RETRACTIVE SPACES AND BOUSFIELD-KAN COMPLETIONS

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ABSTRACT. In this short paper we apply some recent techniques developed by Schonsheck, and subsequently Carr-Harper, in the context of operadic algebras in spectra—on convergence of Bousfield-Kan completions and comparisons with convergence of the Taylor tower of the identity functor in Goodwillie's functor calculus—to the setting of retractive spaces: this arises when working with spaces centered away from the one-point space. Interestingly, in the retractive spaces context, the comparison results are stronger in terms of convergence outside of functor calculus' notion of "radius of (strong) convergence" for analytic functors. In particular, we give a new proof (and generalization to retractive spaces) of the Arone-Kankaanrinta result for convergence of the Taylor tower of the identity functor to various Bousfield-Kan completions; it's notable that no use is made of Snaith splittings—rather, we make extensive use of the kinds of homotopical estimates that appear in earlier work of Dundas and Dundas-Goodwillie-McCarthy.

1. Introduction

This paper is written simplicially so that "space" means "simplicial set" unless otherwise noted; see [7, 17]. In particular, we refer to the category of pointed simplicial sets S_* as pointed spaces; this is equipped with the usual homotopy theory ([7, 17]). Our basic assumption is that Z is a 0-connected fibrant pointed space. Denote by $\mathsf{S}_*^Z \cong \mathrm{id}_Z \downarrow (\mathsf{S}_* \downarrow Z)$ the factorization category ([2, 2.1], [20, 4.9]) of the identity map on Z, called the category of retractive pointed spaces over Z, equipped with the homotopy theory ([2, 2.1], [20, 4.9]) inherited from S_*; in particular, it has the structure of a simplicial cofibrantly generated model structure ([21], [29]) with an action of S_* ([22, 4.2]). The setting of retractive spaces naturally arises in Goodwillie's homotopy functor calculus [18, 19] when working with Taylor towers centered away from the one-point space; see also [24]. When working with Bousfield-Kan completions, we make extensive use of the kinds of homotopical resolutions studied in [5]. We say that a retractive space X over Z is *k-connected relative to Z* (or k-connected (rel. Z)) if the structure map $Z \rightarrow X$ is k-connected.

Here are our main results. In the special case when $Z = *$ (the one-point space), Theorem 1.1 is proved by Bousfield and Hopkins [6] (for $r \geq 1$) and Carlsson [8] (for $r = \infty$) for any 0-connected nilpotent space X, and subsequently in [4] for any 1-connected space X (using different arguments closely related to [3, 11, 13, 14]). Our result in Theorem 1.2, generalizes this to any 0-connected (rel. Z) retractive space F over Z, provided that, furthermore, F fits into an appropriate homotopy pullback square. Our technical approach is motivated by the work in [26], and the subsequent development in [9], for operadic algebras in spectra (where the estimates are different). We make extensive use of (the retractive version of) the homotopical estimates worked out in [4]; these are the kinds of homotopical estimates that appear in earlier work of Dundas [13] and Dundas-Goodwillie-McCarthy [14].

In the following theorems, $\tilde{\Sigma}^r_Z$ (resp. $\tilde{\Omega}^r_Z$) denotes the derived r-fold suspension ([16, 25]) (resp. derived r-fold loops ([16, 25])) in S^Z_*, and $\tilde{\Sigma}^\infty_Z$ (resp. $\tilde{\Omega}^\infty_Z$) denotes derived stabilization ([23]) on S^Z_* (resp. derived 0-th object functor ([23]) on Hovey spectra $\mathsf{Sp}^{\mathbb{N}}(\mathsf{S}^Z_*)$ on S^Z_*).

Theorem 1.1. *Assume that Z is a 0-connected pointed space. Let X be a retractive pointed space over Z. If X is* 1*-connected (rel. Z), then the coaugmentations*

$$X \simeq X^\wedge_{\tilde{\Omega}^r_Z \tilde{\Sigma}^r_Z}, \qquad (1 \leq r \leq \infty)$$

are weak equivalences in retractive pointed spaces over Z.

Theorem 1.2. *Assume that Z is a 0-connected pointed space. If $F \to X \to Y$ is a fibration sequence in retractive pointed spaces over Z and X, Y are 1-connected (rel. Z), then the coaugmentations*

$$F \simeq F^\wedge_{\tilde{\Omega}^r_Z \tilde{\Sigma}^r_Z}, \qquad (1 \leq r \leq \infty)$$

are weak equivalences in retractive pointed spaces over Z. More generally, let

$$\begin{array}{ccc} F & \longrightarrow & X \\ \downarrow & & \downarrow \\ A & \longrightarrow & Y \end{array}$$

be a homotopy pullback square in retractive pointed spaces over Z. If A, X, Y are 1-connected (rel. Z), then the coaugmentations

$$F \simeq F^\wedge_{\tilde{\Omega}^r_Z \tilde{\Sigma}^r_Z}, \qquad (1 \leq r \leq \infty)$$

are weak equivalences in retractive pointed spaces over Z.

In the special case when $Z = *$, Theorem 1.3 is proved in Arone-Kankaanrinta [1] for $r = \infty$ (using closely related, but different, arguments). We generalize their result to spaces centered away from $*$ and for $1 \leq r \leq \infty$. Our technical approach is motivated by the work in [28], and the subsequent development in [9], for operadic algebras in spectra (where the estimates are different). It's notable that no use is made of Snaith splittings—rather (as above) we make extensive use of (the retractive version of) the homotopical estimates worked out in [4], which are similar in spirit to the kinds of homotopical estimates appearing in the earlier work of Dundas [13] and Dundas-Goodwillie-McCarthy [14]; the possibility of giving a proof of Theorem 1.3 (when $Z = *$) along the lines developed here, was suggested in [27].

Theorem 1.3. *Assume that Z is a 0-connected pointed space. Let X be a retractive pointed space over Z. If X is 0-connected (rel. Z), then there are weak equivalences of the form*

$$P^Z_\infty(\mathrm{id})X \simeq X^\wedge_{\tilde{\Omega}^r_Z \tilde{\Sigma}^r_Z}, \qquad (1 \leq r \leq \infty)$$

in retractive pointed spaces over Z; here, $P^Z_n(\mathrm{id})X$ is the n-excisive approximation to the identity functor id *on retractive pointed spaces over Z, evaluated at X, and $P^Z_\infty(\mathrm{id})X$ denotes the homotopy limit of the associated Taylor tower $\{P^Z_n(\mathrm{id})X\}$ of the identity functor* id*, evaluated at X, in Goodwillie's functor calculus* [19].

To keep this paper appropriately concise, we will freely use language from [4].

