & X_c \\ & & \downarrow \\ X_b & \longrightarrow & X_d \end{array} \qquad\qquad \begin{array}{ccc} \lim_{\mathsf{D}} X & \xrightarrow{t_c} & X_c \\ {\scriptstyle t_b}\downarrow & & \downarrow \\ X_b & \longrightarrow & X_d \end{array}$$

in Set. In this case, the limit $\lim_{\mathsf{D}} X \cong X_b \times_{X_d} X_c$ is the pullback of the left-hand diagram; the right-hand limit diagram of X is a pullback diagram in Set. The colimit $\mathrm{colim}_{\mathsf{D}} X \cong X_d$ is the terminal object in the left-hand diagram.

Let $X\colon \mathsf{D} \to \mathsf{Set}$ be a diagram of the left-hand form

$$\begin{array}{ccc} X_a & \longrightarrow & X_c \\ \downarrow & & \\ X_b & & \end{array} \qquad\qquad \begin{array}{ccc} X_a & \longrightarrow & X_c \\ \downarrow & & \downarrow{\scriptstyle i_c} \\ X_b & \xrightarrow[i_b]{} & \mathrm{colim}_{\mathsf{D}} X \end{array}$$

in Set. In this case, the limit $\lim_{\mathsf{D}} X \cong X_a$ is the initial object in the left-hand diagram. The colimit $\mathrm{colim}_{\mathsf{D}} X \cong X_b \amalg_{X_a} X_c$ is the pushout of the left-hand diagram; the right-hand colimit diagram of X is a pushout diagram in Set.

Let $X\colon \mathsf{D} \to \mathsf{Set}$ be a diagram of the left-hand form

$$X_0 \underset{h}{\overset{g}{\rightrightarrows}} X_1 \qquad \lim_{\mathsf{D}} X \xrightarrow{t_0} X_0 \underset{h}{\overset{g}{\rightrightarrows}} X_1 \qquad X_0 \underset{h}{\overset{g}{\rightrightarrows}} X_1 \xrightarrow{i_1} \mathrm{colim}_{\mathsf{D}} X$$

in Set. In this case, the limit $\lim_{\mathsf{D}} X \cong \{x \in X_0 |\, g(x) = h(x)\}$ is the equalizer of the left-hand diagram. The colimit $\mathrm{colim}_{\mathsf{D}} X \cong X_1/\!\sim$ is the coequalizer of the left-hand diagram, where $\sim$ is the equivalence relation on X_1 generated by $g(x) \sim h(x)$ for each $x \in X_1$.

Let $X\colon \mathsf{D} \to \mathsf{Set}$ be a diagram of the form

$$X_0 \xleftarrow{\supset} X_1 \xleftarrow{\supset} X_2 \xleftarrow{\supset} X_3 \xleftarrow{\supset} X_4 \xleftarrow{\supset} \cdots$$

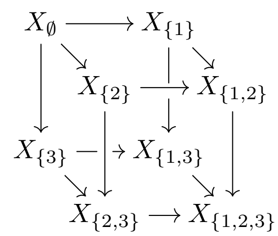
in Set. Assume that each of the maps is an inclusion. In this case, the limit $\lim_{\mathsf{D}} X \cong \bigcap_{i\geq 0} X_i$ is the intersection of the indicated sets. The colimit $\mathrm{colim}_{\mathsf{D}} X \cong X_0$ is the terminal object in the diagram; this remains true, even without the inclusions assumption.

Let $X\colon \mathsf{D} \to \mathsf{Set}$ be a diagram of the form

$$X_0 \xrightarrow{\subset} X_1 \xrightarrow{\subset} X_2 \xrightarrow{\subset} X_3 \xrightarrow{\subset} X_4 \xrightarrow{\subset} \cdots$$

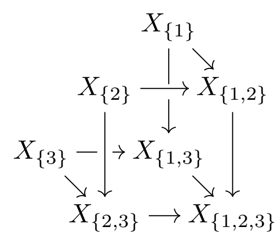
in Set. Assume that each of the maps is an inclusion. In this case, the limit $\lim_{\mathsf{D}} X \cong X_0$ is the initial object in the diagram; this remains true, even without the inclusions assumption. The colimit $\mathrm{colim}_{\mathsf{D}} X \cong \bigcup_{i\geq 0} X_i$ is the union of the indicated sets.

Let $X\colon \mathsf{D} \to \mathsf{Set}$ be a diagram of the form



in Set. In this case, the limit $\lim_{\mathsf{D}} X \cong X_\emptyset$ is the initial object in the diagram and the colimit $\operatorname{colim}_{\mathsf{D}} X \cong X_{\{1,2,3\}}$ is the terminal object in the diagram.

Let $X\colon \mathsf{D} \to \mathsf{Set}$ be a diagram of the form

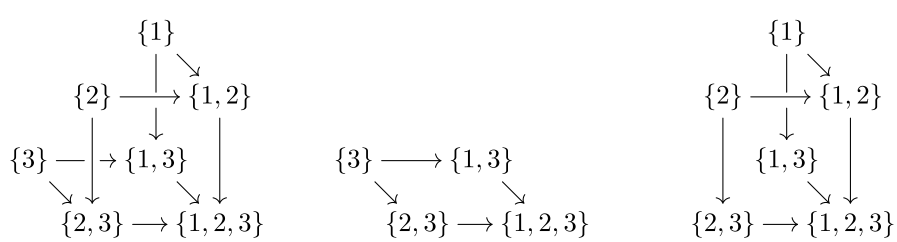


in Set. In this case, the limit $\lim_{\mathsf{D}} X$ is the pullback of the induced diagram of the left-hand form

(1.89)
$$\begin{array}{ccc} & & X_{\{2\}} \times_{X_{\{1,2\}}} X_{\{1\}} \\ & & \downarrow \\ X_{\{3\}} & \longrightarrow & X_{\{2,3\}} \times_{X_{\{1,2,3\}}} X_{\{1,3\}} \end{array} \qquad \begin{array}{ccc} \lim_{\mathsf{D}} X & \longrightarrow & X_{\{2\}} \times_{X_{\{1,2\}}} X_{\{1\}} \\ \downarrow & & \downarrow \\ X_{\{3\}} & \longrightarrow & X_{\{2,3\}} \times_{X_{\{1,2,3\}}} X_{\{1,3\}} \end{array}$$

in Set; i.e., the limit $\lim_{\mathsf{D}} X$ fits into a pullback diagram of the right-hand form. The colimit $\operatorname{colim}_{\mathsf{D}} X \cong X_{\{1,2,3\}}$ is the terminal object in the diagram.

There is a pattern underlying this example which is worth pointing out now. Consider the indexing poset (or category) D for X of the left-hand form



Denote by $\mathcal{A} \subset \mathsf{D}$ the indicated middle poset (or subcategory of D) and by $\mathcal{B} \subset \mathsf{D}$ the indicated right-hand poset (or subcategory of D). Then the poset (or category) $\mathsf{D} = \mathcal{A} \cup \mathcal{B}$ and the right-hand pullback diagram in (1.89) has the form

$$\begin{array}{ccc} \lim_{\mathcal{A}\cup\mathcal{B}} X & \longrightarrow & \lim_{\mathcal{B}} X \\ \downarrow & & \downarrow \\ \lim_{\mathcal{A}} X & \longrightarrow & \lim_{\mathcal{A}\cap\mathcal{B}} X \end{array}$$

This is a special case of the following useful observation. Define the sets $\mathbf{n} := \{1, \ldots, n\}$ for each $n \geq 0$, where $\mathbf{0} := \emptyset$ denotes the empty set. Denote by $\mathcal{P}(\mathbf{n})$ the poset of all subsets of $\mathbf{n}$, ordered by inclusion $\subset$ of sets. We will often regard $\mathcal{P}(\mathbf{n})$ as the category associated to this partial order in the usual way; the objects are the elements of $\mathcal{P}(\mathbf{n})$, and there is a morphism $U \to V$ if and only if $U \subset V$.

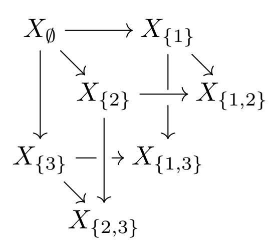
Let $n \geq 1$ and consider any subset $\mathcal{B} \subset \mathcal{P}(\mathbf{n})$. A subset $\mathcal{A} \subset \mathcal{B}$ is *concave* if every element of $\mathcal{B}$ which is greater than an element of $\mathcal{A}$ is in $\mathcal{A}$. The following is proved in [**2**, pp. 317–318].

Proposition 1.11.16. *Let $n \geq 1$. Consider the indexing poset (or category) $\mathsf{D} = \mathcal{P}(\mathbf{n})$ and let $X \colon \mathsf{D} \to \mathsf{Set}$ be a diagram. For any concave subsets $\mathcal{A}, \mathcal{B}$ of $\mathcal{P}(\mathbf{n})$, the diagram*

$$\begin{array}{ccc} \lim_{\mathcal{A}\cup\mathcal{B}} X & \longrightarrow & \lim_{\mathcal{B}} X \\ \downarrow & & \downarrow \\ \lim_{\mathcal{A}} X & \longrightarrow & \lim_{\mathcal{A}\cap\mathcal{B}} X \end{array}$$

is a pullback diagram in Set.

Let $X \colon \mathsf{D} \to \mathsf{Set}$ be a diagram of the form

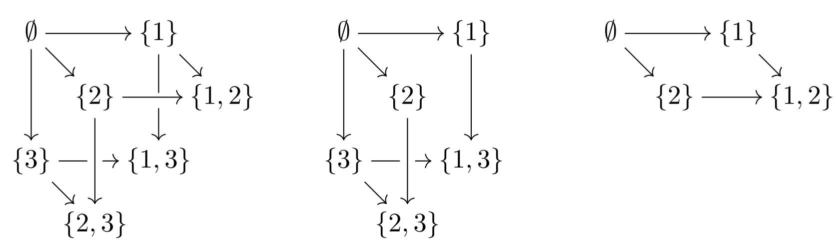


In this case, the limit $\lim_{\mathsf{D}} X \cong X_\emptyset$ is the initial object in the diagram. The colimit $\operatorname{colim}_{\mathsf{D}} X$ is the pushout of the induced diagram of the left-hand form

(1.90)
$$\begin{array}{ccc} X_{\{2\}} \amalg_{X_\emptyset} X_{\{1\}} & \longrightarrow & X_{\{1,2\}} \\ \downarrow & & \\ X_{\{2,3\}} \amalg_{X_{\{3\}}} X_{\{1,3\}} & & \end{array} \qquad \begin{array}{ccc} X_{\{2\}} \amalg_{X_\emptyset} X_{\{1\}} & \longrightarrow & X_{\{1,2\}} \\ \downarrow & & \downarrow \\ X_{\{2,3\}} \amalg_{X_{\{3\}}} X_{\{1,3\}} & \longrightarrow & \operatorname{colim}_{\mathsf{D}} X \end{array}$$

in Set; i.e., the colimit $\operatorname{colim}_{\mathsf{D}} X$ fits into a pushout diagram of the right-hand form.

There is a pattern underlying this example which is worth pointing out now; we will further develop this later: consider the indexing poset (or category) D for X of the left-hand form



Denote by $\mathcal{A} \subset \mathsf{D}$ the indicated middle poset (or subcategory of D) and by $\mathcal{B} \subset \mathsf{D}$ the indicated right-hand poset (or subcategory of D). Then the poset (or category) $\mathsf{D} = \mathcal{A} \cup \mathcal{B}$ and the right-hand pullback diagram in (1.90) has the form

$$\begin{array}{ccc} \operatorname{colim}_{\mathcal{A}\cap\mathcal{B}} X & \longrightarrow & \operatorname{colim}_{\mathcal{B}} X \\ \downarrow & & \downarrow \\ \operatorname{colim}_{\mathcal{A}} X & \longrightarrow & \operatorname{colim}_{\mathcal{A}\cup\mathcal{B}} X \end{array}$$

This is a special case of the following useful observation. Let $n \geq 1$ and consider any subset $\mathcal{B} \subset \mathcal{P}(\mathbf{n})$. A subset $\mathcal{A} \subset \mathcal{B}$ is *convex* if every element of $\mathcal{B}$ which is less than an element of $\mathcal{A}$ is in $\mathcal{A}$. The following is proved in [**2**, pp. 314–315].

