

TOPOLOGICAL QUILLEN LOCALIZATION OF STRUCTURED RING SPECTRA

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ABSTRACT. The aim of this short paper is to construct a TQ-localization functor on algebras over a spectral operad \mathcal{O} , in the general case where no connectivity assumptions are made on the \mathcal{O} -algebras, and to establish the associated TQ-local homotopy theory as a left Bousfield localization of the usual model structure on \mathcal{O} -algebras, which itself is not left proper, in general. In the resulting TQ-local homotopy theory, the weak equivalences are the TQ-homology equivalences, where “TQ-homology” is short for topological Quillen homology. More generally, we establish these results for TQ-homology with coefficients in a spectral algebra \mathcal{A} . A key observation, that goes back to the work of Goerss-Hopkins on moduli problems, is that the usual left properness assumption may be replaced with a strong cofibration condition in the desired subcell lifting arguments: our main result is that the TQ-local homotopy theory can be constructed, without left properness on \mathcal{O} -algebras, by localizing with respect to a set of strong cofibrations that are TQ-equivalences.

1. INTRODUCTION

In this paper we are working in the framework of algebras over an operad in symmetric spectra [23, 32], and more generally, in \mathcal{R} -modules, where $\mathcal{O}[0] = *$ (the trivial \mathcal{R} -module); such \mathcal{O} -algebras are non-unital. Here, \mathcal{R} is any commutative ring spectrum (i.e., any commutative monoid object in the category $(\mathrm{Sp}^{\Sigma}, \otimes_S, S)$ of symmetric spectra, and we denote by $(\mathrm{Mod}_{\mathcal{R}}, \wedge, \mathcal{R})$ the closed symmetric monoidal category of \mathcal{R} -modules.

Topological Quillen homology (or TQ-homology) is the \mathcal{O} -algebra analog of derived abelianization and stabilization; in particular, it is the precise \mathcal{O} -algebra analog of both the integral homology of spaces and the stabilization of spaces. A useful starting point is [15, 29, 30], together with [1, 2, 3] and [10, 25, 26, 27]; see also [8, 9, 13, 14, 19, 31]. In [10, 20] the TQ-completion of \mathcal{O} -algebras is studied; in particular, it is shown in [10] that connected \mathcal{O} -algebras are TQ-complete.

The purpose of this paper is to explore the possibility of removing the connectivity assumptions on \mathcal{O} -algebras—informally, we would like to construct the “part of an \mathcal{O} -algebra X that topological Quillen homology sees” called the TQ-localization of X ; we follow closely the arguments in [5] and [18, 24] (see also [11] for a useful introduction to these ideas, along with [6, 21, 28] in the context of spaces); to make the localization techniques work in the context of \mathcal{O} -algebras, we exploit the cellular ideas in [22]. A potential wrinkle is the failure of \mathcal{O} -algebras to be left proper, in general; we show that exploiting an observation in [16, 17] enables the desired topological Quillen localization to be constructed by localizing with respect to a particular set of *strong* cofibrations that are topological Quillen homology

equivalences; the establishment of this TQ-localization functor and the associated TQ-local homotopy theory are our main results.

To keep this paper appropriately concise, we freely use notation from [20].

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2. TQ-HOMOLOGY OF AN \mathcal{O} -ALGEBRA WITH COEFFICIENTS IN \mathcal{A}

If X is an \mathcal{O} -algebra, then we may factor the map $* \rightarrow X$

$$* \rightarrow \tilde{X} \xrightarrow{\simeq} X$$

as a cofibration followed by an acyclic fibration; we are using the positive flat stable model structure (see, for instance, [20]). In particular, \tilde{X} is a cofibrant replacement of X .

Consider the canonical map of operads $f: \mathcal{O} \rightarrow \tau_1 \mathcal{O}$ and any map $\alpha: \mathcal{O}[1] \rightarrow \mathcal{A}$ of \mathcal{R} -algebras. These maps induce adjunctions of the form

$$(1) \quad \text{Alg}_{\mathcal{O}} \begin{array}{c} \xrightarrow{f_*} \\ \xleftarrow{f^*} \end{array} \text{Alg}_{\tau_1 \mathcal{O}} = \text{Mod}_{\mathcal{O}[1]} \begin{array}{c} \xrightarrow{\alpha_*} \\ \xleftarrow{\alpha^*} \end{array} \text{Mod}_{\mathcal{A}}$$

with left adjoints on top, where $f_*(X) := \tau_1 \mathcal{O} \circ_{\mathcal{O}} (X)$ and f^* denotes restriction along f of the left $\tau_1 \mathcal{O}$ -action, and similarly, $\alpha_*(Y) := \mathcal{A} \wedge_{\mathcal{O}[1]} Y$ and α^* denotes restriction along α of the left \mathcal{A} -action; in other words, f^* and α^* are the indicated forgetful functors. For notational convenience purposes, we denote by $Q := \alpha_* f_*$ the composite of left adjoints in (1) and by $U := f^* \alpha^*$ the composite of right adjoints in (1). It follows that (Q, U) fit into an adjunction of the form

$$(2) \quad \text{Alg}_{\mathcal{O}} \begin{array}{c} \xrightarrow{Q} \\ \xleftarrow{U} \end{array} \text{Mod}_{\mathcal{A}}$$

with left adjoint on top; here, Q is for indecomposable ‘‘quotient’’ and U is the indicated forgetful functor.

Definition 2.1. If X is an \mathcal{O} -algebra, then its TQ-homology is the \mathcal{O} -algebra

$$\text{TQ}(X) := \tau_1 \mathcal{O} \circ_{\mathcal{O}}^{\text{h}} (X) := \text{R}f^*(\text{L}f_*(X)) \simeq \tau_1 \mathcal{O} \circ_{\mathcal{O}} (\tilde{X})$$

and its TQ-homology with coefficients in \mathcal{A} , is the \mathcal{O} -algebra

$$\text{TQ}^{\mathcal{A}}(X) := \text{R}U(\text{L}Q(X)) \simeq Q(\tilde{X}) = \mathcal{A} \wedge_{\mathcal{O}[1]} (\tau_1 \mathcal{O} \circ_{\mathcal{O}} (\tilde{X}))$$

In particular, if the algebra map $\alpha = \text{id}$ on $\mathcal{O}[1]$, then $\text{TQ}^{\mathcal{O}[1]}(X) \simeq \text{TQ}(X)$. Here, TQ-homology is short for ‘‘topological Quillen homology’’.

3. DETECTING TQ ^{\mathcal{A}} -LOCAL \mathcal{O} -ALGEBRAS

Definition 3.1. A map $i: A \rightarrow B$ of \mathcal{O} -algebras is a *strong cofibration* if it is a cofibration between cofibrant objects in $\text{Alg}_{\mathcal{O}}$.

Definition 3.2. Let X be an \mathcal{O} -algebra. We say that X is TQ^A -local if (i) X is fibrant in $\mathbf{Alg}_{\mathcal{O}}$ and (ii) every strong cofibration $A \rightarrow B$ that induces a weak equivalence $\mathrm{TQ}^A(A) \simeq \mathrm{TQ}^A(B)$ on TQ^A -homology, induces a weak equivalence

$$(3) \quad \mathbf{Hom}(A, X) \xleftarrow{\simeq} \mathbf{Hom}(B, X)$$

on mapping spaces in \mathbf{sSet} .

Remark 3.3. The intuition here is that the derived space of maps into a TQ^A -local \mathcal{O} -algebra cannot distinguish between TQ^A -equivalent \mathcal{O} -algebras (Proposition 3.7), up to weak equivalence.

