Exercise 1. Prove Proposition 1.

Proposition 1. Let $G$ be a finite group, $H \subset G$ a subgroup, and $\{X_\alpha\}$ a set of left $G$-spaces. There are isomorphisms

$$(\amalg X_\alpha)^H \cong \amalg (X_\alpha^H)$$

in $\text{Top}$ natural in $X_\alpha$; i.e., the $H$-fixed points functor $(-)^H : \text{Top}^G \to \text{Top}$ commutes with all small coproducts.


Proposition 2. Let $G$ be a finite group and $H \subset G$ a subgroup. If $f : A \to B$ is a closed injective map in $\text{Top}^G$, then the induced map $f^H : A^H \to B^H$ is a closed injective map in $\text{Top}$; i.e., the $H$-fixed points functor $(-)^H : \text{Top}^G \to \text{Top}$ preserves closed injective maps.

Recall the following proposition.

Proposition 3. Let $X, Y, Z$ be spaces. If $Y$ is Hausdorff and locally compact, then there are isomorphisms

$$\text{hom}_{\text{Top}}(X \times Y, Z) \cong \text{hom}_{\text{Top}}(X, \text{Map}(Y, Z))$$

natural in such $X, Y, Z$.


Proposition 4. Let $G$ be a finite group and $X, Y, Z \in \text{Top}^G$. If $Y$ is Hausdorff and locally compact, then there are isomorphisms

$$\text{hom}_{\text{Top}^G}(X \times Y, Z) \cong \text{hom}_{\text{Top}^G}(X, \text{Map}(Y, Z))$$

natural in such $X, Y, Z$. In particular, the functor $- \times I : \text{Top}^G \to \text{Top}^G$ is a left adjoint and hence preserves colimits. Here, $I := [0, 1]$ with trivial left $G$-action.


Proposition 5. Let $G$ be a finite group, $H \subset G$ a subgroup, and $\{j_\alpha : A_\alpha \to B_\alpha\}$ a set of maps in $\text{Top}^G$. Consider the induced map $\amalg j_\alpha : \amalg A_\alpha \to \amalg B_\alpha$ in $\text{Top}^G$ and the induced map $\amalg (j_\alpha^H) : \amalg (A_\alpha^H) \to \amalg (B_\alpha^H)$ in $\text{Top}$.

(a) If each map $j_\alpha$ is closed injective with image $j_\alpha(A_\alpha)$ a strong deformation retract of $B_\alpha$ in $\text{Top}^G$, then the image of the induced map $\amalg j_\alpha$ is a strong deformation retract of $\amalg B_\alpha$ in $\text{Top}^G$.

(b) If each map $j_\alpha$ is closed injective with image $j_\alpha(A_\alpha)$ a strong deformation retract of $B_\alpha$ in $\text{Top}^G$, then the image of the induced map $\amalg (j_\alpha^H)$ is a strong deformation retract of $\amalg (B_\alpha^H)$ in $\text{Top}$.

Recall the following proposition.

Proposition 6. Let $X$ be a space. If $X$ is Hausdorff, then the diagonal $X \subset X \times X$ is a closed subspace of $X \times X$. 

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**Proposition 7.** Let $G$ be a finite group, $H \subset G$ a subgroup, and $X$ a left $G$-space. If $X$ is Hausdorff, then the $H$-fixed points $X^H \subset X$ is a closed subspace of $X$.


**Proposition 8.** Let $G$ be a finite group and $H \subset G$ a subgroup. Consider any left-hand pushout diagram of the form

\[
\begin{array}{ccc}
A & \xrightarrow{f} & C \\
\downarrow i & & \downarrow j \\
B & \xrightarrow{g} & D \\
\end{array}
\]

\[
\begin{array}{ccc}
A^H & \xrightarrow{f^H} & C^H \\
\downarrow i^H & & \downarrow j^H \\
B^H & \xrightarrow{g^H} & D^H \\
\end{array}
\]

in $\text{Top}^G$ and the corresponding right-hand diagram in $\text{Top}$. If $A, B$ are Hausdorff and $i$ is a closed injective map, then the right-hand diagram is a pushout diagram in $\text{Top}$.