2. Proofs of the main results

To get Bousfield-Kan completion into the picture, we work with the kinds of homotopical resolutions studied in [5]. There are adjunctions of the form

$$\mathsf{S}_*^Z \underset{\Omega_Z^r}{\overset{\Sigma_Z^r}{\rightleftarrows}} \mathsf{S}_*^Z \qquad \mathsf{S}_*^Z \underset{\Omega_Z^\infty}{\overset{\Sigma_Z^\infty}{\rightleftarrows}} \mathsf{Sp}^{\mathbb{N}}(\mathsf{S}_*^Z) \qquad (r \geq 1)$$

with left adjoints on top, where Σ_Z^r is given by the pointed spaces action of $S^r := (S^1)^{\wedge r} \in \mathsf{S}_*$ on objects in S_*^Z and $\mathsf{Sp}^{\mathbb{N}}(\mathsf{S}_*^Z)$ denotes Hovey spectra ([23]) on S_*^Z; here, Σ_Z^∞ (resp. Ω_Z^∞) denotes the stabilization (resp. "0-th object") functor. Denote by $\mathrm{id} \rightarrow \Phi$ and $\Phi\Phi \rightarrow \Phi$ the unit and multiplication maps of the fibrant replacement monad Φ on S_*^Z (see [5, 6.1]) and define $\tilde{\Omega}_Z^r := \Omega_Z^r \Phi$. Similarly, denote by $\mathrm{id} \rightarrow F$ and $FF \rightarrow F$ the unit and multiplication maps of the fibrant replacement monad F on $\mathsf{Sp}^{\mathbb{N}}(\mathsf{S}_*^Z)$ (see [5, 6.1]) and define $\tilde{\Omega}_Z^\infty := \Omega_Z^\infty F$. Since every object in S_*^Z is cofibrant, Σ^r is already derived and we define $\tilde{\Sigma}^r := \Sigma^r$. If we iterate the comparison map $\mathrm{id} \rightarrow \tilde{\Omega}_Z^r \tilde{\Sigma}_Z^r$ it follows that we can build a cosimplicial resolution of id with respect to $\tilde{\Omega}_Z^r \tilde{\Sigma}_Z^r$ of the form

$$\mathrm{id} \longrightarrow (\tilde{\Omega}_Z^r \tilde{\Sigma}_Z^r) \rightrightarrows (\tilde{\Omega}_Z^r \tilde{\Sigma}_Z^r)^2 \rightrightarrows (\tilde{\Omega}_Z^r \tilde{\Sigma}_Z^r)^3 \cdots \tag{1}$$

for each $1 \leq r \leq \infty$; these are the types of homotopical resolutions studied in [5]; see also [4, 9, 11]. Here, we are only showing the coface maps. If $X \in \mathsf{S}_*^Z$, the Bousfield-Kan completion of X with respect to $\tilde{\Omega}_Z^r \tilde{\Sigma}_Z^r$ is the homotopy limit

$$X^\wedge_{\tilde{\Omega}_Z^r \tilde{\Sigma}_Z^r} := \mathrm{holim}_\Delta (\tilde{\Omega}_Z^r \tilde{\Sigma}_Z^r)^{\bullet+1}(X) \tag{2}$$

of the Bousfield-Kan cosimplicial resolution (1) evaluated at X. To obtain the Bousfield-Kan completion tower, we filter Δ ([4, 5.22]) by its subcategories $\Delta^{\leq n} \subset \Delta$, $n \geq 0$, and define

$$(\tilde{\Omega}_Z^r \tilde{\Sigma}_Z^r)_n := \mathrm{holim}_{\Delta^{\leq n}} (\tilde{\Omega}_Z^r \tilde{\Sigma}_Z^r)^{\bullet+1}, \qquad n \geq 0$$

to obtain the $\tilde{\Omega}_Z^r \tilde{\Sigma}_Z^r$-completion of X

$$X^\wedge_{\tilde{\Omega}_Z^r \tilde{\Sigma}_Z^r} \simeq \mathrm{holim}\Big((\tilde{\Omega}_Z^r \tilde{\Sigma}_Z^r)_0(X) \leftarrow (\tilde{\Omega}_Z^r \tilde{\Sigma}_Z^r)_1(X) \leftarrow (\tilde{\Omega}_Z^r \tilde{\Sigma}_Z^r)_2(X) \leftarrow \cdots \Big) \tag{3}$$

as the homotopy limit of the completion tower, where

$$(\tilde{\Omega}_Z^r \tilde{\Sigma}_Z^r)_0(X) \simeq (\tilde{\Omega}_Z^r \tilde{\Sigma}_Z^r)(X)$$

For conceptual simplicity and convenience we denote by $*_Z := Z$ the null object in S_*^Z. It will be useful to denote by $*'_Z \simeq *_Z$ an appropriately fattened-up version of the null object $*_Z$ in S_*^Z.

Proposition 2.1. *Let $k \geq 0$ and $1 \leq r \leq \infty$. Let X be a retractive pointed space over Z. If X is k-connected (rel. Z), then the comparison map $X \rightarrow \tilde{\Omega}_Z^r \tilde{\Sigma}_Z^r X$ is $(2k+1)$-connected.*

Proof. Consider the case $r = 1$. Consider a pushout cofibration 2-cube of the form

$$\begin{array}{ccc} X & \longrightarrow & *'_Z \\ \downarrow & & \downarrow \\ *'_Z & \longrightarrow & \tilde{\Sigma}_Z X \end{array}$$

in S_*^Z. By assumption we know that the upper and left-hand 1-faces are $(k+1)$-connected. Since the 2-cube is ∞-cocartesian, it follows that the lower and right-hand 1-faces are $(k+1)$-connected. By higher Blakers-Massey [18, 2.5] for S_*, we know that the 2-cube is l-cartesian where l is the minimum of

$$1 - 2 + l_{\{1,2\}} = -1 + \infty$$
$$1 - 2 + l_{\{1\}} + l_{\{2\}} = -1 + (k+1) + (k+1)$$

Hence $l = 2k + 1$, the 2-cube is $(2k+1)$-cartesian, and the comparison map $X \to \tilde{\Omega}_Z \tilde{\Sigma}_Z X$ is $(2k+1)$-connected. The other cases ($r \geq 2$) follow by repeated application of the $r = 1$ case to $\tilde{\Sigma}_Z X, \tilde{\Sigma}_Z^2 X, \tilde{\Sigma}_Z^3 X, \dots$ in the usual way, and finally, for $r = \infty$ by considering the homotopy colimit of the resulting sequence. □

For $k \geq 1$, this pattern persists for the iterative application of $\mathrm{id} \to \tilde{\Omega}_Z^r \tilde{\Sigma}_Z^r$ to go from 0-cubes to 1-cubes to 2-cubes to 3-cubes, and so forth.

Proposition 2.2. *Let $k \geq 1$ and $1 \leq r \leq \infty$. Let W be a finite set and $\mathcal{X}$ a W-cube in S_*^Z. Let $n = |W|$. If the n-cube $\mathcal{X}$ is $(k(\mathrm{id}+1)+1)$-cartesian in S_*^Z, then so is the $(n+1)$-cube of the form $\mathcal{X} \to \tilde{\Omega}_Z^r \tilde{\Sigma}_Z^r \mathcal{X}$.*

Proof. These estimates are proved in [4, 1.7, 1.8] for the special case of $Z = *$ using higher Blakers-Massey (and its dual) [18, 2.5, 2.6], together with ideas closely related to [11, 13, 14]; and similar to the proof of Proposition 2.1, exactly the same arguments (and estimates) remain true in the more general context of pointed spaces centered at Z; see, for instance, [9] where the analogous passage to the retractive setting is demonstrated in detail for operadic algebras in spectra. □

Proof of Theorem 1.1. To verify that $X \simeq X^\wedge_{\tilde{\Omega}_Z^r \tilde{\Sigma}_Z^r}$, it suffices to verify that the map of the form $X \to (\tilde{\Omega}_Z^r \tilde{\Sigma}_Z^r)_n(X)$ into the n-th stage of the Bousfield-Kan completion tower has connectivity strictly increasing with n. The connectivity of this map is the same as the cartesian-ness of the coface $(n+1)$-cube ([3, 3.13]) of the coaugmented Bousfield-Kan cosimplicial resolution which we calculated (Propositions 2.1 and 2.2) to be $(((n+1)+1)+1) = n+3$, which completes the proof. □

Proposition 2.3. *Let $n \geq 1$. Let X be a retractive pointed space over Z. If X is 0-connected (rel. Z), then the maps*

$$(\tilde{\Omega}_Z \tilde{\Sigma}_Z)^k(X) \xrightarrow{(*)_n} P_n^Z (\tilde{\Omega}_Z \tilde{\Sigma}_Z)^k(X), \qquad k \geq 1$$

are $(n+1)$-connected.