Evaluating the map (3) at level 0 gives a surjection

$$\mathrm{hom}(A, X) \leftarrow \mathrm{hom}(B, X)$$

of sets, since acyclic fibrations in \mathbf{sSet} are necessarily levelwise surjections. This suggests that TQ^A -local \mathcal{O} -algebras X might be detected by a right lifting property and motivates the following classes of maps (Proposition 3.12); compare with [5].

Definition 3.4 (TQ^A -local homotopy theory: Classes of maps). A map $f: X \rightarrow Y$ of \mathcal{O} -algebras is

- (i) a TQ^A -equivalence if it induces a weak equivalence $\mathrm{TQ}^A(X) \simeq \mathrm{TQ}^A(Y)$
- (ii) a TQ^A -cofibration if it is a cofibration in $\mathbf{Alg}_{\mathcal{O}}$
- (iii) a TQ^A -fibration if it has the right lifting property with respect to every cofibration that is a TQ^A -equivalence
- (iv) a weak TQ^A -fibration (or TQ^A -injective fibration) if it has the right lifting property with respect to every strong cofibration that is a TQ^A -equivalence

A cofibration (resp. strong cofibration) is called TQ^A -acyclic if it is also a TQ^A -equivalence.

Remark 3.5. The additional class of maps (iv) naturally arises in the TQ^A -local homotopy theory established below on \mathcal{O} -algebras; this is a consequence of the fact that the model structure on $\mathbf{Alg}_{\mathcal{O}}$ is not left proper, in general. In the special cases where it happens that $\mathbf{Alg}_{\mathcal{O}}$ is left proper, then the class of weak TQ^A -fibrations will be identical to the class of TQ^A -fibrations.

Proposition 3.6. *The following implications are satisfied*

$$\begin{aligned} \text{strong cofibration} &\implies \text{cofibration} \\ \text{weak equivalence} &\implies \mathrm{TQ}^A\text{-equivalence} \\ \mathrm{TQ}^A\text{-fibration} &\implies \text{weak } \mathrm{TQ}^A\text{-fibration} \implies \text{fibration} \end{aligned}$$

for maps of \mathcal{O} -algebras.

Proof. The first implication is immediate and the second is because TQ^A preserves weak equivalences, by construction. The last two implications are because the class of TQ^A -acyclic cofibrations contains the class of TQ^A -acyclic strong cofibrations, which itself contains the class of generating acyclic cofibrations in $\mathbf{Alg}_{\mathcal{O}}$; we have used the fact [34] that the generating acyclic cofibrations in $\mathbf{Alg}_{\mathcal{O}}$ have cofibrant domains. \square

Proposition 3.7. *Let X be a fibrant \mathcal{O} -algebra. Then X is TQ^A -local if and only if every map $f: A \rightarrow B$ between cofibrant \mathcal{O} -algebras that is a TQ^A -equivalence induces a weak equivalence (3) on mapping spaces.*

Proof. It suffices to verify the “only if” direction. Consider any map $f: A \rightarrow B$ between cofibrant \mathcal{O} -algebras that is a TQ^A -equivalence. Factor f as a cofibration i followed by an acyclic fibration p in $\mathrm{Alg}_{\mathcal{O}}$. Since f is a TQ^A -equivalence and p is a weak equivalence, it follows that i is a TQ^A -equivalence. The left-hand commutative diagram induces

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow i & \nearrow p & \\ B' & & \end{array} \quad \begin{array}{ccc} \mathbf{Hom}(A, X) & \xleftarrow{(*)} & \mathbf{Hom}(B, X) \\ (**) \uparrow & \nearrow (\#) & \\ \mathbf{Hom}(B', X) & & \end{array}$$

the right-hand commutative diagram. Since p is a weak equivalence between cofibrant objects and X is fibrant, we know that $(\#)$ is a weak equivalence, hence $(*)$ is a weak equivalence if and only if $(**)$ is a weak equivalence. Since i is a strong cofibration, by construction, this completes the proof. \square

Proposition 3.8. *Consider any map $f: X \rightarrow Y$ of \mathcal{O} -algebras. If X is cofibrant in $\mathrm{Alg}_{\mathcal{O}}$, then the following are equivalent:*

- (i) f is a weak TQ^A -fibration and TQ^A -equivalence
- (ii) f is a TQ^A -fibration and TQ^A -equivalence
- (iii) f is a fibration and weak equivalence

Furthermore, the implications (ii) \Leftrightarrow (iii) remain true without the cofibrancy assumption on X .

Proof. We want to show that (i) \Leftrightarrow (iii). Suppose f is a weak TQ^A -fibration and TQ^A -equivalence; let’s verify that f is an acyclic fibration. We factor f as a cofibration followed by an acyclic fibration $X \xrightarrow{i} \tilde{Y} \xrightarrow{p} Y$ in $\mathrm{Alg}_{\mathcal{O}}$, and since f, p are TQ^A -equivalences, it follows that i is a TQ^A -equivalence. Hence i is a TQ^A -acyclic strong cofibration and the left-hand solid commutative diagram

$$(4) \quad \begin{array}{ccc} X & \xlongequal{\quad} & X \\ \downarrow i & \nearrow \xi & \downarrow f \\ \tilde{Y} & \xrightarrow{p} & Y \end{array} \quad \begin{array}{ccccc} X & \xrightarrow{i} & \tilde{Y} & \xrightarrow{\xi} & X \\ \downarrow f & & \downarrow p & & \downarrow f \\ Y & \xlongequal{\quad} & Y & \xlongequal{\quad} & Y \end{array}$$

has a lift ξ . It follows that the right-hand diagram commutes with upper horizontal composite the identity map; in particular, f is a retract of p which completes the proof of this direction. The converse direction is immediate by Proposition 3.6. Noting that the implications (ii) \Leftrightarrow (iii) are proved using exactly the same argument, completes the proof. \square

The following is proved, for instance, in [10, 7.6].

Proposition 3.9. *If A is an \mathcal{O} -algebra and $K \in \mathbf{sSet}$, then there are isomorphisms $Q(A \otimes K) \cong Q(A) \dot{\otimes} K$ in $\mathrm{Alg}_{\mathcal{O}}$, natural in A, K .*

Proposition 3.10. *If $j: A \rightarrow B$ is a strong cofibration of \mathcal{O} -algebras and $i: K \rightarrow L$ is a cofibration in \mathbf{sSet} , then the pushout corner map*

$$A \dot{\otimes} L \amalg_{A \dot{\otimes} K} B \dot{\otimes} K \rightarrow B \dot{\otimes} L$$

in $\mathbf{Alg}_{\mathcal{O}}$ is a strong cofibration that is a \mathbf{TQ}^A -equivalence if j is a \mathbf{TQ}^A -equivalence.

Proof. We know that the pushout corner map is a strong cofibration by the simplicial model structure on $\mathbf{Alg}_{\mathcal{O}}$ (see, for instance, [20]), hence it suffices to verify that Q applied to this map is a weak equivalence. Since Q is a left Quillen functor, it follows that the pushout corner map

$$Q(A) \dot{\otimes} L \amalg_{Q(A) \dot{\otimes} K} Q(B) \dot{\otimes} K \rightarrow Q(B) \dot{\otimes} L$$

is a cofibration that is a weak equivalence if $Q(A) \rightarrow Q(B)$ is a weak equivalence, and Proposition 3.9 completes the proof. \square

Proposition 3.11. *If $j: A \rightarrow B$ is a \mathbf{TQ}^A -acyclic strong cofibration and $p: X \rightarrow Y$ is a weak \mathbf{TQ}^A -fibration of \mathcal{O} -algebras, then the pullback corner map*

$$(5) \quad \mathbf{Hom}(B, X) \rightarrow \mathbf{Hom}(A, X) \times_{\mathbf{Hom}(A, Y)} \mathbf{Hom}(B, Y)$$

in \mathbf{sSet} is an acyclic fibration.