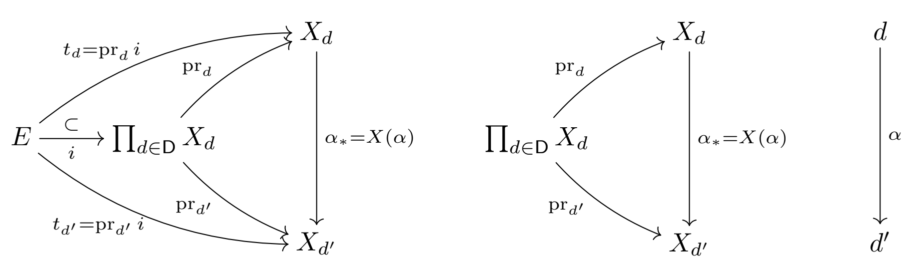
PROPOSITION 1.11.17. *Let $n \geq 1$. Consider the indexing poset (or category) $\mathsf{D} = \mathcal{P}(\mathbf{n})$ and let $X\colon \mathsf{D} \to \mathsf{Set}$ be a diagram. For any convex subsets $\mathcal{A}, \mathcal{B}$ of $\mathcal{P}(\mathbf{n})$, the diagram*

$$\begin{array}{ccc} \operatorname{colim}_{\mathcal{A}\cap\mathcal{B}} X & \longrightarrow & \operatorname{colim}_{\mathcal{B}} X \\ \downarrow & & \downarrow \\ \operatorname{colim}_{\mathcal{A}} X & \longrightarrow & \operatorname{colim}_{\mathcal{A}\cup\mathcal{B}} X \end{array}$$

is a pushout diagram in Set.

PROPOSITION 1.11.18. *Let D be a small category and $X\colon \mathsf{D} \to \mathsf{Set}$ a diagram. Then the limit $\lim_{\mathsf{D}} X$ exists. In other words,* Set *has all small limits.*

PROOF. Here is the classical way to build the limit $\lim_{\mathsf{D}} X$. As a first step, we look for something that has naturally occurring maps into X_d, for each object $d \in \mathsf{D}$; we already know of such a construction, it is the product $\prod_{d\in\mathsf{D}} X_d$. This product exists since D has a set of objects. Giving this a try, we get a diagram of the middle form



for each arrow α in D. There is no reason for the middle diagram to commute, in general, for each arrow α in D. So as a second step, the idea is to force it to commute by restricting to the subset $E \subset \prod_{d\in\mathsf{D}} X_d$ of elements where this is satisfied. Define $E := \{e \in \prod_{d\in\mathsf{D}} X_d | \, \alpha_* \operatorname{pr}_d(e) = \operatorname{pr}_{d'}(e) \text{ for each arrow } \alpha \text{ in } \mathsf{D}\}$. This leads us to the left-hand outer diagram which commutes (we just forced it to); this left-hand outer diagram (involving each arrow α in D) is a limit diagram of X. It is easy to check that $E \cong \lim_{\mathsf{D}} X$: we could either use the classical constructions of products and subsets to verify the universal property of limits in (1.85), or we could work directly with the universal properties of products and subsets to verify (1.85). $\square$

REMARK 1.11.19. Let's reformulate this argument in terms of products and equalizers. Consider a pair of maps of the form

$$\prod_{d\in\mathsf{D}} X_d \overset{g}{\underset{g'}{\rightrightarrows}} \prod_{\substack{\alpha\,:\,d\to d'\\ \text{in }\mathsf{D}}} X_{d'}$$

in Set. Consider the middle diagram above (which does not commute, in general); we want to force it to commute by restricting along an appropriate equalizer. Let's define the map g so that, for each arrow α in D, it picks out the bottom part of the middle diagram above (i.e., the map $\mathrm{pr}_{d'}$). Similarly, let's define the map g' so that, for each arrow α in D, it picks out the other part of the middle diagram above (i.e., the composite $\alpha_* \,\mathrm{pr}_d$). Remember, giving a map into a product of the right-hand form is the same as giving, for each arrow $\alpha\colon\ d\to d'$ in D, a map into X'_d. Hence we can describe g and g' as induced by the following diagram—start with the projections on the right-hand side, then work towards the left-hand projections. The projections pointing up define g and the projections pointing down define g'.

$$\begin{array}{ccc}
X_{d'} & = & X_{d'} \\
\uparrow{\scriptstyle \mathrm{pr}_{d'}} & & \uparrow{\scriptstyle \mathrm{pr}_{\alpha\,:\,d\to d'}} \\
\prod_{d\in\mathsf{D}} X_d & \overset{g}{\underset{g'}{\rightrightarrows}} & \prod_{\substack{\alpha\,:\,d\to d'\\ \text{in }\mathsf{D}}} X_{d'} \\
\downarrow{\scriptstyle \mathrm{pr}_{d}} & & \downarrow{\scriptstyle \mathrm{pr}_{\alpha\,:\,d\to d'}} \\
X_d & \underset{\alpha_*}{\longrightarrow} & X_{d'}
\end{array}$$

Think of g as induced on each projection map $\mathrm{pr}_{\alpha\,:d\to d'}$ via the map that "misses" or "removes" the object d in the arrow $\alpha\colon\ d\to d'$; if we start with the arrow $d\to d'$ and miss or remove the object d, then we are left with the object d' (hence the map $\mathrm{pr}_{d'}$). Similarly, think of g' as induced on each projection map $\mathrm{pr}_{\alpha\,:d\to d'}$ via the map that "misses" or "removes" the object d' in the arrow $\alpha\colon\ d\to d'$; if we start with the arrow $d\to d'$ and miss or remove the object d', then we are left with the object d (hence the map pr_d), and to get from X_d to $X_{d'}$ we compose with the induced map $\alpha_*\colon\ X_d\to X_{d'}$. It is easy to check that $\lim_{\mathsf{D}} X$ can be constructed as the equalizer

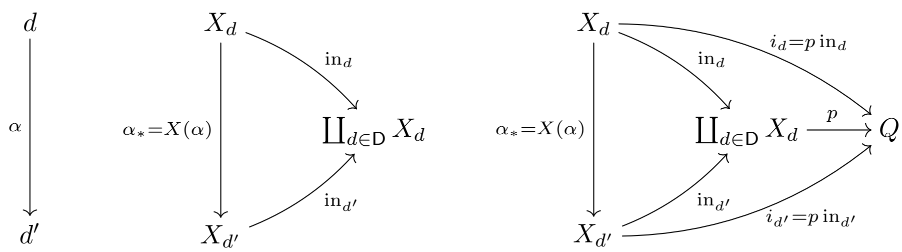
$$\lim_{\mathsf{D}} X \cong \lim\Big(\prod_{d\in\mathsf{D}} X_d \overset{g}{\underset{g'}{\rightrightarrows}} \prod_{\substack{\alpha\,:\,d\to d'\\ \text{in }\mathsf{D}}} X_{d'}\Big)$$

of the pair of maps g, g': we could either use the classical constructions of products and equalizers to recover the description of E above, or we could work directly with the universal properties of products and equalizers. Conceptually, this means that once we know all equalizers and small products exist in Set, then we know that all small limits exist in Set. There is a pattern underlying this construction which is worth remarking on now; we will further develop this later: the pair of maps g, g' is the beginning part of a cosimplicial resolution associated to $\lim_{\mathsf{D}} X$, that will be useful later when we construct the Bousfield-Kan homotopy limit functor.

PROPOSITION 1.11.20. *Let* D *be a small category and* $X\colon\ \mathsf{D}\to\mathsf{Set}$ *a diagram. Then the colimit* $\mathrm{colim}_{\mathsf{D}} X$ *exists. In other words,* Set *has all small colimits.*

PROOF. Here is the classical way to build the colimit $\mathrm{colim}_{\mathsf{D}} X$. As a first step, we look for something that has naturally occurring maps out of X_d, for each object

$d \in \mathsf{D}$; we already know of such a construction, it is the coproduct $\coprod_{d\in\mathsf{D}} X_d$. This coproduct exists since D has a set of objects. Giving this a try, we get a diagram of the middle form

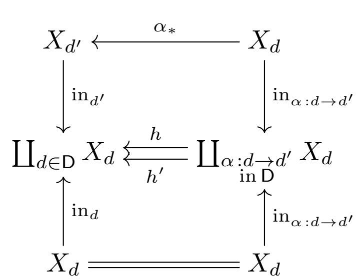


for each arrow α in D. There is no reason for the middle diagram to commute, in general, for each arrow α in D. So as a second step, the idea is to force it to commute by mapping to the quotient set Q of $\coprod_{d\in\mathsf{D}} X_d$ where this is satisfied. Define $Q := \left(\coprod_{d\in\mathsf{D}} X_d\right)/\!\sim$, where $\sim$ is the equivalence relation on $\coprod_{d\in\mathsf{D}} X_d$ generated by $\mathrm{in}_d(x) \sim \mathrm{in}_{d'}\,\alpha_*(x)$ for each arrow α in D and $x \in X_d$. This leads us to the right-hand outer diagram which commutes (we just forced it to); this right-hand outer diagram (involving each arrow α in D) is a colimit diagram of X. It is easy to check that $Q \cong \mathrm{colim}_{\mathsf{D}}\, X$: we could either use the classical constructions of coproducts and quotient sets to verify the universal property of colimits in (1.87), or we could work directly with the universal properties of coproducts and quotient sets to verify (1.87). $\square$

Remark 1.11.21. Let's reformulate this argument in terms of coproducts and coequalizers. Consider a pair of maps of the form

$$\coprod_{d\in\mathsf{D}} X_d \underset{h'}{\overset{h}{\leftleftarrows}} \coprod_{\substack{\alpha\colon d\to d'\\ \text{in } \mathsf{D}}} X_d$$

in Set. Consider the middle diagram above (which does not commute, in general); we want to force it to commute by mapping to an appropriate coequalizer. Let's define the map h so that, for each arrow α in D, it picks out the bottom part of the middle diagram above (i.e., the composite $\mathrm{in}_{d'}\,\alpha_*$). Similarly, let's define the map h' so that, for each arrow α in D, it picks out the other part of the middle diagram above (i.e., the map in_d). Remember, giving a map out of a coproduct of the right-hand form is the same as giving, for each arrow $\alpha\colon d \to d'$ in D, a map out of X_d. Hence we can describe h and h' as induced by the following diagram—start with the inclusions on the right-hand side, then work towards the left-hand inclusions. The inclusions pointing down define h and the inclusions pointing up define h'.



Think of h as induced on each inclusion map $\mathrm{in}_{\alpha\colon d\to d'}$ via the map that "misses" or "removes" the object d in the arrow $\alpha\colon d\to d'$; if we start with the arrow $d\to d'$ and miss or remove the object d, then we are left with the object d' (hence the map $\mathrm{in}_{d'}$), and to get from X_d to $X_{d'}$ we compose with the induced map $\alpha_*\colon X_d\to X_{d'}$. Similarly, think of h' as induced on each inclusion map $\mathrm{in}_{\alpha\colon d\to d'}$ via the map that "misses" or "removes" the object d' in the arrow $\alpha\colon d\to d'$; if we start with the arrow $d\to d'$ and miss or remove the object d', then we are left with the object d (hence the map in_d). It is easy to check that $\mathrm{colim}_{\mathsf{D}} X$ can be constructed as the coequalizer

$$\mathrm{colim}_{\mathsf{D}} X \cong \mathrm{colim}\Big(\textstyle\coprod_{d\in\mathsf{D}} X_d \overset{h}{\underset{h'}{\leftleftarrows}} \coprod_{\substack{\alpha\colon d\to d'\\ \text{in } \mathsf{D}}} X_d \Big)$$

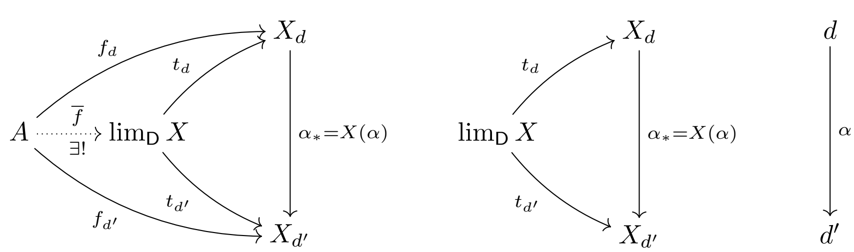
of the pair of maps h, h': we could either use the classical constructions of coproducts and coequalizers to recover the description of Q above, or we could work directly with the universal properties of coproducts and coequalizers. Conceptually, this means that once we know all coequalizers and small coproducts exist in Set, then we know that all small colimits exist in Set. There is a pattern underlying this construction which is worth remarking on now; we will further develop this later: the pair of maps h, h' is the beginning part of a simplicial resolution associated to $\mathrm{colim}_{\mathsf{D}} X$, that will be useful later when we construct the Bousfield-Kan homotopy colimit functor.