Proof. Consider the case of $n = 1$. The 1-excisive approximation $P_1^Z (\tilde{\Omega}_Z \tilde{\Sigma}_Z)^k(X)$ to the functor $(\tilde{\Omega}_Z \tilde{\Sigma}_Z)^k$ on retractive pointed spaces over Z, evaluated at X, is the homotopy colimit of

$$(\tilde{\Omega}_Z \tilde{\Sigma}_Z)^k(X) \xrightarrow{(\#)_1} T_1^Z (\tilde{\Omega}_Z \tilde{\Sigma}_Z)^k(X) \to T_1^Z (T_1^Z (\tilde{\Omega}_Z \tilde{\Sigma}_Z)^k)(X) \to \cdots$$

the indicated sequence ([19, Section 1]); we want to estimate the connectivities of these maps. It follows, by iteratively applying higher Blakers-Massey (and its dual) [18, 2.5, 2.6] for S_* that the maps $(\#)_1$ are 2-connected, and the other maps are higher connected. In more detail: here is the basic idea for the maps $(\#)_1$; estimates for the other (more highly connected) maps are similar. Consider the ∞-cocartesian 2-cube $\mathcal{X}$ of the form

$$\begin{array}{ccc} X & \longrightarrow & *'_Z \\ \downarrow & & \downarrow \\ *'_Z & \longrightarrow & \tilde{\Sigma}_Z X \end{array}$$

in S_*^Z. Since X is 0-connected (rel. Z), we know that $*_Z \to X$ is 0-connected and hence $X \to *_Z$ is 1-connected ([18, 1.5]); therefore we know that $\mathcal{X}$ satisfies: the 1-subcubes are 1-connected and the 2-cube is ∞-cocartesian. Then $\tilde{\Sigma}_Z\mathcal{X}$ satisfies: the 1-subcubes are 2-connected and the 2-subcubes (there is only one) are ∞-cocartesian. By higher Blakers-Massey [18, 2.5] for S_*, we know the 2-cube is k-cartesian where k is the minimum of

$$\begin{aligned} 1-2+k_{\{1,2\}} &= -1+\infty \\ 1-2+k_{\{1\}}+k_{\{2\}} &= -1+2+2 \end{aligned}$$

Hence $k=3$, our 2-cube is 3-cartesian, and $\tilde{\Sigma}_Z\mathcal{X}$ satisfies: the 1-subcubes are 2-connected and the 2-subcubes are 3-cartesian. Then $(\tilde{\Omega}_Z\tilde{\Sigma}_Z)\mathcal{X}$ satisfies: the 1-subcubes are 1-connected and the 2-subcubes are 2-cartesian. Hence we have verifed that the map

$$(\tilde{\Omega}_Z\tilde{\Sigma}_Z)(X) \xrightarrow{(\#)_1} T_1^Z(\tilde{\Omega}_Z\tilde{\Sigma}_Z)(X)$$

is 2-connected. Let's keep going. By higher dual Blakers-Massey [18, 2.6] for S_*, we know the 2-cube is k-cocartesian where k is the minimum of

$$\begin{aligned} 2-1+k_{\{1,2\}} &= 1+2 \\ 2-1+k_{\{1\}}+k_{\{2\}} &= 1+1+1 \end{aligned}$$

Hence $k=3$, our 2-cube is 3-cocartesian, and $(\tilde{\Omega}_Z\tilde{\Sigma}_Z)\mathcal{X}$ satisfies: the 1-subcubes are 1-connected and the 2-subcubes are 3-cocartesian. Then $\tilde{\Sigma}_Z(\tilde{\Omega}_Z\tilde{\Sigma}_Z)\mathcal{X}$ satisfies: the 1-subcubes are 2-connected and the 2-subcubes are 4-cocartesian. By higher Blakers-Massey [18, 2.5] for S_*, we know the 2-cube is k-cartesian where k is the minimum of

$$\begin{aligned} 1-2+k_{\{1,2\}} &= -1+4 \\ 1-2+k_{\{1\}}+k_{\{2\}} &= -1+2+2 \end{aligned}$$

Hence, $k=3$, our 2-cube is 3-cartesian, and $\tilde{\Sigma}_Z(\tilde{\Omega}_Z\tilde{\Sigma}_Z)\mathcal{X}$ satisfies: the 1-subcubes are 2-connected and the 2-subcubes are 3-cartesian. Then $(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^2\mathcal{X}$ satisfies: the 1-subcubes are 1-connected and the 2-subcubes are 2-cartesian. Hence we have verifed that the map

$$(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^2(X) \xrightarrow{(\#)_1} T_1^Z(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^2(X)$$

is 2-connected. Let's keep going. By higher dual Blakers-Massey [18, 2.6] for S_*, we know the 2-cube is k-cocartesian where k is the minimum of

$$\begin{aligned} 2-1+k_{\{1,2\}} &= 1+2 \\ 2-1+k_{\{1\}}+k_{\{2\}} &= 1+1+1 \end{aligned}$$

Hence $k=3$, our 2-cube is 3-cocartesian, and $(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^2\mathcal{X}$ satisfies: the 1-subcubes are 1-connected and the 2-subcubes are 3-cocartesian. Then $\tilde{\Sigma}_Z(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^2\mathcal{X}$ satisfies: the 1-subcubes are 2-connected and the 2-subcubes are 4-cocartesian. By higher Blakers-Massey [18, 2.5] for S_*, we know the 2-cube is k-cartesian where k is the minimum of

$$\begin{aligned} 1-2+k_{\{1,2\}} &= -1+4 \\ 1-2+k_{\{1\}}+k_{\{2\}} &= -1+2+2 \end{aligned}$$

Hence, $k=3$, our 2-cube is 3-cartesian, and $\tilde{\Sigma}_Z(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^2\mathcal{X}$ satisfies: the 1-subcubes are 2-connected and the 2-subcubes are 3-cartesian. Then $(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^3\mathcal{X}$ satisfies: the 1-subcubes are 1-connected and the 2-subcubes are 2-cartesian. Hence we have verifed that the map

$$(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^3(X) \xrightarrow{(\#)_1} T_1^Z(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^3(X)$$

is 2-connected; notice how the subcube estimates have stabilized at each respective step. And so forth. Hence it follows that the maps

$$(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^k(X) \xrightarrow{(\#)_1} T_1^Z(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^k(X), \qquad k\geq 1$$

are 2-connected. Consider the case of $n=2$. The 2-excisive approximation $P_2^Z(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^k(X)$ to the functor $(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^k$ on retractive pointed spaces over Z, evaluated at X, is the homotopy colimit of

$$(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^k(X) \xrightarrow{(\#)_2} T_2^Z(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^k(X) \rightarrow T_2^Z(T_2^Z(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^k)(X) \rightarrow \cdots$$