Proof. Consider any cofibration $i: K \rightarrow L$ in \mathbf{sSet} . We want to show that the pullback corner map (5) satisfies the right lifting property with respect to i .

$$\begin{array}{ccc} K & \longrightarrow & \mathbf{Hom}(B, X) \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ L & \longrightarrow & \mathbf{Hom}(A, X) \times_{\mathbf{Hom}(A, Y)} \mathbf{Hom}(B, Y) \end{array} \qquad \begin{array}{ccc} A \dot{\otimes} L \amalg_{A \dot{\otimes} K} B \dot{\otimes} K & \longrightarrow & X \\ \downarrow (*) & \nearrow \text{dotted} & \downarrow \\ B \dot{\otimes} L & \longrightarrow & Y \end{array}$$

The left-hand solid commutative diagram has a lift if and only if the corresponding right-hand solid commutative diagram has a lift. Noting that $(*)$ is a \mathbf{TQ}^A -acyclic strong cofibration (Proposition 3.10) completes the proof. \square

Proposition 3.12 (Detecting \mathbf{TQ}^A -local \mathcal{O} -algebras: Part 1). *Let X be a fibrant \mathcal{O} -algebra. Then X is \mathbf{TQ}^A -local if and only if $X \rightarrow *$ satisfies the right lifting property with respect to every \mathbf{TQ}^A -acyclic strong cofibration $A \rightarrow B$ of \mathcal{O} -algebras.*

Proof. Suppose X is \mathbf{TQ}^A -local and let $i: A \rightarrow B$ be a \mathbf{TQ}^A -acyclic strong cofibration. Let's verify that $X \rightarrow *$ satisfies the right lifting property with respect to i . We know that the induced map of simplicial sets (3) is an acyclic fibration, hence evaluating the induced map (3) at level 0 gives a surjection

$$\mathbf{hom}(A, X) \leftarrow \mathbf{hom}(B, X)$$

of sets, which verifies the desired lift exists. Conversely, consider any \mathbf{TQ}^A -acyclic strong cofibration $A \rightarrow B$ of \mathcal{O} -algebras. Let's verify that the induced map (3) is an acyclic fibration. It suffices to verify the right lifting property with respect to any generating cofibration $\partial\Delta[n] \rightarrow \Delta[n]$ in \mathbf{sSet} . Consider any left-hand solid

commutative diagram of the form

$$\begin{array}{ccc}
\partial\Delta[n] & \longrightarrow & \mathbf{Hom}(B, X) \\
\downarrow & \nearrow & \downarrow \\
\Delta[n] & \longrightarrow & \mathbf{Hom}(A, X)
\end{array}
\quad
\begin{array}{ccc}
A \dot{\otimes} \Delta[n] \amalg_{A \dot{\otimes} \partial\Delta[n]} B \dot{\otimes} \partial\Delta[n] & \longrightarrow & X \\
\downarrow (*) & \nearrow & \downarrow \\
B \dot{\otimes} \Delta[n] & \longrightarrow & *
\end{array}$$

in \mathbf{sSet} . Then the left-hand lift exists in \mathbf{sSet} if and only if the corresponding right-hand lift exists in $\mathbf{Alg}_{\mathcal{O}}$. The map $(*)$ is a \mathbf{TQ}^A -acyclic strong cofibration by Proposition 3.10, hence, by assumption, the lift in the right-hand diagram exists, which completes the proof. \square

Remark 3.13. Since the generating acyclic cofibrations in $\mathbf{Alg}_{\mathcal{O}}$ have cofibrant domains, the fibrancy assumption on X in Proposition 3.12 could be dropped; we keep it in, however, to motivate later closely related statements.

4. CELL \mathcal{O} -ALGEBRAS AND THE SUBCELL LIFTING PROPERTY

Suppose we start with an \mathcal{O} -algebra A . It may not be cofibrant, so we can run the small object argument with respect to the set of generating cofibrations in $\mathbf{Alg}_{\mathcal{O}}$ for the map $* \rightarrow A$. This gives a factorization in $\mathbf{Alg}_{\mathcal{O}}$ as $* \rightarrow \hat{A} \rightarrow A$ a cofibration followed by an acyclic fibration. In particular, this construction builds \hat{A} by attaching cells; we would like to think of \hat{A} as a “cell \mathcal{O} -algebra”, and we will want to work with a useful notion of “subcell \mathcal{O} -algebra” obtained by only attaching a subset of the cells above. Since every \mathcal{O} -algebra can be replaced by such a cell \mathcal{O} -algebra, up to weak equivalence, the idea is that this should provide a convenient class of \mathcal{O} -algebras to reduce to when constructing the \mathbf{TQ}^A -localization functor; this reduction strategy—to work with cellular objects—is one of the main themes in Hirschhorn [22], and it plays a key role in this paper. The first step is to recall the generating cofibrations for $\mathbf{Alg}_{\mathcal{O}}$ and to make these cellular ideas more precise in the particular context of \mathcal{O} -algebras needed for this paper.

Recall from [20, 7.10] that the generating cofibrations for the positive flat stable model structure on \mathcal{R} -modules is given by the set of maps of the form

$$\mathcal{R} \otimes G_m^H \partial\Delta[k]_+ \xrightarrow{i_m^{H,k}} \mathcal{R} \otimes G_m^H \Delta[k]_+ \quad (m \geq 1, k \geq 0, H \subset \Sigma_m \text{ subgroup})$$

in \mathcal{R} -modules. For ease of notational purposes, it will be convenient to denote this set of maps using the more concise notation

$$S_m^{H,k} \xrightarrow{i_m^{H,k}} D_m^{H,k} \quad (m \geq 1, k \geq 0, H \subset \Sigma_m \text{ subgroup})$$

where $S_m^{H,k}$ are $D_m^{H,k}$ are intended to remind the reader of “sphere” and “disk”, respectively. In terms of this notation, recall from [20, 7.15] that the generating cofibrations for the positive flat stable model structure on \mathcal{O} -algebras is given by the set of maps of the form

$$(6) \quad \mathcal{O} \circ (S_m^{H,k}) \xrightarrow{\text{id} \circ (i_m^{H,k})} \mathcal{O} \circ (D_m^{H,k}) \quad (m \geq 1, k \geq 0, H \subset \Sigma_m \text{ subgroup})$$

in \mathcal{O} -algebras.

Definitions 4.1–4.4 below appear in Hirschhorn [22, 10.5.8, 10.6] in the more general context of cellular model categories; we have tailored the definitions to exactly what is needed for this paper; i.e., in the context of \mathcal{O} -algebras.