What happens if we replace Set with Top in our above discussion? We have already worked out what we need. Let M be a category; the reader should be thinking of the special case $\mathsf{M} = \mathsf{Top}$, for now, in which case the phrase "an object in M" can be replaced with the phrase "a topological space".

A D-*shaped diagram* in M is a functor $X\colon \mathsf{D}\to\mathsf{M}$; here, D is a category which we sometimes call the indexing category for the diagram X. Let $X\colon \mathsf{D}\to\mathsf{M}$ be a diagram. A *limit* of X, denoted $\lim_{\mathsf{D}} X$, is an object in M with the following mapping properties: (i) (Cone): there is a collection $\{t_d\}$ of maps

$$t_d\colon \lim_{\mathsf{D}} X \to X_d \qquad d\in\mathsf{D}$$

in M, indexed on the objects $d\in\mathsf{D}$, which make the middle diagram

(1.91)


commute (i.e., such that $\alpha_*\, t_d = t_{d'}$) for each arrow α in D (such a collection $\{t_d\}$ is sometimes called a *cone* into X) and (ii) (Universal property): the cone $\{t_d\}$ is terminal with respect to all such cones into X; i.e., for any object A in M and collection $\{f_d\}$ of maps

$$f_d\colon A\to X_d \qquad d\in\mathsf{D}$$

in M, indexed on the objects $d\in\mathsf{D}$, which make the left-hand outer diagram commute (i.e., such that $\alpha_*\, f_d = f_{d'}$) for each arrow α in D, there exists a unique map $\overline{f}$ in M which makes the diagram commute; i.e., such that $t_d \overline{f} = f_d$ for each $d \in \mathsf{D}$. We call the cone $\{t_d\}$ the *limiting cone* of X (or the terminal cone into X).

Remark 1.11.22. In other words, property (ii) states that every cone $\{f_d\}$ into X factors uniquely through the limiting cone $\{t_d\}$ of X.

Remark 1.11.23. The upshot is: giving a map $\overline{f}\colon A \to \lim_{\mathsf{D}} X$ in M is the same as giving a cone $\{f_d\}$ into X. For this reason, sometimes $\overline{f}$ is written as $\overline{f} = (f_d)_{d\in\mathsf{D}}$ or simply $\overline{f} = (f_d)$.

Remark 1.11.24. Arguing as above, the mapping properties in (1.91) characterize the limit $\lim_{\mathsf{D}} X$, up to isomorphism, provided that it exists.

A *limit diagram* in M is a commutative diagram of the middle form (involving each arrow α in D) in (1.91) in M, that satisfies the universal property of the left-hand form in (1.91).

Remark 1.11.25. A diagram that is naturally isomorphic to a limit diagram in M, is a limit diagram in M.

Proposition 1.11.26. *Let* D *be a small category and* $X\colon \mathsf{D} \to \mathsf{Top}$ *a diagram. Then the limit* $\lim_{\mathsf{D}} X$ *exists. In other words,* Top *has all small limits.*

Proof. We already know that Top has all equalizers and small products. Hence the argument in Remark 1.11.19, with Set replaced by Top, shows that $\lim_{\mathsf{D}} X$ can be constructed as the equalizer

$$\lim_{\mathsf{D}} X \cong \lim\Big(\textstyle\prod_{d\in\mathsf{D}} X_d \overset{g}{\underset{g'}{\rightrightarrows}} \prod_{\substack{\alpha\colon d\to d' \\ \text{in } \mathsf{D}}} X_{d'} \Big)$$

of the pair of maps g, g'. □

Proposition 1.11.27. *Let* M *be a category with all equalizers and small (resp. finite) products. Let* $X\colon \mathsf{D} \to \mathsf{M}$ *a small (resp. finite) diagram. Then the limit* $\lim_{\mathsf{D}} X$ *exists. In other words,* M *has all small (resp. finite) limits.*

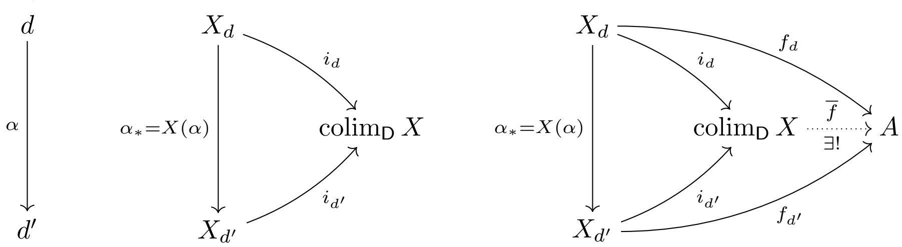
Proof. This follows from the proof of Proposition 1.11.26, with Top replaced by M. □

Let $X\colon \mathsf{D} \to \mathsf{M}$ be a diagram. A *colimit* of X, denoted $\operatorname{colim}_{\mathsf{D}} X$, is an object in M with the following mapping properties: (i) (Cone): there is a collection $\{i_d\}$ of maps

$$i_d\colon X_d \to \operatorname{colim}_{\mathsf{D}} X \qquad d \in \mathsf{D}$$

in M, indexed on the objects $d \in \mathsf{D}$, which make the middle diagram

(1.92)



commute (i.e., such that $i_{d'}\,\alpha_* = i_d$) for each arrow α in D (such a collection $\{i_d\}$ is sometimes called a *cone* out of X) and (ii) (Universal property): the cone $\{i_d\}$ is initial with respect to all such cones out of X; i.e., for any object A in M and collection $\{f_d\}$ of maps

$$f_d\colon\ X_d \to A \qquad d \in \mathsf{D}$$

in M, indexed on the objects $d \in \mathsf{D}$, which make the right-hand outer diagram commute (i.e., such that $f_{d'}\,\alpha_* = f_d$) for each arrow α in D, there exists a unique map $\overline{f}$ in M which makes the diagram commute; i.e., such that $\overline{f}\,i_d = f_d$ for each $d \in \mathsf{D}$. We call the cone $\{i_d\}$ the *colimiting cone* of X (or the initial cone out of X).

Remark 1.11.28. In other words, property (ii) states that every cone $\{f_d\}$ out of X factors uniquely through the colimiting cone $\{i_d\}$ of X.

Remark 1.11.29. The upshot is: giving a map $\overline{f}\colon \operatorname{colim}_{\mathsf{D}} X \to A$ in M is the same as giving a cone $\{f_d\}$ out of X. For this reason, sometimes $\overline{f}$ is written as $\overline{f} = (f_d)_{d\in\mathsf{D}}$ or simply $\overline{f} = (f_d)$.

Proposition 1.11.30. *Let* D *be a small category and* $X\colon\ \mathsf{D} \to \mathsf{Top}$ *a diagram. Then the colimit* $\operatorname{colim}_{\mathsf{D}} X$ *exists. In other words,* Top *has all small colimits.*

Proof. We already know that Top has all coequalizers and small coproducts. Hence the argument in Remark 1.11.21, with Set replaced by Top, shows that $\operatorname{colim}_{\mathsf{D}} X$ can be constructed as the coequalizer

$$\operatorname{colim}_{\mathsf{D}} X \cong \operatorname{colim}\Big(\textstyle\coprod_{d\in\mathsf{D}} X_d \overset{h}{\underset{h'}{\longleftleftarrows}} \coprod_{\substack{\alpha\colon d\to d' \\ \text{in } \mathsf{D}}} X_d \Big)$$

of the pair of maps h, h'. ☐

Proposition 1.11.31. *Let* M *be a category with all coequalizers and small (resp. finite) coproducts. Let* $X\colon\ \mathsf{D} \to \mathsf{M}$ *a small (resp. finite) diagram. Then the colimit* $\operatorname{colim}_{\mathsf{D}} X$ *exists. In other words,* M *has all small (resp. finite) colimits.*

Proof. This follows from the proof of Proposition 1.11.30, with Top replaced by M. ☐

1.12. Ends and coends

Ends and coends provide a form of hom and tensor for diagrams. Let's develop this idea. Let M be a category; the reader should be thinking of the special case $\mathsf{M} = \mathsf{Set}$, for now, in which case the phrase "an object in M" can be replaced with the phrase "a set". Let D be a category. The *opposite* category D^{op} is the category with the same objects as D, but with one arrow $a \xleftarrow{\alpha^{\mathrm{op}}} b$ for each arrow $a \xrightarrow{\alpha} b$ in D. For instance, if D is the category of the form $b \to d \leftarrow c$, then D^{op} is the category of the form $b \leftarrow d \to c$. In this case, a diagram $A\colon\ \mathsf{D}^{\mathrm{op}} \to \mathsf{M}$ (resp. $B\colon\ \mathsf{D} \to \mathsf{M}$) has the form

$$A_b \longleftarrow A_d \longrightarrow A_c \qquad \Big(\text{resp.}\quad B_b \longrightarrow B_d \longleftarrow B_c\Big)$$

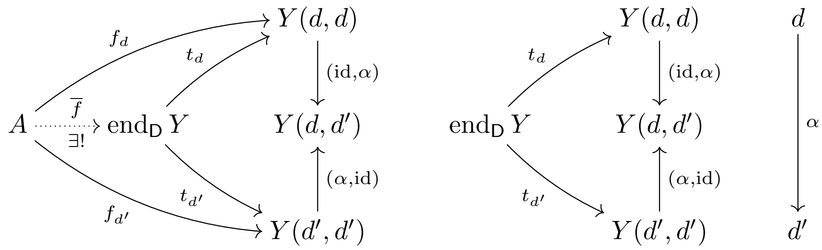
in M. If D is a category, note that $(\mathsf{D}^{\mathrm{op}})^{\mathrm{op}} = \mathsf{D}$. Ends and coends are closely related to limits and colimits, respectively—they involve wedges instead of cones. Let's develop these ideas. Let D be a category and $Y\colon\ \mathsf{D}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{M}$ a diagram.

An *end* of Y, denoted $\mathrm{end}_{\mathsf{D}}\, Y$ or Y^{D}, is an object in M with the following mapping properties: (i) (Wedge): there is a collection $\{t_d\}$ of maps

$$t_d\colon\ \mathrm{end}_{\mathsf{D}}\, Y \to Y(d,d) \qquad d \in \mathsf{D}$$

in M, indexed on the objects $d \in \mathsf{D}$, which make the middle diagram

(1.93)



commute (i.e., such that $(\mathrm{id},\alpha)\, t_d = (\alpha,\mathrm{id})\, t_{d'}$) for each arrow α in D (such a collection $\{t_d\}$ is sometimes called a *wedge* into Y) and (ii) (Universal property): the wedge $\{t_d\}$ is terminal with respect to all such wedges into Y; i.e., for any object A in M and collection $\{f_d\}$ of maps

$$f_d\colon\ A \to Y(d,d) \qquad d \in \mathsf{D}$$

in M, indexed on the objects $d \in \mathsf{D}$, which make the left-hand outer diagram commute (i.e., such that $(\mathrm{id},\alpha)\, f_d = (\alpha,\mathrm{id})\, f_{d'}$) for each arrow α in D, there exists a unique map $\overline{f}$ in M which makes the diagram commute; i.e., such that $t_d\, \overline{f} = f_d$ for each $d \in \mathsf{D}$. We call the wedge $\{t_d\}$ the *ending wedge* of Y (or the terminal wedge into Y).

REMARK 1.12.1. In other words, property (ii) states that every wedge $\{f_d\}$ into Y factors uniquely through the ending wedge $\{t_d\}$ of Y.

REMARK 1.12.2. The upshot is: giving a map $\overline{f}\colon A \to \mathrm{end}_{\mathsf{D}}\, Y$ in M is the same as giving a wedge $\{f_d\}$ into Y. For this reason, sometimes $\overline{f}$ is written as $\overline{f} = (f_d)_{d\in\mathsf{D}}$ or simply $\overline{f} = (f_d)$.

To work effectively with the end $\mathrm{end}_{\mathsf{D}}\, Y$ of the diagram Y, we need to understand how to verify that a pair of maps into it are identical.