**Definition 9.** Let $G$ be a finite group. The $G$-equivariant model structure on $\text{Top}^G$ is defined by the following three classes of maps: a map $f: X \to Y$ in $\text{Top}^G$ is

(i) a weak equivalence if the map $f^H: X^H \to Y^H$ is a weak equivalence in $\text{Top}$ for each subgroup $H \subset G$,

(ii) a fibration if the map $f^H: X^H \to Y^H$ is a fibration in $\text{Top}$ for each subgroup $H \subset G$,

(iii) a cofibration if it has the left lifting property with respect to all acyclic fibrations.


**Proposition 10.** Let $G$ be a finite group and $H \subset G$ a subgroup. Let $X$ be a non-empty left $G$-space and consider the diagram of the form

\[
\begin{array}{ccc}
X & \xrightarrow{i_1} & G^1 \\
\downarrow i_\infty & & \downarrow i_1 \\
colim_k G^k & \xrightarrow{i_3} & G^3 \\
\end{array}
\]

constructed in the proof of MC5(ii) (factorization axiom) for a map $p: X \to Y$ in $\text{Top}^G$ with the $G$-equivariant model structure.

(a) The map $(i_k)^H: (G^{k-1})^H \to (G^k)^H$ is a weak equivalence in $\text{Top}$ for each $k \geq 1$.

(b) The map $(i_\infty)^H: X^H \to (\colim_k G^k)^H$ is a weak equivalence in $\text{Top}$.

**Definition 11.** Let $G$ be a finite group. A map $f: A \to X$ in $\text{Top}^G$ is a relative $G$-CW inclusion if there is a sequence of maps of the form

\[
A = X^{-1} \xrightarrow{i_0} X^0 \xrightarrow{i_1} X^1 \xrightarrow{i_2} X^2 \to \cdots
\]

in $\text{Top}^G$ such that

(i) $X \cong \colim_n X^n$ under $A$ in $\text{Top}^G$, and
(ii) each map $i_n$ fits into a pushout diagram of the form
\[
\bigsqcup_{H} \bigsqcup_{A_H} G/H \times S^{n-1} \rightarrow X^{n-1}
\]
\[
\bigsqcup_{H} \bigsqcup_{A_H} G/H \times D^n \rightarrow X^n
\]

in $\text{Top}^G$; in other words, $X^n$ is obtained from $X^{n-1}$ by attaching $G$-cells $G/H \times D^n$. Here, the outer coproduct is indexed over all subgroups $H \subset G$, and $A_H$ denotes an indexing set (possibly empty).

A left $G$-space $X$ is a $G$-$CW$ complex if the map $\emptyset \rightarrow X$ is a relative $G$-$CW$ inclusion.

**Exercise 8.** Prove Proposition 12.

**Proposition 12.** Let $G$ be a finite group and consider $\text{Top}^G$ with the $G$-equivariant model structure.

(a) If $f: A \rightarrow X$ in $\text{Top}^G$ is a relative $G$-$CW$ inclusion, then $f$ is a cofibration in $\text{Top}^G$.

(b) If $X$ is a $G$-$CW$ complex, then $X$ is cofibrant in $\text{Top}^G$.

**Definition 13.** Let $G$ be a finite group. A map $f: A \rightarrow X$ in $\text{Top}^G$ is a generalized relative $G$-$CW$ inclusion if there is a sequence of maps of the form
\[
A = G^0 \rightarrow^i G^1 \rightarrow^i G^2 \rightarrow^i G^3 \rightarrow \ldots
\]
in $\text{Top}^G$ such that

(i) $X \cong \text{colim}_k G^k$ under $A$ in $\text{Top}^G$, and

(ii) each map $i_k$ fits into a pushout diagram of the form
\[
\bigsqcup_{n, H} \bigsqcup_{A_{n, H}} G/H \times S^{n-1} \rightarrow G^{k-1}
\]
\[
\bigsqcup_{n, H} \bigsqcup_{A_{n, H}} G/H \times D^n \rightarrow G^k
\]
in $\text{Top}^G$; in other words, $G^k$ is obtained from $G^{k-1}$ by attaching $G$-cells $G/H \times D^n$. Here, the outer coproduct is indexed over all $n \geq 0$ and subgroups $H \subset G$, and $A_{n, H}$ denotes an indexing set (possibly empty).