Definition 4.1. A map $\alpha: W \rightarrow Z$ in $\mathbf{Alg}_{\mathcal{O}}$ is a *relative cell \mathcal{O} -algebra* if it can be constructed as a transfinite composition of maps of the form

$$W = Z_0 \rightarrow Z_1 \rightarrow Z_2 \rightarrow \cdots \rightarrow Z_\infty := \operatorname{colim}_n Z_n \cong Z$$

such that each map $Z_n \rightarrow Z_{n+1}$ is built from a pushout diagram of the form

$$(7) \quad \begin{array}{ccc} \coprod_{i \in I_n} \mathcal{O} \circ (S_{m_i}^{H_i, k_i}) & \xrightarrow{(*)} & Z_n \\ \Pi_{i \in I_n} \operatorname{id}_{\mathcal{O}}(i_{m_i}^{H_i, k_i}) \downarrow & & \downarrow \\ \coprod_{i \in I_n} \mathcal{O} \circ (D_{m_i}^{H_i, k_i}) & \longrightarrow & Z_{n+1} \end{array}$$

in $\mathbf{Alg}_{\mathcal{O}}$, for each $n \geq 0$. A choice of such a transfinite composition of pushouts is a *presentation* of $\alpha: W \rightarrow Z$ as a relative cell \mathcal{O} -algebra. With respect to such a presentation, the *set of cells* in α is the set $\sqcup_{n \geq 0} I_n$ and the *number of cells* in α is the cardinality of its set of cells; here, \sqcup denotes disjoint union of sets.

Remark 4.2. We often drop explicit mention of the choice of presentation of a relative cell \mathcal{O} -algebra, for ease of reading purposes, when no confusion can result.

Definition 4.3. An \mathcal{O} -algebra Z is a *cell \mathcal{O} -algebra* if $* \rightarrow Z$ is a relative cell \mathcal{O} -algebra. The *number of cells* in Z , denoted $\#Z$, is the number of cells in $* \rightarrow Z$ (with respect to a choice of presentation of $* \rightarrow Z$).

Definition 4.4. Let Z be a cell \mathcal{O} -algebra. A *subcell \mathcal{O} -algebra* of Z is a cell \mathcal{O} -algebra Y built by a subset of cells in Z (with respect to a choice of presentation of $* \rightarrow Z$). More precisely, $Y \subset Z$ is a subcell \mathcal{O} -algebra if $* \rightarrow Y$ can be constructed as a transfinite composition of maps of the form

$$* = Y_0 \rightarrow Y_1 \rightarrow Y_2 \rightarrow \cdots \rightarrow Y_\infty := \operatorname{colim}_n Y_n \cong Y$$

such that each map $Y_n \rightarrow Y_{n+1}$ is built from a pushout diagram of the form

$$\begin{array}{ccc} \coprod_{j \in J_n} \mathcal{O} \circ (S_{m_j}^{H_j, k_j}) & \xrightarrow{(**)} & Y_n \\ \Pi_{j \in J_n} \operatorname{id}_{\mathcal{O}}(i_{m_j}^{H_j, k_j}) \downarrow & & \downarrow \\ \coprod_{j \in J_n} \mathcal{O} \circ (D_{m_j}^{H_j, k_j}) & \longrightarrow & Y_{n+1} \end{array}$$

in $\mathbf{Alg}_{\mathcal{O}}$, where $J_n \subset I_n$ and the attaching map $(**)$ is the restriction of the corresponding attaching map $(*)$ in (7) (taking $W = *$), for each $n \geq 0$.

Definition 4.5. Let Z be a cell \mathcal{O} -algebra. A subcell \mathcal{O} -algebra $Y \subset Z$ is *finite* if $\#Y$ is finite (with respect to a choice of presentation of $* \rightarrow Z$); in this case we say that Y has finitely many cells.

Remark 4.6. Let Z be a cell \mathcal{O} -algebra. A subcell \mathcal{O} -algebra $Y \subset Z$ can be described by giving a compatible collection of subsets $J_n \subset I_n$, $n \geq 0$, (with respect to a choice of presentation for $* \rightarrow Z$); here, *compatible* means that the corresponding

attaching maps are well-defined. It follows that the resulting subcell \mathcal{O} -algebra inclusion $Y \subset Z$ can be constructed stage-by-stage

$$\begin{array}{ccccccccccc} * = Y_0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & \dots & \longrightarrow & Y_\infty & \xrightarrow{\cong} & Y \\ \parallel & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ * = Z_0 & \longrightarrow & Z_1 & \longrightarrow & Z_2 & \longrightarrow & \dots & \longrightarrow & Z_\infty & \xrightarrow{\cong} & Z \end{array}$$

as the indicated colimit.

Proposition 4.7. *Let Z be a cell \mathcal{O} -algebra. If $A \subset Z$ and $B \subset Z$ are subcell \mathcal{O} -algebras, then there is a pushout diagram of the form*

$$(8) \quad \begin{array}{ccc} A \cap B & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & A \cup B \end{array}$$

in $\text{Alg}_{\mathcal{O}}$, which is also a pullback diagram, where the indicated arrows are subcell \mathcal{O} -algebra inclusions.

Proof. This is proved in Hirschhorn [22, 12.2.2] in a more general context, but here is the basic idea: Consider $* \rightarrow Z$ with presentation as in (7) (taking $W = *$). Suppose that $S_n \subset I_n$ and $T_n \subset I_n$, $n \geq 0$, correspond to the subcell \mathcal{O} -algebras $A \subset Z$ and $B \subset Z$, respectively. Then it follows (by induction on n) that $S_n \cap T_n \subset I_n$ and $S_n \cup T_n \subset I_n$, $n \geq 0$, are compatible collections of subsets and taking $A \cap B \subset Z$ and $A \cup B \subset Z$ to be the corresponding subcell \mathcal{O} -algebras, respectively, completes the proof. Here, we are using the fact that every cofibration of \mathcal{O} -algebras is, in particular, a monomorphism of underlying symmetric spectra, and hence an effective monomorphism [22, 12.2] of \mathcal{O} -algebras. \square

The following is proved in [7, I.2.4, I.2.5].

Proposition 4.8. *Let \mathcal{M} be a model category (see, for instance, [12, 3.3]).*

(a) *Consider any commutative diagram of the form*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow i & & \downarrow \\ C & \xrightarrow{g} & D \end{array}$$

in \mathcal{M} , where A, B, C are cofibrant and i is a cofibration. If f is a weak equivalence, then g is a weak equivalence.

(b) *Consider any commutative diagram of the form*

$$\begin{array}{ccccc} A_0 & \longleftarrow & A_1 & \longrightarrow & A_2 \\ \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow \\ B_0 & \longleftarrow & B_1 & \longrightarrow & B_2 \end{array}$$

in \mathcal{M} , where A_i, B_i are cofibrant for each $0 \leq i \leq 2$, the vertical maps are weak equivalences, and $A_0 \leftarrow A_1$ is a cofibration. If either $B_0 \leftarrow B_1$ or $B_1 \rightarrow B_2$ is a cofibration, then the induced map

$$A_0 \amalg_{A_1} A_2 \xrightarrow{\simeq} B_0 \amalg_{B_1} B_2$$

is a weak equivalence.

The following proposition, which is an exercise left to the reader, has been exploited, for instance, in [4, 2.1] and [22, 13.2.1]; it is closely related to the usual induced model structures on over-categories and under-categories; see, for instance, [12, 3.10].