PROPOSITION 1.12.3. *Let* $Y\colon \mathsf{D}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{M}$ *be a diagram. Assume that its end* $\mathrm{end}_{\mathsf{D}}\, Y$ (1.93) *exists. Consider any pair of maps of the form*

$$A \underset{g}{\overset{f}{\rightrightarrows}} \mathrm{end}_{\mathsf{D}}\, Y$$

in M. *Then f and g are identical if and only if their corresponding wedges into Y are identical; i.e., in other words, $f = g$ if and only if $t_d\, f = t_d\, g$ for each $d \in \mathsf{D}$.*

PROOF. This follows from the universal property of ends. □

Here is an equivalent way to formulate this observation; note how the change in notation makes the statement even more obvious.

PROPOSITION 1.12.4. *Let* $Y\colon \mathsf{D}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{M}$ *be a diagram. Assume that its end* $\mathrm{end}_{\mathsf{D}}\, Y$ (1.93) *exists. Consider any pair of maps of the form*

$$A \underset{(g_d)}{\overset{(f_d)}{\rightrightarrows}} \mathrm{end}_{\mathsf{D}}\, Y$$

in M*. Then* (f_d) *and* (g_d) *are identical if and only if their corresponding wedges into* Y *are identical; i.e., in other words,* $(f_d) = (g_d)$ *if and only if* $f_d = g_d$ *for each* $d \in \mathsf{D}$*.*

For instance, consider any wedges $\{p_d\}$ and $\{h_d\}$ into Y of the indicated left-hand form in (1.93); i.e., such that $(\mathrm{id}, \alpha)\, p_d = (\alpha, \mathrm{id})\, p_{d'}$ and $(\mathrm{id}, \alpha)\, h_d = (\alpha, \mathrm{id})\, h_{d'}$ for each arrow α in D. Then the left-hand diagram of the form

$$
\begin{array}{ccc}
A & \xrightarrow{g} & C \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle (p_d)} \\
B & \xrightarrow[(h_d)]{} & \mathrm{end}_{\mathsf{D}}\, Y
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & \xrightarrow{g} & C \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p_d} \\
B & \xrightarrow[h_d]{} & Y(d,d)
\end{array}
\tag{1.94}
$$

in M commutes if and only if the corresponding right-hand diagram in M commutes for each $d \in \mathsf{D}$.

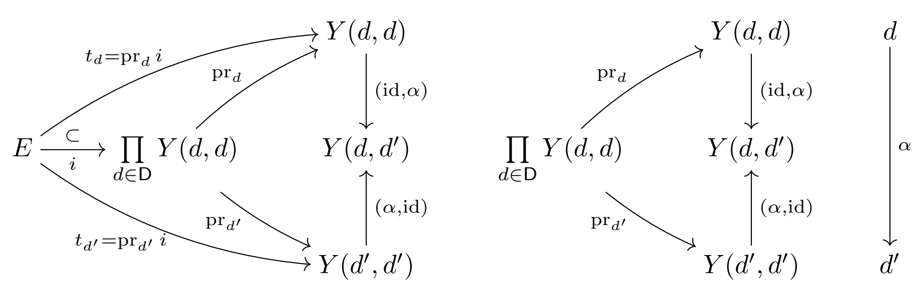
REMARK 1.12.5. Arguing as above, the mapping properties in (1.93) characterize the end $\mathrm{end}_{\mathsf{D}}\, Y$, up to isomorphism, provided that it exists.

An *end diagram* in M is a commutative diagram of the middle form (involving each arrow α in D) in (1.93) in M, that satisfies the universal property of the left-hand form in (1.93).

REMARK 1.12.6. A diagram that is naturally isomorphic to an end diagram in M, is an end diagram in M.

PROPOSITION 1.12.7. *Let* D *be a small category and* $Y\colon \mathsf{D}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{Set}$ *a diagram. Then the end* $\mathrm{end}_{\mathsf{D}}\, X$ *exists. In other words,* Set *has all small ends.*

PROOF. Here is the classical way to build the end $\mathrm{end}_{\mathsf{D}}\, Y$. As a first step, we look for something that has naturally occurring maps into $Y(d,d)$, for each object $d \in \mathsf{D}$; we already know of such a construction, it is the product $\prod_{d\in\mathsf{D}} Y(d,d)$. This product exists since D has a set of objects. Giving this a try, we get a diagram of the middle form



for each arrow α in D. There is no reason for the middle diagram to commute, in general, for each arrow α in D. So as a second step, the idea is to force it to commute by restricting to the subset $E \subset \prod_{d\in\mathsf{D}} Y(d,d)$ of elements where this is satisfied. Define $E := \{e \in \prod_{d\in\mathsf{D}} Y(d,d) |\, (\mathrm{id}, \alpha)\, \mathrm{pr}_d(e) = (\alpha, \mathrm{id})\, \mathrm{pr}_{d'}(e)$ for each arrow α in $\mathsf{D}\}$. This leads us to the left-hand outer diagram which commutes (we just forced it to); this left-hand outer diagram (involving each arrow α in D) is a limit diagram of X. It is easy to check that $E \cong \mathrm{end}_{\mathsf{D}}\, Y$: we could either use the classical constructions of products and subsets to verify the universal property of limits in (1.93), or we could work directly with the universal properties of products and subsets to verify (1.93). ☐

Remark 1.12.8. This argument can be reformulated in terms of products and equalizers as follows. Consider a pair of maps of the form

$$\prod_{d\in \mathsf{D}} Y(d,d) \underset{g'}{\overset{g}{\rightrightarrows}} \prod_{\substack{\alpha\,:d\to d' \\ \text{in } \mathsf{D}}} Y(d,d')$$

in Set. Consider the middle diagram above (which does not commute, in general); we want to force it to commute by restricting along an appropriate equalizer. Let's define the map g so that, for each arrow α in D, it picks out the bottom part of the middle diagram above (i.e., the composite $(\alpha,\mathrm{id})\,\mathrm{pr}_{d'}$). Similarly, let's define the map g' so that, for each arrow α in D, it picks out the other part of the middle diagram above (i.e., the composite $(\mathrm{id},\alpha)\,\mathrm{pr}_d$). Remember, giving a map into a product of the right-hand form is the same as giving, for each arrow $\alpha\colon d\to d'$ in D, a map into $Y(d,d')$. Hence we can describe g and g' as induced by the following diagram—start with the projections on the right-hand side, then work towards the left-hand projections. The projections pointing up define g and the projections pointing down define g'.

$$\begin{array}{ccc}
Y(d',d') & \xrightarrow{(\alpha,\mathrm{id})} & Y(d,d') \\
\Big\uparrow{\scriptstyle \mathrm{pr}_{d'}} & & \Big\uparrow{\scriptstyle \mathrm{pr}_{\alpha\,:d\to d'}} \\
\prod_{d\in \mathsf{D}} Y(d,d) & \underset{g'}{\overset{g}{\rightrightarrows}} & \prod_{\substack{\alpha\,:d\to d' \\ \text{in } \mathsf{D}}} Y(d,d') \\
\Big\downarrow{\scriptstyle \mathrm{pr}_{d}} & & \Big\downarrow{\scriptstyle \mathrm{pr}_{\alpha\,:d\to d'}} \\
Y(d,d) & \xrightarrow[(\mathrm{id},\alpha)]{} & Y(d,d')
\end{array}$$

Think of g as induced on each projection map $\mathrm{pr}_{\alpha\,:d\to d'}$ via the map that "misses" or "removes" the object d in the arrow $\alpha\colon d\to d'$; if we start with the arrow $d\to d'$ and miss or remove the object d, then we are left with the object d' (hence the map $\mathrm{pr}_{d'}$), and to get from $Y(d',d')$ to $Y(d,d')$ we compose with the induced map $(\alpha,\mathrm{id})\colon Y(d',d')\to Y(d,d')$. Similarly, think of g' as induced on each projection map $\mathrm{pr}_{\alpha\,:d\to d'}$ via the map that "misses" or "removes" the object d' in the arrow $\alpha\colon d\to d'$; if we start with the arrow $d\to d'$ and miss or remove the object d', then we are left with the object d (hence the map pr_d), and to get from $Y(d,d)$ to $Y(d,d')$ we compose with the induced map $(\mathrm{id},\alpha)\colon Y(d,d)\to Y(d,d')$. It is easy to check that $\mathrm{end}_{\mathsf{D}}\, Y$ can be constructed as the equalizer

$$\mathrm{end}_{\mathsf{D}}\, Y \cong \lim\Big(\prod_{d\in \mathsf{D}} Y(d,d) \underset{g'}{\overset{g}{\rightrightarrows}} \prod_{\substack{\alpha\,:d\to d' \\ \text{in } \mathsf{D}}} Y(d,d') \Big)$$

of the pair of maps g,g': we could either use the classical constructions of products and equalizers to recover the description of E above, or we could work directly with the universal properties of products and equalizers. Conceptually, this means that once we know all equalizers and small products exist in Set, then we know that all small ends exist in Set.

Proposition 1.12.9. *Let* D *be a small category and* $Y\colon \mathsf{D}^{\mathrm{op}}\times\mathsf{D}\to\mathsf{Top}$ *a diagram. Then the end* $\mathrm{end}_{\mathsf{D}}\, Y$ *exists. In other words,* Top *has all small ends.*

Proof. We already know that Top has all equalizers and small products. Hence the argument in Remark 1.12.8, with Set replaced by Top, shows that $\mathrm{end}_{\mathsf{D}} Y$ can be constructed as the equalizer

$$\mathrm{end}_{\mathsf{D}} Y \cong \lim\Big(\prod_{d\in\mathsf{D}} Y(d,d) \overset{g}{\underset{g'}{\rightrightarrows}} \prod_{\substack{\alpha: d\to d'\\ \text{in } \mathsf{D}}} Y(d,d') \Big)$$

of the pair of maps g, g'. □

Proposition 1.12.10. *Let* M *be a category with all equalizers and small (resp. finite) products. Let* $Y\colon \mathsf{D}^{\mathrm{op}}\times\mathsf{D}\to\mathsf{M}$ *a small (resp. finite) diagram. Then the end* $\mathrm{end}_{\mathsf{D}} Y$ *exists. In other words,* M *has all small (resp. finite) ends.*

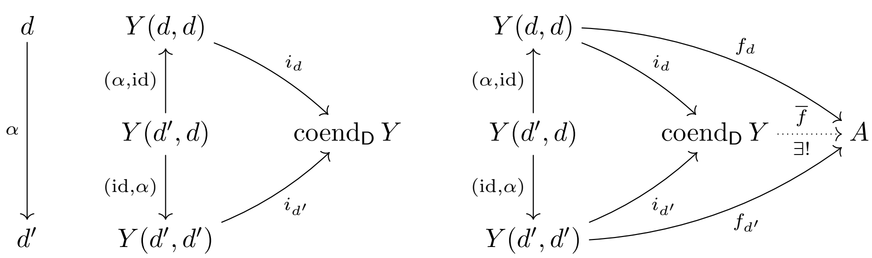
Proof. This follows from the proof of Proposition 1.12.9, with Top replaced by M. □

If we reverse all the arrows in an end diagram and its mapping properties, we are naturally led to the mapping properties of a coend diagram: let's develop this idea. Let D be a category and $Y\colon \mathsf{D}^{\mathrm{op}}\times\mathsf{D}\to\mathsf{M}$ a diagram. A *coend* of Y, denoted $\mathrm{coend}_{\mathsf{D}} Y$ or Y_{D}, is an object in M with the following mapping properties: (i) (Wedge): there is a collection $\{i_d\}$ of maps

$$i_d\colon\ Y(d,d)\to \mathrm{coend}_{\mathsf{D}} Y \qquad d\in\mathsf{D}$$

in M, indexed on the objects $d\in\mathsf{D}$, which make the middle diagram

(1.95)



commute (i.e., such that $i_d\,(\alpha,\mathrm{id}) = i_{d'}\,(\mathrm{id},\alpha)$) for each arrow α in D (such a collection $\{i_d\}$ is sometimes called a *wedge* out of Y) and (ii) (Universal property): the wedge $\{i_d\}$ is initial with respect to all such wedges out of Y; i.e., for any object A in M and collection $\{f_d\}$ of maps

$$f_d\colon\ Y(d,d)\to A \qquad d\in\mathsf{D}$$

in M, indexed on the objects $d\in\mathsf{D}$, which make the right-hand outer diagram commute (i.e., such that $f_d\,(\alpha,\mathrm{id}) = f_{d'}\,(\mathrm{id},\alpha)$) for each arrow α in D, there exists a unique map $\overline{f}$ in M which makes the diagram commute; i.e., such that $\overline{f}\,i_d = f_d$ for each $d\in\mathsf{D}$. We call the wedge $\{i_d\}$ the *coending wedge* of Y (or the initial wedge out of Y).

