In the exercises and propositions below, we are working in a fixed model category $C$. References for the following include [1, Sections 5-6] and [2, Chapter 7].

**Exercise 1.** Prove Proposition 1.

**Proposition 1.** Given a map $f : X \rightarrow Y$ in $C$, there exists a map $Qf$ which makes the diagram

\[
\begin{array}{ccc}
QX & \xrightarrow{Qf} & QY \\
p_X & \sim & p_Y \\
X & \xrightarrow{f} & Y
\end{array}
\]

commute. Furthermore:

(a) The map $Qf$ is a weak equivalence if and only if the map $f$ is a weak equivalence.

(b) If a map $\tilde{f} : QX \rightarrow QY$ also makes the diagram commute (i.e., $py\tilde{f} = fp_X$), then $\tilde{f} \sim Qf$ and $\tilde{f} \sim Qf$.

(c) If $Y$ is fibrant and $[f'] = [f]$ in $\pi^l(X,Y)$ for some map $f' : X \rightarrow Y$, then $Qf' \sim Qf$ and $Qf' \sim Qf$.

**Exercise 2.** Use duality in model categories to obtain a corresponding proposition involving existence of a map $Rf$ which makes the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
R_X & \sim & R_Y \\
RX & \xrightarrow{Rf} & RY
\end{array}
\]

commute.

**Exercise 3.** Prove Proposition 2.

**Proposition 2.** The restriction of the functor $Q : C \rightarrow \pi C_c$ to $C_f$ induces a functor $Q' : \pi C_f \rightarrow \pi C_{cf}$. The restriction of the functor $R : C \rightarrow \pi C_f$ to $C_c$ induces a functor $R' : \pi C_c \rightarrow \pi C_{cf}$.

**Exercise 4.** Prove Proposition 3.

**Proposition 3.** If $F, G : \text{Ho}(C) \rightarrow D$ is a pair of functors and $t : F\gamma \rightarrow G\gamma$ is a natural transformation, then $t$ also gives a natural transformation from $F$ to $G$.

**Exercise 5.** Prove Proposition 4.

**Proposition 4.** Let $C$ be a model category and $F : C \rightarrow D$ a functor which sends weak equivalences in $C$ to isomorphisms in $D$. If $f \sim g : A \rightarrow X$ or $f \sim g : A \rightarrow X$, then $F(f) = F(g)$ in $D$.

**Exercise 6.** Prove Proposition 5.
Proposition 5. Suppose that $A$ is a cofibrant object of $C$ and $X$ is a fibrant object of $C$. Then the map $\gamma : \text{hom}_C(A, X) \to \text{hom}_{\text{Ho}(C)}(A, X)$ is surjective, and induces a bijection $\gamma : \pi(A, X) \cong \text{hom}_{\text{Ho}(C)}(A, X)$.


Definition 6. Let $C$ be a category and $W \subseteq C$ a class of morphisms. A localization of $C$ with respect to $W$ is a category $C[W^{-1}]$ with the following mapping properties: (1) there is a functor $\gamma : C \to C[W^{-1}]$ such that $\gamma(f)$ is an isomorphism for each $f \in W$, (2) (universal property) the functor $\gamma$ is initial with respect to all such functors; i.e., for any category $D$ and functor $G : C \to D$ such that $G(f)$ is an isomorphism for each $f \in W$, then there exists a unique functor $\overline{G}$ which makes the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{G} & D \\
\gamma \downarrow & & \exists ! \overline{G} \downarrow \\
C[W^{-1}] & & \\
\end{array}
\]

commute.

Theorem 7. Let $C$ be a model category and $W \subseteq C$ the class of weak equivalences. Then the functor $\gamma : C \to \text{Ho}(C)$ is a localization of $C$ with respect to $W$.

References for the following include [1, Section 9] and [2, Chapter 7].


Proposition 8. Let $C$ be a model category and $F : C_c \to D$ a functor such that $F(f)$ is an isomorphism whenever $f$ is an acyclic cofibration between objects of $C_c$. Suppose that $f, g : A \to B$ are maps in $C_c$ such that $f$ is right homotopic to $g$ in $C$. Then $F(f) = F(g)$.

Exercise 9. Use duality in model categories to obtain a corresponding proposition involving acyclic fibrations between objects of $C_f$.

Exercise 10. Prove Proposition 9

Proposition 9 (K. Brown’s lemma). Let $F : C \to D$ be a functor between model categories. If $F$ sends acyclic cofibrations between cofibrant objects to weak equivalences, then $F$ preserves all weak equivalences between cofibrant objects.

Exercise 11. Use duality in model categories to obtain a corresponding proposition involving acyclic fibrations between fibrant objects.

Exercise 12. Please read [1, Sections 5-6].

References