Proposition 4.9 (Factorization category of a map). *Let \mathbf{M} be a model category and $z: A \rightarrow Y$ a map in \mathbf{M} . Denote by $\mathbf{M}(z)$ the category with objects the factorizations $\mathbf{X}: A \rightarrow X \rightarrow Y$ of z in \mathbf{M} and morphisms $\xi: \mathbf{X} \rightarrow \mathbf{X}'$ the commutative diagrams of the form*

$$\begin{array}{ccccc} \mathbf{X}: & A & \longrightarrow & X & \longrightarrow & Y \\ & \downarrow \xi & & \downarrow \xi & & \\ \mathbf{X}': & A & \longrightarrow & X' & \longrightarrow & Y \end{array}$$

in \mathbf{M} . Define a map $\xi: \mathbf{X} \rightarrow \mathbf{X}'$ to be a weak equivalence (resp. fibration, resp. cofibration) if $\xi: X \rightarrow X'$ is a weak equivalence (resp. fibration, resp. cofibration) in \mathbf{M} . With these three classes of maps, $\mathbf{M}(z)$ inherits a naturally occurring model structure from \mathbf{M} . Since the initial object (resp. terminal object) in $\mathbf{M}(z)$ has the form $A = A \xrightarrow{z} Y$ (resp. $A \xrightarrow{z} Y = Y$), it follows that \mathbf{X} is cofibrant (resp. fibrant) if and only if $A \rightarrow X$ is a cofibration (resp. $X \rightarrow Y$ is a fibration) in \mathbf{M} .

Proof. This appears in [4, 2.1] and is closely related to [12, 3.10] and [30, II.2.8]. \square

The following subcell lifting property can be thought of as an \mathcal{O} -algebra analog of Hirschhorn [22, 13.2.1] as a key step in establishing localizations in left proper cellular model categories. One technical difficulty with Proposition 3.12 for detecting TQ^A -local \mathcal{O} -algebras is that it involves a lifting condition with respect to a collection of maps, instead of a set of maps. Proposition 4.10 provides our first reduction towards eventually refining the lifting criterion for TQ^A -local \mathcal{O} -algebras to a set of maps. Even though the left properness assumption in [22, 13.2.1] is not satisfied by \mathcal{O} -algebras, in general, a key observation, that goes back to the work of Goerss-Hopkins [17, 1.5] on moduli problems, is that the subcell lifting argument only requires an appropriate pushout diagram to be a homotopy pushout diagram—this is ensured by the strong cofibration condition in Proposition 4.10.

Proposition 4.10 (Subcell lifting property). *Let $p: X \rightarrow Y$ be a fibration of \mathcal{O} -algebras. Then the following are equivalent:*

- (a) *The map p has the right lifting property with respect to every strong cofibration $A \rightarrow B$ of \mathcal{O} -algebras that is a TQ^A -equivalence.*
- (b) *The map p has the right lifting property with respect to every subcell \mathcal{O} -algebra inclusion $A \subset B$ that is a TQ^A -equivalence.*

Proof. Since every subcell \mathcal{O} -algebra inclusion $A \subset B$ is a strong cofibration, the implication (a) \Rightarrow (b) is immediate. Conversely, suppose p has the right lifting property with respect to every subcell \mathcal{O} -algebra inclusion that is a TQ^A -equivalence. Let $i: A \rightarrow B$ be a strong cofibration of \mathcal{O} -algebras that is a TQ^A -equivalence and

consider any solid commutative diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{g} & X \\ i \downarrow & \nearrow \xi & \downarrow p \\ B & \xrightarrow{h} & Y \end{array}$$

in $\mathbf{Alg}_{\mathcal{O}}$. We want to verify that a lift ξ exists. The first step is to get subcell \mathcal{O} -algebras into the picture. Running the small object argument with respect to the generating cofibrations in $\mathbf{Alg}_{\mathcal{O}}$, we first functorially factor the map $* \rightarrow A$ as a cofibration followed by an acyclic fibration $* \rightarrow A' \xrightarrow{a} A$, and then we functorially factor the composite map $A' \rightarrow A \rightarrow B$ as a cofibration followed by an acyclic fibration $A' \xrightarrow{i'} B' \xrightarrow{b} B$. Putting it all together, we get a commutative diagram of the form

$$\begin{array}{ccccc} A' & \xrightarrow{a} & A & \xrightarrow{g} & X \\ i' \downarrow & & \downarrow i & & \downarrow p \\ B' & \xrightarrow{b} & B & \xrightarrow{h} & Y \end{array}$$

where i' is a subcell \mathcal{O} -algebra inclusion, by construction. Furthermore, since i is a TQ^A -equivalence and a, b are weak equivalences, it follows that i' is a TQ^A -equivalence. Denote by M the pushout of the upper left-hand corner maps i' and a , and consider the induced maps c, d, α of the form

$$\begin{array}{ccccc} A' & \xrightarrow{a} & A & \xrightarrow{g} & X \\ i' \downarrow & & \downarrow i & & \downarrow p \\ B' & \xrightarrow{b} & B & \xrightarrow{h} & Y \end{array} \quad \begin{array}{c} \nearrow c \\ \nearrow d \\ \nearrow \alpha \end{array} \quad \begin{array}{c} \nearrow \xi' \\ \nearrow \xi \end{array}$$

Since B', A', A are cofibrant and i' is a cofibration, we know that M is a homotopy pushout (Proposition 4.8); in particular, since a is a weak equivalence, it follows that c is a weak equivalence. Since c, b are weak equivalences, we know that α is a weak equivalence. By assumption, p has the right lifting property with respect to i' , and hence with respect to its pushout d . In particular, a lift ξ' exists such that $\xi' d = g$ and $p \xi' = h \alpha$. It turns out this is enough to conclude that a lift ξ exists such that $\xi i = g$ and $p \xi = h$. Here is why: Consider the factorization category $\mathbf{Alg}_{\mathcal{O}}(pg)$ (Proposition 4.9) of the map pg , together with the objects

$$\mathbf{B} : A \xrightarrow{i} B \xrightarrow{h} Y, \quad \mathbf{X} : A \xrightarrow{g} X \xrightarrow{p} Y, \quad \mathbf{M} : A \xrightarrow{d} M \xrightarrow{h\alpha} Y$$

Note that giving the desired lift ξ is the same as giving a map of the form

$$\begin{array}{ccccc} \mathbf{X} : & A & \longrightarrow & X & \longrightarrow & Y \\ \uparrow \xi & \parallel & & \uparrow \xi & & \parallel \\ \mathbf{B} : & A & \longrightarrow & B & \longrightarrow & Y \end{array}$$

in $\text{Alg}_{\mathcal{O}}(pg)$. Also, we know from above that a lift ξ' exists; i.e., we have shown there is a map of the form

$$\begin{array}{ccccc} \mathbf{X} : & A & \longrightarrow & X & \longrightarrow & Y \\ & \uparrow & & \uparrow & & \\ & \xi' & & \xi' & & \\ \mathbf{M} : & A & \longrightarrow & M & \longrightarrow & Y \end{array}$$

in $\text{Alg}_{\mathcal{O}}(pg)$. We also know from above that the map α is a weak equivalence, and hence we have a weak equivalence of the form

$$\begin{array}{ccccc} \mathbf{M} : & A & \longrightarrow & M & \longrightarrow & Y \\ \simeq \downarrow \alpha & \parallel & & \simeq \downarrow \alpha & & \parallel \\ \mathbf{B} : & A & \longrightarrow & B & \longrightarrow & Y \end{array}$$

in $\text{Alg}_{\mathcal{O}}(pg)$. Since i, d are cofibrations, we know that \mathbf{B}, \mathbf{M} are cofibrant in $\text{Alg}_{\mathcal{O}}(pg)$, and since p is a fibration, we know that \mathbf{X} is fibrant in $\text{Alg}_{\mathcal{O}}(pg)$ (Proposition 4.9). It follows that the weak equivalence $\alpha: \mathbf{M} \rightarrow \mathbf{B}$ induces an isomorphism

$$[\mathbf{M}, \mathbf{X}] \xleftarrow{\cong} [\mathbf{B}, \mathbf{X}]$$

on homotopy classes of maps in $\text{Alg}_{\mathcal{O}}(pg)$, and since the left-hand side is non-empty, it follows that the right-hand side is also non-empty; in other words, there exists a map $[\xi] \in [\mathbf{B}, \mathbf{X}]$. Hence we have verified there exists a map of the form $\xi: \mathbf{B} \rightarrow \mathbf{X}$ in $\text{Alg}_{\mathcal{O}}(pg)$; in other words, we have shown that the desired lift ξ exists. This completes the proof of the implication $(b) \Rightarrow (a)$. \square

