Exercise 1. Prove Proposition 1.

Proposition 1. Let $C$ be a model category.

(a) The cofibrations in $C$ are the maps which have the left lifting property with respect to acyclic fibrations.
(b) The acyclic cofibrations in $C$ are the maps which have the left lifting property with respect to fibrations.
(c) The fibrations in $C$ are the maps which have the right lifting property with respect to acyclic cofibrations.
(d) The acyclic fibrations in $C$ are the maps which have the right lifting property with respect to cofibrations.


Proposition 2. Let $C$ be a model category.

(a) The class of cofibrations in $C$ is closed under pushouts.
(b) The class of acyclic cofibrations in $C$ is closed under pushouts.
(c) The class of fibrations in $C$ is closed under pullbacks.
(d) The class of acyclic fibrations in $C$ is closed under pullbacks.

If $C$ is a category and $D$ is a small category, denote by $C^D$ the diagram category with objects the diagrams $X : D \rightarrow C$ and morphisms their natural transformations. If the colimit $\text{colim} X$ exists for every diagram $X \in C^D$, then it follows easily—from the universal property of colimits—that the objects $\text{colim} X$ give a well-defined functor $\text{colim} : C^D \rightarrow C$.


Proposition 3. Let $C$ be a model category and let $f : X \rightarrow Y$ be a map in $C^D$.

(a) If $f_a, f_b$ are acyclic cofibrations, $f_c$ is a weak equivalence, and the induced map $Y_b \amalg_{X_b} X_c \rightarrow Y_c$ is a cofibration, then $\text{colim} f$ is an acyclic cofibration.
(b) If $f_a$ and the induced map $Y_b \amalg_{X_b} X_c \rightarrow Y_c$ are cofibrations, then $\text{colim} f$ is a cofibration.

Exercise 4. Use duality in model categories to obtain a corresponding proposition for the limit functor $\text{lim} : C^{D^\text{op}} \rightarrow C$. Note that $D^\text{op}$ is the category $\{a \rightarrow b \leftarrow c\}$. 

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Let $C$ be a category and let $D$ be the category $\{0 \to 1 \to 2 \to 3 \to \cdots\}$ with objects the non-negative integers and a single morphism $i \to j$ for each $i \leq j$. Then a morphism $f: X \to Y$ in $C^D$ is a collection of maps $f_0, f_1, f_2, f_3, \ldots$ which makes the left-hand diagram

\[ \begin{array}{cccccc}
X_0 & \rightarrow & X_1 & \rightarrow & X_2 & \rightarrow & \cdots \\
\downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow \cdots \\
Y_0 & \rightarrow & Y_1 & \rightarrow & Y_2 & \rightarrow & \cdots \\
\end{array} \]

in $C$ commute, and colim $f$ is the induced map $\text{colim } X \rightarrow \text{colim } Y$ on colimits. The following proposition explores some homotopical properties of the colimit functor $\text{colim}: C^D \to C$. We will establish additional homotopical properties after developing certain results on left (and right) homotopic maps.

**Proposition 4.** Let $C$ be a model category and let $f: X \to Y$ be a map in $C^D$. Assume that $C$ has all small colimits.

(a) If $f_0$ and each of the induced maps $Y_i \amalg X_i, X_{i+1} \rightarrow Y_{i+1}$ ($i \geq 0$) is an acyclic cofibration, then colim $f$ is an acyclic cofibration.

(b) If $f_0$ and each of the induced maps $Y_i \amalg X_i, X_{i+1} \rightarrow Y_{i+1}$ ($i \geq 0$) is a cofibration, then colim $f$ is a cofibration.

Exercise 6. Use duality in model categories to obtain a corresponding proposition for the limit functor $\text{lim}: C^{D^{op}} \to C$. Note that $D^{op} = \{0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots\}$ is the category with objects the non-negative integers and a single morphism $i \leftarrow j$ for each $i \leq j$.

Exercise 7. Prove Proposition 5

**Proposition 5.** Let $C$ be a model category.

(a) The class of cofibrations contains all isomorphisms.

(b) The class of fibrations contains all isomorphisms.

(c) The class of weak equivalences contains all isomorphisms.


**Proposition 6.** Let $C$ be a model category. Assume that $C$ has all small coproducts and products.

(a) The class of cofibrations in $C$ is closed under coproducts.

(b) The class of acyclic cofibrations in $C$ is closed under coproducts.

(c) The class of fibrations in $C$ is closed under products.

(d) The class of acyclic fibrations in $C$ is closed under products.

In the exercises and propositions which follow, we are working in a fixed model category $C$; references for this material include [1, Section 4] and [2, Chapter 7]. Recall the following proposition.

**Proposition 7.** If $A$ is cofibrant and $A \wedge I$ is a good cylinder object for $A$, then the maps $i_0, i_1: A \rightarrow A \wedge I$ are acyclic cofibrations.

Exercise 9. Use duality in model categories to obtain a corresponding proposition involving path objects.

Proposition 8. If \( f \sim g: A \to X \), then there exists a good left homotopy from \( f \) to \( g \). If in addition \( X \) is fibrant, then there exists a very good left homotopy from \( f \) to \( g \).

Exercise 11. Use duality in model categories to obtain a corresponding proposition involving right homotopies.


Proposition 9. If \( A \) is cofibrant, then \( \sim \) is an equivalence relation on \( \text{hom}_C(A, X) \).

Exercise 13. Use duality in model categories to obtain a corresponding proposition involving right homotopy.


Proposition 10. If \( A \) is cofibrant and \( p: Y \to X \) is an acyclic fibration, then composition with \( p \) induces a bijection:
\[
p_*: \pi_l(A, Y) \xrightarrow{\sim} \pi_l(A, X), \quad [A \xrightarrow{f} Y] \mapsto [A \xrightarrow{f} Y \xrightarrow{p} X].
\]

Exercise 15. Use duality in model categories to obtain a corresponding proposition involving right homotopy classes of maps.


Proposition 11. Suppose that \( X \) is fibrant, that \( f \sim g: A \to X \), and that \( h: A' \to A \) is a map. Then \( fh \sim gh \).

Exercise 17. Use duality in model categories to obtain a corresponding proposition involving right homotopy.

Exercise 18. Prove Proposition 12.

Proposition 12. If \( X \) is fibrant, then the composition in \( C \) induces a map:
\[
\pi_l(A', A) \times \pi_l(A, X) \to \pi_l(A', X), \quad ([A' \xrightarrow{h} A], [A \xrightarrow{f} X]) \mapsto [A' \xrightarrow{h} A \xrightarrow{f} X].
\]

Exercise 19. Use duality in model categories to obtain a corresponding proposition involving right homotopy classes of maps.


Proposition 13. Let \( f, g: A \to X \) be maps.
\begin{itemize}
  \item[(a)] If \( A \) is cofibrant and \( f \sim g \), then \( f \sim g \).
  \item[(b)] If \( X \) is fibrant and \( f \sim g \), then \( f \sim g \).
\end{itemize}


The following proposition is a key result of this section.

Proposition 14. Suppose that \( f: A \to X \) is a map between objects \( A \) and \( X \) which are both fibrant and cofibrant. Then \( f \) is a weak equivalence if and only if \( f \) has a homotopy inverse; i.e., if and only if there exists a map \( g: X \to A \) such that \( gf \sim \text{id} \) and \( fg \sim \text{id} \).

Exercise 22. Please read [1, Sections 3-4].
References