the indicated sequence ([19, Section 1]); we want to estimate the connectivities of these maps. It follows, by iteratively applying higher Blakers-Massey (and its dual) [18, 2.5, 2.6] for S_* that the maps $(\#)_2$ are 3-connected, and the other maps are higher connected. In more detail: here is the basic idea for the maps $(\#)_2$; estimates for the other (more highly connected) maps are similar. Consider a strongly ∞-cocartesian 3-cube $\mathcal{X}$ satisfying: the 1-subcubes are 1-connected, the 2-subcubes are ∞-cocartesian, and the 3-subcubes (there is only one) are ∞-cocartesian. Then $\tilde{\Sigma}_Z\mathcal{X}$ satisfies: the 1-subcubes are 2-connected, the 2-subcubes are ∞-cocartesian, and the 3-subcubes are ∞-cocartesian. By higher Blakers-Massey [18, 2.5] for S_*, we know the 3-cube is k-cartesian where k is the minimum of

$$\begin{aligned} 1-3+k_{\{1,2,3\}} &= -2+\infty \\ 1-3+k_{\{1,2\}}+k_{\{3\}} &= -2+\infty+2 \\ 1-3+k_{\{1\}}+k_{\{2\}}+k_{\{3\}} &= -2+2+2+2 \end{aligned}$$

Hence $k=4$, our 3-cube is 4-cartesian, and $\tilde{\Sigma}_Z\mathcal{X}$ satisfies: the 1-subcubes are 2-connected, the 2-subcubes are 3-cartesian, and the 3-subcubes are 4-cartesian.

Then $(\tilde{\Omega}_Z\tilde{\Sigma}_Z)\mathcal{X}$ satisfies: the 1-subcubes are 1-connected, the 2-subcubes are 2-cartesian, and the 3-subcubes are 3-cartesian. Hence we have verifed that the map

$$(\tilde{\Omega}_Z\tilde{\Sigma}_Z)(X) \xrightarrow{(\#)_2} T_2^Z(\tilde{\Omega}_Z\tilde{\Sigma}_Z)(X)$$

is 3-connected. Let's keep going. By higher dual Blakers-Massey [18, 2.6] for S_*, we know the 3-cube is k-cocartesian where k is the minimum of

$$\begin{aligned} 3-1+k_{\{1,2,3\}} &= 2+3 \\ 3-1+k_{\{1,2\}}+k_{\{3\}} &= 2+2+1 \\ 3-1+k_{\{1\}}+k_{\{2\}}+k_{\{3\}} &= 2+1+1+1 \end{aligned}$$

Hence $k=5$, our 3-cube is 5-cocartesian, and $(\tilde{\Omega}_Z\tilde{\Sigma}_Z)\mathcal{X}$ satisfies: the 1-subcubes are 1-connected, the 2-subcubes are 3-cocartesian, and the 3-subcubes are 5-cocartesian. Then $\tilde{\Sigma}_Z(\tilde{\Omega}_Z\tilde{\Sigma}_Z)\mathcal{X}$ satisfies: the 1-subcubes are 2-connected, the 2-subcubes are 4-cocartesian, and the 3-subcubes are 6-cocartesian. By higher Blakers-Massey [18, 2.5] for S_*, we know the 3-cube is k-cartesian where k is the minimum of

$$\begin{aligned} 1-3+k_{\{1,2,3\}} &= -2+6 \\ 1-3+k_{\{1,2\}}+k_{\{3\}} &= -2+4+2 \\ 1-3+k_{\{1\}}+k_{\{2\}}+k_{\{3\}} &= -2+2+2+2 \end{aligned}$$

Hence, $k=4$, our 3-cube is 4-cartesian, and $\tilde{\Sigma}_Z(\tilde{\Omega}_Z\tilde{\Sigma}_Z)\mathcal{X}$ satisfies: the 1-subcubes are 2-connected, the 2-subcubes are 3-cartesian, and the 3-subcubes are 4-cartesian. Then $(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^2\mathcal{X}$ satisfies: the 1-subcubes are 1-connected, the 2-subcubes are 2-cartesian, and the 3-subcubes are 3-cartesian. Hence we have verifed that the map

$$(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^2(X) \xrightarrow{(\#)_2} T_2^Z(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^2(X)$$

is 3-connected. Let's keep going. By higher dual Blakers-Massey [18, 2.6] for S_*, we know the 3-cube is k-cocartesian where k is the minimum of

$$\begin{aligned} 3-1+k_{\{1,2,3\}} &= 2+3 \\ 3-1+k_{\{1,2\}}+k_{\{3\}} &= 2+2+1 \\ 3-1+k_{\{1\}}+k_{\{2\}}+k_{\{3\}} &= 2+1+1+1 \end{aligned}$$

Hence $k=5$, our 3-cube is 5-cocartesian, and $(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^2\mathcal{X}$ satisfies: the 1-subcubes are 1-connected, the 2-subcubes are 3-cocartesian, and the 3-subcubes are 5-cocartesian. Then $\tilde{\Sigma}_Z(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^2\mathcal{X}$ satisfies: the 1-subcubes are 2-connected, the 2-subcubes are 4-cocartesian, and the 3-subcubes are 6-cocartesian. By higher Blakers-Massey [18, 2.5] for S_*, we know the 3-cube is k-cartesian where k is the minimum of

$$\begin{aligned} 1-3+k_{\{1,2,3\}} &= -2+6 \\ 1-3+k_{\{1,2\}}+k_{\{3\}} &= -2+4+2 \\ 1-3+k_{\{1\}}+k_{\{2\}}+k_{\{3\}} &= -2+2+2+2 \end{aligned}$$

Hence, $k=4$, our 3-cube is 4-cartesian, and $\tilde{\Sigma}_Z(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^2\mathcal{X}$ satisfies: the 1-subcubes are 2-connected, the 2-subcubes are 3-cartesian, and the 3-subcubes are 4-cartesian. Then $(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^3\mathcal{X}$ satisfies: the 1-subcubes are 1-connected, the 2-subcubes are 2-cartesian, and the 3-subcubes are 3-cartesian. Hence we have verifed that the map

$$(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^3(X) \xrightarrow{(\#)_2} T_2^Z(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^3(X)$$

is 3-connected; notice how the subcube estimates have stabilized at each respective step. And so forth. Hence it follows that the maps

$$(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^k(X) \xrightarrow{(\#)_2} T_2^Z(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^k(X), \qquad k \geq 1$$

are 3-connected. And so forth. □

Proposition 2.4. *Let $n \geq 1$ and $1 \leq r \leq \infty$. Let X be a retractive pointed space over Z. If X is 0-connected (rel. Z), then the maps*

$$(\tilde{\Omega}_Z^r\tilde{\Sigma}_Z^r)^k(X) \xrightarrow{(*)_n} P_n^Z(\tilde{\Omega}_Z^r\tilde{\Sigma}_Z^r)^k(X), \qquad k \geq 1$$

are $(n+1)$-connected.

Proof. A detailed proof of the $r = 1$ case is given above (Proposition 2.3), and the other cases are similar. In the case of $r = \infty$, several of the steps are easier since $\tilde{\Sigma}_Z^\infty$ preserves cocartesian-ness, $\tilde{\Omega}_Z^\infty$ preserves cartesian-ness, and the stable estimates in [10, 3.10] are available for each estimate step following the application of $\tilde{\Sigma}_Z^\infty$. □

Proposition 2.5. *Let $n \geq 1$. Let X be a retractive pointed space over Z. If X is 0-connected (rel. Z), then the maps*

$$(\tilde{\Omega}_Z\tilde{\Sigma}_Z)_k(X) \xrightarrow{(**)_n} P_{n+k}^Z(\tilde{\Omega}_Z\tilde{\Sigma}_Z)_k(X), \qquad k \geq 0$$

are $(n+1)$-connected.