Remark 1.12.11. In other words, property (ii) states that every wedge $\{f_d\}$ out of Y factors uniquely through the coending wedge $\{i_d\}$ of Y.

Remark 1.12.12. The upshot is: giving a map $\overline{f}\colon \mathrm{coend}_{\mathsf{D}} Y\to A$ in M is the same as giving a wedge $\{f_d\}$ out of Y. For this reason, sometimes $\overline{f}$ is written as $\overline{f} = (f_d)_{d\in\mathsf{D}}$ or simply $\overline{f} = (f_d)$

To work effectively with the coend $\mathrm{coend}_{\mathsf{D}} Y$ of the diagram Y, we need to understand how to verify that a pair of maps out of it are identical.

PROPOSITION 1.12.13. *Let $Y : \mathsf{D}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{M}$ be a diagram. Assume that its coend* $\mathrm{coend}_{\mathsf{D}} Y$ (1.95) *exists. Consider any pair of maps of the form*

$$\mathrm{coend}_{\mathsf{D}} Y \underset{g}{\overset{f}{\rightrightarrows}} A$$

in M. Then f and g are identical if and only if their corresponding wedges out of Y are identical; i.e., in other words, $f = g$ if and only if $f\, i_d = g\, i_d$ for each $d \in \mathsf{D}$.

PROOF. This follows from the universal property of coends. □

Here is an equivalent way to formulate this observation; note how the change in notation makes the statement even more obvious.

PROPOSITION 1.12.14. *Let $Y : \mathsf{D}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{M}$ be a diagram. Assume that its coend* $\mathrm{coend}_{\mathsf{D}} Y$ (1.95) *exists. Consider any pair of maps of the form*

$$\mathrm{coend}_{\mathsf{D}} Y \underset{(g_d)}{\overset{(f_d)}{\rightrightarrows}} A$$

in M. Then (f_d) and (g_d) are identical if and only if their corresponding wedges out of Y are identical; i.e., in other words, $(f_d) = (g_d)$ if and only if $f_d = g_d$ for each $d \in \mathsf{D}$.

For instance, consider any wedges (k_d) and (n_d) out of Y of the indicated right-hand form in (1.95); i.e., such that $k_d\,(\alpha, \mathrm{id}) = k_{d'}\,(\mathrm{id}, \alpha)$ and $n_d\,(\alpha, \mathrm{id}) = n_{d'}\,(\mathrm{id}, \alpha)$ for each arrow α in D. Then the left-hand diagram of the form

(1.96)
$$\begin{array}{ccc} \mathrm{coend}_{\mathsf{D}} Y & \xrightarrow{(n_d)} & C \\ {\scriptstyle (k_d)}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[h]{} & D \end{array} \qquad \begin{array}{ccc} Y(d,d) & \xrightarrow{n_d} & C \\ {\scriptstyle k_d}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[h]{} & D \end{array}$$

in M commutes if and only if the corresponding right-hand diagram in Set commutes for each $d \in \mathsf{D}$.

REMARK 1.12.15. Arguing as above, the mapping properties in (1.95) characterize the coend $\mathrm{coend}_{\mathsf{D}} Y$, up to isomorphism, provided that it exists.

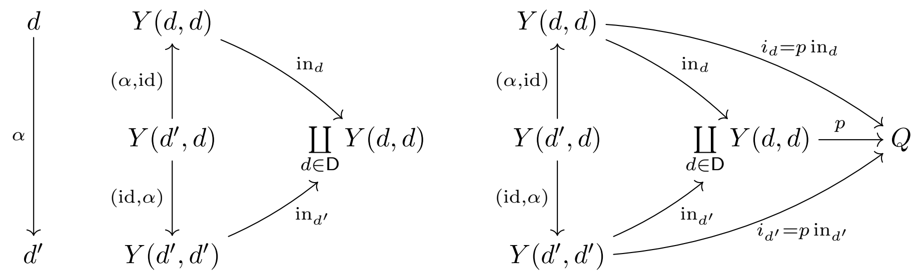
A *coend diagram* in M is a commutative diagram of the middle form (involving each arrow α in D) in (1.95) in M, that satisfies the universal property of the right-hand form in (1.95).

REMARK 1.12.16. A diagram that is naturally isomorphic to a coend diagram in M, is a coend diagram in M.

PROPOSITION 1.12.17. *Let D be a small category and $Y : \mathsf{D}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{Set}$ a diagram. Then the coend* $\mathrm{coend}_{\mathsf{D}} Y$ *exists. In other words,* Set *has all small coends.*

PROOF. Here is the classical way to build the coend $\mathrm{coend}_{\mathsf{D}} Y$. As a first step, we look for something that has naturally occurring maps out of $Y(d, d)$, for each object $d \in \mathsf{D}$; we already know of such a construction, it is the coproduct

$\coprod_{d\in\mathsf{D}} Y(d,d)$. This coproduct exists since D has a set of objects. Giving this a try, we get a diagram of the middle form

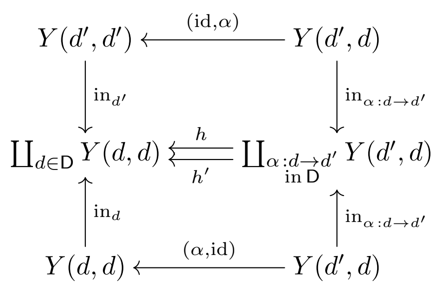


for each arrow α in D. There is no reason for the middle diagram to commute, in general, for each arrow α in D. So as a second step, the idea is to force it to commute by mapping to the quotient set Q of $\coprod_{d\in\mathsf{D}} Y(d,d)$ where this is satisfied. Define $Q := \left(\coprod_{d\in\mathsf{D}} Y(d,d)\right)/\!\sim$, where $\sim$ is the equivalence relation on $\coprod_{d\in\mathsf{D}} Y(d,d)$ generated by $\mathrm{in}_d\,(\alpha,\mathrm{id})(y) \sim \mathrm{in}_{d'}\,(\mathrm{id},\alpha)(y)$ for each arrow α in D and $y \in Y(d',d)$. This leads us to the right-hand outer diagram which commutes (we just forced it to); this right-hand outer diagram (involving each arrow α in D) is a coend diagram of Y. It is easy to check that $Q \cong \mathrm{coend}_{\mathsf{D}}\, Y$: we could either use the classical constructions of coproducts and quotient sets to verify the universal property of coends in (1.95), or we could work directly with the universal properties of coproducts and quotient sets to verify (1.95). $\square$

Remark 1.12.18. This argument can be reformulated in terms of coproducts and coequalizers as follows. Consider a pair of maps of the form

$$\coprod_{d\in\mathsf{D}} Y(d,d) \underset{h'}{\overset{h}{\leftleftarrows}} \coprod_{\substack{\alpha\colon d\to d'\\ \text{in } \mathsf{D}}} Y(d',d)$$

in Set. Consider the middle diagram above (which does not commute, in general); we want to force it to commute by mapping to an appropriate coequalizer. Let's define the map h so that, for each arrow α in D, it picks out the bottom part of the middle diagram above (i.e., the composite $\mathrm{in}_{d'}\,(\mathrm{id},\alpha)$). Similarly, let's define the map h' so that, for each arrow α in D, it picks out the other part of the middle diagram above (i.e., the composite $\mathrm{in}_d\,(\alpha,\mathrm{id})$). Remember, giving a map out of a coproduct of the right-hand form is the same as giving, for each arrow $\alpha\colon d\to d'$ in D, a map out of $Y(d',d)$. Hence we can describe h and h' as induced by the following diagram—start with the inclusions on the right-hand side, then work towards the left-hand inclusions. The inclusions pointing down define h and the inclusions pointing up define h'.



Think of h as induced on each inclusion map $\mathrm{in}_{\alpha\colon d\to d'}$ via the map that "misses" or "removes" the object d in the arrow $\alpha\colon d \to d'$; if we start with the arrow $d \to d'$ and miss or remove the object d, then we are left with the object d' (hence the map $\mathrm{in}_{d'}$), and to get from $Y(d', d)$ to $Y(d', d')$ we compose with the induced map $(\mathrm{id}, \alpha)\colon Y(d', d) \to Y(d', d')$. Similarly, think of h' as induced on each inclusion map $\mathrm{in}_{\alpha\colon d\to d'}$ via the map that "misses" or "removes" the object d' in the arrow $\alpha\colon d \to d'$; if we start with the arrow $d \to d'$ and miss or remove the object d', then we are left with the object d (hence the map in_d), and to get from $Y(d', d)$ to $Y(d, d)$ we compose with the induced map $(\alpha, \mathrm{id})\colon Y(d', d) \to Y(d, d)$. It is easy to check that $\mathrm{coend}_{\mathsf{D}} Y$ can be constructed as the coequalizer

$$\mathrm{coend}_{\mathsf{D}} Y \cong \mathrm{colim}\Big(\textstyle\coprod_{d\in\mathsf{D}} Y(d,d) \overset{h}{\underset{h'}{\Longleftarrow}} \coprod_{\substack{\alpha\colon d\to d' \\ \text{in } \mathsf{D}}} Y(d', d) \Big)$$

of the pair of maps h, h': we could either use the classical constructions of coproducts and coequalizers to recover the description of Q above, or we could work directly with the universal properties of coproducts and coequalizers. Conceptually, this means that once we know all coequalizers and small coproducts exist in Set, then we know that all small coends exist in Set.

Proposition 1.12.19. *Let* D *be a small category and* $Y\colon \mathsf{D}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{Top}$ *a diagram. Then the coend* $\mathrm{coend}_{\mathsf{D}} Y$ *exists. In other words,* Top *has all small coends.*

Proof. We already know that Top has all coequalizers and small coproducts. Hence the argument in Remark 1.12.18, with Set replaced by Top, shows that $\mathrm{coend}_{\mathsf{D}} Y$ can be constructed as the coequalizer

$$\mathrm{coend}_{\mathsf{D}} Y \cong \mathrm{colim}\Big(\textstyle\coprod_{d\in\mathsf{D}} Y(d,d) \overset{h}{\underset{h'}{\Longleftarrow}} \coprod_{\substack{\alpha\colon d\to d' \\ \text{in } \mathsf{D}}} Y(d', d) \Big)$$

of the pair of maps h, h'. □

Proposition 1.12.20. *Let* M *be a category with all coequalizers and small (resp. finite) coproducts. Let* $Y\colon \mathsf{D}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{M}$ *a small (resp. finite) diagram. Then the coend* $\mathrm{coend}_{\mathsf{D}} Y$ *exists. In other words,* M *has all small (resp. finite) coends.*

Proof. This follows from the proof of Proposition 1.12.19, with Top replaced by M. □

Let $B, C\colon \mathsf{D} \to \mathsf{M}$ be a pair of diagrams. A *map* $f\colon B \to C$ of diagrams is a natural transformation; i.e., a collection $\{f_d\}$ of maps

$$f_d\colon B_d \to C_d \qquad d \in \mathsf{D}$$

in M, indexed on the objects $d \in \mathsf{D}$, which make the right-hand diagram

$$\begin{array}{ccc} d & \qquad & B_d \xrightarrow{\;f_d\;} C_d \\ {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\alpha_* = B(\alpha)}\big\downarrow \qquad \big\downarrow{\scriptstyle\alpha_* = C(\alpha)} \\ d' & & B_{d'} \xrightarrow[\;f_{d'}\;]{} C_{d'} \end{array}$$

commute (i.e., such that $\alpha_* f_d = f_{d'} \alpha_*$) for each arrow α in D. If D is a small category, we denote by M^{D} the category of D-shaped diagrams in M and their maps. For instance, if D is the category of the form

$$0 \longleftarrow 1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4 \longleftarrow \cdots$$

(with objects the non-negative integers and a single morphism $i \leftarrow j$ for each $i \leq j$), then a map $f\colon\ B \to C$ in $\mathsf{Set}^{\mathsf{D}}$ is a collection of maps $f_0, f_1, f_2, \ldots$ which make the diagram

$$\begin{array}{ccccccccccc} B_0 & \longleftarrow & B_1 & \longleftarrow & B_2 & \longleftarrow & B_3 & \longleftarrow & B_4 & \longleftarrow & \cdots \\ {\scriptstyle f_0}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & \\ C_0 & \longleftarrow & C_1 & \longleftarrow & C_2 & \longleftarrow & C_3 & \longleftarrow & C_4 & \longleftarrow & \cdots \end{array}$$

in Set commute.