A left $G$-space $X$ is a generalized $G$-$CW$ complex if the map $\emptyset \rightarrow X$ is a generalized relative $G$-$CW$ inclusion.

**Exercise 9.** Prove Proposition 14.

**Proposition 14.** Let $G$ be a finite group and consider $\text{Top}^G$ with the $G$-equivariant model structure.

(a) Every left $G$-space is fibrant in $\text{Top}^G$.

(b) A map $f: A \rightarrow X$ in $\text{Top}^G$ is a cofibration if and only if $f$ is a retract of a generalized relative $G$-$CW$ inclusion in $\text{Top}^G$.

(c) A left $G$-space $X$ is cofibrant in $\text{Top}^G$ if and only if $X$ is a retract of a generalized $G$-$CW$ complex in $\text{Top}^G$.  


Proposition 15. Let $C$ be a category with all small limits and colimits. Let $D$ be a small category. There are adjunctions

$$
\begin{array}{c}
C & \xrightarrow{\Delta} & \text{colim}_D \\
\text{lim}_D & \xleftarrow{\Delta} & C
\end{array}
$$

with left adjoints on top. Here, $\Delta$ is the “diagonal” functor with $\Delta(X) \in \text{C}^D$ the constant diagram with value $X$.

Exercise 11. Prove Proposition 16.

Proposition 16. Let $C$ be a category with all small limits and colimits. Let $I, J$ small categories.

(a) The diagram category $C^J$ has all small limits and colimits, and they are calculated objectwise.

(b) There are natural isomorphisms of diagram categories

$$(C^I)^J \cong C^{I \times J} \cong (C^J)^I.$$  

(c) If $X \in C^{I \times J}$, there are natural isomorphisms

$$\text{colim}_J(\text{colim}_I X) \cong \text{colim}_{I \times J} X \cong \text{colim}(\text{colim}_J X).$$

(d) If $X \in C^{I \times J}$, there are natural isomorphisms

$$\text{lim}_I(\text{lim}_J X) \cong \text{lim}_{I \times J} X \cong \text{lim}(\text{lim}_J X).$$


Proposition 17. Let $J$ be a small category. Every left-hand adjunction

$$
\begin{array}{c}
C & \xrightarrow{F} & D \\
G & \xleftarrow{\_} & \text{C}^J
\end{array}
$$

with left adjoint on top, induces the right-hand adjunction on diagram categories with left adjoint on top. Here, the induced functors $F$ and $G$ are defined objectwise; i.e., $F(X)(j) := F(X(j))$ and $G(Y)(j) := G(Y(j))$ for each $X \in C^J$ and $Y \in D^J$.


Proposition 18. Consider any left-hand pair of adjunctions of the form

$$
\begin{array}{c}
C & \xrightarrow{F} & D \\
G & \xleftarrow{\_} & \text{C}^J
\end{array}
\quad \quad
\begin{array}{c}
D & \xrightarrow{F'} & E \\
G' & \xleftarrow{\_} & \text{C}^J
\end{array}
$$

with left adjoints on top. Then the right-hand pair of composite functors is an adjunction with left adjoint on top.


Proposition 19. Consider any adjunctions of the form

$$
\begin{array}{c}
C & \xrightarrow{F} & D \\
G & \xleftarrow{\_} & \text{C}^J \\
\end{array}
\quad \quad
\begin{array}{c}
C & \xrightarrow{F'} & D \\
G' & \xleftarrow{\_} & \text{C}^J
\end{array}
$$

with left adjoints on top. Then there are isomorphisms of functors $F' \cong F$ and $G' \cong G$; in other words, left adjoints (resp. right adjoints) are unique up to isomorphism.