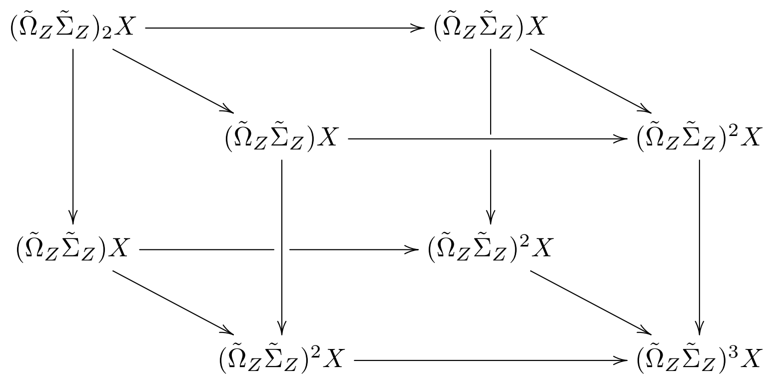
Proof. Consider the case of $k = 0$. Then the map $(**)_n$ is $(n+1)$-connected by Proposition 2.4. Consider the case of $k = 1$. By definition, $(\tilde{\Omega}_Z\tilde{\Sigma}_Z)_1 X$ fits into an ∞-cartesian 2-cube of the form ([4, 5.26])

$$\begin{array}{ccc}
(\tilde{\Omega}_Z\tilde{\Sigma}_Z)_1 X & \longrightarrow & (\tilde{\Omega}_Z\tilde{\Sigma}_Z) X \\
\downarrow & & \downarrow \\
(\tilde{\Omega}_Z\tilde{\Sigma}_Z) X & \longrightarrow & (\tilde{\Omega}_Z\tilde{\Sigma}_Z)^2 X
\end{array}$$

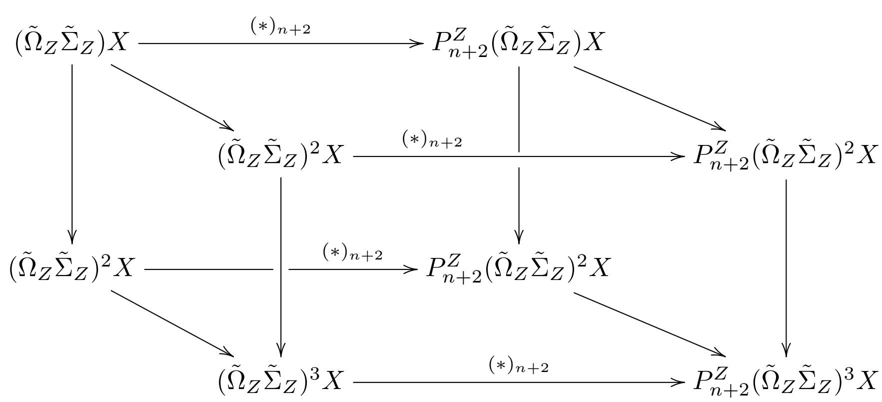
and therefore the map $(**)_n$ fits into a 3-cube of the form

$$\begin{array}{ccccccc}
(\tilde{\Omega}_Z\tilde{\Sigma}_Z)_1 X & & \longrightarrow & & (\tilde{\Omega}_Z\tilde{\Sigma}_Z) X & & \\
\downarrow & \searrow^{(**)_n} & & & \downarrow & \searrow^{(*)_{n+1}} & \\
& & P_{n+1}^Z(\tilde{\Omega}_Z\tilde{\Sigma}_Z)_1 X & & \longrightarrow & & P_{n+1}^Z(\tilde{\Omega}_Z\tilde{\Sigma}_Z) X \\
(\tilde{\Omega}_Z\tilde{\Sigma}_Z) X & & \downarrow & \longrightarrow & (\tilde{\Omega}_Z\tilde{\Sigma}_Z)^2 X & & \downarrow \\
& \searrow^{(*)_{n+1}} & & & & \searrow^{(*)_{n+1}} & \\
& & P_{n+1}^Z(\tilde{\Omega}_Z\tilde{\Sigma}_Z) X & & \longrightarrow & & P_{n+1}^Z(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^2 X
\end{array} \tag{4}$$

Several applications of [18, 1.6] show that the map $(**)_n$ is $(n+1)$-connected. In more detail: the back 2-face is ∞-cartesian, hence the front 2-face is ∞-cartesian ([19, 1.7]). Therefore, the 3-cube is ∞-cartesian. By Proposition 2.4, the maps $(*)_{n+1}$ are $(n+2)$-connected, hence the right-hand 2-face is $(n+1)$-cartesian. Since the 3-cube is ∞-cartesian, we therefore know the left-hand 2-face is $(n{+}1)$-cartesian; and hence, since the map $(*)_{n+1}$ is $(n+2)$-connected, therefore we know the map $(**)_n$ is $(n+1)$-connected. Consider the case of $k = 2$. By definition, $(\tilde{\Omega}_Z\tilde{\Sigma}_Z)_2 X$ fits into an ∞-cartesian 3-cube $\mathcal{X}$ of the form

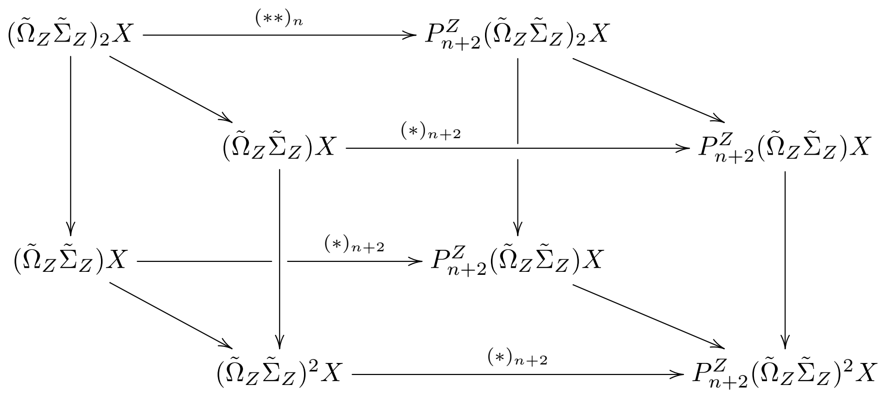


and therefore the map $(**)_n$ fits into a 4-cube of the form $\mathcal{X} \to P^Z_{n+2}\mathcal{X}$. Several applications of [18, 1.6] show that the map $(**)_n$ is $(n+1)$-connected. In more detail: $\mathcal{X}$ is ∞-cartesian, hence $P^Z_{n+2}\mathcal{X}$ is ∞-cartesian ([19, 1.7]). Therefore, the 4-cube is ∞-cartesian. Consider the 3-face of the form



By Proposition 2.4, the maps $(*)_{n+2}$ are $(n+3)$-connected, hence the top and bottom 2-faces are $(n+2)$-cartesian, and therefore the 3-face is $(n+1)$-cartesian.