If M is a category and $B, C \in \mathsf{M}$ is a pair of objects in M, then $\hom_{\mathsf{M}}(B, C)$ denotes the set of maps $B \to C$ in M; maps (or arrows, or morphisms) in M are sometimes called homomorphisms in M; hence the "hom" notation for the indicated collection of maps. There are functors of the form

$$\hom_{\mathsf{M}}(B, -)\colon\ \mathsf{M} \to \mathsf{Set}, \qquad Z \mapsto \hom_{\mathsf{M}}(B, Z) \tag{1.97}$$

$$\hom_{\mathsf{M}}(-, C)\colon\ \mathsf{M}^{\mathrm{op}} \to \mathsf{Set}, \qquad Y \mapsto \hom_{\mathsf{M}}(Y, C) \tag{1.98}$$

$$\hom_{\mathsf{M}}(-, -)\colon\ \mathsf{M}^{\mathrm{op}} \times \mathsf{M} \to \mathsf{Set}, \qquad (Y, Z) \mapsto \hom_{\mathsf{M}}(Y, Z) \tag{1.99}$$

Remark 1.12.21. We will sometimes drop the decoration M from the notation and simply write hom in place of $\hom_{\mathsf{M}}$; this will not cause any confusion. Let's remind ourselves how the induced maps are defined—they are the naturally occurring ones:

$$\begin{array}{ccc} Z \xrightarrow{\alpha} Z' & \mapsto & \hom(B, Z) \xrightarrow{(\mathrm{id}, \alpha)} \hom(B, Z') \\ & & \left(B \xrightarrow{\xi} Z \quad \mapsto \quad B \xrightarrow{\xi} Z \xrightarrow{\alpha} Z'\right) \\ Y \xleftarrow{\beta} Y' & \mapsto & \hom(Y, C) \xrightarrow{(\beta, \mathrm{id})} \hom(Y', C) \\ & & \left(Y \xrightarrow{\xi} C \quad \mapsto \quad Y' \xrightarrow{\beta} Y \xrightarrow{\xi} C\right) \end{array}$$

In other words, $(id, \alpha)(\xi) = \alpha\xi$ and $(\beta, \mathrm{id})(\xi) = \xi\beta$. Similarly, for α, β a pair of maps in M, the right-hand diagram

$$\begin{array}{ccccc} Y & & & & \hom(Y, Z) \xrightarrow{(\mathrm{id}, \alpha)} \hom(Y, Z') \\ {\scriptstyle \beta}\uparrow & & Z \xrightarrow{\ \alpha\ } Z' & & {\scriptstyle (\beta, \mathrm{id})}\downarrow \qquad\qquad \downarrow{\scriptstyle (\beta, \mathrm{id})} \\ Y' & & & & \hom(Y', Z) \xrightarrow[(\mathrm{id}, \alpha)]{} \hom(Y', Z') \end{array}$$

in Set commutes; i.e., $(\beta, \mathrm{id})(\mathrm{id}, \alpha) = (\mathrm{id}, \alpha)(\beta, \mathrm{id})$. We denote the composite by

$$\begin{array}{c} \hom(Y, Z) \xrightarrow{(\beta, \alpha)} \hom(Y', Z') \\ \left(Y \xrightarrow{\xi} Z \quad \mapsto \quad Y' \xrightarrow{\beta} Y \xrightarrow{\xi} Z \xrightarrow{\alpha} Z'\right) \end{array}$$

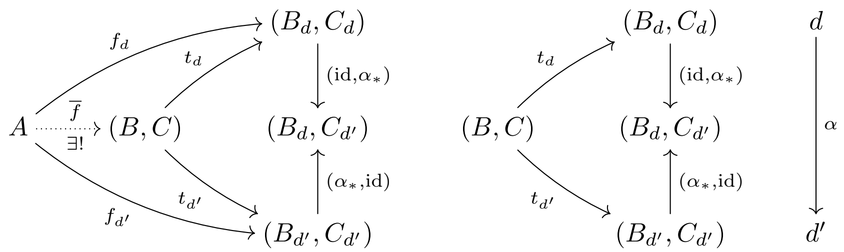
In other words, $(\beta, \alpha)\xi = \alpha\xi\beta$.

Proposition 1.12.22. *Let* D *be a small category. If* $B, C \in \mathsf{M}^{\mathsf{D}}$*, then the set of maps* $\hom_{\mathsf{M}^{\mathsf{D}}}(B, C)$ *is naturally isomorphic to the end*

$$\hom_{\mathsf{M}^{\mathsf{D}}}(B, C) \cong \hom_{\mathsf{M}}(B, C)^{\mathsf{D}}$$

of the diagram $\hom_{\mathsf{M}}(B, C)\colon \mathsf{D}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{Set}$.

Proof. The basic idea is to (i) look for a naturally occuring wedge into $\hom_{\mathsf{M}}(B, C)$ and (ii) verify it is terminal with respect to all such wedges. Consider the middle diagram of the form



in Set; here, we use the shorthand notation $(B_d, C_d) = \hom_{\mathsf{M}}(B_d, C_d)$ and $(B, C) = \hom_{\mathsf{M}^{\mathsf{D}}}(B, C)$. The map t_d sends $B \to C$ to $B_d \to C_d$; i.e., it is the "evaluate at d" map. Consider any wedge $\{f_d\}$ of the left-hand form. Noting that each $a \in A$ picks out a map $B \to C$ of diagrams (i.e., understanding what it means for the outer left-hand diagram to commute, for each arrow α in D) completes the proof—in other words, uniqueness is forced on us, since the diagram commutes implies that $\overline{f}(a)$ is the map $B \to C$ of diagrams given by the collection $\{f_d(a)\}$ of maps, for each $a \in A$, and existence follows since this is a well-defined map in Set. □

Proposition 1.12.23. *Let* D *be a small category and* $Z\colon \mathsf{D} \to \mathsf{Set}$ *a diagram. Then the limit* $\lim_{\mathsf{D}} Z$ *is naturally isomorphic to the end*

$$\lim_{\mathsf{D}} Z \cong \hom_{\mathsf{Set}}(*, Z)^{\mathsf{D}} \cong \hom_{\mathsf{Set}^{\mathsf{D}}}(*, Z)$$

Here, $*\colon \mathsf{D} \to \mathsf{Set}$ *denotes the constant diagram with value the terminal object* $*$ *in* Set.

Proof. This is because $Z_d \cong \hom_{\mathsf{Set}}(*, Z_d)$ for each object $d \in \mathsf{D}$; in other words, the terminal wedge of the indicated end reduces to the same information as the terminal cone of the indicated limit. □

Remark 1.12.24. In other words, the limit $\lim_{\mathsf{D}} Z$ can be calculated by taking "hom" (as D-shaped diagrams) of $*$ into Z. This observation plays a role later when we develop the Bousfield-Kan homotopy limit functor.

Proposition 1.12.25. *Let* D *be a small category and* $X\colon \mathsf{D}^{\mathrm{op}} \to \mathsf{Set}$ *a diagram. Then the colimit* $\operatorname{colim}_{\mathsf{D}^{\mathrm{op}}} X$ *is naturally isomorphic to the coend*

$$\operatorname{colim}_{\mathsf{D}^{\mathrm{op}}} X \cong X \times_{\mathsf{D}} *$$

Here, the diagram $*\colon \mathsf{D} \to \mathsf{Set}$ *denotes the constant diagram with value* $*$ *(the terminal object) in* Set.

Proof. This is because $X_d \cong X_d \times *$ for each object $d \in \mathsf{D}$; in other words, the initial wedge of the indicated coend reduces to the same information as the initial cone of the indicated colimit. □

REMARK 1.12.26. In other words, the colimit $\operatorname{colim}_{\mathsf{D}^{\mathrm{op}}} X$ can be calculated by "tensoring" (as D-shaped diagrams) X with $*$. This observation plays a role later when we develop the Bousfield-Kan homotopy colimit functor.

The following is similar, but let's point it out anyways.

PROPOSITION 1.12.27. *Let* M *be a category with all small limits and colimits. Let* $Y\colon \mathsf{D}\to\mathsf{M}$ *a small diagram. Then the limit* $\lim_{\mathsf{D}} Y$ *(resp. colimit* $\operatorname{colim}_{\mathsf{D}} Y$*) is naturally isomorphic to the end (resp. coend)*

$$\lim_{\mathsf{D}} Y \cong (*\times Y)^{\mathsf{D}} \qquad \Big(\textit{resp.}\quad \operatorname{colim}_{\mathsf{D}} Y \cong * \times_{\mathsf{D}} Y\Big)$$

Here, the diagram $*\colon \mathsf{D}^{\mathrm{op}}\to\mathsf{M}$ *denotes the constant diagram with value* $*$ *(the terminal object) in* M.

PROOF. This is because $Y_d \cong * \times Y_d$ for each object $d\in\mathsf{D}$; in other words, the terminal wedge (resp. initial wedge) of the indicated end (resp. coend) reduces to the same information as the terminal cone (resp. initial cone) of the indicated limit (resp. colimit). □

1.13. Yoneda lemma

PROPOSITION 1.13.1 (Yoneda lemma: D-shaped diagrams in Set). *Let* D *be a small category,* $b\in\mathsf{D}$*, and consider the associated diagram* $\hom(b,-)\colon \mathsf{D}\to\mathsf{Set}$*. If* $X\colon \mathsf{D}\to\mathsf{Set}$ *is a diagram, then there exists an isomorphism*

$$\hom_{\mathsf{Set}^{\mathsf{D}}}(\hom(b,-),X)\cong X_b$$

in Set*, natural in* b, X*. This conclusion remains true, even if we drop the assumption that* D *is small—in this case,* $\mathsf{Set}^{\mathsf{D}}$ *(as a large category) may only have hom-classes, instead of hom-sets, in general, but the bijection remains true and* X_b *is a set, hence this particular hom-class is a hom-set.*

PROOF. Here is the basic idea: a map of diagrams out of $\hom(b,-)$ has so much structure forced on it, that it is already completely determined by where the identity map is sent. In more detail: consider any map $f\colon \hom(b,-)\to X$ of diagrams. Then f consists of a collection $\{f_d\}$ of maps in Set, indexed on the objects $d\in\mathsf{D}$, which makes the middle diagram

$$\begin{array}{ccccccc} d & & \hom(b,d) & \xrightarrow{f_d} & X_d & \quad \hom(b,d) & \xrightarrow{f} & X_d \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle(\mathrm{id},\alpha)} & & \downarrow{\scriptstyle X(\alpha)} & \quad \downarrow{\scriptstyle(\mathrm{id},\alpha)} & & \downarrow{\scriptstyle X(\alpha)} \\ d' & & \hom(b,d') & \xrightarrow[f_{d'}]{} & X_{d'} & \quad \hom(b,d') & \xrightarrow[f]{} & X_{d'} \end{array} \tag{1.100}$$

commute for each arrow α in D. For notational convenience, let's drop the subscripts on f; then we can write the middle diagram as the right-hand diagram. In particular, for each arrow $\alpha\colon b\to d$ in D, the right-hand diagram

$$\begin{array}{ccccc} b & & \hom(b,b) & \xrightarrow{f} & X_b \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle(\mathrm{id},\alpha)} & & \downarrow{\scriptstyle X(\alpha)} \\ d & & \hom(b,d) & \xrightarrow[f]{} & X_d \end{array} \tag{1.101}$$

in Set commutes; here, we obtained (1.101) by replacing α in (1.100) with an element $\alpha \in \hom(b,d)$. Chasing the identity map $\mathrm{id} \in \hom(b,b)$ around the diagram verifies

$$\begin{array}{ccccc} b \xrightarrow{\mathrm{id}} b & \mapsto & f(\mathrm{id}) & \mapsto & X(\alpha)f(\mathrm{id}) \\ b \xrightarrow{\mathrm{id}} b & \mapsto & b \xrightarrow{\mathrm{id}} b \xrightarrow{\alpha} d & \mapsto & f(\alpha) \end{array}$$

that $f(\alpha) = X(\alpha)f(\mathrm{id})$. The upshot is that f is completely determined by its value $f(\mathrm{id})$ on the identity map $\mathrm{id}\colon b \to b$. It is easy to check that the *Yoneda* map

$$\hom_{\mathsf{Set}^{\mathsf{D}}}(\hom(b,-),X) \xrightarrow{\cong} X_b, \qquad f \mapsto f(\mathrm{id})$$

is a bijection, natural in b, X.