Proposition 4.11 (Detecting TQ^A -local \mathcal{O} -algebras: Part 2). *Let X be a fibrant \mathcal{O} -algebra. Then X is TQ^A -local if and only if $X \rightarrow *$ satisfies the right lifting property with respect to every subcell \mathcal{O} -algebra inclusion $A \subset B$ that is a TQ^A -equivalence.*

Proof. This follows immediately from Proposition 4.10. \square

5. CONSTRUCTING THE TQ^A -LOCALIZATION FUNCTOR

The purpose of this section is to establish versions of Propositions 4.10 and 4.11 that include a bound on how many cells B has. Once this is accomplished, we can run the small object argument to construct the TQ^A -localization functor on \mathcal{O} -algebras and the associated TQ^A -local homotopy theory. Our argument can be thought of as an \mathcal{O} -algebra analog of the bounded cofibration property in Bousfield [5, 11.2], Goerss-Jardine [18, X.2.13], and Jardine [24, 5.2], mixed together with the subcell inclusion ideas in Hirschhorn [22, 2.3.7].

Proposition 5.1. *Let $i: A \rightarrow B$ be a strong cofibration and consider the pushout diagram of the form*

$$(9) \quad \begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow & & \downarrow \\ * & \longrightarrow & B//A \end{array}$$

in $\text{Alg}_{\mathcal{O}}$. Then there is an associated cofibration sequence of the form

$$\text{TQ}^A(A) \rightarrow \text{TQ}^A(B) \rightarrow \text{TQ}^A(B//A)$$

in $\text{Mod}_{\mathcal{A}}$ and corresponding long exact sequence of abelian groups of the form

$$(10) \quad \dots \text{TQ}_{s+1}^A(B//A) \rightarrow \text{TQ}_s^A(A) \rightarrow \text{TQ}_s^A(B) \rightarrow \text{TQ}_s^A(B//A) \rightarrow \text{TQ}_{s-1}^A(A) \rightarrow \dots$$

where $\text{TQ}_s^A(X) := \pi_s \text{TQ}^A(X)$ denotes the s -th TQ^A -homology group of an \mathcal{O} -algebra X and π_* denotes the derived (or true) homotopy groups of a symmetric spectrum [32, 33].

Proof. This is because Q is a left Quillen functor and hence preserves cofibrations and pushout diagrams. \square

Definition 5.2. Let κ be a large enough (infinite) regular cardinal such that

$$\kappa > \left| \bigoplus_{s,m,k} \bigoplus_H \text{TQ}_s^A(\mathcal{O} \circ (D_m^{H,k}/S_m^{H,k})) \right|$$

where the first direct sum is indexed over all $s \in \mathbb{Z}$, $m \geq 1$, $k \geq 0$ and the second direct sum is indexed over all subgroups $H \subset \Sigma_m$.

Remark 5.3. The significance of this choice of regular cardinal κ arises from the cofiber sequence of the form

$$\text{TQ}^A(Z_n) \rightarrow \text{TQ}^A(Z_{n+1}) \rightarrow \coprod_{i \in I_n} \text{TQ}^A(\mathcal{O} \circ (D_{m_i}^{H_i, k_i}/S_{m_i}^{H_i, k_i}))$$

in $\text{Mod}_{\mathcal{A}}$ associated to the pushout diagram (7).

Proposition 5.4. Let Z be a cell \mathcal{O} -algebra with less than κ cells (with respect to a choice of presentation $* \rightarrow Z$). Then

$$\left| \bigoplus_s \text{TQ}_s^A(Z) \right| < \kappa$$

where the direct sum is indexed over all $s \in \mathbb{Z}$.

Proof. Using the presentation notation in (7) (taking $W = *$), this follows from Remark 5.3, together with Proposition 5.1, by induction on n . In more detail: Since $Z_0 = *$ we know that $\left| \bigoplus_s \text{TQ}_s^A(Z_0) \right| < \kappa$. Let $n \geq 0$ and assume that

$$(11) \quad \left| \bigoplus_s \text{TQ}_s^A(Z_n) \right| < \kappa$$

We want to show that $\left| \bigoplus_s \text{TQ}_s^A(Z_{n+1}) \right| < \kappa$. Consider the long exact sequence in TQ^A -homology groups of the form

$$(12) \quad \dots \rightarrow \text{TQ}_s^A(Z_n) \rightarrow \text{TQ}_s^A(Z_{n+1}) \rightarrow \bigoplus_{i \in I_n} \text{TQ}_s^A(\mathcal{O} \circ (D_{m_i}^{H_i, k_i}/S_{m_i}^{H_i, k_i})) \rightarrow \dots$$

associated to the cofiber sequence in Remark 5.3. It follows easily that

$$\left| \text{TQ}_s^A(Z_{n+1}) \right| \leq \left| \text{TQ}_s^A(Z_n) \oplus \bigoplus_{i \in I_n} \text{TQ}_s^A(\mathcal{O} \circ (D_{m_i}^{H_i, k_i}/S_{m_i}^{H_i, k_i})) \right| < \kappa$$

and hence $\left| \bigoplus_s \text{TQ}_s^A(Z_{n+1}) \right| < \kappa$. Hence we have verified, by induction on n , that (11) is true for every $n \geq 0$; noting that $Z \cong Z_\infty = \text{colim}_n Z_n$ (by definition) completes the proof. \square

Proposition 5.5 (Bounded subcell property). Let M be a cell \mathcal{O} -algebra and $L \subset M$ a subcell \mathcal{O} -algebra. If $L \neq M$ and $L \subset M$ is a TQ^A -equivalence, then there exists $A \subset M$ subcell \mathcal{O} -algebra such that

- (i) A has less than κ cells
- (ii) $A \not\subset L$
- (iii) $L \subset L \cup A$ is a TQ^A -equivalence

Proof. The main idea is to develop a TQ^A -homology analog for \mathcal{O} -algebras of the closely related argument in Bousfield's localization of spaces work [5]; we have benefitted from the subsequent elaboration in Goerss-Jardine [18, X.3]. We are effectively replacing arguments in terms of adding on non-degenerate simplices with arguments in terms of adding on subcell \mathcal{O} -algebras; this idea to work with cellular structures appears in Hirschhorn [22] assuming left properness; however, the techniques can be made to work without the left properness assumption as indicated below.