Since the 4-cube is ∞-cartesian, we therefore know the opposite 3-face of the form



is $(n+1)$-cartesian; and hence, since the maps $(*)_{n+2}$ are $(n+3)$-connected, we know the bottom 2-face is $(n+2)$-cartesian, it follows that the top face is $(n+1)$-cartesian, and since the map $(*)_{n+2}$ is $(n+3)$-connected, therefore we know the map $(**)_n$ is $(n+1)$-connected. The other cases similarly follow by repeated applications of [18, 1.6]. □

Proposition 2.6. *Let $n \geq 1$ and $1 \leq r \leq \infty$. Let X be a retractive pointed space over Z. If X is 0-connected (rel. Z), then the maps*

$$(\tilde{\Omega}^r_Z \tilde{\Sigma}^r_Z)_k(X) \xrightarrow{(**)_n} P^Z_{n+k}(\tilde{\Omega}^r_Z \tilde{\Sigma}^r_Z)_k(X), \qquad k \geq 0$$

are $(n+1)$-connected.

Proof. A detailed proof of the $r = 1$ case is given above (Proposition 2.5), and the other cases are similar; the estimates are identical (Proposition 2.4). □

Proof of Theorem 1.3. We follow the basic proof strategy in [28], and the subsequent development in [9], for operadic algebras in spectra (where the estimates are different). Here is the basic idea. Consider the case of $r = 1$. We start with the Bousfield-Kan completion tower of the form

$$(\tilde{\Omega}_Z \tilde{\Sigma}_Z)_0 \longleftarrow (\tilde{\Omega}_Z \tilde{\Sigma}_Z)_1 \longleftarrow (\tilde{\Omega}_Z \tilde{\Sigma}_Z)_2 \cdots$$

and resolve each term by its Taylor tower to produce the tower of towers diagram

$$\begin{array}{ccccc} P^Z_3(\tilde{\Omega}_Z \tilde{\Sigma}_Z)_0(X) & \longleftarrow & P^Z_3(\tilde{\Omega}_Z \tilde{\Sigma}_Z)_1(X) & \longleftarrow & P^Z_3(\tilde{\Omega}_Z \tilde{\Sigma}_Z)_2(X) \cdots \\ \downarrow & & \downarrow & & \downarrow \\ P^Z_2(\tilde{\Omega}_Z \tilde{\Sigma}_Z)_0(X) & \longleftarrow & P^Z_2(\tilde{\Omega}_Z \tilde{\Sigma}_Z)_1(X) & \longleftarrow & P^Z_2(\tilde{\Omega}_Z \tilde{\Sigma}_Z)_2(X) \cdots \\ \downarrow & & \downarrow & & \downarrow \\ P^Z_1(\tilde{\Omega}_Z \tilde{\Sigma}_Z)_0(X) & \longleftarrow & P^Z_1(\tilde{\Omega}_Z \tilde{\Sigma}_Z)_1(X) & \longleftarrow & P^Z_1(\tilde{\Omega}_Z \tilde{\Sigma}_Z)_2(X) \cdots \end{array} \tag{5}$$

By our uniformity estimates (Propositions 2.1 and 2.2), it follows immediately that $\mathrm{id} \rightarrow (\tilde{\Omega}_Z \tilde{\Sigma}_Z)_n$ satisfies O_{n+1} ([19, 1.2]) for each $n \geq 0$; in other words, via this map the functors id and $(\tilde{\Omega}_Z \tilde{\Sigma}_Z)_n$ agree to order $n+1$ and hence the maps

$$P^Z_m(\mathrm{id})(X) \xrightarrow{\simeq} P^Z_m(\tilde{\Omega}_Z \tilde{\Sigma}_Z)_n(X), \qquad 1 \leq m \leq n+1$$

are weak equivalences for every $n \geq 0$. It follows that holim applied horizontally produces the Taylor tower of the identity functor $\{P_n^Z(\mathrm{id})\}$ and hence applying holim first horizontally and then vertically produces

$$\operatorname*{holim}_{\mathsf{vert}} \operatorname*{holim}_{\mathsf{horiz}} (5) \simeq P_\infty^Z(\mathrm{id})(X)$$

What about the other way? By our estimates (Proposition 2.6), it follows immediately that holim applied vertically produces the Bousfield-Kan completion tower

$$(\tilde{\Omega}_Z\tilde{\Sigma}_Z)_0(X) \longleftarrow (\tilde{\Omega}_Z\tilde{\Sigma}_Z)_1(X) \longleftarrow (\tilde{\Omega}_Z\tilde{\Sigma}_Z)_2(X)\cdots$$

and hence applying holim first vertically and then horizontally produces

$$\operatorname*{holim}_{\mathsf{horiz}} \operatorname*{holim}_{\mathsf{vert}} (5) \simeq X^\wedge_{\tilde{\Omega}_Z\tilde{\Sigma}_Z}$$

Hence we have verified that

$$P_\infty^Z(\mathrm{id})(X) \simeq X^\wedge_{\tilde{\Omega}_Z\tilde{\Sigma}_Z}$$

The other cases are similar; the estimates are identical (Proposition 2.6). □

Proposition 2.7. *If $n \geq -1$, then $(\tilde{\Omega}_Z\tilde{\Sigma}_Z)$ preserves $(\mathrm{id}{+}1)$-cartesian $(n{+}1)$-cubes in S_*^Z.*

Proof. The cases for $n = -1, 0$ are trivial. Now that we know the desired behavior is satisfied on 0-subcubes, we will not continue to indicate their estimates below when verifying the $(\mathrm{id}+1)$-cartesian property. Consider the case of $n = 1$. Assume that $\mathcal{X}$ is an $(\mathrm{id}+1)$-cartesian 2-cube in S_*^Z. Then $\mathcal{X}$ satisfies: the 1-subcubes are 2-connected, and the 2-subcubes (there is only one) are 3-cartesian. By higher dual Blakers-Massey [18, 2.6] for S_*, we know the 2-cube is k-cocartesian where k is the minimum of

$$2 - 1 + k_{\{1,2\}} = 1 + 3$$
$$2 - 1 + k_{\{1\}} + k_{\{2\}} = 1 + 2 + 2$$

Hence $k = 4$, our 3-cube is 4-cocartesian, and $\mathcal{X}$ satisfies: the 1-subcubes are 2-connected, and the 2-subcubes are 4-cocartesian. Then $\tilde{\Sigma}_Z\mathcal{X}$ satisfies: the 1-subcubes are 3-connected, and the 2-subcubes are 5-cocartesian. By higher Blakers-Massey [18, 2.5] for S_*, we know the 2-cube is k-cartesian where k is the minimum of

$$1 - 2 + k_{\{1,2\}} = -1 + 5$$
$$1 - 2 + k_{\{1\}} + k_{\{2\}} = -1 + 3 + 3$$

Hence, $k = 4$, our 2-cube is 4-cartesian, and $\tilde{\Sigma}_Z\mathcal{X}$ satisfies: the 1-subcubes are 3-connected, and the 2-subcubes are 4-cartesian. Then $\tilde{\Omega}_Z\tilde{\Sigma}_Z\mathcal{X}$ satisfies: the 1-subcubes are 2-connected, and the 2-subcubes are 3-cartesian. Hence we have verified that $(\tilde{\Omega}_Z\tilde{\Sigma}_Z)\mathcal{X}$ is $(\mathrm{id}+1)$-cartesian in S_*^Z. Consider the case of $n = 2$. Assume that $\mathcal{X}$ is an $(\mathrm{id}{+}1)$-cartesian 3-cube in S_*^Z. Then $\mathcal{X}$ satisfies: the 1-subcubes are 2-connected, the 2-subcubes are 3-cartesian, and the 3-subcubes (there is only one) are 4-cartesian. By higher dual Blakers-Massey [18, 2.5] for S_*, we know the