In more detail: let's verify it is a surjection. Choose any $a \in X_b$ and define $f\colon \hom(b,-) \to X$ by $f(\mathrm{id}) = a$ and $f(\alpha) = X(\alpha)f(\mathrm{id})$ for each arrow $\alpha\colon b \to d$ in D. It follows that f is a well-defined map of diagrams and hence the Yoneda map is a surjection. Let's verify it is an injection. Suppose $f, f'\colon \hom(b,-) \to X$ is a pair of maps of diagrams. Assume that $f(\mathrm{id}) = f'(\mathrm{id})$. Then $f(\alpha) = X(\alpha)f(\mathrm{id}) = X(\alpha)f'(\mathrm{id}) = f'(\alpha)$ for each arrow $\alpha\colon b \to d$ in D and hence the Yoneda map is an injection. Hence we have verified the Yoneda map is an isomorphism in Set. □

COROLLARY 1.13.2. *Let* D *be a category. If* $b, b' \in \mathsf{D}$*, then there exists an isomorphism* $\hom_{\mathsf{Set}^{\mathsf{D}}}(\hom(b,-),\hom(b',-)) \cong \hom(b',b)$ *in* Set*, natural in* b, b'.

REMARK 1.13.3. The upshot is: giving a map $f\colon \hom(b,-) \to \hom(b',-)$ of diagrams is the same as giving a map $b \leftarrow b'$ in D.

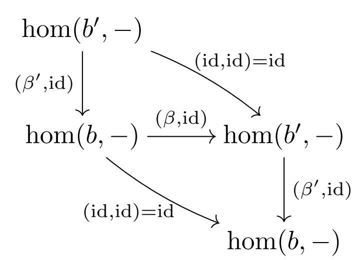
It will be useful to further elaborate this.

COROLLARY 1.13.4. *Let* D *be a category. If* $b, b' \in \mathsf{D}$*, then each map*

$$f\colon \hom(b,-) \to \hom(b',-)$$

of diagrams has the form $f = (\beta, \mathrm{id})$ *for some map* $b \xleftarrow{\beta} b'$ *in* D *(i.e.,* $\beta = f(\mathrm{id})$*); furthermore,* f *is an isomorphism if and only if* β *is an isomorphism in* D.

PROOF. Define $\beta = f(\mathrm{id})\colon b' \to b$ in D. Then $f(\alpha) = (\mathrm{id},\alpha)f(\mathrm{id}) = (\mathrm{id},\alpha)\beta = \alpha\beta = (\beta,\mathrm{id})\alpha$ for each arrow $\alpha\colon b \to d$ in D. Hence we have verified that $f = (\beta, \mathrm{id})$. Let's verify that f is an isomorphism if and only if β is an isomorphism in D. Suppose f is an isomorphism. Then there is a commutative diagram of the form



for some arrow $\beta'\colon b \to b'$ in D. It follows that $\beta'\beta = \mathrm{id}$ and $\beta\beta' = \mathrm{id}$ and hence β is an isomorphism in D. Conversely, if β is an isomorphism, then so is $(\beta, \mathrm{id}) = f$. □

Once we get adjunctions into the picture, this sometimes provides an efficient method for showing that a pair of particular objects $b, b' \in \mathsf{D}$ are isomorphic.

PROPOSITION 1.13.5 (Yoneda lemma: D^{op}-shaped diagrams in Set). *Let* D *be a small category,* $c \in \mathsf{D}$*, and consider the associated diagram* $\hom(-,c)\colon \mathsf{D}^{\mathrm{op}} \to \mathsf{Set}$*. If* $Y\colon \mathsf{D}^{\mathrm{op}} \to \mathsf{Set}$ *is a diagram, then there exists an isomorphism*

$$\hom_{\mathsf{Set}^{\mathsf{D}^{\mathrm{op}}}}(\hom(-,c), Y) \cong Y_c$$

in Set*, natural in* c, Y*. This conclusion remains true, even if we drop the assumption that* D *is small—in this case,* $\mathsf{Set}^{\mathsf{D}^{\mathrm{op}}}$ *(as a large category) may only have hom-classes, instead of hom-sets, in general, but the bijection remains true and* Y_c *is a set, hence this particular hom-class is a hom-set.*

PROOF. Here is the basic idea: a map of diagrams out of $\hom(-,c)$ has so much structure forced on it, that it is already completely determined by where the identity map is sent. In more detail: consider any map $f\colon \hom(-,c) \to Y$ of diagrams. Then f consists of a collection $\{f_d\}$ of maps in Set, indexed on the objects $d \in \mathsf{D}$, which makes the middle diagram

(1.102)
$$\begin{array}{ccc} d & \quad & \hom(d,c) & \xrightarrow{f_d} & Y_d & \quad & \hom(d,c) & \xrightarrow{f} & Y_d \\ {\scriptstyle\alpha}\uparrow & & {\scriptstyle(\alpha,\mathrm{id})}\downarrow & & \downarrow{\scriptstyle Y(\alpha)} & & {\scriptstyle(\alpha,\mathrm{id})}\downarrow & & \downarrow{\scriptstyle Y(\alpha)} \\ d' & & \hom(d',c) & \xrightarrow[f_{d'}]{} & Y_{d'} & & \hom(d',c) & \xrightarrow[f]{} & Y_{d'} \end{array}$$

commute for each arrow α in D. For notational convenience, let's drop the subscripts on f; then we can write the middle diagram as the right-hand diagram. In particular, for each arrow $\alpha\colon d \to c$ in D, the right-hand diagram

(1.103)
$$\begin{array}{ccc} c & \quad & \hom(c,c) & \xrightarrow{f} & Y_c \\ {\scriptstyle\alpha}\uparrow & & {\scriptstyle(\alpha,\mathrm{id})}\downarrow & & \downarrow{\scriptstyle Y(\alpha)} \\ d & & \hom(d,c) & \xrightarrow[f]{} & Y_d \end{array}$$

in Set commutes; here, we obtained (1.103) by replacing α in (1.102) with an element $\alpha \in \hom(d,c)$. Chasing the identity map $\mathrm{id} \in \hom(c,c)$ around the diagram verifies

$$\begin{array}{ccccccc} c \xrightarrow{\mathrm{id}} c & \mapsto & f(\mathrm{id}) & \mapsto & Y(\alpha)f(\mathrm{id}) \\ c \xrightarrow{\mathrm{id}} c & \mapsto & d \xrightarrow{\alpha} c \xrightarrow{\mathrm{id}} c & \mapsto & f(\alpha) \end{array}$$

that $f(\alpha) = Y(\alpha)f(\mathrm{id})$. The upshot is that f is completely determined by its value $f(\mathrm{id})$ on the identity map $\mathrm{id}\colon c \to c$. It is easy to check that the *Yoneda* map

$$\hom_{\mathsf{Set}^{\mathsf{D}^{\mathrm{op}}}}(\hom(-,c), Y) \xrightarrow{\cong} Y_c, \qquad f \mapsto f(\mathrm{id})$$

is a bijection, natural in c, Y.

In more detail: let's verify it is a surjection. Choose any $a \in Y_c$ and define $f\colon \hom(-,c) \to Y$ by $f(\mathrm{id}) = a$ and $f(\alpha) = Y(\alpha)f(\mathrm{id})$ for each arrow $\alpha\colon d \to c$ in D. It follows that f is a well-defined map of diagrams and hence the Yoneda map is a surjection. Let's verify it is an injection. Suppose $f, f'\colon \hom(-,c) \to Y$ is a pair of maps of diagrams. Assume that $f(\mathrm{id}) = f'(\mathrm{id})$. Then $f(\alpha) = Y(\alpha)f(\mathrm{id}) = Y(\alpha)f'(\mathrm{id}) = f'(\alpha)$ for each arrow $\alpha\colon d \to c$ in D and hence the Yoneda map is an injection. Hence we have verified the Yoneda map is an isomorphism in Set. $\square$

COROLLARY 1.13.6. *Let* D *be a category. If* $c, c' \in \mathsf{D}$*, then there exists an isomorphism* $\hom_{\mathsf{Set}^{\mathsf{D}^{\mathrm{op}}}}(\hom(-,c), \hom(-,c')) \cong \hom(c,c')$ *in* Set*, natural in* c, c'*.*

REMARK 1.13.7. The upshot is: giving a map $f\colon \hom(-,c) \to \hom(-,c')$ of diagrams is the same as giving a map $c \to c'$ in D.

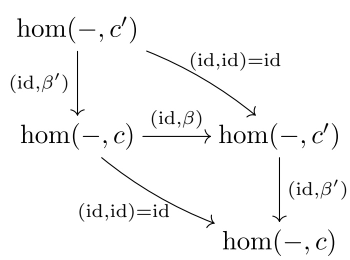
It will be useful to further elaborate this.

COROLLARY 1.13.8. *Let* D *be a category. If* $c, c' \in \mathsf{D}$*, then each map*

$$f\colon \hom(-,c) \to \hom(-,c')$$

of diagrams has the form $f = (\mathrm{id}, \beta)$ *for some map* $c \xrightarrow{\beta} c'$ *in* D *(i.e.,* $\beta = f(\mathrm{id})$*); furthermore,* f *is an isomorphism if and only if* β *is an isomorphism in* D*.*

PROOF. Define $\beta = f(\mathrm{id})\colon c \to c'$ in D. Then $f(\alpha) = (\alpha, \mathrm{id})f(\mathrm{id}) = (\alpha, \mathrm{id})\beta = \beta\alpha = (\mathrm{id}, \beta)\alpha$ for each arrow $\alpha\colon d \to c$ in D. Hence we have verified that $f = (\mathrm{id}, \beta)$. Let's verify that f is an isomorphism if and only if β is an isomorphism in D. Suppose f is an isomorphism. Then there is a commutative diagram of the form



for some arrow $\beta'\colon c' \to c$ in D. It follows that $\beta\beta' = \mathrm{id}$ and $\beta'\beta = \mathrm{id}$ and hence β is an isomorphism in D. Conversely, if β is an isomorphism, then so is $(\mathrm{id}, \beta) = f$. $\square$

Once we get adjunctions into the picture, this sometimes provides an efficient method for verifying that a pair of particular objects $c, c' \in \mathsf{D}$ are isomorphic.

1.14. Colimits of representables

Suppose D is a small category and let $Y\colon \mathsf{D}^{\mathrm{op}} \to \mathsf{Set}$ be a diagram. Consider the functor $\hom(-,-)\colon \mathsf{D} \to \mathsf{Set}^{\mathsf{D}^{\mathrm{op}}}$ defined objectwise by $c \mapsto \hom(-,c)$.

REMARK 1.14.1. We know by Yoneda that giving a map $f\colon \hom(-,c) \to Y$ of diagrams is the same as giving a map $y\colon * \to Y_c$ in Set, where $*$ is the terminal object in Set (i.e., a one point set) and we label the map by the element that it picks out—in this case, it picks out the element $y = f(\mathrm{id}) \in Y_c$. When working with maps of diagrams of the form $f\colon \hom(-,c) \to Y$, we often only need to keep track of the element $y = f(\mathrm{id})$ since it uniquely determines the map $f\colon \hom(-,c) \to Y$. In what follows, for notational convenience reasons we will often use the same label for both the map $y\colon * \to Y_c$ in Set (i.e., the map that picks out the element $y \in Y_c$) and its corresponding map $f\colon \hom(-,c) \to Y$ of diagrams (where $f(\mathrm{id}) = y$); this will not cause any confusion. For instance, in this notation, the map $y\colon \hom(-,c) \to Y$ of diagrams corresponds to the map $y\colon * \to Y_c$ in Set and the middle diagram

$$
\begin{array}{c}
c\\
{\scriptstyle\alpha}\downarrow\\
c'
\end{array}
\qquad
\begin{array}{ccc}
\hom(-,c) & \xrightarrow{y} & Y\\
{\scriptstyle(\mathrm{id},\alpha)}\downarrow & & \|\\
\hom(-,c') & \xrightarrow[y']{} & Y
\end{array}
\qquad
\begin{array}{ccc}
* & \xrightarrow{y} & Y_c\\
\| & & \uparrow{\scriptstyle Y(\alpha)}\\
* & \xrightarrow[y']{} & Y_{c'}
\end{array}
$$

in $\mathsf{Set}^{\mathsf{D}^{\mathrm{op}}}$ commutes if and only if the corresponding right-hand diagram in Set commutes; here, $\alpha\colon c \to c'$ denotes a map in D.