To start, choose any $A_0 \subset M$ subcell \mathcal{O} -algebra such that

- (i) A_0 has less than κ cells
- (ii) $A_0 \not\subset L$

Here is the main idea, which is essentially a small object argument idea: We would like $L \subset L \cup A_0$ to be a TQ^A -equivalence (i.e., we would like $\mathrm{TQ}_*^A(L \cup A_0 // L) = 0$), but it might not be. So we do the next best thing. We build $A_1 \supset A_0$ such that when we consider the following pushout diagrams in $\mathbf{Alg}_{\mathcal{O}}$

$$\begin{array}{ccccc} L & \longrightarrow & L \cup A_0 & \longrightarrow & L \cup A_1 \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & L \cup A_0 // L & \xrightarrow{(\#)} & L \cup A_1 // L \end{array}$$

which are also homotopy pushout diagrams in $\mathbf{Alg}_{\mathcal{O}}$, the map $(\#)$ induces

$$(13) \quad \mathrm{TQ}_*^A(L \cup A_0 // L) \rightarrow \mathrm{TQ}_*^A(L \cup A_1 // L)$$

the zero map; in other words, we construct A_1 by killing off elements in the TQ^A -homology groups $\mathrm{TQ}_*^A(L \cup A_0 // L)$ by attaching subcell \mathcal{O} -algebras to A_0 , but in a controlled manner. Since $L \cup A_0 \subset M$ is a subcell \mathcal{O} -algebra, it follows that M is weakly equivalent to the filtered homotopy colimit

$$M \cong \operatorname{colim}_{F_i \subset M} (L \cup A_0 \cup F_i) \simeq \operatorname{hocolim}_{F_i \subset M} (L \cup A_0 \cup F_i)$$

indexed over all finite $F_i \subset M$ subcell \mathcal{O} -algebras and hence

$$0 = \mathrm{TQ}_*^A(M // L) \cong \operatorname{colim}_{F_i \subset M} \mathrm{TQ}_*^A(L \cup A_0 \cup F_i // L)$$

where the left-hand side is trivial by assumption. Hence for each $0 \neq x \in \mathrm{TQ}_*^A(L \cup A_0 // L)$ there exists a finite $F_x \subset M$ subcell \mathcal{O} -algebra such that the induced map

$$\mathrm{TQ}_*^A(L \cup A_0 // L) \rightarrow \mathrm{TQ}_*^A(L \cup A_0 \cup F_x // L)$$

sends x to zero. Define $A_1 := (A_0 \cup \cup_{x \neq 0} F_x) \subset M$ subcell \mathcal{O} -algebra. By construction the induced map (13) on TQ^A -homology groups is the zero map. Furthermore, the pushout diagram in $\mathbf{Alg}_{\mathcal{O}}$

$$\begin{array}{ccc} L \cap A_0 & \longrightarrow & L \\ \downarrow & & \downarrow \\ A_0 & \longrightarrow & L \cup A_0 \end{array}$$

implies that $L \cup A_0 // L \cong A_0 // L \cap A_0$, hence from the cofiber sequence of the form

$$L \cap A_0 \rightarrow A_0 \rightarrow L \cup A_0 // L$$

in $\text{Alg}_{\mathcal{O}}$ and its associated long exact sequence in TQ_*^A it follows that $A_1 \subset M$ subcell \mathcal{O} -algebra satisfies

- (i) A_1 has less than κ cells
- (ii) $A_1 \not\subset L$

Now we repeat the main idea above, but replacing A_0 with A_1 : We would like $L \subset L \cup A_1$ to be a TQ_*^A -equivalence (i.e., we would like $\text{TQ}_*^A(L \cup A_1 // L) = 0$), but it might not be. So we do the next best thing. We build $A_2 \supset A_1$ such that the induced map $\text{TQ}_*^A(L \cup A_1 // L) \rightarrow \text{TQ}_*^A(L \cup A_2 // L)$ is zero by attaching subcell \mathcal{O} -algebras to A_1 , but in a controlled manner, \dots , and so on: By induction we construct, exactly as above, a sequence of subcell \mathcal{O} -algebras

$$(14) \quad A_0 \subset A_1 \subset \dots \subset A_n \subset A_{n+1} \subset \dots$$

satisfying ($n \geq 0$)

- (i) A_n has less than κ cells
- (ii) $A_n \not\subset L$
- (iii) $\text{TQ}_*^A(L \cup A_n // L) \rightarrow \text{TQ}_*^A(L \cup A_{n+1} // L)$ is the zero map

Define $A := \cup_n A_n$. Let's verify that $L \subset L \cup A$ is a TQ_*^A -equivalence; this is the same as checking that $\text{TQ}_*^A(L \cup A // L) = 0$. Since (14) is a sequence of subcell \mathcal{O} -algebras, it follows that $L \cup A$ is weakly equivalent to the filtered homotopy colimit

$$L \cup A \cong \text{colim}_n (L \cup A_n) \simeq \text{hocolim}_n (L \cup A_n)$$

and hence

$$\text{TQ}_*^A(L \cup A // L) \cong \text{colim}_n \text{TQ}_*^A(L \cup A_n // L)$$

In particular, each $x \in \text{TQ}_*^A(L \cup A // L)$ is represented by an element in $\text{TQ}_*^A(L \cup A_n // L)$ for some n , and hence it is in the image of the composite map

$$\text{TQ}_*^A(L \cup A_n // L) \rightarrow \text{TQ}_*^A(L \cup A_{n+1} // L) \rightarrow \text{TQ}_*^A(L \cup A // L)$$

Since the left-hand map is the zero map by construction, this verifies that $x = 0$. Hence we have verified $L \subset L \cup A$ is a TQ_*^A -equivalence, which completes the proof. \square

The following is closely related to [5, 11.3], [18, X.2.14], and [24, 5.4], together with the subcell ideas in [22, 2.3.8].

Proposition 5.6 (Bounded subcell lifting property). *Let $p: X \rightarrow Y$ be a fibration of \mathcal{O} -algebras. Then the following are equivalent:*

- (a) *the map p has the right lifting property with respect to every strong cofibration $A \rightarrow B$ of \mathcal{O} -algebras that is a TQ_*^A -equivalence.*
- (b) *the map p has the right lifting property with respect to every subcell \mathcal{O} -algebra inclusion $A \subset B$ that is a TQ_*^A -equivalence and such that B has less than κ cells.*

Proof. The implication (a) \Rightarrow (b) is immediate. Conversely, suppose p has the right lifting property with respect to every subcell \mathcal{O} -algebra inclusion $A \subset B$ that is a TQ^A -equivalence and such that B has less than κ cells. We want to verify that p satisfies the lifting conditions in (a); by the subcell lifting property, it suffices to verify that p satisfies the lifting conditions in Proposition 4.10(b). Let $A \subset B$ be a subcell \mathcal{O} -algebra inclusion that is a TQ^A -equivalence and consider any left-hand solid commutative diagram of the form

$$(15) \quad \begin{array}{ccc} A & \xrightarrow{g} & X \\ \downarrow \subset & \nearrow \xi & \downarrow p \\ B & \xrightarrow{h} & Y \end{array} \quad \begin{array}{ccc} A & \xrightarrow{g} & X \\ \downarrow \subset & \nearrow \xi_s & \downarrow p \\ A_s & \xrightarrow{\subset} B & \xrightarrow{h} Y \end{array}$$

in $\mathrm{Alg}_{\mathcal{O}}$. We want to verify that a lift ξ exists. The idea is to use a Zorn's lemma argument on an appropriate poset Ω of partial lifts, together with Proposition 5.5, following closely [18, X.2.14] and [22, 2.3.8]. Denote by Ω the poset of all pairs (A_s, ξ_s) such that (i) $A_s \subset B$ is a subcell \mathcal{O} -algebra inclusion that is a TQ^A -equivalence and (ii) $\xi_s: A_s \rightarrow X$ is a map in $\mathrm{Alg}_{\mathcal{O}}$ that makes the right-hand diagram in (15) commute (i.e., $\xi_s|_A = g$ and $p\xi_s = h|_{A_s}$), where Ω is ordered by the following relation: $(A_s, \xi_s) \leq (A_t, \xi_t)$ if $A_s \subset A_t$ is a subcell \mathcal{O} -algebra inclusion and $\xi_t|_{A_s} = \xi_s$. Then by Zorn's lemma, this set Ω has a maximal element (A_m, ξ_m) .