3-cube is k-cocartesian where k is the minimum of

$$\begin{aligned} 3-1+k_{\{1,2,3\}} &= 2+4 \\ 3-1+k_{\{1,2\}}+k_{\{3\}} &= 2+3+2 \\ 3-1+k_{\{1\}}+k_{\{2\}}+k_{\{3\}} &= 2+2+2+2 \end{aligned}$$

Hence $k=6$, our 3-cube is 6-cocartesian, and $\mathcal{X}$ satisfies: the 1-subcubes are 2-connected, the 2-subcubes are 4-cocartesian, and the 3-subcubes are 6-cocartesian. Then $\tilde{\Sigma}_Z\mathcal{X}$ satisfies: the 1-subcubes are 3-connected, the 2-subcubes are 5-cocartesian, and the 3-subcubes are 7-cocartesian. By higher Blakers-Massey [18, 2.5] for S_*, we know the 3-cube is k-cartesian where k is the minimum of

$$\begin{aligned} 1-3+k_{\{1,2,3\}} &= -2+7 \\ 1-3+k_{\{1,2\}}+k_{\{3\}} &= -2+5+3 \\ 1-3+k_{\{1\}}+k_{\{2\}}+k_{\{3\}} &= -2+3+3+3 \end{aligned}$$

Hence, $k=5$, our 3-cube is 5-cartesian, and $\tilde{\Sigma}_Z\mathcal{X}$ satisfies: the 1-subcubes are 3-connected, the 2-subcubes are 4-cartesian, and the 3-subcubes are 5-cartesian. Then $(\tilde{\Omega}_Z\tilde{\Sigma}_Z)\mathcal{X}$ satisfies: the 1-subcubes are 2-connected, the 2-subcubes are 3-cartesian, and the 3-subcubes are 4-cartesian. Hence we have verified that $(\tilde{\Omega}_Z\tilde{\Sigma}_Z)\mathcal{X}$ is $(\mathrm{id}+1)$-cartesian in S_*^Z. And so forth. □

Proposition 2.8. *Let $1 \leq r \leq \infty$. If $n \geq -1$, then $(\tilde{\Omega}_Z^r\tilde{\Sigma}_Z^r)$ preserves $(\mathrm{id}+1)$-cartesian $(n+1)$-cubes in S_*^Z.*

Proof. A detailed proof of the $r=1$ case is given above (Proposition 2.7), and the other cases are similar. In the case of $r=\infty$, several of the steps are easier since $\tilde{\Sigma}_Z^\infty$ preserves cocartesian-ness, $\tilde{\Omega}_Z^\infty$ preserves cartesian-ness, and the stable estimates in [10, 3.10] are available for each estimate step following the application of $\tilde{\Sigma}_Z^\infty$. □

Proof of Theorem 1.2. We follow the basic proof strategy in [26], and the subsequent development in [9], for operadic algebras in spectra (where the estimates are different). Here is the basic idea. Consider the case $r=1$. Start with the Bousfield-Kan cosimplicial resolution

$$\mathrm{id} \longrightarrow (\tilde{\Omega}_Z\tilde{\Sigma}_Z) \rightrightarrows (\tilde{\Omega}_Z\tilde{\Sigma}_Z)^2 \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} (\tilde{\Omega}_Z\tilde{\Sigma}_Z)^3 \cdots \tag{6}$$

of the identity functor and consider the fibration sequence $F \to E \to B$ in S_*^Z. Since we know that E, B are 1-connected (rel. Z) by assumption, this means that we have the homotopical estimates in Propositions 2.1 and 2.2 available. With this in mind, let's resolve E, B with respect to the Bousfield-Kan cosimplicial resolution

$$\begin{array}{ccccccc}
F & \longrightarrow & \tilde{F}^0 & \rightrightarrows & \tilde{F}^1 & \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} & \tilde{F}^2 \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E & \longrightarrow & (\tilde{\Omega}_Z\tilde{\Sigma}_Z)(E) & \rightrightarrows & (\tilde{\Omega}_Z\tilde{\Sigma}_Z)^2(E) & \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} & (\tilde{\Omega}_Z\tilde{\Sigma}_Z)^3(E) \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B & \longrightarrow & (\tilde{\Omega}_Z\tilde{\Sigma}_Z)(B) & \rightrightarrows & (\tilde{\Omega}_Z\tilde{\Sigma}_Z)^2(B) & \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} & (\tilde{\Omega}_Z\tilde{\Sigma}_Z)^3(B) \cdots
\end{array}$$

and take homotopy fibers vertically to define the coaugmented cosimplicial diagram of the form $F \to \tilde{F}$. By construction the columns are homotopy fiber sequences in S^Z_*, and since E, B are 1-connected (rel Z), we know from Theorem 1.1 that

$$E \xrightarrow{\simeq} \mathrm{holim}_\Delta (\tilde{\Omega}_Z\tilde{\Sigma}_Z)^{\bullet+1}(E)$$
$$B \xrightarrow{\simeq} \mathrm{holim}_\Delta (\tilde{\Omega}_Z\tilde{\Sigma}_Z)^{\bullet+1}(B)$$

Since homotopy limits commute with homotopy fibers, it follows that

$$F \simeq \ \mathrm{holim}_\Delta \tilde{F}$$

We want to show that $F \simeq F^\wedge_{\tilde{\Omega}_Z\tilde{\Sigma}_Z}$. To get Bousfield-Kan completion into the picture, let's resolve each term in $F \to \tilde{F}$ with respect to (6) to obtain the cosimplicial resolution of coaugmented cosimplicial diagrams of the form

$$(7)\qquad
\begin{array}{ccccccc}
(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^3F & \xrightarrow{(\#)} & (\tilde{\Omega}_Z\tilde{\Sigma}_Z)^3\tilde{F}^0 & \rightrightarrows & (\tilde{\Omega}_Z\tilde{\Sigma}_Z)^3\tilde{F}^1 & \Rrightarrow & (\tilde{\Omega}_Z\tilde{\Sigma}_Z)^3\tilde{F}^2\cdots \\
\uparrow\uparrow\uparrow & & \uparrow\uparrow\uparrow & & \uparrow\uparrow\uparrow & & \uparrow\uparrow\uparrow \\
(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^2F & \xrightarrow{(\#)} & (\tilde{\Omega}_Z\tilde{\Sigma}_Z)^2\tilde{F}^0 & \rightrightarrows & (\tilde{\Omega}_Z\tilde{\Sigma}_Z)^2\tilde{F}^1 & \Rrightarrow & (\tilde{\Omega}_Z\tilde{\Sigma}_Z)^2\tilde{F}^2\cdots \\
\uparrow\uparrow & & \uparrow\uparrow & & \uparrow\uparrow & & \uparrow\uparrow \\
(\tilde{\Omega}_Z\tilde{\Sigma}_Z)F & \xrightarrow{(\#)} & (\tilde{\Omega}_Z\tilde{\Sigma}_Z)\tilde{F}^0 & \rightrightarrows & (\tilde{\Omega}_Z\tilde{\Sigma}_Z)\tilde{F}^1 & \Rrightarrow & (\tilde{\Omega}_Z\tilde{\Sigma}_Z)\tilde{F}^2\cdots \\
\uparrow & & \uparrow{\scriptstyle (**)} & & \uparrow{\scriptstyle (**)} & & \uparrow{\scriptstyle (**)} \\
F & \xrightarrow{(\#)} & \tilde{F}^0 & \rightrightarrows & \tilde{F}^1 & \Rrightarrow & \tilde{F}^2\cdots
\end{array}$$