Let's use the notational convention in Remark 1.14.1 for what follows. The *category of elements* of Y, denoted $\hom(-,-) \downarrow Y$, is the category with objects all pairs (c, y) where $c \in \mathsf{D}$ and $y\colon \hom(-,c) \to Y$ is a map of diagrams, and with maps $\alpha\colon (c,y) \to (c',y')$ those maps $\alpha\colon c \to c'$ in D that make the diagram

$$
\begin{array}{c}
c\\
{\scriptstyle\alpha}\downarrow\\
c'
\end{array}
\qquad
\begin{array}{ccc}
\hom(-,c) & \xrightarrow{y} & Y\\
{\scriptstyle(\mathrm{id},\alpha)}\downarrow & & \|\\
\hom(-,c') & \xrightarrow[y']{} & Y
\end{array}
$$

in $\mathsf{Set}^{\mathsf{D}^{\mathrm{op}}}$ commute. We often refer to the object (c, y) simply by y, when c is clear from the context; this will not cause any confusion. The *diagram of elements* of Y is the diagram defined objectwise by

$$\hom(-,-) \downarrow Y \to \mathsf{Set}^{\mathsf{D}^{\mathrm{op}}}, \qquad \hom(-,c) \xrightarrow{y} Y \quad \mapsto \quad \hom(-,c)$$

In other words, it is the projection functor (or forgetful functor) that sends y to its domain.

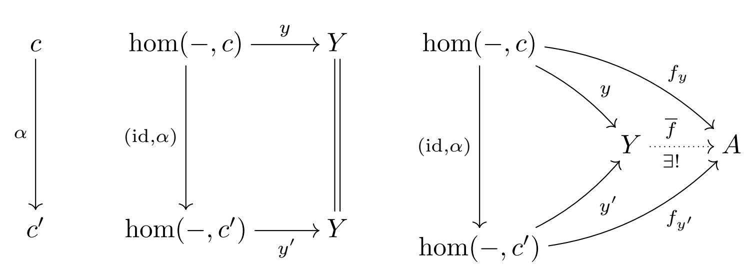
Proposition 1.14.2. *If $Y\colon \mathsf{D}^{\mathrm{op}} \to \mathsf{Set}$ is a diagram, then the natural map*

$$\operatorname*{colim}_{y\colon \hom(-,c)\to Y} \hom(-,c) \xrightarrow{\cong} Y$$

in $\mathsf{Set}^{\mathsf{D}^{\mathrm{op}}}$ is an isomorphism; here, the indicated colimit is indexed over all objects $y\colon \hom(-,c) \to Y$ in the category $\hom(-,-) \downarrow Y$ of elements of Y.

Proof. The basic idea is that Y should be the colimit of its elements when appropriately glued together—the diagram of elements of Y makes this precise. Let's verify that Y is the indicated colimit: it suffices to verify the universal property of colimits. The first step is to look for a naturally occurring cone into Y, but that is built into the indexing category via the notion of a map from y to y' as indicated

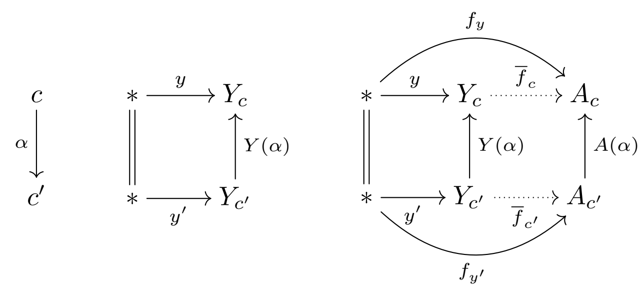
(1.104)



in the middle diagram. Consider any $A \in \mathsf{Set}^{\mathsf{D}^{\mathrm{op}}}$ and collection $\{f_y\}$ of maps

$$f_y\colon \hom(-,c) \to A \qquad y \in \hom(-,-) \downarrow Y$$

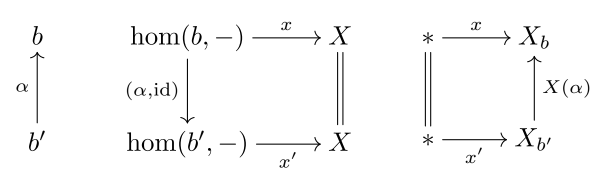
in $\mathsf{Set}^{\mathsf{D}^{\mathrm{op}}}$, indexed on the objects $y \in \hom(-,-) \downarrow Y$, which make the right-hand outer diagram commute (i.e., such that $f_{y'}(\mathrm{id}, \alpha) = f_y$) for each arrow α in $\hom(-,-) \downarrow Y$ (indicated in the middle diagram). We want to show: there exists a unique map $\overline{f}$ in $\mathsf{Set}^{\mathsf{D}^{\mathrm{op}}}$ which makes the right-hand diagram commute (i.e., such that $\overline{f}\,y = f_y$ for each $y \in \hom(-,-) \downarrow Y$. To understand the meaning of this, let's rewrite the diagrams in (1.104) (see Remark 1.14.1) in their equivalent forms as the diagrams



in Set. Uniqueness is forced on us, since the right-hand diagram in (1.104) commutes implies that $\overline{f}_c\,y = f_y$ for each $y \in \hom(-,-) \downarrow Y$. Existence follows since $\overline{f}$ is a well-defined map of diagrams: in more detail, we want to verify that $\overline{f}_c\,Y(\alpha) = A(\alpha)\overline{f}_{c'}$. Choose any $y' \in Y_{c'}$ and consider the corresponding map $y'\colon * \to Y_{c'}$. Let $y = Y(\alpha)y'$. Then we know that $\overline{f}_c\,Y(\alpha)y' = \overline{f}_c\,y = f_y = A(\alpha)f_{y'} = A(\alpha)\overline{f}_{c'}y'$. Hence we have verified that $\overline{f}$ is a well-defined map of diagrams. $\square$

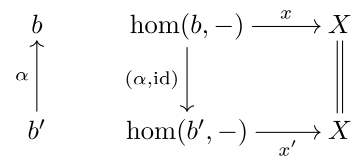
Suppose D is a small category and let $X\colon \mathsf{D} \to \mathsf{Set}$ be a diagram. Consider the functor $\hom(-,-)\colon \mathsf{D}^{\mathrm{op}} \to \mathsf{Set}^{\mathsf{D}}$ defined objectwise by $b \mapsto \hom(b,-)$.

REMARK 1.14.3. We know by Yoneda that giving a map $f\colon \hom(b,-) \to X$ of diagrams is the same as giving a map $x\colon * \to X_b$ in Set, where $*$ is the terminal object in Set (i.e., a one point set) and we label the map by the element that it picks out—in this case, it picks out the element $x = f(\mathrm{id}) \in X_b$. When working with maps of diagrams of the form $f\colon \hom(b,-) \to X$, we often only need to keep track of the element $x = f(\mathrm{id})$ since it uniquely determines the map $f\colon \hom(b,-) \to X$. In what follows, for notational convenience reasons, we will often use the same label for both the map $x\colon * \to X_b$ in Set (i.e., the map that picks out the element $x \in X_b$) and its corresponding map $f\colon \hom(b,-) \to X$ of diagrams (where $f(\mathrm{id}) = x$); this will not cause any confusion. For instance, in this notation, the map $x\colon \hom(-,b) \to X$ of diagrams corresponds to the map $x\colon * \to X_b$ in Set and the middle diagram



in $\mathsf{Set}^{\mathsf{D}}$ commutes if and only if the corresponding right-hand diagram in Set commutes; here, $\alpha\colon b' \to b$ denotes a map in D.

Let's use the notational convention in Remark 1.14.3 for what follows. The *category of elements* of X, denoted $\hom(-,-) \downarrow X$, is the category with objects all pairs (b, x) where $b \in \mathsf{D}$ and $x \colon \hom(b,-) \to X$ is a map of diagrams, and with maps $\alpha \colon (b, x) \to (b', x')$ those maps $\alpha \colon b' \to b$ in D that make the diagram



in $\mathsf{Set}^{\mathsf{D}}$ commute. We often refer to the object (b, x) simply by x, when x is clear from the context; this will not cause any confusion. The *diagram of elements* of X is the diagram defined objectwise by

$$\hom(-,-) \downarrow X \to \mathsf{Set}^{\mathsf{D}}, \qquad \hom(b,-) \xrightarrow{x} X \quad \mapsto \quad \hom(b,-)$$

In other words, it is the projection functor (or forgetful functor) that sends x to its domain.

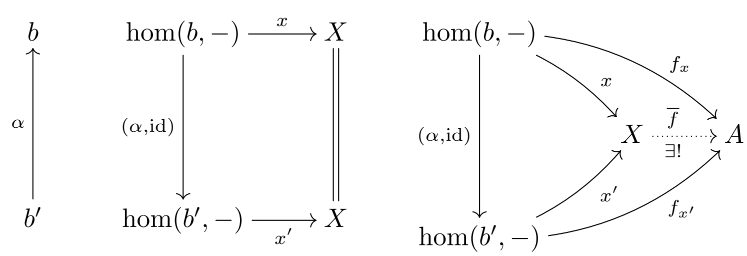
Proposition 1.14.4. *If $X \colon \mathsf{D} \to \mathsf{Set}$ is a diagram, then the natural map*

$$\operatorname*{colim}_{x \colon \hom(b,-) \to X} \hom(b,-) \xrightarrow{\cong} X$$

in $\mathsf{Set}^{\mathsf{D}}$ is an isomorphism; here, the indicated colimit is indexed over all objects $x \colon \hom(b,-) \to X$ in the category $\hom(-,-) \downarrow X$ of elements of X.

Proof. The basic idea is that X should be the colimit of its elements when appropriately glued together—the diagram of elements of X makes this precise. Let's verify that X is the indicated colimit: it suffices to verify the universal property of colimits. The first step is to look for a naturally occurring cone into X, but that is built into the indexing category via the notion of a map from x to x' as indicated

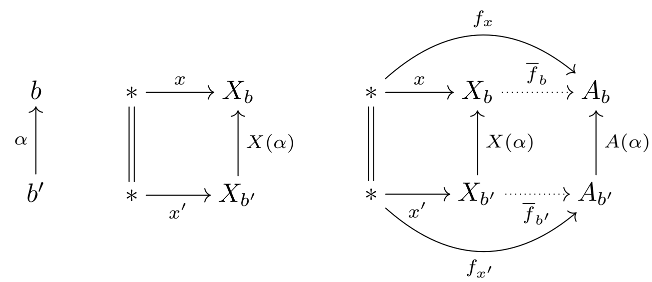
(1.105)



in the middle diagram. Consider any $A \in \mathsf{Set}^{\mathsf{D}}$ and collection $\{f_x\}$ of maps

$$f_x \colon \hom(b,-) \to A \qquad x \in \hom(-,-) \downarrow X$$

in $\mathsf{Set}^{\mathsf{D}}$, indexed on the objects $x \in \hom(-,-) \downarrow X$, which make the right-hand outer diagram commute (i.e., such that $f_{x'}(\alpha, \mathrm{id}) = f_x$) for each arrow α in $\hom(-,-) \downarrow X$ (indicated in the middle diagram). We want to show: there exists a unique map $\overline{f}$ in $\mathsf{Set}^{\mathsf{D}}$ which makes the right-hand diagram commute (i.e., such that $\overline{f}\, x = f_x$ for each $x \in \hom(-,-) \downarrow X$. To understand the meaning of this, let's rewrite the diagrams in (1.105) (see Remark 1.14.3) in their equivalent forms as the diagrams



in Set. Uniqueness is forced on us, since the right-hand diagram in (1.104) commutes implies that $\overline{f}_b\, x = f_x$ for each $x \in \hom(-,-) \downarrow X$. Existence follows since $\overline{f}$ is a well-defined map of diagrams: in more detail, we want to verify that $\overline{f}_b\, X(\alpha) = A(\alpha)\overline{f}_{b'}$. Choose any $x' \in X_{b'}$ and consider the corresponding map $x' \colon * \to X_{b'}$. Let $x = X(\alpha)x'$. Then we know that $\overline{f}_b\, X(\alpha)x' = \overline{f}_b\, x = f_x = A(\alpha)f_{x'} = A(\alpha)\overline{f}_{b'}x'$. Hence we have verified that $\overline{f}$ is a well-defined map of diagrams. □

NEXT STEPS: FINISH TYPING UP PENCIL AND PAPER NOTES FOR THE REMAINING SECTIONS IN CHAPTER 1. ADD REFERENCES. START TYPING UP PENCIL AND PAPER NOTES FOR CHAPTERS 2 AND 3.

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