We want to show that $A_m = B$. Suppose not. Then $A_m \neq B$ and $A_m \subset B$ is a TQ^A -equivalence, hence by the bounded subcell property (Proposition 5.5) there exists $K \subset B$ subcell \mathcal{O} -algebra such that

- (i) K has less than κ cells
- (ii) $K \not\subset A_m$
- (iii) $A_m \subset A_m \cup K$ is a TQ^A -equivalence

We have a pushout diagram of the left-hand form

$$\begin{array}{ccc} A_m \cap K & \longrightarrow & A_m \\ \downarrow & & \downarrow \\ K & \longrightarrow & A_m \cup K \end{array} \quad \begin{array}{ccc} A_m \cap K & \longrightarrow & A_m \xrightarrow{\xi_m} X \\ \downarrow & \nearrow \xi & \downarrow p \\ K & \longrightarrow & B \xrightarrow{h} Y \end{array}$$

in $\mathrm{Alg}_{\mathcal{O}}$ where the indicated maps are inclusions, and by assumption on p , the right-hand solid commutative diagram in $\mathrm{Alg}_{\mathcal{O}}$ has a lift ξ . It follows that the induced map $\xi_m \cup \xi$ makes the following diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & X \\ \downarrow & \nearrow \xi_m & \downarrow p \\ A_m & \longrightarrow & A_m \cup K \longrightarrow B \xrightarrow{h} Y \end{array}$$

in $\mathrm{Alg}_{\mathcal{O}}$ commute, where the unlabeled arrows are the natural inclusions. In particular, since $K \not\subset A_m$, then $A_m \neq A_m \cup K$, and hence we have constructed an element $(A_m \cup K, \xi_m \cup \xi)$ of the set Ω that is strictly greater than the maximal element (A_m, ξ_m) , which is a contradiction. Therefore $A_m = B$ and the desired lift $\xi = \xi_m$ exists, which completes the proof. \square

Proposition 5.7 (Detecting TQ^A -local \mathcal{O} -algebras: Part 3). *Let X be a fibrant \mathcal{O} -algebra. Then X is TQ^A -local if and only if $X \rightarrow *$ satisfies the right lifting property with respect to every subcell \mathcal{O} -algebra inclusion $A \subset B$ that is a TQ^A -equivalence and such that B has less than κ cells.*

Proof. This follows immediately from Proposition 5.6. \square

Proposition 5.8. *If f is a retract of g and g is a TQ^A -acyclic strong cofibration, then so is f .*

Proof. This is because strong cofibrations and weak equivalences are closed under retracts and Q is a left Quillen functor. \square

Proposition 5.9. *Consider any pushout diagram of the form*

$$(16) \quad \begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & & \downarrow j \\ B & \longrightarrow & Y \end{array}$$

in $\mathrm{Alg}_{\mathcal{O}}$. If X is cofibrant and i is a TQ^A -acyclic strong cofibration, then j is a TQ^A -acyclic strong cofibration.

Proof. Applying Q to the diagram (16) gives a pushout diagram of the form

$$\begin{array}{ccc} Q(A) & \longrightarrow & Q(X) \\ (*) \downarrow & & \downarrow (**) \\ Q(B) & \longrightarrow & Q(Y) \end{array}$$

in $\mathrm{Alg}_{\mathcal{O}}$. Since $(*)$ is an acyclic cofibration by assumption, it follows that $(**)$ is an acyclic cofibration, which completes the proof. \square

Proposition 5.10. *The class of TQ -acyclic strong cofibrations is (i) closed under all small coproducts and (ii) closed under all (possibly transfinite) compositions.*

Proof. Part (i) is because strong cofibrations are closed under all small coproducts and Q is a left Quillen functor, and part (ii) is because strong cofibrations are closed under all (possibly transfinite) compositions and Q is a left Quillen functor. \square

Definition 5.11. Denote by I_{TQ^A} the set of generating cofibrations in $\mathrm{Alg}_{\mathcal{O}}$ and by J_{TQ^A} the set of generating acyclic cofibrations in $\mathrm{Alg}_{\mathcal{O}}$ union the set of TQ^A -acyclic strong cofibrations consisting of one representative of each isomorphism class of subcell \mathcal{O} -algebra inclusions $A \subset B$ that are TQ^A -equivalences and such that B has less than κ cells.

Theorem 5.12. *Any map $X \rightarrow Y$ of \mathcal{O} -algebras with X cofibrant can be factored as $X \rightarrow X' \rightarrow Y$ a TQ^A -acyclic strong cofibration followed by a weak TQ^A -fibration.*

Proof. We know by [22, 12.4] that the set J_{TQ^A} permits the small object argument [22, 10.5.15], and running the small object argument for the map $X \rightarrow Y$ with respect to J_{TQ^A} produces a functorial factorization of the form

$$X \xrightarrow{j} X' \xrightarrow{p} Y$$

in $\mathbf{Alg}_{\mathcal{O}}$. We know that j is a \mathbf{TQ}^A -acyclic strong cofibration by Propositions 5.9 and 5.10. Since $J_{\mathbf{TQ}^A}$ contains the set of generating acyclic cofibrations for $\mathbf{Alg}_{\mathcal{O}}$, we know that p is a fibration of \mathcal{O} -algebras, and hence it follows from Proposition 5.6 that p is a weak \mathbf{TQ}^A -fibration, which completes the proof. \square

Let X be an \mathcal{O} -algebra and run the small object argument with respect to the set $I_{\mathbf{TQ}^A}$ for the map $* \rightarrow X$; this gives a functorial factorization in $\mathbf{Alg}_{\mathcal{O}}$ as a cofibration followed by an acyclic fibration $* \rightarrow \tilde{X} \xrightarrow{\simeq} X$; in particular, \tilde{X} is cofibrant. Now run the small object argument with respect to the set $J_{\mathbf{TQ}^A}$ for the map $\tilde{X} \rightarrow *$; this gives a functorial factorization in $\mathbf{Alg}_{\mathcal{O}}$ as $\tilde{X} \rightarrow L(\tilde{X}) \rightarrow * \rightarrow X$ a \mathbf{TQ} -acyclic strong cofibration followed by a weak \mathbf{TQ} -fibration; in particular, $L(\tilde{X})$ is \mathbf{TQ} -local and the natural zigzag $X \simeq \tilde{X} \rightarrow L(\tilde{X})$ is a \mathbf{TQ} -equivalence. Hence we have verified the following theorem.

Theorem 5.13. *If X is an \mathcal{O} -algebra, then (i) there is a natural zigzag of \mathbf{TQ} -equivalences of the form $X \simeq \tilde{X} \rightarrow L_{\mathbf{TQ}^A}(\tilde{X})$ with \mathbf{TQ} -local codomain, and if furthermore X is cofibrant, then (ii) there is a natural \mathbf{TQ} -equivalence of the form $X \rightarrow L_{\mathbf{TQ}^A}(X)$ with \mathbf{TQ} -local codomain.*

Proof. Taking $L_{\mathbf{TQ}^A}(\tilde{X}) := L(\tilde{X})$ for part (i) and $L_{\mathbf{TQ}^A}(X) := L(X)$ for part (ii) completes the proof. \square

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