We know by our homotopical estimates (Propositions 2.1 and 2.2) that the coface $(n+1)$-cubes ([3, 3.13]) associated to

$$E \to (\tilde{\Omega}_Z\tilde{\Sigma}_Z)^{\bullet+1}(E)$$
$$B \to (\tilde{\Omega}_Z\tilde{\Sigma}_Z)^{\bullet+1}(B)$$

are $((\mathrm{id}+1)+1)$-cartesian in S^Z_*, and hence it follows by several applications of [18, 1.6, 1.18] that the coface $(n+1)$-cubes associated to $F \to \tilde{F}$ are $(\mathrm{id}+1)$-cartesian in S^Z_*. We know, by Proposition 2.8, that $(\tilde{\Omega}_Z\tilde{\Sigma}_Z)$ preserves $(\mathrm{id}+1)$-cartesian $(n+1)$-cubes in S^Z_* for each $n \geq -1$. Therefore, the coface $(n+1)$-cubes ([3, 3.13]) associated to

$$(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^kF \to (\tilde{\Omega}_Z\tilde{\Sigma}_Z)^k\tilde{F}, \qquad k \geq 0$$

are $(\mathrm{id}+1)$-cartesian in S^Z_* for each $n \geq -1$, and hence each of the maps

$$(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^kF \to \mathrm{holim}_{\Delta^{\leq n}}(\tilde{\Omega}_Z\tilde{\Sigma}_Z)^k\tilde{F}, \qquad k \geq 0$$

is $(n+2)$-connected. Therefore applying holim_Δ horizontally to the maps $(\#)$ induces a weak equivalence, and hence applying holim_Δ first horizontally and then vertically produces

$$\operatorname*{holim}_{\mathsf{vert}}\operatorname*{holim}_{\mathsf{horiz}}(7) \simeq F^\wedge_{\tilde{\Omega}_Z\tilde{\Sigma}_Z}$$

What about the other way? Since the $(**)$ columns have extra codegeneracy maps s^{-1} [15, 6.2] (by formal reasons: $\tilde{\Omega}_Z$ commutes with homotopy fibers), applying

$\operatorname{holim}_\Delta$ vertically produces [12, 3.16] the coaugmented cosimplicial diagram

$$F \longrightarrow \tilde{F}^0 \rightrightarrows \tilde{F}^1 \Rrightarrow \tilde{F}^2 \cdots$$

and hence applying $\operatorname{holim}_\Delta$ first vertically and then horizontally produces

$$\operatorname*{holim}_{\mathsf{horiz}} \operatorname*{holim}_{\mathsf{vert}} (7) \simeq F$$

Hence we have verified that the coaugmentation

$$F \simeq F^{\wedge}_{\tilde{\Omega}_Z \tilde{\Sigma}_Z}$$

is a weak equivalence. The other cases are similar (the estimates are identical). Consider the case of the homotopy pullback square; then $F \to \tilde{F}$ is constructed by taking homotopy pullbacks instead of homotopy fibers and the above arguments complete the proof. □

References

[1] G. Arone and M. Kankaanrinta. A functorial model for iterated Snaith splitting with applications to calculus of functors. In *Stable and unstable homotopy (Toronto, ON, 1996)*, volume 19 of *Fields Inst. Commun.*, pages 1–30. Amer. Math. Soc., Providence, RI, 1998.

[2] K. Bauer, B. Johnson, and R. McCarthy. Cross effects and calculus in an unbased setting. *Trans. Amer. Math. Soc.*, 367(9):6671–6718, 2015. With an appendix by R. Eldred.

[3] J. R. Blomquist and J. E. Harper. Integral chains and Bousfield-Kan completion. *Homology Homotopy Appl.*, 21(2):29–58, 2019.

[4] J. R. Blomquist and J. E. Harper. Higher stabilization and higher Freudenthal suspension. *Trans. Amer. Math. Soc.*, 375(11):8193–8240, 2022.

[5] A. J. Blumberg and E. Riehl. Homotopical resolutions associated to deformable adjunctions. *Algebr. Geom. Topol.*, 14(5):3021–3048, 2014.

[6] A. K. Bousfield. On the homology spectral sequence of a cosimplicial space. *Amer. J. Math.*, 109(2):361–394, 1987.

[7] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*. Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin, 1972.

[8] G. Carlsson. Equivariant stable homotopy and Sullivan's conjecture. *Invent. Math.*, 103(3):497–525, 1991.

[9] M. B. Carr and J. E. Harper. Functor calculus completions for retractive operadic algebras in spectra. `arXiv:2407.01819 [math.AT]`, 2024.

[10] M. Ching and J. E. Harper. Higher homotopy excision and Blakers-Massey theorems for structured ring spectra. *Adv. Math.*, 298:654–692, 2016.

[11] M. Ching and J. E. Harper. Derived Koszul duality and TQ-homology completion of structured ring spectra. *Adv. Math.*, 341:118–187, 2019.

[12] E. Dror and W. G. Dwyer. A long homology localization tower. *Comment. Math. Helv.*, 52(2):185–210, 1977.

[13] B. I. Dundas. Relative K-theory and topological cyclic homology. *Acta Math.*, 179(2):223–242, 1997.

[14] B. I. Dundas, T. G. Goodwillie, and R. McCarthy. *The local structure of algebraic K-theory*, volume 18 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2013.

[15] W. G. Dwyer, H. R. Miller, and J. Neisendorfer. Fibrewise completion and unstable Adams spectral sequences. *Israel J. Math.*, 66(1-3):160–178, 1989.

[16] W. G. Dwyer and J. Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, Amsterdam, 1995.

[17] P. G. Goerss and J. F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.

[18] T. G. Goodwillie. Calculus. II. Analytic functors. *K-Theory*, 5(4):295–332, 1991/92.

[19] T. G. Goodwillie. Calculus. III. Taylor series. *Geom. Topol.*, 7:645–711 (electronic), 2003.

[20] J. E. Harper and Y. Zhang. Topological Quillen localization of structured ring spectra. *Tbilisi Math. J.*, 12(3):69–91, 2019.

[21] P. S. Hirschhorn. Overcategories and undercategories of model categories. arXiv:1507.01624 [math.AT], 2015.
[22] M. Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.
[23] M. Hovey. Spectra and symmetric spectra in general model categories. *J. Pure Appl. Algebra*, 165(1):63–127, 2001.
[24] N. J. Kuhn. Goodwillie towers and chromatic homotopy: an overview. In *Proceedings of the Nishida Fest (Kinosaki 2003)*, volume 10 of *Geom. Topol. Monogr.*, pages 245–279. Geom. Topol. Publ., Coventry, 2007.
[25] D. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin, 1967.
[26] N. Schonsheck. Fibration theorems for TQ-completion of structured ring spectra. *Tbilisi Math. J.: Special Issue on Homotopy Theory, Spectra, and Structured Ring Spectra*, pages 1–15, 2020.
[27] N. Schonsheck. Personal communication. 2021.
[28] N. Schonsheck. TQ-completion and the Taylor tower of the identity functor. *J. Homotopy Relat. Struct.*, 17(2):201–216, 2022.
[29] S. Schwede. Spectra in model categories and applications to the algebraic cotangent complex. *J. Pure Appl. Algebra*, 120(1):77–104, 1997.

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